Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(6): 17382–17398.
DOI: 10.3934/math.2026711
Received: 10 March 2026
Revised: 21 May 2026
Accepted: 08 June 2026
Published: 15 June 2026

Research article

On the five-parameter Mittag-Leffler matrix function and its properties

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Abstract: In this paper, we introduce and study a matrix analog of the five-parameter Mittag-Leffler function. We establish the absolute convergence of the series defining this function on the unit circle $|\varpi| = 1$ under certain spectral conditions. Several fundamental properties are derived, including integral representations, derivative formulas, and differential recurrence relations. Furthermore, we obtain a variety of finite summation formulas for this matrix function and its related Fox-Wright matrix analog. Finally, we investigate the composition of the function with generalized fractional calculus operators introduced by Katugampola, deriving closed-form expressions for both fractional integrals and derivatives. The results presented here extend the theory of special matrix functions and contribute to the field of fractional calculus.

Keywords: Mittag-Leffler functions; matrix functional calculus; generalized fractional calculus
Mathematics Subject Classification: 15A15, 26A33, 33C15, 33E12

1. Introduction

In the last 20 years, special matrix functions have become a major focus in mathematical analysis due to their critical applications in the study of Lie algebras, Lie groups, and mathematical physics. In both pure and applied mathematics, hypergeometric functions of a matrix argument have many uses, including in statistics and random matrix theory [1–4].

Numerous researchers have contributed to the study of matrix analogs of special functions. Notably, Abdalla [5,6], Dwivedi and Sahai [7,8], Jódar and Cortés [9–11], Jatav et al. [12], Jatav and Shukla [13, 14], and Pal et al. [15] have investigated a variety of their properties. These include differential and integral representations, finite summation formulas, and generating functions. Such matrix functions are instrumental in solving a wide array of problems across the mathematical sciences, engineering, probability theory, and statistics. Matrix calculus has also seen many developments in solving problems involving large, dispersed matrices, including the following: Extended global Arnoldi algorithms for solving large-scale linear systems of ordinary differential equations [16], the extended block Arnoldi method for solving generalized differential Sylvester equations [17], and the Krylov method for large-scale systems of differential equations [18].

Throughout this paper, we work in the space $C^{l\times l}$ of square complex matrices of order l. For a given matrix $\mathcal{V} \in C^{l\times l}$, we denote its spectrum (the set of all eigenvalues) by $\sigma(\mathcal{V})$. We define the quantities

$$\mathsf{d}(\mathcal{V}) = \max\left\{\Re(\varpi) : \varpi \in \sigma(\mathcal{V})\right\}, \quad \mathsf{c}(\mathcal{V}) = \min\left\{\Re(\varpi) : \varpi \in \sigma(\mathcal{V})\right\}, \tag{1.1}$$

where $\mathsf{d}(\mathcal{V})$ is the spectral abscissa of $\mathcal{V}$, and it follows that $\mathsf{c}(\mathcal{V}) = -\mathsf{d}(-\mathcal{V})$.

A Hermitian matrix $\mathcal{V} \in C^{l\times l}$ is said to be *positive stable* if every eigenvalue $\lambda \in \sigma(\mathcal{V})$ satisfies $\Re(\lambda) > 0$.

The Euclidean norm of a vector $\mathsf{x} \in C^l$ is defined as $\|\mathsf{x}\|_2 = (\mathsf{x}^*\mathsf{x})^{1/2}$, where x^* denotes the conjugate transpose. The induced two-norm of a matrix $\mathcal{V}$ is then given by

$$\|\mathcal{V}\| = \sup_{\mathsf{x}\neq 0} \frac{\|\mathcal{V}\mathsf{x}\|_2}{\|\mathsf{x}\|_2} = \max\left\{\sqrt{\lambda} : \lambda \in \sigma(\mathcal{V}^*\mathcal{V})\right\}. \tag{1.2}$$

In what follows, $\mathcal{I}$ and O denote the identity matrix and the null matrix in $C^{l\times l}$, respectively.

Using the Schur decomposition of a matrix $\mathcal{V}$ [19], we recall the following estimate for the matrix exponential:

$$\left\|e^{\mathcal{V}t}\right\| \leq e^{t\mathsf{d}(\mathcal{V})} \sum_{\vartheta=0}^{\mathsf{r}-1} \frac{\left(\|\mathcal{V}\|\,\mathsf{r}^{1/2} t\right)^{\vartheta}}{\vartheta!}, \quad (t \geq 0). \tag{1.3}$$

This leads to the inequality

$$\left\|t^{\mathcal{V}}\right\| \leq \left\|e^{\mathcal{V}\ln t}\right\| \leq t^{\mathsf{d}(\mathcal{V})} \sum_{\vartheta=0}^{\mathsf{r}-1} \frac{\left(\|\mathcal{V}\|\,\mathsf{r}^{1/2}\ln t\right)^{\vartheta}}{\vartheta!}, \quad (t \geq 1). \tag{1.4}$$

Finally, by the properties of matrix functional calculus [20], if $h_1(\varpi)$ and $h_2(\varpi)$ are holomorphic functions defined on an open set w of the complex plane, and if $\mathcal{V} \in C^{l\times l}$ satisfies $\sigma(\mathcal{V}) \subset \mathsf{w}$, then

$$h_1(\mathcal{V})h_2(\mathcal{V}) = h_2(\mathcal{V})h_1(\mathcal{V}). \tag{1.5}$$

In addition, if $\mathcal{R} \in C^{l\times l}$ with $\sigma(\mathcal{R}) \subset \mathsf{w}$, and if $\mathcal{V}\mathcal{R} = \mathcal{R}\mathcal{V}$, then

$$h_1(\mathcal{V})h_2(\mathcal{R}) = h_2(\mathcal{R})h_1(\mathcal{V}). \tag{1.6}$$

According to [21,22], the logarithmic norm $\mathsf{w}(\mathcal{V})$ of $\mathcal{V} \in C^{l\times l}$ is

$$\mathsf{w}(\mathcal{V}) := \lim_{\mathsf{k}\to 0} \frac{\|\mathcal{I} + \mathsf{k}\mathcal{V}\| - 1}{\mathsf{k}} \tag{1.7}$$

$$= \max\left\{\varpi \mid \varpi \in \sigma\left(\frac{\mathcal{V}+\mathcal{V}^*}{2}\right)\right\}. \tag{1.8}$$

Let $\tilde{\mathsf{w}}(\mathcal{V})$ be such that

$$\tilde{\mathsf{w}}(\mathcal{V}) = -\mathsf{w}(-\mathcal{V}) = \min\left\{\varpi \mid \varpi \in \sigma\left(\frac{\mathcal{V}+\mathcal{V}^*}{2}\right)\right\}. \tag{1.9}$$

The image of $\Gamma^{-1}(\varpi)$ acting on $\mathcal{V}$, denoted by $\Gamma^{-1}(\mathcal{V})$, is a well-defined matrix. The reciprocal gamma function

$$\Gamma^{-1}(\varpi) = \frac{1}{\Gamma(\varpi)},$$

is an entire function of the complex variable ϖ. If $\mathcal{V} + j\mathcal{I}$ is invertible for all integers $j \geq 0$, then the reciprocal gamma function of a matrix is defined as follows (see [10]):

$$\Gamma^{-1}(\mathcal{V}) = \mathcal{V}(\mathcal{V}+I)\ldots(\mathcal{V}+(j-1)\mathcal{I})\,\Gamma^{-1}(\mathcal{V}+j\mathcal{I}), \quad j \geq 1. \tag{1.10}$$

The Pochhammer symbol [10] for $\mathcal{V} \in C^{l\times l}$ is

$$(\mathcal{V})_j = \begin{cases} \mathcal{I}, & \text{if } j = 0 \\ \mathcal{V}(\mathcal{V}+\mathcal{I})\ldots(\mathcal{V}+(j-1)\mathcal{I}), & \text{if } j \geq 1, \end{cases} \tag{1.11}$$

which provides

$$(\mathcal{V})_j = \Gamma^{-1}(\mathcal{V})\,\Gamma(\mathcal{V}+j\mathcal{I}), \quad j \geq 1. \tag{1.12}$$

If $\mathcal{V} \in C^{l\times l}$ is a positive stable matrix and $j \geq 1$ is an integer, then the gamma matrix function admits the following limit representation [9]:

$$\Gamma(\mathcal{V}) = \lim_{j\to\infty}(j-1)!\,(\mathcal{V})_j^{-1}\,j^{\mathcal{V}}. \tag{1.13}$$

For any matrix $\mathcal{V} \in C^{l\times l}$, one gets the following relation due to Jodar and Cortés [9]:

$$(1-t)^{-\mathcal{V}} = \sum_{j=0}^{\infty} \frac{(\mathcal{V})_j}{j!} t^j, \qquad |t| < 1. \tag{1.14}$$

Now, let $\mathcal{V}, \mathcal{R} \in C^{l\times l}$ be two positive stable matrices. The gamma matrix function $\Gamma(\mathcal{V})$ and the beta matrix function $\mathbf{B}(\mathcal{V}, \mathcal{R})$ [9, 10] are defined by

$$\Gamma(\mathcal{V}) = \int_0^\infty e^{-\vartheta}\,\vartheta^{\mathcal{V}-1} d\vartheta; \quad \vartheta^{\mathcal{V}-1} = \exp((\mathcal{V}-\mathcal{I})\ln\vartheta), \tag{1.15}$$

and

$$\mathbf{B}(\mathcal{V}, \mathcal{R}) = \int_0^1 \vartheta^{\mathcal{V}-1}\,(1-\vartheta)^{\mathcal{R}-1} d\vartheta. \tag{1.16}$$

According to [10], let $\mathcal{V}, \mathcal{R} \in C^{l\times l}$ be two commuting matrices such that $\mathcal{V}+j\mathcal{I}$, $\mathcal{R}+j\mathcal{I}$, and $\mathcal{V}+\mathcal{R}+j\mathcal{I}$ are invertible for every integer $j \geq 0$.

$$\mathbf{B}(\mathcal{V}, \mathcal{R}) = \Gamma(\mathcal{V})\Gamma(\mathcal{R}) \left[\Gamma(\mathcal{V} + \mathcal{R})\right]^{-1}. \tag{1.17}$$

In the recent studies of special matrix functions, Jatav et al. [12] introduced Shively's pseudo Laguerre-type matrix polynomial and defined it as

$$R_n^{(Q,\lambda,\Theta)}(z) = \frac{\Gamma(Q + 2nI)}{n!} \sum_{k=0}^{n} \frac{(-nI)_k \, \lambda^k z^k}{k!} \Gamma^{-1}(Q + (\Theta k + n)I), \tag{1.18}$$

where $Q \in C^{l\times l}$ is a positive stable matrix, and $\Theta \in \mathbb{Z}^+$, $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$.
In this paper, we introduce and investigate a matrix analog of the five-parameter Mittag-Leffler matrix function, which was defined by Özarslan and Fernandez [23] in 2021:

$$\mathrm{E}^{\mathsf{w}}_{\mathsf{w}_1,\mathsf{w}_2,\gamma_1,\gamma_2}(\varpi) = \sum_{j=0}^{\infty} \frac{(\mathsf{w})_j}{\Gamma(\mathsf{w}_1 j + \gamma_1)\Gamma(\mathsf{w}_2 j + \gamma_2)} \frac{\varpi^j}{j!}, \qquad \varpi \in \mathbb{C}, \tag{1.19}$$

where $\mathsf{w}_1, \mathsf{w}_2, \gamma_1, \gamma_2, \mathsf{w} \in \mathbb{C}$, and $\Re(\mathsf{w}_1 + \mathsf{w}_2) > 0$. For $\mathsf{w} = 1$, this reduces to the general Fox-Wright function studied by Luchko [24]:

$$\mathsf{W}_{(\mathsf{w}_1,\gamma_1),(\mathsf{w}_2,\gamma_2)}(\varpi) = \sum_{j=0}^{\infty} \frac{\varpi^j}{\Gamma(\mathsf{w}_1 j + \gamma_1)\Gamma(\mathsf{w}_2 j + \gamma_2)}. \tag{1.20}$$

2. The five-parameter Mittag-Leffler matrix function

In this section, we present the five-parameter Mittag-Leffler matrix function. We study its convergence on the unit circle $|\varpi| = 1$ and establish its key analytical properties.

Definition 1. *Let $\mathcal{V}, \mathcal{V}_1, \mathcal{V}_2 \in C^{l\times l}$ be positive stable matrices such that $\mathcal{V}_1 + \gamma_1 k\mathcal{I}$ and $\mathcal{V}_2 + \gamma_2 k\mathcal{I}$ are invertible for all integers $k \geq 0$. Then, the five-parameter Mittag-Leffler matrix function is defined as follows:*

$$\mathrm{E}^{\mathcal{V}}_{\gamma_1,\gamma_2,\mathcal{V}_1,\mathcal{V}_2}(\varpi) = \Gamma^{-1}(\mathcal{V}) \sum_{j=0}^{\infty} \Gamma(\mathcal{V} + jI)\Gamma^{-1}(\mathcal{V}_1 + \gamma_1 j\mathcal{I})\Gamma^{-1}(\mathcal{V}_2 + \gamma_2 j\mathcal{I}) \frac{\varpi^j}{j!}, \tag{2.1}$$

where $\varpi \in \mathbb{C}, \gamma_1, \gamma_2 \in \mathbf{R}^+$.

For $\mathcal{V} = \mathcal{I}$, using (1.12) and $(1)_j = j!$, (2.1) reduces to the four-parameter Fox-Wright matrix function:

$$\mathsf{W}_{(\gamma_1,\mathcal{V}_1),(\gamma_2,\mathcal{V}_2)}(\varpi) = \sum_{j=0}^{\infty} \Gamma^{-1}(\mathcal{V}_1 + \gamma_1 j\mathcal{I})\Gamma^{-1}(\mathcal{V}_2 + \gamma_2 j\mathcal{I}) \, \varpi^j. \tag{2.2}$$

We now establish the absolute convergence of the series defined in (2.1) on the circle $|\varpi| = 1$. The following theorem presents the corresponding convergence result for the matrix analog of the five-parameter Mittag-Leffler function.

Theorem 1. *Let $\mathcal{V}, \mathcal{V}_1, \mathcal{V}_2 \in C^{l\times l}$ be positive stable matrices satisfying*

$$\mathsf{d}(\mathcal{V}_1) + \mathsf{c}(\mathcal{V}_2) > \mathsf{d}(\mathcal{V}),$$

where the matrix analog of the five-parameter Mittag-Leffler function defined in Eq (2.1) converges absolutely for all $|\varpi| = 1$.

Proof. By the assumption $\mathsf{d}(\mathcal{V}_1) + \mathsf{c}(\mathcal{V}_2) > \mathsf{d}(\mathcal{V})$, we can choose a positive constant δ satisfying

$$\mathsf{d}(\mathcal{V}_1) + \mathsf{c}(\mathcal{V}_2) - \mathsf{d}(\mathcal{V}) = 2\delta. \tag{2.3}$$

Now we can write

$$\begin{aligned} & j^{1+\delta}\left[\Gamma^{-1}(\mathcal{V})\Gamma(\mathcal{V}+j\mathcal{I})\Gamma^{-1}(\mathcal{V}_1+\gamma_1 j\mathcal{I})\Gamma^{-1}(\mathcal{V}_2+\gamma_2 j\mathcal{I})\frac{1}{j!}\right] \\ &= \frac{j^{\delta}}{[(j-1)!]^2}\left(\frac{j^{-\mathcal{V}}\Gamma^{-1}(\mathcal{V})\Gamma(\mathcal{V}+j\mathcal{I})}{(j-1)!}\right) j^{\mathcal{V}} \times \left(j^{-\mathcal{V}_1}(j-1)!\,\Gamma^{-1}(\mathcal{V}_1+\gamma_1 j\mathcal{I})\right) j^{\mathcal{V}_1} \\ &\times \left(j^{-\mathcal{V}_2}(j-1)!\,\Gamma^{-1}(\mathcal{V}_2+\gamma_2 j\mathcal{I})\right) j^{\mathcal{V}_2}. \end{aligned} \tag{2.4}$$

Using (1.4) and $\mathsf{d}(-\mathcal{V}) = -\mathsf{c}(\mathcal{V})$, we obtain

$$\begin{aligned} \left\|j^{\mathcal{V}}\right\|\left\|j^{-\mathcal{V}_1}\right\|\left\|j^{-\mathcal{V}_2}\right\| &\leq j^{\mathsf{d}(\mathcal{V})-\mathsf{d}(\mathcal{V}_1)-\mathsf{c}(\mathcal{V}_2)}\left\{\sum_{\vartheta=0}^{\mathsf{r}-1}\frac{\left(\|\mathcal{V}_1\|\,\mathsf{r}^{1/2}\ln j\right)^{\vartheta}}{\vartheta!}\right\} \\ &\times\left\{\sum_{\vartheta=0}^{\mathsf{r}-1}\frac{\left(\|\mathcal{V}_2\|\,\mathsf{r}^{1/2}\ln j\right)^{\vartheta}}{\vartheta!}\right\}\left\{\sum_{\vartheta=0}^{\mathsf{r}-1}\frac{\left(\|\mathcal{V}\|\,\mathsf{r}^{1/2}\ln j\right)^{\vartheta}}{\vartheta!}\right\} \\ &\leq j^{-2\delta}\left\{\sum_{\vartheta=0}^{\mathsf{r}-1}\frac{\left(\max\{\|\mathcal{V}\|,\|\mathcal{V}_1\|,\|\mathcal{V}_2\|\}\,\mathsf{r}^{1/2}\ln j\right)^{\vartheta}}{\vartheta!}\right\}^3. \end{aligned} \tag{2.5}$$

Using (2.4) and (2.5) and passing to the limit $j\to\infty$ for $|\varpi| = 1$, it follows that

$$\begin{aligned} & \lim_{j\to\infty} j^{1+\delta}\left\|j^{1+\delta}\left[\Gamma^{-1}(\mathcal{V})\Gamma(\mathcal{V}+j\mathcal{I})\Gamma^{-1}(\mathcal{V}_1+\gamma_1 j\mathcal{I})\Gamma^{-1}(\mathcal{V}_2+\gamma_2 j\mathcal{I})\frac{1}{j!}\right]\right\| \\ &\leq \lim_{j\to\infty}\frac{j^{\delta}}{[(j-1)!]^2}\left\|\frac{j^{-\mathcal{V}}\Gamma^{-1}(\mathcal{V})\Gamma(\mathcal{V}+j\mathcal{I})}{(j-1)!}\right\|\left\|j^{\mathcal{V}}\right\|\left\|j^{\mathcal{V}_1}(j-1)!\,\Gamma^{-1}(\mathcal{V}_1+\gamma_1 j\mathcal{I})\right\|\left\|j^{-\mathcal{V}_1}\right\| \\ &\times\left\|j^{\mathcal{V}_2}(j-1)!\,\Gamma^{-1}(\mathcal{V}_2+\gamma_2 j\mathcal{I})\right\|\left\|j^{-\mathcal{V}_2}\right\| \\ &\leq \lim_{j\to\infty}\frac{j^{-\delta}}{[(j-1)!]^2}\left\|\frac{j^{-\mathcal{V}}\Gamma^{-1}(\mathcal{V})\Gamma(\mathcal{V}+j\mathcal{I})}{(j-1)!}\right\|\left\|j^{\mathcal{V}_1}(j-1)!\,\Gamma^{-1}(\mathcal{V}_1+\gamma_1 j\mathcal{I})\right\| \\ &\times\left\|j^{\mathcal{V}_2}(j-1)!\,\Gamma^{-1}(\mathcal{V}_2+\gamma_2 j\mathcal{I})\right\|\left\{\sum_{\vartheta=0}^{\mathsf{r}-1}\frac{\left(\max\{\|\mathcal{V}\|,\|\mathcal{V}_1\|,\|\mathcal{V}_2\|\}\,\mathsf{r}^{1/2}\ln j\right)^{\vartheta}}{\vartheta!}\right\}^3 = 0. \end{aligned}$$

Thus, we conclude that series (2.1) converges absolutely on the unit circle $|\varpi| = 1$.

Further, using the Cauchy-Hadamard formula, we get

$$\mathsf{R}^{-1} = \limsup_{j\to\infty} \|\mathsf{A}_j\|^{1/j} = 0,$$

where

$$\mathsf{A}_j = \Gamma^{-1}(\mathcal{V})\Gamma(\mathcal{V} + j\mathcal{I})\Gamma^{-1}(\mathcal{V}_1 + \gamma_1 j\mathcal{I})\Gamma^{-1}(\mathcal{V}_2 + \gamma_2 j\mathcal{I})\frac{1}{j!}.$$

Hence, the radius of convergence satisfies $\mathsf{R} = \infty$. Consequently, the series in (2.1) converges absolutely for all $\varpi \in \mathbb{C}$, and therefore defines an entire matrix-valued function. □

Theorem 2. *Let $\mathcal{V}, \mathcal{V}_1, \mathcal{V}_2 \in C^{l\times l}$ be positive stable matrices. Suppose that $\mathcal{V}_1$ and $\mathcal{V}_2$ are commuting matrices and assume further that $\mathcal{V}_1 + \mathcal{V}_2$ is also positive stable and $\gamma_1, \gamma_2 \in \mathbf{R}^+$. Then, we get*

$$\begin{aligned}&\int_{\mathsf{t}}^{\mathsf{x}} (\mathsf{x} - \vartheta)^{\mathcal{V}_2 - \mathcal{I}} (\vartheta - \mathsf{t})^{\mathcal{V}_1 - \mathcal{I}}\, \mathrm{E}^{\mathcal{V}}_{\gamma_1,\gamma_2,\mathcal{V}_1,\mathcal{V}_2} [\mathsf{w}(\vartheta - \mathsf{t})^{\gamma_1}]\, \mathsf{d}\vartheta \\ &= (\mathsf{x} - \mathsf{t})^{\mathcal{V}_1+\mathcal{V}_2-\mathcal{I}}\, \Gamma(\mathcal{V}_2)\Gamma^{-1}(\mathcal{V}) \times \mathrm{E}^{\mathcal{V}}_{\gamma_1,\gamma_2,\mathcal{V}_1+\mathcal{V}_2,\mathcal{V}_2} [\mathsf{w}(\mathsf{x} - \mathsf{t})^{\gamma_1}]\,. \end{aligned} \tag{2.6}$$

Proof. Consider the left-hand side of (2.6). Using the series Definition 1, we obtain

$$\begin{aligned}&\int_{\mathsf{t}}^{\mathsf{x}} (\mathsf{x} - \vartheta)^{\mathcal{V}_2 - \mathcal{I}} (\vartheta - \mathsf{t})^{\mathcal{V}_1 - \mathcal{I}}\, \mathrm{E}^{\mathcal{V}}_{\gamma_1,\gamma_2,\mathcal{V}_1,\mathcal{V}_2} [\mathsf{w}(\vartheta - \mathsf{t})^{\gamma_1}]\, \mathsf{d}\vartheta \\ &= \Gamma^{-1}(\mathcal{V}) \sum_{j=0}^{\infty} \Gamma(\mathcal{V} + j\mathcal{I})\Gamma^{-1}(\mathcal{V}_1 + \gamma_1 j\mathcal{I})\Gamma^{-1}(\mathcal{V}_2 + \gamma_1 j\mathcal{I})\frac{\mathsf{w}^j}{j!} \\ &\quad \times \int_{\mathsf{t}}^{\mathsf{x}} (\mathsf{x} - \vartheta)^{\mathcal{V}_2 - \mathcal{I}} (\vartheta - \mathsf{t})^{\mathcal{V}_1 + (\gamma_1 j - 1)\mathcal{I}}\, \mathsf{d}\vartheta. \end{aligned}$$

Applying the change of variable $\mathsf{s} = \frac{\vartheta - \mathsf{t}}{\mathsf{x} - \mathsf{t}}$, the integral transforms into

$$\begin{aligned}&\int_{\mathsf{t}}^{\mathsf{x}} (\mathsf{x} - \vartheta)^{\mathcal{V}_2 - \mathcal{I}} (\vartheta - \mathsf{t})^{\mathcal{V}_1 - \mathcal{I}}\, \mathrm{E}^{\mathcal{V}}_{\gamma_1,\gamma_2,\mathcal{V}_1,\mathcal{V}_2} [\mathsf{w}(\vartheta - \mathsf{t})^{\gamma_1}]\, \mathsf{d}\vartheta \\ &= (\mathsf{x} - \mathsf{t})^{\mathcal{V}_1+\mathcal{V}_2-\mathcal{I}}\Gamma^{-1}(\mathcal{V}) \sum_{j=0}^{\infty} \Gamma(\mathcal{V} + j\mathcal{I})\Gamma^{-1}(\mathcal{V}_1 + \gamma_1 j\mathcal{I})\Gamma^{-1}(\mathcal{V}_2 + \gamma_1 j\mathcal{I})\frac{[\mathsf{w}(\mathsf{x} - \mathsf{t})^{\gamma_1}]^j}{j!} \\ &\quad \times \int_0^1 (1 - \mathsf{s})^{\mathcal{V}_2 - \mathcal{I}} \mathsf{s}^{\mathcal{V}_1 + (\gamma_1 j - 1)\mathcal{I}}\, \mathsf{ds}. \end{aligned}$$

Evaluating the beta integral yields

$$\begin{aligned}&= (\mathsf{x} - \mathsf{t})^{\mathcal{V}_1+\mathcal{V}_2-\mathcal{I}}\Gamma(\mathcal{V}_2)\Gamma^{-1}(\mathcal{V}) \\ &\quad \times \sum_{j=0}^{\infty} \Gamma(\mathcal{V} + j\mathcal{I})\Gamma^{-1}(\mathcal{V}_1 + \mathcal{V}_2 + \gamma_1 j\mathcal{I})\Gamma^{-1}(\mathcal{V}_2 + \gamma_1 j\mathcal{I})\frac{[\mathsf{w}(\mathsf{x} - \mathsf{t})^{\gamma_1}]^j}{j!}. \end{aligned}$$

The right-hand side of the above expression is precisely the series representation of the matrix analog of the five-parameter Mittag-Leffler function, which completes the proof. □

Corollary 1. *Under the hypotheses of Theorem 2, and taking $\mathcal{V} = \mathcal{I}$, we obtain*

$$\int_{t}^{x} (x-\vartheta)^{\mathcal{V}_2-\mathcal{I}}(\vartheta-t)^{\mathcal{V}_1-\mathcal{I}}\, W_{(\gamma_1,\mathcal{V}_1),(\gamma_2,\mathcal{V}_2)}\left(w(\vartheta-t)^{\gamma_1}\right) d\vartheta$$
$$= (x-t)^{\mathcal{V}_1+\mathcal{V}_2-\mathcal{I}}\,\Gamma(\mathcal{V}_2)\times W_{(\gamma_1,\mathcal{V}_1+\mathcal{V}_2),(\gamma_2,\mathcal{V}_2)}\left(w(x-t)^{\gamma_1}\right). \tag{2.7}$$

Theorem 3. *Let $\mathcal{V},\mathcal{V}_1,\mathcal{V}_2 \in C^{l\times l}$ be positive stable matrices such that $\mathcal{V}_1+\gamma_1 k\mathcal{I}$ and $\mathcal{V}_2+\gamma_2 k\mathcal{I}$ are invertible for all integers $k \geq 0$ and $\varpi \in \mathbb{C}$, $\gamma_1,\gamma_2 \in \mathbf{R}^+$, and the series converges uniformly on compact subsets for all $\varpi \in \mathbb{C}$. Then, the following derivative formulas hold:*

$$\left(\frac{d}{d\varpi}\right)^{\zeta}\left[\varpi^{\mathcal{V}_1-\mathcal{I}}\,\mathrm{E}^{\mathcal{V}}_{\gamma_1,\gamma_2,\mathcal{V}_1,\mathcal{V}_2}\left(w\varpi^{\gamma_1}\right)\right] = \varpi^{\mathcal{V}_1-(\zeta+1)\mathcal{I}}\,\mathrm{E}^{\mathcal{V}}_{\gamma_1,\gamma_2,\mathcal{V}_1-\zeta\mathcal{I},\mathcal{V}_2}(w\varpi^{\gamma_1}), \tag{2.8}$$

$$\left(\frac{d}{d\varpi}\right)^{\zeta}\left[\varpi^{\mathcal{V}_2-\mathcal{I}}\,\mathrm{E}^{\mathcal{V}}_{\gamma_1,\gamma_2,\mathcal{V}_1,\mathcal{V}_2}\left(w\varpi^{\gamma_2}\right)\right] = \varpi^{\mathcal{V}_2-(\zeta+1)\mathcal{I}}\,\mathrm{E}^{\mathcal{V}}_{\gamma_1,\gamma_2,\mathcal{V}_1,\mathcal{V}_2-\zeta\mathcal{I}}(w\varpi^{\gamma_2}). \tag{2.9}$$

Proof. We first prove (2.8). Starting from the left-hand side and using the series definition (2.1), we have

$$\left(\frac{d}{d\varpi}\right)^{\zeta}\left[\varpi^{\mathcal{V}_1-\mathcal{I}}\,\mathrm{E}^{\mathcal{V}}_{\gamma_1,\gamma_2,\mathcal{V}_1,\mathcal{V}_2}\left(w\varpi^{\gamma_1}\right)\right]$$
$$= \Gamma^{-1}(\mathcal{V})\sum_{j=0}^{\infty}\Gamma(\mathcal{V}+j\mathcal{I})\Gamma^{-1}(\mathcal{V}_1+\gamma_1 j\mathcal{I})\Gamma^{-1}(\mathcal{V}_2+\gamma_1 j\mathcal{I})\frac{w^j}{j!}$$
$$\times\left(\frac{d}{d\varpi}\right)^{\zeta}\varpi^{\mathcal{V}_1+(\gamma_1 j-1)\mathcal{I}}.$$

Differentiating term by term under the summation sign yields

$$\left(\frac{d}{d\varpi}\right)^{\zeta}\left[\varpi^{\mathcal{V}_1-\mathcal{I}}\,\mathrm{E}^{\mathcal{V}}_{\gamma_1,\gamma_2,\mathcal{V}_1,\mathcal{V}_2}\left(w\varpi^{\gamma_1}\right)\right]$$
$$= \varpi^{\mathcal{V}_1-(\zeta+1)\mathcal{I}}\Gamma^{-1}(\mathcal{V})\sum_{j=0}^{\infty}\Gamma(\mathcal{V}+j\mathcal{I})\Gamma^{-1}(\mathcal{V}_1+(\gamma_1 j-\zeta)\mathcal{I})\Gamma^{-1}(\mathcal{V}_2+\gamma_1 j\mathcal{I})\frac{(w\varpi^{\gamma_1})^j}{j!}.$$

The right-hand side of the above expression is precisely the series representation of $\varpi^{\mathcal{V}_1-(\zeta+1)\mathcal{I}}\mathrm{E}^{\mathcal{V}}_{\gamma_1,\gamma_2,\mathcal{V}_1-\zeta\mathcal{I},\mathcal{V}_2}(w\varpi^{\gamma_1})$, which establishes (2.8). The proof of (2.9) follows by an analogous argument. □

Corollary 2. *Under the hypotheses of Theorem 3, and taking $\mathcal{V} = \mathcal{I}$, we obtain the following derivative formulas:*

$$\left(\frac{d}{d\varpi}\right)^{\zeta}\left[\varpi^{\mathcal{V}_1-\mathcal{I}}\,W_{(\gamma_1,\mathcal{V}_1),(\gamma_2,\mathcal{V}_2)}\left(w\varpi^{\gamma_1}\right)\right] = \varpi^{\mathcal{V}_1-(\zeta+1)\mathcal{I}}\,W_{(\gamma_1,\mathcal{V}_1-\zeta\mathcal{I}),(\gamma_2,\mathcal{V}_2)}(w\varpi^{\gamma_1}), \tag{2.10}$$

$$\left(\frac{d}{d\varpi}\right)^{\zeta}\left[\varpi^{\mathcal{V}_2-\mathcal{I}}\,W_{(\gamma_1,\mathcal{V}_1),(\gamma_2,\mathcal{V}_2)}\left(w\varpi^{\gamma_2}\right)\right] = \varpi^{\mathcal{V}_2-(\zeta+1)\mathcal{I}}\,W_{(\gamma_1,\mathcal{V}_1),(\gamma_2,\mathcal{V}_2-\zeta\mathcal{I})}(w\varpi^{\gamma_2}). \tag{2.11}$$

Theorem 4. *Let $\mathcal{V}, \mathcal{V}_1, \mathcal{V}_2 \in C^{l\times l}$ be positive stable matrices such that $\mathcal{V}_1 + \gamma_1 k\mathcal{I}$ and $\mathcal{V}_2 + \gamma_2 k\mathcal{I}$ are invertible for all integers $k \geq 0$, and $\varpi \in \mathbb{C}$, $\gamma_1, \gamma_2 \in \mathbf{R}^+$. Then the following differential recurrence relations hold:*

$$\mathrm{E}^{\mathcal{V}}_{\gamma_1,\gamma_2,\mathcal{V}_1,\mathcal{V}_2}(\varpi) = \mathcal{V}_1\, \mathrm{E}^{\mathcal{V}}_{\gamma_1,\gamma_2,\mathcal{V}_1+\mathcal{I},\mathcal{V}_2}(\varpi) + \gamma_1 \varpi \frac{\mathsf{d}}{d\varpi} \mathrm{E}^{\mathcal{V}}_{\gamma_1,\gamma_2,\mathcal{V}_1+\mathcal{I},\mathcal{V}_2}(\varpi), \tag{2.12}$$

$$\mathrm{E}^{\mathcal{V}}_{\gamma_1,\gamma_2,\mathcal{V}_1,\mathcal{V}_2}(\varpi) = \mathcal{V}_2\, \mathrm{E}^{\mathcal{V}}_{\gamma_1,\gamma_2,\mathcal{V}_1,\mathcal{V}_2+\mathcal{I}}(\varpi) + \gamma_2 \varpi \frac{\mathsf{d}}{d\varpi} \mathrm{E}^{\mathcal{V}}_{\gamma_1,\gamma_2,\mathcal{V}_1,\mathcal{V}_2+\mathcal{I}}(\varpi). \tag{2.13}$$

Proof. Analyzing the right-hand side of (2.12) and using (2.1), we have

$$\begin{aligned}
&\mathcal{V}_1 \mathrm{E}^{\mathcal{V}}_{\gamma_1,\gamma_2,\mathcal{V}_1+\mathcal{I},\mathcal{V}_2}(\varpi) + \gamma_1 \varpi \frac{\mathsf{d}}{d\varpi} \mathrm{E}^{\mathcal{V}}_{\gamma_1,\gamma_2,\mathcal{V}_1+\mathcal{I},\mathcal{V}_2}(\varpi) \\
&= \mathcal{V}_1 \mathrm{E}^{\mathcal{V}}_{\gamma_1,\gamma_2,\mathcal{V}_1+\mathcal{I},\mathcal{V}_2}(\varpi) + \gamma_1 j \sum_{j=0}^{\infty} \Gamma^{-1}(\mathcal{V})\Gamma(\mathcal{V} + j\mathcal{I})\Gamma^{-1}(\mathcal{V}_1 + (\gamma_1 j + 1)\mathcal{I})\Gamma^{-1}(\mathcal{V}_2 + \gamma_2 j\mathcal{I}) \frac{\varpi^j}{j} \\
&= \Gamma^{-1}(\mathcal{V}) \sum_{j=0}^{\infty} \Gamma(\mathcal{V} + jI)\Gamma^{-1}(\mathcal{V}_1 + \gamma_1 j\mathcal{I})\Gamma^{-1}(\mathcal{V}_2 + \gamma_1 j\mathcal{I}) \frac{\varpi^j}{j!}.
\end{aligned}$$

This leads straight to the proof of the result (2.12). Similarly, we can prove the result (2.13). □

Corollary 3. *Under the hypotheses of Theorem 3, and taking $\mathcal{V} = \mathcal{I}$, we obtain the following differential recurrence relation:*

$$\mathsf{W}_{(\gamma_1,\mathcal{V}_1),(\gamma_2,\mathcal{V}_2)}(\varpi) = \mathcal{V}_1 \mathsf{W}_{(\gamma_1,\mathcal{V}_1+\mathcal{I}),(\gamma_2,\mathcal{V}_2)}(\varpi) + \gamma_1 \varpi \frac{\mathsf{d}}{d\varpi} \mathsf{W}_{(\gamma_1,\mathcal{V}_1+\mathcal{I}),(\gamma_2,\mathcal{V}_2)}(\varpi), \tag{2.14}$$

$$\mathsf{W}_{(\gamma_1,\mathcal{V}_1),(\gamma_2,\mathcal{V}_2)}(\varpi) = \mathcal{V}_2 \mathsf{W}_{(\gamma_1,\mathcal{V}_1),(\gamma_2,\mathcal{V}_2+\mathcal{I})}(\varpi) + \gamma_2 \varpi \frac{\mathsf{d}}{d\varpi} \mathsf{W}_{(\gamma_1,\mathcal{V}_1),(\gamma_2,\mathcal{V}_2+\mathcal{I})}(\varpi). \tag{2.15}$$

3. Finite summation formulas

In this section, we establish several interesting finite summation formulas involving the matrix analog of the five-parameter Mittag-Leffler function defined in (2.1). The following lemma, due to Rainville [25], will be useful in our analysis.

Lemma 1. *Let $\mathcal{V}(k, \zeta)$ and $\mathcal{R}(k, \zeta)$ be matrices in $C^{l\times l}$. Then the following series relations hold:*

$$\sum_{\zeta=0}^{\infty} \sum_{k=0}^{\infty} \mathcal{V}(k, \zeta) = \sum_{\zeta=0}^{\infty} \sum_{k=0}^{\zeta} \mathcal{V}(k, \zeta - k),$$

$$\sum_{\zeta=0}^{\infty} \sum_{k=0}^{\zeta} \mathcal{R}(k, \zeta) = \sum_{\zeta=0}^{\infty} \sum_{k=0}^{\infty} \mathcal{R}(k, \zeta + k).$$

Theorem 5. *Let $\mathcal{V}, \mathcal{V}_1, \mathcal{V}_2 \in C^{l\times l}$ be positive stable matrices such that $\mathcal{V}_1 + \gamma_1 k\mathcal{I}$ and $\mathcal{V}_2 + \gamma_2 k\mathcal{I}$ are invertible for all integers $k \geq 0$ and $\varpi \in \mathbb{C}$. Then the following finite summation formula holds*

$$\sum_{k=0}^{j} \binom{j}{k} \Gamma(\mathcal{V} - k\mathcal{I})\, \mathrm{E}^{\mathcal{V}-k\mathcal{I}}_{\gamma_1,\gamma_1,\mathcal{V}_1-\gamma_1 k\mathcal{I},\mathcal{V}_2-\gamma_2 k\mathcal{I}}(\varpi)\, \mathsf{w}^k = e^{\mathsf{w}\varpi}\, \Gamma(\mathcal{V})\, \mathrm{E}^{\mathcal{V}}_{\gamma_1,\gamma_2,\mathcal{V}_1,\mathcal{V}_2}(\varpi). \tag{3.1}$$

Proof. Analyzing the left-hand side of (3.1) and using (2.1), we have

$$\sum_{k=0}^{j}\binom{j}{k}\Gamma(\mathcal{V}-k\mathcal{I})\,\mathrm{E}^{\mathcal{V}-k\mathcal{I}}_{\gamma_1,\gamma_2,\mathcal{V}_1-\gamma_1 k\mathcal{I},\mathcal{V}_2-\gamma_2 k\mathcal{I}}(\varpi)\,\mathsf{w}^k$$
$$=\sum_{j=0}^{\infty}\sum_{k=0}^{j}\Gamma(\mathcal{V}+(j-k)\mathcal{I})\Gamma^{-1}(\mathcal{V}_1+\gamma_1(j-k)\mathcal{I})\Gamma^{-1}(\mathcal{V}_2+\gamma_2(j-k)\mathcal{I})\frac{\varpi^j\mathsf{w}^k}{k!(j-k)!}.$$

Utilizing Lemma 1, we arrive at

$$\sum_{k=0}^{j}\binom{j}{k}\Gamma(\mathcal{V}-k\mathcal{I})\,\mathrm{E}^{\mathcal{V}-k\mathcal{I}}_{\gamma_1,\gamma_2,\mathcal{V}_1-\gamma_1 k\mathcal{I},\mathcal{V}_2-\gamma_2 k\mathcal{I}}(\varpi)\,\mathsf{w}^k$$
$$=\sum_{j=0}^{\infty}\sum_{k=0}^{j}\Gamma(\mathcal{V}+j\mathcal{I})\Gamma^{-1}(\mathcal{V}_1+\gamma_1 j\mathcal{I})\Gamma^{-1}(\mathcal{V}_2+\gamma_2 j\mathcal{I})\frac{\varpi^{j+k}\mathsf{w}^k}{k!\,j!}$$
$$=\left(\sum_{k=0}^{\infty}\frac{(\varpi\mathsf{w})^k}{k!}\right)\left(\sum_{j=0}^{\infty}\Gamma(\mathcal{V}+j\mathcal{I})\Gamma^{-1}(\mathcal{V}_1+\gamma_1 j\mathcal{I})\Gamma^{-1}(\mathcal{V}_2+\gamma_2 j\mathcal{I})\frac{\varpi^j}{j!}\right)$$
$$=e^{\mathsf{w}\varpi}\left(\sum_{j=0}^{\infty}\Gamma(\mathcal{V}+j\mathcal{I})\Gamma^{-1}(\mathcal{V}_1+\gamma_1 j\mathcal{I})\Gamma^{-1}(\mathcal{V}_2+\gamma_2 j\mathcal{I})\frac{\varpi^j}{j!}\right).$$

This leads directly to the intended outcome (3.1) according to Definition 1. □

Corollary 4. *Under the hypotheses of Theorem 5, and taking* $\mathcal{V}=\mathcal{I}$, *we obtain the following finite summation formula:*

$$\sum_{k=0}^{j}\binom{j}{k}\mathsf{W}_{(\gamma_1,\mathcal{V}_1-\gamma_1 k\mathcal{I}),(\gamma_2,\mathcal{V}_2-\gamma_2 k\mathcal{I})}(\varpi)\,\mathsf{w}^k=e^{\mathsf{w}\varpi}\,\mathsf{W}_{(\gamma_1,\mathcal{V}_1),(\gamma_2,\mathcal{V}_2)}(\varpi). \tag{3.2}$$

Theorem 6. *Let* $\mathcal{V},\mathcal{V}_1,\mathcal{V}_2,\mathcal{S}\in C^{l\times l}$ *be positive stable matrices such that* $\mathcal{V}_1+\gamma_1 k\mathcal{I}$ *and* $\mathcal{V}_2+\gamma_2 k\mathcal{I}$ *are invertible for all integers* $k\geq 0$, *and assume* $|\varpi\mathsf{w}|<1$. *Then, we have*

$$\sum_{k=0}^{j}\binom{j}{k}\Gamma(\mathcal{V}-k\mathcal{I})\,(\mathcal{S})_k\,\mathrm{E}^{\mathcal{V}-k\mathcal{I}}_{\gamma_1,\gamma_2,\mathcal{V}_1-\gamma_1 k\mathcal{I},\mathcal{V}_2-\gamma_2 k\mathcal{I}}(\varpi)\,\mathsf{w}^k=(1-\varpi\mathsf{w})^{-\mathcal{S}}\,\Gamma(\mathcal{V})\,\mathrm{E}^{\mathcal{V}}_{\gamma_1,\gamma_2,\mathcal{V}_1,\mathcal{V}_2}(\varpi). \tag{3.3}$$

Proof. Analyzing the left-hand side of (3.3) and using (2.1), we get

$$\sum_{k=0}^{j}\binom{j}{k}\Gamma(\mathcal{V}-k\mathcal{I})\,(\mathcal{S})_k\,\mathrm{E}^{\mathcal{V}-k\mathcal{I}}_{\gamma_1,\gamma_2,\mathcal{V}_1-\gamma_1 k\mathcal{I},\mathcal{V}_1-\gamma_1 k\mathcal{I}}(\varpi)\,\mathsf{w}^k$$
$$=\sum_{j=0}^{\infty}\sum_{k=0}^{j}(\mathcal{S})_k\Gamma(\mathcal{V}+(j-k)\mathcal{I})\Gamma^{-1}(\mathcal{V}_1+\gamma_1(j-k)\mathcal{I})\Gamma^{-1}(\mathcal{V}_2+\gamma_2(j-k)\mathcal{I})\frac{\varpi^j\mathsf{w}^k}{k!(j-k)!}.$$

Utilizing the Lemma 1, we arrive at

$$\sum_{k=0}^{j}\binom{j}{k}\Gamma(\mathcal{V}-k\mathcal{I})\,(\mathcal{S})_k\,\mathrm{E}^{\mathcal{V}-k\mathcal{I}}_{\gamma_1,\gamma_2,\mathcal{V}_1-\gamma_1 k\mathcal{I},\mathcal{V}_2-\gamma_2 k\mathcal{I}}(\varpi)\,\mathsf{w}^k$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{j}(\mathcal{S})_k\Gamma(\mathcal{V}+j\mathcal{I})\Gamma^{-1}(\mathcal{V}_1+\gamma_1 j\mathcal{I})\Gamma^{-1}(\mathcal{V}_2+\gamma_2 j\mathcal{I})\frac{\varpi^{j+k}\mathsf{w}^k}{k!j!}$$

$$= \left(\sum_{k=0}^{\infty}\frac{(\mathcal{S})_k(\varpi\mathsf{w})^k}{k!}\right)\left(\sum_{j=0}^{\infty}\Gamma(\mathcal{V}+j\mathcal{I})\Gamma^{-1}(\mathcal{V}_1+\gamma_1 j\mathcal{I})\Gamma^{-1}(\mathcal{V}_2+\gamma_2 j\mathcal{I})\frac{\varpi^j}{j!}\right)$$

$$= (1-\varpi\mathsf{w})^{-\mathcal{S}}\left(\sum_{j=0}^{\infty}\Gamma(\mathcal{V}+j\mathcal{I})\Gamma^{-1}(\mathcal{V}_1+\gamma_1 j\mathcal{I})\Gamma^{-1}(\mathcal{V}_2+\gamma_2 j\mathcal{I})\frac{\varpi^j}{j!}\right).$$

This leads directly to the intended outcome (3.3) according to Definition 1. □

Corollary 5. *Under the hypotheses of Theorem 6, and taking $\mathcal{V}=\mathcal{I}$, we obtain*

$$\sum_{k=0}^{j}\binom{j}{k}(\mathcal{S})_k\,\mathsf{W}_{(\gamma_1,\mathcal{V}_1-\gamma_1 k\mathcal{I}),(\gamma_2,\mathcal{V}_2-\gamma_2 k\mathcal{I})}(\varpi)\,\mathsf{w}^k = (1-\varpi\mathsf{w})^{-\mathcal{S}}\,\mathsf{W}_{(\gamma_1,\mathcal{V}_1),(\gamma_2,\mathcal{V}_2)}(\varpi). \tag{3.4}$$

Theorem 7. *Let $\mathcal{V},\mathcal{V}_1,\mathcal{V}_2\in C^{l\times l}$ be positive stable matrices such that $\mathcal{V}_1+\gamma_1 k\mathcal{I}$ and $\mathcal{V}_2+\gamma_2 k\mathcal{I}$ are invertible for all integers $k\geq 0$ and $\varpi\in\mathbb{C}$. Then, the following finite summation formula involving derivatives holds:*

$$\sum_{k=0}^{j}\frac{(\mathsf{w})^k}{k!}\Gamma(\mathcal{V}-k\mathcal{I})\,\frac{\mathsf{d}^k}{\mathsf{d}\varpi^k}\left(\mathrm{E}^{\mathcal{V}-k\mathcal{I}}_{\gamma_1,\gamma_2,\mathcal{V}_1-\gamma_1 k\mathcal{I},\mathcal{V}_2-\gamma_2 k\mathcal{I}}(\varpi)\right) = e^{\mathsf{w}}\,\Gamma(\mathcal{V})\,\mathrm{E}^{\mathcal{V}}_{\gamma_1,\gamma_2,\mathcal{V}_1,\mathcal{V}_2}(\varpi). \tag{3.5}$$

Proof. Analyzing the left-hand side of (3.5) and using (2.1), we have

$$\sum_{k=0}^{j}\frac{(\mathsf{w})^k}{k!}\Gamma(\mathcal{V}-k\mathcal{I})\,\frac{\mathsf{d}^k}{\mathsf{d}\varpi^k}\left(\mathrm{E}^{\mathcal{V}-k\mathcal{I}}_{\gamma_1,\gamma_2,\mathcal{V}_1-\gamma_1 k\mathcal{I},\mathcal{V}_2-\gamma_2 k\mathcal{I}}(\varpi)\right)$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{j}\Gamma(\mathcal{V}+(j-k)\mathcal{I})\Gamma^{-1}(\mathcal{V}_1+\gamma_1(j-k)\mathcal{I})\Gamma^{-1}(\mathcal{V}_2+\gamma_2(j-k)\mathcal{I})\frac{\varpi^{j-k}\mathsf{w}^k}{k!(j-k)!}.$$

Utilizing Lemma 1, we arrive at

$$\sum_{k=0}^{j}\frac{(\mathsf{w})^k}{k!}\Gamma(\mathcal{V}-k\mathcal{I})\,\frac{\mathsf{d}^k}{\mathsf{d}\varpi^k}\left(\mathrm{E}^{\mathcal{V}-k\mathcal{I}}_{\gamma_1,\gamma_2,\mathcal{V}_1-\gamma_1 k\mathcal{I},\mathcal{V}_2-\gamma_2 k\mathcal{I}}(\varpi)\right)$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{j}\Gamma(\mathcal{V}+j\mathcal{I})\Gamma^{-1}(\mathcal{V}_1+\gamma_1 j\mathcal{I})\Gamma^{-1}(\mathcal{V}_2+\gamma_2 j\mathcal{I})\frac{\varpi^j\mathsf{w}^k}{k!j!}$$

$$= \left(\sum_{k=0}^{\infty}\frac{(\mathsf{w})^k}{k!}\right)\left(\sum_{j=0}^{\infty}\Gamma(\mathcal{V}+j\mathcal{I})\Gamma^{-1}(\mathcal{V}_1+\gamma_1 j\mathcal{I})\Gamma^{-1}(\mathcal{V}_2+\gamma_2 j\mathcal{I})\frac{\varpi^j}{j!}\right)$$

$$= e^{\mathsf{w}}\left(\sum_{j=0}^{\infty}\Gamma(\mathcal{V}+j\mathcal{I})\Gamma^{-1}(\mathcal{V}_1+\gamma_1 j\mathcal{I})\Gamma^{-1}(\mathcal{V}_2+\gamma_2 j\mathcal{I})\frac{\varpi^j}{j!}\right).$$

This leads directly to the intended outcome (3.5) according to Definition 1. □

Corollary 6. *Under the hypotheses of Theorem 7, and taking $\mathcal{V} = \mathcal{I}$, we obtain the following finite summation formula:*

$$\sum_{k=0}^{j} \frac{(\mathsf{w})^k}{k!} \frac{\mathsf{d}^k}{\mathsf{d}\varpi^k} \left(\mathsf{W}_{(\gamma_1,\mathcal{V}_1-\gamma_1 k\mathcal{I}),(\gamma_2,\mathcal{V}_2-\gamma_2 k\mathcal{I})}(\varpi) \right) = e^{\mathsf{w}}\, \mathsf{W}_{(\gamma_1,\mathcal{V}_1),(\gamma_2,\mathcal{V}_2)}(\varpi). \tag{3.6}$$

Theorem 8. *Let $\mathcal{V}, \mathcal{V}_1, \mathcal{V}_2, \mathcal{S} \in C^{l\times l}$ be positive stable matrices such that $\mathcal{V}_1 + \gamma_1 k\mathcal{I}$ and $\mathcal{V}_2 + \gamma_2 k\mathcal{I}$ are invertible for all integers $k \geq 0$, and assume $|\varpi \mathsf{w}| < 1$. Then, we get*

$$\begin{aligned}\sum_{k=0}^{j} \frac{(\mathsf{w})^k}{k!} (\mathcal{S})_k\, \Gamma(\mathcal{V} - k\mathcal{I}) \frac{\mathsf{d}^k}{\mathsf{d}\varpi^k} \left(\mathrm{E}^{\mathcal{V}-k\mathcal{I}}_{\gamma_1,\gamma_2,\mathcal{V}_1-\gamma_1 k\mathcal{I},\mathcal{V}_2-\gamma_2 k\mathcal{I}}(\varpi) \right)& \\ = (1-\varpi\mathsf{w})^{-\mathcal{S}}\, \Gamma(\mathcal{V})\, \mathrm{E}^{\mathcal{V}}_{\gamma_1,\gamma_2,\mathcal{V}_1,\mathcal{V}_2}(\varpi).&\end{aligned} \tag{3.7}$$

Proof. Analyzing the left-hand side of (3.7) and using (2.1), we get

$$\begin{aligned}&\sum_{k=0}^{j} \frac{(\mathsf{w})^k}{k!} (\mathcal{S})_k\, \Gamma(\mathcal{V} - k\mathcal{I}) \frac{\mathsf{d}^{\varpi}}{\mathsf{d}\varpi^k} \left(\mathrm{E}^{\mathcal{V}-k\mathcal{I}}_{\gamma_1,\gamma_2,\mathcal{V}_1-\gamma_1 k\mathcal{I},\mathcal{V}_2-\gamma_2 k\mathcal{I}}(\varpi) \right) \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{j} \Gamma(\mathcal{V}+(j-k)\mathcal{I})\Gamma^{-1}(\mathcal{V}_1+\gamma_1(j-k)\mathcal{I})\Gamma^{-1}(\mathcal{V}_2+\gamma_2(j-k)\mathcal{I}) \frac{\varpi^j \mathsf{w}^k}{k!(j-k)!}.\end{aligned}$$

Utilizing Lemma 1, we arrive at

$$\begin{aligned}&\sum_{k=0}^{j} \frac{(\mathsf{w})^k}{k!} (\mathcal{S})_k\, \Gamma(\mathcal{V} - k\mathcal{I}) \frac{\mathsf{d}^k}{\mathsf{d}\varpi^k} \left(\mathrm{E}^{\mathcal{V}-k\mathcal{I}}_{\gamma_1,\gamma_2,\mathcal{V}_1-\gamma_1 k\mathcal{I},\mathcal{V}_2-\gamma_2 k\mathcal{I}}(\varpi) \right) \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{j} (\mathcal{S})_k\, \Gamma(\mathcal{V}+j\mathcal{I})\Gamma^{-1}(\mathcal{V}_1+\gamma_1 j\mathcal{I})\Gamma^{-1}(\mathcal{V}_2+\gamma_2 j\mathcal{I}) \frac{\varpi^{j+k}\mathsf{w}^k}{k!\,j!} \\ &= \left(\sum_{k=0}^{\infty} \frac{(\varpi)^k (\mathsf{w})^k (\mathcal{S})_k}{k!} \right) \left(\sum_{j=0}^{\infty} \Gamma(\mathcal{V}+j\mathcal{I})\Gamma^{-1}(\mathcal{V}_1+\gamma_1 j\mathcal{I})\Gamma^{-1}(\mathcal{V}_2+\gamma_2 j\mathcal{I}) \frac{\varpi^j}{j!} \right) \\ &= (1-\varpi\mathsf{w})^{-\mathcal{S}} \left(\sum_{j=0}^{\infty} \Gamma(\mathcal{V}+j\mathcal{I})\Gamma^{-1}(\mathcal{V}_1+\gamma_1 j\mathcal{I})\Gamma^{-1}(\mathcal{V}_2+\gamma_2 j\mathcal{I}) \frac{\varpi^j}{j!} \right).\end{aligned}$$

This leads directly to the intended outcome (3.7) according to Definition 1. □

Corollary 7. *Under the hypotheses of Theorem 8, and taking $\mathcal{V} = \mathcal{I}$, we obtain*

$$\sum_{k=0}^{j} \frac{(\mathsf{w})^k}{k!} (\mathcal{S})_k \frac{\mathsf{d}^k}{\mathsf{d}\varpi^k} \left(\mathsf{W}_{(\gamma_1,\mathcal{V}_1-\gamma_1 k\mathcal{I}),(\gamma_2,\mathcal{V}_2-\gamma_2 k\mathcal{I})}(\varpi) \right) = (1-\varpi\mathsf{w})^{-\mathcal{S}}\, \mathsf{W}_{(\gamma_1,\mathcal{V}_1),(\gamma_2,\mathcal{V}_2)}(\varpi). \tag{3.8}$$

4. Composition of fractional calculus with the five-parameter Mittag-Leffler matrix function

In this section, we derive the composition formulas involving generalized fractional calculus operators and the five-parameter Mittag-Leffler matrix function. To this end, we first recall the definitions of the generalized fractional integral and derivative due to Katugampola [26, 27].

Definition 2. *[26, 27] Let $h_1 \in X_c^p(c,d)$. The generalized fractional integral ${}^{\beta}\mathcal{I}_{0+}^{\alpha}$ of order $\alpha \in \mathbf{C}$ is defined by*

$$\left({}^{\beta}\mathcal{I}_{0+}^{\alpha} h_1\right)(x) = \frac{\beta^{1-\alpha}}{\Gamma(\alpha)} \int_0^x \frac{u^{\beta-1} h_1(u)}{(x^{\beta} - u^{\beta})^{1-\alpha}}\, du, \quad x > 0,\ \Re(\alpha) > 0,\ \beta > 0, \tag{4.1}$$

where $X_c^p(c,d)$ (with $c \in \mathbf{R}$, $1 \leq p \leq \infty$) denotes the space of complex-valued Lebesgue measurable functions on $[c,d]$.

Definition 3. *[26, 27] The generalized fractional derivative ${}^{\beta}\mathcal{D}_{0+}^{\alpha}$ of order $\alpha \in \mathbf{C}$ for a function $h_1 \in X_c^p(c,d)$ is defined by*

$$\left({}^{\beta}\mathcal{D}_{0+}^{\alpha} h_1\right)(x) = \left(x^{1-\beta} \frac{d}{dx}\right)^{\zeta} \left({}^{\beta}\mathcal{I}_{0+}^{\zeta-\alpha} h_1\right)(x), \quad \Re(\alpha) \geq 0,\ \beta > 0, \tag{4.2}$$

where $\zeta = [\Re(\alpha)] + 1$.

Remark 1. *In the limit $\beta \to 1$, the generalized fractional integral and derivative of order α defined by (4.1) and (4.2) reduce to the classical Riemann-Liouville fractional integral and derivative of order α [28]:*

$$\lim_{\beta \to 1} \left({}^{\beta}\mathcal{I}_{0+}^{\alpha} h_1\right)(x) = \left(\mathcal{I}_{0+}^{\alpha} h_1\right)(x), \tag{4.3}$$

$$\lim_{\beta \to 1} \left({}^{\beta}\mathcal{D}_{0+}^{\alpha} h_1\right)(x) = \left(\mathcal{D}_{0+}^{\alpha} h_1\right)(x), \tag{4.4}$$

for $x > 0$ and $\Re(\alpha) > 0$.

Lemma 2. *Let $\mathcal{V} \in \mathbf{C}^{l \times l}$ be a positive stable matrix, and let $\Re(\alpha) > 0$. Then, the generalized fractional integral of a power function admits the representation*

$${}^{\beta}\mathcal{I}_{0+}^{\alpha}\left(x^{\mathcal{V}-\mathcal{I}}\right) = \beta^{-\alpha}\, x^{\mathcal{V}+(\beta\alpha-1)\mathcal{I}}\, \Gamma\left(\frac{\mathcal{V} + \mathcal{I}(\beta-1)}{\beta}\right) \Gamma^{-1}\left(\frac{\mathcal{V} + \mathcal{I}(\beta-1)}{\beta} + \alpha\mathcal{I}\right), \tag{4.5}$$

with $\beta > 0$.

Proof. Using Definition 2 and applying the change of variable $u = tx$, we obtain

$${}^{\beta}\mathcal{I}_{0+}^{\alpha}\left(x^{\mathcal{V}-\mathcal{I}}\right) = \frac{\beta^{1-\alpha} x^{\beta\alpha+\mathcal{V}-\mathcal{I}}}{\Gamma(\alpha)} \int_0^1 t^{\mathcal{V}-\mathcal{I}+\beta-1} \left(1 - t^{\beta}\right)^{\alpha-1} dt.$$

Now setting $t^{\beta} = z$, we have

$$\begin{aligned} {}^{\beta}\mathcal{I}_{0+}^{\alpha}\left(x^{\mathcal{V}-\mathcal{I}}\right) &= \frac{\beta^{-\alpha} x^{\beta\alpha+\mathcal{V}-\mathcal{I}}}{\Gamma(\alpha)} \mathbf{B}\left(\frac{\mathcal{V} + \mathcal{I}(\beta-1)}{\beta},\ \alpha\mathcal{I}\right) \\ &= \beta^{-\alpha}\, x^{\beta\alpha+\mathcal{V}-\mathcal{I}}\, \Gamma\left(\frac{\mathcal{V} + \mathcal{I}(\beta-1)}{\beta}\right) \Gamma^{-1}\left(\frac{\mathcal{V} + \mathcal{I}(\beta-1)}{\beta} + \alpha\mathcal{I}\right). \end{aligned}$$

□

Remark 2. *Under the hypotheses of Lemma 2, taking the limit $\beta \to 1$ yields the following result due to Bakhet et al. [29]:*

$$\lim_{\beta\to 1} {}^{\beta}\mathcal{I}^{\alpha}_{0+}\left(x^{\mathcal{V}-\mathcal{I}}\right) = x^{\mathcal{V}+(\alpha-1)\mathcal{I}}\,\Gamma\left(\mathcal{V}\right)\Gamma^{-1}\left(\mathcal{V}+\alpha\mathcal{I}\right) = \mathcal{I}^{\alpha}_{0+}\left(x^{\mathcal{V}-\mathcal{I}}\right), \tag{4.6}$$

where $\Re(\alpha) > 0$.

Theorem 9. *Let $\mathcal{V}_1, \mathcal{V}_2 \in \mathbf{C}^{l\times l}$ be positive stable matrices, and let $\beta > 0$ and $\Re(\alpha) > 0$. Then, the generalized fractional integral of the five-parameter Mittag-Leffler matrix function is*

$${}^{\beta}\mathcal{I}^{\alpha}_{0+}\left[x^{\mathcal{V}_2-I}\mathrm{E}^{\mathcal{V}}_{\gamma_1,\frac{\gamma_2}{\beta},\mathcal{V}_1,\frac{\mathcal{V}_2+I(\beta-1)}{\beta}}(\omega x^{\gamma_2})\right] = \beta^{-\alpha}x^{\mathcal{V}_2+(\beta\alpha-1)I}\,\mathrm{E}^{\mathcal{V}}_{\gamma_1,\frac{\gamma_2}{\beta},\mathcal{V}_1,\frac{\mathcal{V}_2+I(\beta-1)}{\beta}+\alpha I}(\omega x^{\gamma_2}).$$

Proof. On using Definition 2 and (2.1), we have

$$\begin{aligned}
&{}^{\beta}\mathcal{I}^{\alpha}_{0+}\left[x^{\mathcal{V}_2-I}\mathrm{E}^{\mathcal{V}}_{\gamma_1,\frac{\gamma_2}{\beta},\mathcal{V}_1,\frac{\mathcal{V}_2+I(\beta-1)}{\beta}}(\omega x^{\gamma_2})\right]\\
&\quad= \frac{\beta^{1-\alpha}}{\Gamma(\alpha)}\int_0^x \frac{u^{\beta-1}}{(x^\beta-u^\beta)^{1-\alpha}}\,u^{\mathcal{V}_2-I}\mathrm{E}^{\mathcal{V}}_{\gamma_1,\frac{\gamma_2}{\beta},\mathcal{V}_1,\frac{\mathcal{V}_2+I(\beta-1)}{\beta}}(\omega u^{\gamma_2})du\\
&\quad= \Gamma^{-1}(\mathcal{V})\sum_{j=0}^{\infty}\Gamma(\mathcal{V}+jI)\Gamma^{-1}(\mathcal{V}_1+\gamma_1 j\mathcal{I})\,\Gamma^{-1}\left(\frac{\mathcal{V}_2+I(\beta-1)}{\beta}+\frac{\gamma_2}{\beta}jI\right)\frac{\omega^j}{j!}\\
&\quad\quad\times{}^{\beta}\mathcal{I}^{\alpha}_{0+}\left(x^{\mathcal{V}_2+(\gamma_2 j-1)I}\right).
\end{aligned}$$

Upon using Lemma 2, we have

$$\begin{aligned}
&{}^{\beta}\mathcal{I}^{\alpha}_{0+}\left[x^{\mathcal{V}_2-I}\mathrm{E}^{\mathcal{V}}_{\gamma_1,\frac{\gamma_2}{\beta},\mathcal{V}_1,\frac{\mathcal{V}_2+I(\beta-1)}{\beta}}(\omega x^{\gamma_2})\right]\\
&= \Gamma^{-1}(\mathcal{V})\sum_{j=0}^{\infty}\Gamma(\mathcal{V}+jI)\Gamma^{-1}(\mathcal{V}_1+\gamma_1 j\mathcal{I})\,\Gamma^{-1}\left(\frac{\mathcal{V}_2+I(\beta-1)}{\beta}+\frac{\gamma_2}{\beta}jI\right)\frac{\omega^j}{j!}\\
&\quad\times\beta^{-\alpha}\,x^{\mathcal{V}_2+\gamma_2 jI+(\beta\alpha-1)I}\,\Gamma\left(\frac{\mathcal{V}_2+I(\beta-1)}{\beta}+\frac{\gamma_2}{\beta}jI\right)\Gamma^{-1}\left(\frac{\mathcal{V}_2+I(\beta-1)}{\beta}+\alpha I+\frac{\gamma_2}{\beta}jI\right)\\
&= \Gamma^{-1}(\mathcal{V})\sum_{j=0}^{\infty}\Gamma(\mathcal{V}+jI)\,\Gamma^{-1}(\mathcal{V}_1+\gamma_1 j\mathcal{I})\,\Gamma^{-1}\left(\frac{\mathcal{V}_2+I(\beta-1)}{\beta}+\alpha I+\frac{\gamma_2}{\beta}jI\right)\frac{\omega^j}{j!}\\
&\quad\times\beta^{-\alpha}\,x^{\mathcal{V}_2+\gamma_2 jI+(\beta\alpha-1)I}.
\end{aligned}$$

Now in accordance with (2.1), this yields the proof of Theorem 9. □

Corollary 8. *Under the hypotheses of Theorem 9, taking the limit $\beta \to 1$ yields*

$$\mathcal{I}^{\alpha}_{0+}\left[x^{\mathcal{V}_2-\mathcal{I}}\,\mathrm{E}^{\mathcal{V}}_{\gamma_1,\gamma_2,\mathcal{V}_1,\mathcal{V}_2}(\omega x^{\gamma_2})\right] = x^{\mathcal{V}_2+(\alpha-1)\mathcal{I}}\,\mathrm{E}^{\mathcal{V}}_{\gamma_1,\gamma_2,\mathcal{V}_1,\mathcal{V}_2+\alpha\mathcal{I}}(\omega x^{\gamma_2}), \tag{4.7}$$

where $\Re(\alpha) > 0$.

Theorem 10. *Let $\mathcal{V}_1, \mathcal{V}_2 \in \mathbb{C}^{l\times l}$ be positive stable matrices, and let $\beta > 0$ and $\Re(\alpha) > 0$. Then, the generalized fractional derivative of the five-parameter Mittag-Leffler matrix function admits the following representation:*

$$ {}^{\beta}\mathcal{D}^{\alpha}_{0+}\left[x^{\mathcal{V}_2-\mathcal{I}}\,\mathrm{E}^{\mathcal{V}}_{\gamma_1,\frac{\gamma_2}{\beta},\mathcal{V}_1,\frac{\mathcal{V}_2+\mathcal{I}(\beta-1)}{\beta}}(\omega x^{\gamma_2})\right] = \beta^{-(\zeta-\alpha)}\,x^{\mathcal{V}_2-(\beta\alpha+1)\mathcal{I}}\,\mathrm{E}^{\mathcal{V}}_{\gamma_1,\frac{\gamma_2}{\beta},\mathcal{V}_1,\frac{\mathcal{V}_2+\mathcal{I}(\beta-1)}{\beta}-\alpha\mathcal{I}}(\omega x^{\gamma_2}). \tag{4.8}$$

Proof. From Definition 3, we get

$$ {}^{\beta}\mathcal{D}^{\alpha}_{0+}\left[x^{\mathcal{V}_2-I}\mathrm{E}^{\mathcal{V}}_{\gamma_1,\frac{\gamma_2}{\beta},\mathcal{V}_1,\frac{\mathcal{V}_2+I(\beta-1)}{\beta}}(\omega x^{\gamma_2})\right] = \left(x^{1-\beta}\frac{d}{dx}\right)^{\zeta}\left[{}^{\beta}\mathcal{I}^{\zeta-\alpha}_{0+}\left(x^{\mathcal{V}_2-I}\mathrm{E}^{\mathcal{V}}_{\gamma_1,\frac{\gamma_2}{\beta},\mathcal{V}_1,\frac{\mathcal{V}_2+I(\beta-1)}{\beta}}(\omega x^{\gamma_2})\right)\right].$$

Using Theorem 9 gives

$$\begin{aligned} &{}^{\beta}\mathcal{D}^{\alpha}_{0+}\left[x^{\mathcal{V}_2-I}\mathrm{E}^{\mathcal{V}}_{\gamma_1,\frac{\gamma_2}{\beta},\mathcal{V}_1,\frac{\mathcal{V}_2+I(\beta-1)}{\beta}}(\omega x^{\gamma_2})\right] \\ &= x^{\zeta(1-\beta)}\times\left(\frac{d}{dx}\right)^{\zeta}\left[\beta^{-(\zeta-\alpha)}x^{\mathcal{V}_2+(\beta(\zeta-\alpha)-1)I}\,\mathrm{E}^{\mathcal{V}}_{\gamma_1,\frac{\gamma_2}{\beta},\mathcal{V}_1,\frac{\mathcal{V}_2+I(\beta-1)}{\beta}+(\zeta-\alpha)I}(\omega x^{\gamma_2})\right]. \end{aligned}$$

From Theorem 3, this immediately yields the desired proof of Theorem 10. □

Corollary 9. *Under the hypotheses of Theorem 10, taking the limit $\beta \to 1$ yields*

$$\mathcal{D}^{\alpha}_{0+}\left[x^{\mathcal{V}_2-I}\mathrm{E}^{\mathcal{V}}_{\gamma_1,\gamma_2,\mathcal{V}_1,\mathcal{V}_2}(\omega x^{\gamma_2})\right] = x^{\mathcal{V}_2-(\alpha+1)I}\,\mathrm{E}^{\mathcal{V}}_{\gamma_1,\gamma_2,\mathcal{V}_1,\mathcal{V}_2-\alpha\mathcal{I}}(\omega x^{\gamma_2}), \tag{4.9}$$

where $\Re(\alpha) > 0$.

5. Conclusions

In this paper, we have introduced and systematically investigated a matrix analogue of the five-parameter Mittag-Leffler function. We established the absolute convergence of the defining series on the unit circle $|\varpi| = 1$ under specific spectral conditions involving the positive stable matrices $\mathcal{V}, \mathcal{V}_1$, and $\mathcal{V}_2$. Several fundamental properties of this matrix function were derived, including:

- Integral representations: A key integral formula was proved, linking the function to beta-type integrals.
- Derivative formulas: Explicit formulas for repeated differentiation were obtained, demonstrating how the parameters shift within the function.
- Differential recurrence relations: Simple recurrence relations connecting the function to its derivatives were established.
- Finite summation formulas: A variety of new finite sums involving the matrix function and the Fox-Wright matrix analog were derived using classical summation techniques.

Furthermore, we explored the interaction of this new matrix function with fractional calculus. Specifically, we examined the composition of the generalized (Katugampola) fractional integral and derivative operators with the five-parameter Mittag-Leffler matrix function, obtaining closed-form expressions that naturally extend the function's parameters.

The results presented in this work contribute to the growing theory of special matrix functions and their applications. Future research could focus on investigating further properties such as Laplace transforms, applications to solving fractional differential equations of matrix order, and exploring connections with other orthogonal matrix polynomials.

Author contributions

Salma Aljawi, Vinod Kumar Jatav and Ankit Pal: Conceptualization, methodology, writing–original draft, writing–review and editing. All authors of this article have contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors express their gratitude to Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2026R514), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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