Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(6): 17191–17207.
DOI:10.3934/math.2026705
Received: 20 January 2026
Revised: 02 June 2026
Accepted: 04 June 2026
Published: 15 June 2026

Research article

On the small Schröder semigroup $\mathcal{SS}'_n$

Muhammad Mansur Zubairu[1,*], **Abdullahi Umar**[2] **and Fatma Salim Al-Kharousi**[3]

[1] Department of Mathematical Sciences, Bayero University Kano, P. M. B. 3011, Kano, Nigeria
[2] Department of Mathematical Sciences, Khalifa University, P. O. Box 127788, Abu Dhabi, UAE
[3] Department of Mathematics, College of Science, Sultan Qaboos University, P. O. Box 36, PC123 Al Khod, Muscat, Seeb Oman

* **Correspondence:** Email: mmzubairu.mth@buk.edu.ng.

Abstract: Let $[n]$ be a finite n-chain $\{1, 2, \ldots, n\}$, and let $\mathcal{LS}_n$ be the large Schröder monoid, consisting of all isotone and order-decreasing partial transformations on $[n]$. Furthermore, let $\mathcal{SS}'_n = \{\alpha \in \mathcal{LS}_n : 1 \notin \mathrm{Dom}\,\alpha\}$ be the subsemigroup of $\mathcal{LS}_n$, consisting of all transformations in $\mathcal{LS}_n$, not containing 1 in their domains. For $1 \leq p \leq k \leq n$, let $I(n, k) = \{\alpha \in \mathcal{SS}'_n : |\mathrm{Im}\ \alpha| \leq k\}$ be the two-sided ideal of $\mathcal{SS}'_n$, consisting of transformations of height at most k, and let $RSS'_n(p)$ denote the Rees quotient of $I(n, k)$. It is shown in this article that the object $\mathcal{SS}'_n$ is a *left monoid* but not a *right monoid*. Moreover, it is shown that for any $p \leq k$ and any $S \in \{\mathcal{SS}'_n, I(n, k), RSS'_n(p)\}$, S is right abundant for all values of n, but not left abundant for all $n \geq 2$. In addition, the rank of the Rees quotient $RSS'_n(p)$ is shown to be equal to the rank of the two-sided ideal $I(n, p)$, which is equal to $\binom{n-1}{p-1} + \sum_{r=p}^{n-1} \binom{n-1}{r}\binom{r-1}{p-1}$. Finally, the rank of $\mathcal{SS}'_n$ is determined to be $3n - 4$.

Keywords: isotone maps; order decreasing; abundant semigroup; rank properties
Mathematics Subject Classification: 20M20

1. Introduction and preliminaries

A partial transformation defined on an n-chain $[n] = \{1, 2, \ldots, n\}$ is a map whose domain and co-domain are both subsets of $[n]$. A transformation α is said to be an *isotone* map (resp., an *anti-tone* map) if (for all $x, y \in \mathrm{Dom}\,\alpha$) $x \leq y$ implies $x\alpha \leq y\alpha$ (resp., $x\alpha \geq y\alpha$); and *order-decreasing* if (for all $x \in \mathrm{Dom}\,\alpha$) $x\alpha \leq x$. Let $\mathcal{LS}_n$ denote the large Schröder monoid consisting of all isotone and order-decreasing partial transformations. This monoid can be obtained as the intersection of the monoid of all order-decreasing partial transformations, $\mathcal{DP}_n$, and the monoid of all isotone partial transformations on $[n]$, $\mathcal{OP}_n$, i.e.,

$$\mathcal{LS}_n = \mathcal{OP}_n \cap \mathcal{DP}_n. \tag{1.1}$$

The monoids $\mathcal{OP}_n$, $\mathcal{DP}_n$, and $\mathcal{LS}_n$ have been studied in various contexts. The algebraic and combinatorial properties of the monoids $\mathcal{OP}_n$ and $\mathcal{DP}_n$ have been investigated in [19–21]. More recently, the rank and maximal subsemigroups of the monoid $\mathcal{LS}_n$ have been investigated in [3], its combinatorial properties have been studied in [18], and subsequently its algebraic properties have been explored in [23]. Algebraic, rank, and combinatorial properties of various types of transformation semigroups/monoids have been investigated over the years; see for example [1, 2, 8] for algebraic results and [6, 7] for combinatorial results.

Now let

$$\mathcal{SS}_n = \{\alpha \in \mathcal{LS}_n : 1 \in \text{Dom } \alpha\} \quad \text{and} \quad \mathcal{SS}'_n = \{\alpha \in \mathcal{LS}_n : 1 \notin \text{Dom } \alpha\}, \tag{1.2}$$

where Dom α denotes the domain set of α. The above sets first appeared in [18], where they were shown to be subsemigroups of $\mathcal{LS}_n$, interestingly having the same order:

$$s_0 = 1, \quad s_n = \frac{1}{2(n+1)} \sum_{r=0}^{n} \binom{n+1}{n-r}\binom{n+r}{r} \ (n \geq 1).$$

This number is the well-known (*small*) *Schröder number*. In [18], the monoid $\mathcal{SS}_n$ is referred to as the (*small*) *Schröder monoid*, and it has been shown to be an idempotent-generated abundant monoid. Its rank has been shown to be $2n-1$, and the ranks of its two-sided ideals and their corresponding Rees quotients were all found in [23]. Even though both $\mathcal{SS}_n$ and $\mathcal{SS}'_n$ have the same order, they are algebraically different. Indeed, they have different ranks, as it will be demonstrated in this paper. The algebraic properties and rank properties of the semigroup $\mathcal{SS}'_n$ do not seem to have been investigated. This paper addresses the aforementioned questions. It is worth noting that $\mathcal{SS}'_1 = \emptyset$; therefore, we shall consider $\mathcal{SS}'_n$ for $n \geq 2$ henceforth. It is also worth mentioning that $\mathcal{SS}'_n$ has no identity element; thus, in line with [18], we shall refer to it as the *small Schröder semigroup*. It is noteworthy that $\mathcal{LS}_{n-1} \subset \mathcal{SS}'_n \subset \mathcal{LS}_n$.

Now, given any two elements α and β in $\mathcal{SS}'_n$, we will use the notation $\alpha\beta$ to mean $\alpha \circ \beta$. We shall adopt the right-hand composition of functions; that is, for any given $x \in \text{Dom } \alpha$, we have $x\alpha\beta = ((x)\alpha)\beta$. Moreover, the *set of fixed points* of α, the *image* set of α, the *number of fixed points* of α, the *height* of α, and the *identity* on any subset A of $[n]$ shall respectively be denoted by

$$F(\alpha) = \{x \in \text{Dom } \alpha : \ x\alpha = x\}, \quad \text{Im } \alpha, \quad f(\alpha) = |F(\alpha)|, \quad h(\alpha) = |\text{Im } \alpha|, \quad \text{and} \quad 1_A.$$

Furthermore, for $0 \leq p \leq k \leq n-1$, let

$$I(n,k) = \{\alpha \in \mathcal{SS}'_n : |\text{Im } \alpha| \leq k\} \quad \text{and} \quad RSS'_n(p) = I(n,p)/I(n,p-1) \quad (p \geq 2). \tag{1.3}$$

For each k, $I(n,k)$ is a two-sided ideal of $\mathcal{SS}'_n$ consisting of maps in $\mathcal{SS}'_n$, each with a height at most k, while for each $0 \leq p \leq k \leq n-1$, $RSS'_n(p)$ is the Rees quotient semigroup of $I(n,p)$. The elements in $RSS'_n(p)$ can be considered as elements of exactly height p in $\mathcal{SS}'_n$ with the following product:

$$\alpha\beta = \begin{cases} \alpha\beta, & \text{if } h(\alpha\beta) = p, \\ 0, & \text{if } h(\alpha\beta) < p. \end{cases}$$

As in [16], an element $\alpha \in \mathcal{SS}'_n$ of height $1 \le p \le n-1$ can always be represented as

$$\alpha = \begin{pmatrix} D_1 & \cdots & D_p \\ y_1 & \cdots & y_p \end{pmatrix} \ (1 \le p \le n-1), \tag{1.4}$$

where $y_i \le \min D_i$ for all $1 \le i \le p$ and in particular $1 < \min D_1$; so also, $D_i < D_j$ if and only if $i < j$ ($i < j$ if and only if $a < b$ for all $a \in D_i$ and $b \in D_j$) by the decreasing and isotone properties, respectively. We shall write **Ker** $\alpha = \{D_1, \dots, D_p\}$ to be the partition of Dom $\alpha \subseteq \{2, \dots, n\}$ by the equivalence relation $\ker \alpha = \{(x, y) \in \text{Dom } \alpha \times \text{Dom } \alpha : x\alpha = y\alpha\}$. Clearly, $1 \le y_1 \le \ldots \le y_p \le n$ by the isotone property. Furthermore, we shall adopt the usual notation $\mathcal{P}_n$ to denote the semigroup of all partial transformations $[n]$. For basic definitions and concepts in semigroup theory, we recommend to the reader the books of Howie [13] and Higgins [17].

Now for $0 \le p \le n-1$, let $J^*_p = \{\alpha \in \mathcal{SS}'_n : h(\alpha) = p\}$. Then evidently, $\mathcal{SS}'_n = J^*_0 \cup \cdots \cup J^*_{n-1}$. An element a in a semigroup S is called an *idempotent* if $a^2 = a$, and as usual the set of all idempotents of any subset A of a semigroup S shall be denoted by $E(A)$. In particular, in the semigroup $\mathcal{SS}'_n$, $E(J^*_{n-1}) = \left\{\begin{pmatrix} 2 & \cdots & n \\ 2 & \cdots & n \end{pmatrix}\right\}$.

2. Algebraic properties of the small Schröder semigroup

We would like to begin the section by introducing the following definition:

Definition 2.1. *A semigroup S is called a* left monoid *if S has a unique left identity; it is called a* right monoid *if it has a unique right identity.*

By way of remark, it means that every monoid is a both left and right monoid, but the converse may not be true. We will prove in the next theorem that the small Schröder semigroup possesses the property of being a left monoid.

Theorem 2.1. *The small Schröder semigroup $\mathcal{SS}'_n$ is a left monoid, but not a monoid.*

Proof. Take $\alpha \in \mathcal{SS}'_n$ as expressed in Eq (1.4), and take $\epsilon = \begin{pmatrix} 2 & \cdots & n \\ 2 & \cdots & n \end{pmatrix}$, which is the idempotent in $E(J^*_{n-1})$. Observe that

$$\epsilon\alpha = \begin{pmatrix} 2 & \cdots & n \\ 2 & \cdots & n \end{pmatrix}\begin{pmatrix} D_1 & \cdots & D_p \\ y_1 & \cdots & y_p \end{pmatrix} = \begin{pmatrix} D_1 & \cdots & D_p \\ y_1 & \cdots & y_p \end{pmatrix} = \alpha.$$

Now if $\alpha = \emptyset$, so is $\epsilon\alpha = \emptyset = \alpha$. However, for all $n \ge 2$, the map $\alpha = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is in $\mathcal{SS}'_n$. Notice that

$$\alpha\epsilon = \begin{pmatrix} 2 \\ 1 \end{pmatrix}\begin{pmatrix} 2 & \cdots & n \\ 2 & \cdots & n \end{pmatrix} = \emptyset \ne \alpha.$$

Thus, ϵ is not a right identity. Therefore, no right identity exists, because if a right identity exists, it must coincide with the left identity ϵ. Clearly, ϵ is unique since it is the only idempotent element of height $n-1$, and so it is a unique left identity, as required. □

If a semigroup S does not have an identity, then an identity can be adjoined to S, and the new set, denoted as $S^1 = S \cup \{1\}$, is a monoid (see [13, p. 2]). Given any new semigroup, the usual algebraic enquiry about the semigroup is the characterization of its Green's relations, provided such a semigroup has some idempotent elements. The five Green's equivalences introduced by J. A Green [9] are $\mathcal{R}$, $\mathcal{L}$, $\mathcal{D}$, $\mathcal{H}$, and $\mathcal{J}$, which are defined as

$$(a, b) \in \mathcal{L} \Leftrightarrow S^1 a = S^1 b,$$

$$(a, b) \in \mathcal{R} \Leftrightarrow aS^1 = bS^1,$$

$$(a, b) \in \mathcal{J} \Leftrightarrow S^1 aS^1 = S^1 bS^1,$$

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R},$$

while

$$\mathcal{D} = \mathcal{L} \circ \mathcal{R}.$$

Elements α and β of height $1 \leq p \leq n-1$ in $\mathcal{SS}'_n$, can be represented as

$$\alpha = \begin{pmatrix} A_1 & \cdots & A_p \\ a_1 & \cdots & a_p \end{pmatrix} \text{and}\, \beta = \begin{pmatrix} B_1 & \cdots & B_p \\ b_1 & \cdots & b_p \end{pmatrix}, \tag{2.1}$$

where $1 \leq a_1 < \cdots < a_p \leq n$ and $1 \leq b_1 < \cdots < b_p \leq n$ due to the isotone property of α and β, and also $a_i \leq \min A_i$ and $b_i \leq \min B_i$ for all $1 \leq i \leq p$, by the order-decreasing property. We now state the following result.

Theorem 2.2. *Let $\alpha, \beta \in \mathcal{SS}'_n$ be as in Eq (2.1). Then $\alpha \mathcal{L} \beta$ if and only if* $\mathrm{Im}\ \alpha = \mathrm{Im}\ \beta$ (*i.e., $a_i = b_i$ for* $1 \leq i \leq p$) *and* $\min A_i = \min B_i$ *for all* $1 \leq i \leq p$.

Proof. The proof, going forward, resembles the proof in [21, Lemma 2.2.1(2)].

Conversely, suppose that $\mathrm{Im}\ \alpha = \mathrm{Im}\ \beta$ and $\min A_i = \min B_i$ for all $1 \leq i \leq p$. Notice that the assumption $\mathrm{Im}\ \alpha = \mathrm{Im}\ \beta$ and the isotone property on α and β, ensures that $a_i = b_i$ for all $1 \leq i \leq p$. Now let $t_i = \min A_i = \min B_i$ for $1 \leq i \leq p$, and define γ_1, γ_2 as

$$\gamma_1 = \begin{pmatrix} A_1 & \dots & A_p \\ t_1 & \dots & t_p \end{pmatrix} \text{ and } \gamma_2 = \begin{pmatrix} B_1 & \dots & B_p \\ t_1 & \dots & t_p \end{pmatrix}. \tag{2.2}$$

Clearly, $\gamma_1, \gamma_2 \in \mathcal{SS}'_n$. Moreover, it easily follows that $\alpha = \gamma_1\beta$ and $\beta = \gamma_2\alpha$. Thus, $(\alpha, \beta) \in \mathcal{L}$, as required. □

Notice that the small Schröder semigroup $\mathcal{SS}'_n$ is a subsemigroup of $\mathcal{LS}_n$, and since by [23, Theorem 4.2], $\mathcal{LS}_n$ is $\mathcal{R}$-trivial, so is $\mathcal{SS}'_n$; it is not difficult to see that the characterizations of the relations $\mathcal{H}$ and $\mathcal{D}$ in $\mathcal{LS}_n$ as demonstrated in [23] are the same as in $\mathcal{SS}'_n$. Thus, we have

$$(\alpha, \beta) \in \mathcal{R} \Leftrightarrow \alpha = \beta,$$

$$\mathcal{H} = \mathcal{R},$$

$$\mathcal{J} = \mathcal{D} = \mathcal{L}\ \ (\mathcal{J} = \mathcal{D} \text{ since } \mathcal{SS}'_n \text{ is a finite semigroup}).$$

An element a in a semigroup S is called *regular* if $a = aba$ for some $b \in S$. The following is a well-known result about regularity in an $\mathcal{R}$-trivial semigroup.

Proposition 2.1. *[24, Lemma 4] In an $\mathcal{R}$-trivial semigroup S, an element a in S is regular if and only if a is an idempotent.*

Since $\mathcal{SS}'_n$ is an $\mathcal{R}$-trivial semigroup, we deduce the following lemma.

Lemma 2.1. *An element $\alpha \in \mathcal{SS}'_n$ is regular if and only if α is an idempotent.*

A direct implication of the above theorem is that $\mathcal{SS}'_n$ is non-regular. We conclude that based on the characterization of Green's relations on $\mathcal{SS}'_n$, for any semigroup S in $\{RSS'_n(p), I(n,k)\}$, if $\mathcal{K} \in \{\mathcal{L}, \mathcal{R}, \mathcal{D}, \mathcal{H}\}$, then $\mathcal{K}(S)$ has the same characterization as $\mathcal{K}$ on $\mathcal{SS}'_n$, and hence for $p \geq 2$, S is not a regular semigroup. Non-regular semigroups can be studied structurally via starred Green's relations; these are relations (see Fountain [4, 5]) $\mathcal{R}^*$, $\mathcal{L}^*$, $\mathcal{H}^*$, $\mathcal{D}^*$, and $\mathcal{J}^*$. The relation $\mathcal{L}^*$ is defined as (for all $a, b \in S$) $a\mathcal{L}^*b$ if and only if a, b are related by Green's $\mathcal{L}$ relation in some oversemigroup of S. The relation $\mathcal{R}^*$ is defined dually, while the relation $\mathcal{D}^*$ is defined as the join of the relations $\mathcal{L}^*$ and $\mathcal{R}^*$. The intersection of $\mathcal{L}^*$ and $\mathcal{R}^*$ is denoted by $\mathcal{H}^*$. The relations $\mathcal{R}^*$ and $\mathcal{L}^*$ have the following interpretations:

$$\mathcal{R}^* = \{(a, b) \in S \times S \ : \ (\text{for all } x, y \in S^1) \ xa = ya \iff xb = yb\}, \tag{2.3}$$

and

$$\mathcal{L}^* = \{(a, b) \in S \times S \ : \ (\text{for all } x, y \in S^1) \ ax = ay \iff bx = by\}, \tag{2.4}$$

respectively. It is a known fact that in a finite non-regular semigroup, $\mathcal{R}^*$ and $\mathcal{L}^*$ may not commute. A semigroup S is said to be: *left abundant* if every $\mathcal{L}^*$-class contains an idempotent; *right abundant* if every $\mathcal{R}^*$-class contains an idempotent; and *abundant* if it is both left and right abundant. The concept of *left and right inverse ideals* in semigroups can be traced back to Umar [22], where it is defined that a subsemigroup U of a semigroup S is said to be a *right inverse ideal* of S if $\forall u \in U$, $\exists\, u' \in S$ such that $uu'u = u$ and $uu' \in U$; on the other hand, if for $u \in U$, there exists $u' \in S$ such that $uu'u = u$ and $u'u \in U$, then it is said to be a *left inverse ideal* of S; and it is an *inverse ideal* of S if it is both left and right inverse ideal. In fact, the following results have been proved in [22].

Lemma 2.2. *[22, Lemma 3.1.8] If U is a right inverse ideal of S, then U is a right abundant semigroup.*

Lemma 2.3. *If U is a right inverse ideal of S, then $\mathcal{R}^*(U)$ and $\mathcal{R}(S) \cap (U \times U)$ are equal.*

We can now prove the following result.

Lemma 2.4. *For $n \geq 2$, the small Schröder semigroup $\mathcal{SS}'_n$ is a right inverse ideal of the semigroup of all partial transformations $\mathcal{P}_n$, but not a left inverse ideal.*

Proof. Consider an element $\alpha \in \mathcal{SS}'_n$ as expressed in Eq (1.4). Now denote $\min D_i = c_i$ for all $i \in \{1, \ldots, p\}$. Define a map β from $\operatorname{Im} \alpha$ to the set $\{c_i : 1 \leq i \leq p\}$ by $y_i\beta = c_i$ for all $1 \leq i \leq p$. Then observe that for each $1 \leq i \leq p$

$$D_i\alpha\beta\alpha = (D_i\alpha)\beta\alpha = y_i\beta\alpha = c_i\alpha = D_i\alpha.$$

Thus, $\alpha\beta\alpha = \alpha$. Moreover, for each $1 \leq i \leq p$,

$$c_i\alpha\beta = y_i\beta = c_i.$$

This ensures that the blocks D_i are *stationary* (i.e., $c_i \in D_i$ for each i), and so $\alpha\beta \in E(\mathcal{SS}'_n)$, proving the first claim. For the second claim, observe that the map $\alpha = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an element of $\mathcal{SS}'_n$ for all $n \geq 2$. Notice that if for any $\beta \in \mathcal{P}_n$, $\alpha\beta\alpha = \alpha$, then it is necessary that $1 \in \operatorname{Dom}\beta$. This ensures that $1 \in \operatorname{Dom}\beta\alpha$, violating the fact that 1 does not belong to the domain of an element in $\mathcal{SS}'_n$. The result now follows. □

By virtue of Lemmas 2.2 and 2.4, we have actually proved the following result.

Theorem 2.3. *For $n \geq 2$, the small Schröder semigroup $\mathcal{SS}'_n$ is right abundant.*

Moving forward in this section, we will denote α and β in $\mathcal{SS}'_n$ as

$$\alpha = \begin{pmatrix} D_1 & \cdots & D_p \\ y_1 & \cdots & y_p \end{pmatrix}, \text{ and } \beta = \begin{pmatrix} E_1 & \cdots & E_p \\ x_1 & \cdots & x_p \end{pmatrix} \ (1 \leq p \leq n-1). \tag{2.5}$$

One of the applications of the above theorem is the following characterization of the starred Green's relation.

Theorem 2.4. *For $\alpha, \beta \in \mathcal{SS}'_n$, we have*

(i) $\alpha\mathcal{R}^*\beta \Leftrightarrow \ker\alpha = \ker\beta$*;*
(ii) $\alpha\mathcal{L}^*\beta \Leftrightarrow \operatorname{Im}\alpha = \operatorname{Im}\beta$*;*
(iii) $\alpha\mathcal{H}^*\beta \Leftrightarrow \alpha = \beta$*;*
(iv) $\alpha\mathcal{D}^*\beta \Leftrightarrow |\operatorname{Im}\alpha| = |\operatorname{Im}\beta|$.

Proof. (i) Both the forward and the converse are direct consequences of Lemmas 2.3 and 2.4, along with [13, Exercise 2.6.17].

(ii) Let $\alpha, \beta \in \mathcal{SS}'_n$ such that $\alpha\mathcal{L}^*\beta$. Now if $1 \notin \operatorname{Im}\alpha$, define $\gamma = 1_{\operatorname{Im}\alpha}$ (in $\mathcal{SS}'_n$). Then by Eq (2.4), we see that

$$\alpha\gamma = \alpha 1_{[n]} \iff \beta\gamma = \beta 1_{[n]},$$

which implies that $\operatorname{Im}\beta = \operatorname{Im}\alpha$.

Now if $1 \in \operatorname{Im}\alpha$, then define $\gamma = 1_{\operatorname{Im}\alpha\setminus\{1\}}$ (in $\mathcal{SS}'_n$). Notice that $1_{[n]\setminus\{1\}} = \begin{pmatrix} 2 & \cdots & n \\ 2 & \cdots & n \end{pmatrix} = \epsilon$ is the unique left identity in $\mathcal{SS}'_n$ which belongs to $\mathcal{SS}'^{\,1}_n$, and so by Eq (2.4), we see that

$$\alpha\gamma = \alpha\epsilon \iff \beta\gamma = \beta\epsilon,$$

which also implies that $\operatorname{Im}\beta = \operatorname{Im}\alpha$.

Conversely, if $\operatorname{Im}\alpha = \operatorname{Im}\beta$, then by [13, Exercise 2.6.17], $\alpha\mathcal{L}^{\mathcal{P}_n}\beta$ (meaning that α and β are $\mathcal{L}$ related in $\mathcal{P}_n$), and so we conclude $\alpha\mathcal{L}^*\beta$ by the definition of $\mathcal{L}^*$.

(iii) This follows from (i) and (ii) along with the fact that α and β are both isotone maps.

(iv) The forward implication is the direct application of [13, Proposition 1.5.11] together with (i) and (ii).

Conversely, suppose $|\operatorname{Im}\beta| = |\operatorname{Im}\alpha|$, where

$$\alpha = \begin{pmatrix} D_1 & \cdots & D_p \\ y_1 & \cdots & y_p \end{pmatrix}, \text{ and } \beta = \begin{pmatrix} E_1 & \cdots & E_p \\ x_1 & \cdots & x_p \end{pmatrix}. \tag{2.6}$$

Now define

$$\delta = \begin{pmatrix} D_1 & \cdots & D_p \\ 1 & \cdots & p \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} E_1 & \cdots & E_p \\ 1 & \cdots & p \end{pmatrix}.$$

The elements δ and γ are clearly in $\mathcal{SS}'_n$; furthermore $\ker\alpha = \ker\delta$, $\operatorname{Im}\delta = \operatorname{Im}\gamma$, and $\ker\gamma = \ker\beta$. So, using (i) and (ii), it follows that $\alpha\mathcal{R}^*\delta\mathcal{L}^*\gamma\mathcal{R}^*\beta$. Hence, $\alpha\mathcal{D}^*\beta$ follows from [13, Proposition 1.5.11], as required.

□

Lemma 2.5. *The Schröder semigroup $\mathcal{SS}'_n$ has the following property: for $n \geq 3$, $\mathcal{D}^* = \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^* = \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^*$ and $\mathcal{D}^* = \mathcal{L}^* \circ \mathcal{R}^* \neq \mathcal{R}^* \circ \mathcal{L}^*$. Moreover, for $n = 2$, $\mathcal{D}^* = \mathcal{L}^* \circ \mathcal{R}^* = \mathcal{R}^* \circ \mathcal{L}^*$.*

Proof. The first equality is derived from the converse of the proof of (iv) in the aforementioned theorem.

For the second equality, consider α and β as expressed in Eq (2.6), and define $\delta = \begin{pmatrix} n-p+1 & \cdots & n \\ y_1 & \cdots & y_p \end{pmatrix}$ and $\gamma = \begin{pmatrix} n-p+1 & \cdots & n \\ x_1 & \cdots & x_p \end{pmatrix}$. Clearly, δ and $\gamma \in \mathcal{SS}'_n$. Also observe that $\operatorname{Im}\alpha = \operatorname{Im}\delta$, $\ker\delta = \ker\gamma$, and $\operatorname{Im}\gamma = \operatorname{Im}\beta$. Now using Theorem 2.4(i) and (ii), it follows that

$$\alpha\mathcal{L}^*\delta\mathcal{R}^*\gamma\mathcal{L}^*\beta.$$

To establish the necessity, we need to demonstrate that $\mathcal{L}^* \circ \mathcal{R}^* \neq \mathcal{R}^* \circ \mathcal{L}^*$. To do so, for any $n \geq 3$, take

$$\alpha = \begin{pmatrix} 2 & \cdots & n-1 \\ 2 & \cdots & n-1 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 3 & \cdots & n \\ 3 & \cdots & n \end{pmatrix}.$$

Now define $\delta = \begin{pmatrix} 3 & 4 & \cdots & n \\ 2 & 3 & \cdots & n-1 \end{pmatrix}$. Thus, it is evident that $\operatorname{Im}\alpha = \operatorname{Im}\delta$ and $\ker\delta = \ker\beta$, which implies that $\alpha\mathcal{L}^*\delta\mathcal{R}^*\beta$. In other words, $(\alpha,\beta) \in \mathcal{L}^* \circ \mathcal{R}^*$.

Conversely, if we have $(\alpha,\beta) \in \mathcal{R}^* \circ \mathcal{L}^*$, it implies that there is a $\gamma \in \mathcal{SS}'_n$ such that $\alpha\mathcal{R}^*\gamma\mathcal{L}^*\beta$. This implies that $\ker\alpha = \ker\gamma = \{2, \ldots, n-1\}$ and $\operatorname{Im}\gamma = \operatorname{Im}\beta = \{3, \ldots, n\}$, which is not possible. Thus, the conclusion follows.

The last part of the lemma is trivial. □

By way of remark we have the following.

Remark 2.1. *For any semigroup $S \in \{RSS'_n(p), I(n,k)\}$ where $1 \leq p \leq k \leq n-1$, we have $\mathcal{D}^* = \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^* = \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^*$.*

Lemma 2.6. *The Schröder semigroup $\mathcal{SS}'_n$ is not left abundant for all $n \geq 2$.*

Proof. Consider $\alpha = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \in \mathcal{SS}'_n$ for all $n \geq 2$. Now the $\mathcal{L}^*$-class of α is

$$L^*{}_{\alpha} = \left\{ \begin{pmatrix} D \\ 1 \end{pmatrix} \mid D \subseteq \{2, \ldots, n\} \right\},$$

which clearly has no idempotent element. The result now follows. □

In accordance with [5], to define the relation $\mathcal{J}^*$ on a semigroup S, we first denote the $\mathcal{L}^*$-class that contains the element $a \in S$ as L_a^*. Similar notation applies to the classes of other relations. A *left* (or *right*) $*$-*ideal* of a semigroup S is defined as a *left* (or *right*) ideal I of S such that $L_a^* \subseteq I$ (or $R_a^* \subseteq I$) for every $a \in I$. A subset I of S is said to be a $*$-*ideal* of S if it is both a left and a right $*$-ideal of S. The *principal* $*$-*ideal* $J^*(a)$ generated by the element $a \in S$ is defined as the intersection of all $*$-ideals of S containing a. The relation $\mathcal{J}^*$ is defined such that $a\mathcal{J}^*b$ if and only if $J^*(a) = J^*(b)$, where $J^*(a)$ is the principal $*$-ideal generated by a. We state the following results without proofs because they are similar to the proofs of [23, Lemmas 2.16–2.18].

Lemma 2.7. *Let* $\alpha, \beta \in \mathcal{SS}'_n$. *Then*

(i) $\alpha \in J^*(\beta)$ *implies* $|\text{Im } \alpha| \leq |\text{Im } \beta|$;
(ii) In the small Schröder semigroup $\mathcal{SS}'_n$, *it holds that* $\mathcal{J}^* = \mathcal{D}^*$;
(iii) In the semigroup S *within* $\{\mathcal{SS}'_n, I(n,k)\}$, *each* $\mathcal{R}^*$*-class contains a unique idempotent.*

We can now make the following useful remarks.

Remark 2.2. *(i) It is now evident that for each* $1 \leq p \leq n$, *the number of* $\mathcal{R}^*$*-classes in* $J_p^* = \{\alpha \in \mathcal{SS}'_n : |\text{Im } \alpha| = p\}$ *is equal to the number of all possible partially ordered partitions of* $[n] \setminus \{1\}$ *into* p *parts. This quantity matches the number of* $\mathcal{R}$*-classes in* $\{\alpha \in \mathcal{OP}_n : |\text{Im } \alpha| = p\}$ *(i.e.,* $\sum_{r=p}^{n} \binom{n}{r}\binom{r-1}{p-1}$*, as shown in [20, Lemma 4.1]), minus the number of* $\mathcal{R}$*-classes in* $\{\alpha \in \mathcal{SS}_n : |\text{Im } \alpha| = p\}$ *(i.e.,* $\sum_{r=p}^{n} \binom{n-1}{r-1}\binom{r-1}{p-1}$*, as shown in [23, Lemma 2.20]). This value is clearly equal to* $\sum_{r=p}^{n-1} \binom{n-1}{r}\binom{r-1}{p-1}$.

(ii) If $S \in \{RSS'_n(p), I(n,k)\}$ *where* $p \leq k$*, then the characterizations of the starred Green's relations stated in Theorem 2.4 also apply to* S.

Now it becomes clear that for each $p \leq k$, $I(n,k)$ possesses $\sum_{r=p}^{n-1} \binom{n-1}{r}\binom{r-1}{p-1}$ $\mathcal{R}^*$-classes and $\binom{n}{p}$ $\mathcal{L}^*$-classes within each J_p^*. Thus, the Rees quotient semigroup $RSS'_n(p)$ has $\sum_{r=p}^{n-1} \binom{n-1}{r}\binom{r-1}{p-1} + 1$ $\mathcal{R}^*$-classes and $\binom{n}{p} + 1$ $\mathcal{L}^*$-classes (the additional 1 accounts for the singleton class that contains the zero element in all cases).

Before we present the next result, we first note that

$$|E(\mathcal{LS}_n)| = \frac{3^n + 1}{2} \text{ and } |E(\mathcal{SS}_n)| = 3^{n-1},$$

as found in [19, Proposition 3.5] and [23, Theorem 2.22], respectively. Thus, we present the following result.

Theorem 2.5. *Let* $\mathcal{SS}'_n$ *be defined as in Eq* (1.2)*. Then* $|E(\mathcal{SS}'_n)| = \frac{3^{n-1}+1}{2}$.

Proof. It is important to note that $E(\mathcal{SS}'_n) \cap E(\mathcal{SS}_n) = \emptyset$ and that $E(\mathcal{SS}'_n) \cup E(\mathcal{SS}_n) = E(\mathcal{LS}_n)$. Therefore, the conclusion follows from the fact that $|E(\mathcal{SS}'_n)| = |E(\mathcal{LS}_n) \setminus E(\mathcal{SS}_n)| = |E(\mathcal{LS}_n)| - |E(\mathcal{SS}_n)|$. □

We deduce the following identity.

Corollary 2.1. *For any natural number* $n \geq 2$, $1 + \sum_{p=1}^{n-1} \sum_{r=p}^{n-1} \binom{n-1}{r}\binom{r-1}{p-1} = \frac{3^{n-1}+1}{2}$.

3. The generators and rank properties in $\mathcal{SS}'_n$

For a semigroup S and a nonempty subset of S, say A, the *smallest subsemigroup* of S that contains A, denoted by $\langle A\rangle$, is referred to as the *subsemigroup generated by* A. If A is a finite subset of S such that $\langle A\rangle$ equals S, then S is called a *finitely-generated semigroup*. The *rank* of a finitely generated semigroup S is denoted and defined as

$$\text{rank}(S) = \min\{|A| : \langle A\rangle = S\}.$$

If $\langle A\rangle = S$, and the set A consists only of idempotents in S, then S is said to be an *idempotent-generated semigroup* (equivalently, a *semiband*), and the idempotent-rank is denoted by idrank(S). For a more detailed explanation of the rank properties of a semigroup, we refer the reader to [10, 14, 15]. There are various classes of transformation semigroups, whose ranks have been investigated; see for example [11, 12, 16]. In [23], the ranks of the large and small Schröder monoids $\mathcal{LS}_n$ and $\mathcal{SS}_n$, as well as the ranks of their respective certain two-sided ideals and their Rees quotients, have been investigated. In fact, these semigroups have been shown to be idempotent-generated. Our aim is to compute the rank of the two-sided ideal $I(n, p)$ of the Schröder semigroup $\mathcal{SS}'_n$, thereby obtaining the rank of $\mathcal{SS}'_n$ as a special case. First, we note the following well-known result about decreasing maps [21].

Lemma 3.1. *For all order-decreasing partial maps α and β on $A \subseteq [n]$, $F(\alpha\beta) = F(\alpha) \cap F(\beta) = F(\beta\alpha)$.*

Proof. If α or β is zero (i.e., the empty map), the result follows. When α and β are nonzero order-decreasing partial maps, the proof is the same as the proof of Lemma 2.1.3 in [21]. □

We initiate our examination of the rank properties of the semigroup $S \in \{I(n, p), \mathcal{SS}'_n\}$ by first introducing the following definition about injective elements in J^*_p for any $1 \le p \le n - 1$.

Definition 3.1. *An injective map α in J^*_p is said to be a* requisite element *if it is of the form:*

$$\alpha_i = \begin{pmatrix} 2 & \cdots & i & a_i & \cdots & a_p \\ 1 & \cdots & i-1 & a_i & \cdots & a_p \end{pmatrix},$$

where $1 < i < a_i < a_{i+1} < \cdots < a_p \le n$.

Remark 3.1. *If α_i is a requisite element, then observe that for each $1 \le i \le p$,*

$$\text{Dom } \alpha_i = \{2, \ldots, i, a_i, \ldots, a_p\} = \{2, \ldots, i\} \cup F(\alpha_i).$$

Moreover, α_i is unique in $L^*_{\alpha_i}$, in the sense that no two α_i's belong to the same $\mathcal{L}^*$-class. In other words, an $\mathcal{L}^*$-class L^*_α, contains an idempotent if and only if $1 \notin$ Im α. Otherwise, it contains a unique requisite element that is maximal in this class with respect to the $\mathcal{L}$-order as will be proved in Lemma 3.2. However, an $\mathcal{R}^*$-class may contain more than one requisite element. In fact, in J^*_{n-1}, all the $n - 1$ requisite elements belong to a single $\mathcal{R}^*$-class.

We immediately have the following lemma.

Lemma 3.2. *If α_i is the unique requisite element in L^*_α in J^*_p $(1 \le p \le n - 1)$, then there exists $\beta \in R^*_\alpha$ such that $\alpha = \beta\alpha_i$.*

Proof. Let α_i be the unique requisite element in L^*_α in J^*_p, where α is as expressed in Eq (1.4). Now since $\alpha_i \in L^*_\alpha$, Im α = Im α_i, and so $y_j = j$ for all $1 \leq j \leq i-1$, so that α and α_i are

$$\alpha = \begin{pmatrix} D_1 & \cdots & D_{i-1} & D_i & \cdots & D_p \\ 1 & \cdots & i-1 & y_i & \cdots & y_p \end{pmatrix} \text{ and } \alpha_i = \begin{pmatrix} 2 & \cdots & i & y_i & \cdots & y_p \\ 1 & \cdots & i-1 & y_i & \cdots & y_p \end{pmatrix},$$

respectively. By order preservedness, it is clear that $\min D_j < \min D_{j+1}$ for all $1 \leq j \leq i-1$, and since $2 \leq \min D_1$, it follows that $j+1 \leq \min D_j$. Thus, the map β, defined as

$$\beta = \begin{pmatrix} D_1 & \cdots & D_{i-1} & D_i & \cdots & D_p \\ 2 & \cdots & i & y_i & \cdots & y_p \end{pmatrix} \in \mathcal{SS}'_n.$$

Furthermore, notice that ker α = ker β, and so $\beta \in R^*_\alpha$. Clearly, $\beta\alpha_i = \alpha$.

The proof is now complete. □

Theorem 3.1. *Let $\alpha \in \mathcal{SS}'_n$ be as expressed in Eq* (1.4). *Then,*

(i) if $y_1 \neq 1$, then α is idempotent-generated;
(ii) if $y_1 = 1$, then α is a product of idempotents and the unique requisite element in L^*_α.

Proof. Let $\alpha \in \mathcal{SS}'_n$ be as expressed in Eq (1.4).
(i) Suppose $y_1 \neq 1$, and let $U = \{\alpha \in \mathcal{SS}'_n : 1 \notin \text{Im } \alpha\}$. Then it is not difficult to see that U is a subsemigroup of $\mathcal{SS}'_n$, which is isomorphic to $\mathcal{LS}_{n-1}$. Hence, it is not difficult to see that by [23, Lemma 3.2], U is idempotent-generated.
(ii) Now suppose $y_1 = 1$. Thus, by Lemma 3.2, α can be expressed as

$$\alpha = \beta\alpha_i,$$

for some $\beta \in R^*_\alpha$, where α_i is the unique requisite element in L^*_α. To be precise,

$$\beta = \begin{pmatrix} D_1 & \cdots & D_{i-1} & D_i & \cdots & D_p \\ 2 & \cdots & i & y_i & \cdots & y_p \end{pmatrix} \text{ and } \alpha_i = \begin{pmatrix} 2 & \cdots & i & y_i & \cdots & y_p \\ 1 & \cdots & i-1 & y_i & \cdots & y_p \end{pmatrix}.$$

Notice that in Im β, $y_1 = 2 \neq 1$, and so $\beta \in U$. Thus, by (i), β is idempotent-generated, as stipulated. □

Consequently, we have the following corollary.

Corollary 3.1. *The semigroup $\mathcal{SS}'_n$ is generated by its idempotents and requisite elements.*

Notice that in the proof of Theorem 3.1, $|\text{Im } \alpha| = h(\alpha) = h(\beta) = h(\alpha_i) = p$ for all $1 \leq i \leq p$. Based on this, we have the following result, which generalizes [23, Lemma 3.3].

Lemma 3.3. *Every element in $\mathcal{SS}'_n$ of height p can be expressed as a product of idempotents and requisite elements in $\mathcal{SS}'_n$, each of height p.*

Now let $M(p) = \{\alpha \in RSS'_n(p) : \alpha \text{ is a requisite element}\}$ and $E(RSS'_n(p) \setminus \{0\})$ be the collection of all nonzero idempotents in $RSS'_n(p)$. Then we have the following lemma.

Lemma 3.4. *For $1 \leq p \leq n-1$, we have* *(i)* $|M(p)| = \binom{n-1}{p-1}$;

(ii) $|E(RSS'_n(p)) \setminus \{0\}| = \sum_{r=p}^{n-1} \binom{n-1}{r}\binom{r-1}{p-1}$.

Proof. **(i)** Given that 1 must be included in every image set that contains p images, we can then choose the remaining $p-1$ images from the available $n-1$ elements in $\binom{n-1}{p-1}$ distinct ways, as stipulated.
(ii) The result follows from Remark 2.2. □

Now, let $G(p) = M(p) \cup (E(RSS'_n(p)) \setminus \{0\})$. The next result shows that the subset $G(p)$ of $RSS'_n(p)$ is the minimum generating set of $RSS'_n(p) \setminus \{0\}$.

Lemma 3.5. *Let α, β be elements in $RSS'_n(p) \setminus \{0\}$. Then $\alpha\beta \in G(p)$ if and only if $\alpha, \beta \in G(p)$ and $\alpha\beta = \alpha$ or $\alpha\beta = \beta$.*

Proof. Suppose $\alpha\beta \in G(p)$. Thus, either $\alpha\beta \in E(RSS'_n(p) \setminus \{0\})$ or $\alpha\beta \in M(p)$. We consider the two cases separately.
Case i. Suppose $\alpha\beta \in E(RSS'_n(p) \setminus \{0\})$. Then,

$$p = f(\alpha\beta) \leq f(\alpha) \leq |\mathrm{Im}\ \alpha| = p,$$

$$p = f(\alpha\beta) \leq f(\beta) \leq |\mathrm{Im}\ \beta| = p.$$

This ensures that

$$F(\alpha) = F(\alpha\beta) = F(\beta),$$

and so $\alpha, \beta \in E(RSS'_n(p) \setminus \{0\}) \subset G(p)$, which implies that $\alpha, \beta \in G(p)$ and $\alpha\beta = \alpha$.
Case ii. Now suppose $\alpha\beta \in M(p)$. Thus $\alpha\beta$ is a requisite element that has the form

$$\alpha\beta = \begin{pmatrix} 2 & \cdots & i & y_i & \cdots & y_p \\ 1 & \cdots & i-1 & y_i & \cdots & y_p \end{pmatrix},$$

where $1 < i < y_i < y_{i+1} < \cdots < y_p \leq n$. This means that Dom α = Dom $\alpha\beta$ and Im β = Im $\alpha\beta$. Thus,

$$\alpha = \begin{pmatrix} 2 & \cdots & i & y_i & \cdots & y_p \\ 2\alpha & \cdots & i\alpha & y_i & \cdots & y_p \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 2\alpha & \cdots & i\alpha & y_i & \cdots & y_p \\ 1 & \cdots & i-1 & y_i & \cdots & y_p \end{pmatrix},$$

since Im α = Dom β. The claim here is that α must be an idempotent. Notice that either $2\alpha = 1$ or $2\alpha = 2$. In the former, we see that $1 = 2\alpha \in$ Dom β, which is a contradiction. Therefore, $2\alpha = 2$, which implies $j\alpha = j$ for all $2 \leq j \leq i$. Thus α is an idempotent, that is,

$$\alpha = \begin{pmatrix} 2 & \cdots & i & y_i & \cdots & y_p \\ 2 & \cdots & i & y_i & \cdots & y_p \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 2 & \cdots & i & y_i & \cdots & y_p \\ 1 & \cdots & i-1 & y_i & \cdots & y_p \end{pmatrix}.$$

Therefore, $\beta \in M(p) \subset G(p)$ and $\alpha \in E(RSS'_n(p) \setminus \{0\}) \subset G(p)$, and also $\alpha\beta = \beta$. The converse is obvious. □

We have now established a key result of this section.

Theorem 3.2. *Let $RSS'_n(p)$ be as defined in Eq* (1.3). *Then*

$$\text{rank } RSS'_n(p) = \binom{n-1}{p-1} + \sum_{r=p}^{n-1} \binom{n-1}{r}\binom{r-1}{p-1}.$$

Proof. The proof follows from Lemmas 3.4 and 3.5. □

The next lemma is crucial in determining the ranks of the Schröder semigroup $\mathcal{SS}'_n$ and its two sided-ideal $I(n,k)$. Now for $1 \leq p \leq n-1$ let

$$J^*_p = \{\alpha \in \mathcal{SS}'_n : |\text{Im } \alpha| = p\}.$$

Moreover, for $1 \leq p \leq n-1$, let $M(p)$ be the collection of all requisite elements in J^*_p, and let

$$G(p) = M(p) \cup E(J^*_p).$$

Then we have the following lemmas.

Lemma 3.6. *For* $1 \leq p \leq n-1$, $|G(p)| = \binom{n-1}{p-1} + \sum_{r=p}^{n-1} \binom{n-1}{r}\binom{r-1}{p-1}$.

Proof. The result follows from Lemma 3.4. □

Lemma 3.7. *For* $0 \leq p \leq n-3$, $J^*_p \subset \langle J^*_{p+1} \rangle$. *In other words, if* $\alpha \in J^*_p$, *then* $\alpha \in \langle J^*_{p+1} \rangle$ *for* $1 \leq p \leq n-3$.

Proof. Using Theorem 3.1, it suffices to prove that every element in $G(p)$ can be expressed as a product of elements in $G(p+1)$. That is to say, every idempotent of height p can be expressed as a product of idempotents of height $p+1$, and every requisite element, can be expressed as a product of requisite elements of height $p+1$. Thus, we consider the elements of $E(J^*_p)$ and $M(p)$ separately.

i. The elements in $E(J^*_p)$.

Let $\epsilon \in E(J^*_p)$ be expressed as

$$\epsilon = \begin{pmatrix} D_1 & \cdots & D_p \\ d_1 & \cdots & d_p \end{pmatrix},$$

where $d_i = \min D_i$ for all $1 \leq i \leq p$. Notice that since $1 \notin \text{Dom } \epsilon$, $1 \notin D_1$, and so $d_1 \neq 1$. This implies that $\epsilon \in E(U)$, where $U = \{\alpha \in \mathcal{SS}'_n : 1 \notin \text{Im } \alpha\}$, which is isomorphic to the large Schröder monoid $\mathcal{LS}_{n-1}$, and so, the proof follows from [23, Lemma 3.7].

ii. The elements in $M(p)$.

Let α_i be a requisite element of height p, which has the form:

$$\alpha_i = \begin{pmatrix} 2 & \cdots & i & y_i & \cdots & y_p \\ 1 & \cdots & i-1 & y_i & \cdots & y_p \end{pmatrix},$$

where $1 < i < y_i < y_{i+1} < \cdots < y_p \leq n$. Now, since $p \leq n-3$, it implies that $(\text{Dom } \alpha_i \cup \text{Im } \alpha_i)^c$ contains at least two distinct elements, say g and h. Let $A = \text{Dom } \alpha_i \cup \{h\}$ and $B = \text{Dom } \alpha_i \cup \{g\}$. Now define β and γ as follows: For $x \in A$ and $y \in B$,

$$x\beta = x \text{ and } y\gamma = \begin{cases} y\alpha_i, & \text{if } y \neq g, \\ g, & \text{if } y = g. \end{cases}$$

Notice that $\beta \in E(J^*_{p+1}) \subset G(p+1)$, and it is not difficult to see that γ is a requisite element in $M(p+1) \subset G(p+1)$. One can now easily show that $\alpha_i = \beta\gamma$.

The proof of the lemma is now complete. □

Consequently, we have the following result.

Theorem 3.3. *Let $I(n,k)$ be as defined in Eq* (1.3)*. Then for* $1 \le p \le k \le n-2$*, we have the*

$$\text{rank } I(n,p) = \binom{n-1}{p-1} + \sum_{r=p}^{n-1} \binom{n-1}{r}\binom{r-1}{p-1}.$$

Proof. Notice that by Lemma 3.7, $\langle J_p^* \rangle = I(n,p)$ for all $1 \le p \le n-2$. Notice also that, $\langle E(RSS'_n(p) \setminus \{0\}) \cup M(p) \rangle = J_p^*$. The result now follows from Theorem 3.2. □

It is important to note that Lemma 3.7 does not cover the case $p = n-2$, meaning that the assertion $J_{n-2}^* \subset \langle J_{n-1}^* \rangle$ is not true. This can be demonstrated by first noting that every element α in J_{n-1}^* is an injective map, because Dom $\alpha = [n] \setminus \{1\}$. However, in J_{n-2}^*, there are maps that are not necessarily injective. For example, a map α with Dom $\alpha = [n] \setminus \{1\}$ and $|\text{Im } \alpha| = n-2$ cannot be generated by injective maps, since injective maps alone can only generate injective maps. Therefore, we need to investigate the generating set of J_{n-1}^*, and its relationship with $G(n-2)$.

The elements of J_{n-1}^* are not only injective, but are isotone and decreasing maps from $[n] \setminus \{1\}$ into $[n]$, and are all requisite elements α_i of the form

$$\alpha_i = \begin{pmatrix} 2 & 3 & \cdots & i-1 & i & i+1 & \cdots & n \\ 1 & 2 & \cdots & i-2 & i-1 & i+1 & \cdots & n \end{pmatrix} \ (2 \le i \le n), \tag{3.1}$$

and the unique idempotent

$$\epsilon = \begin{pmatrix} 2 & \cdots & n \\ 2 & \cdots & n \end{pmatrix},$$

which is the unique left identity on $\mathcal{SS}'_n$, but not an identity element. It is also clear that J_{n-1}^* has only 1 $\mathcal{R}^*$-class and n $\mathcal{L}^*$-classes, of which there are $n-1$ of them, each containing a unique requisite element. Thus, $G(n-1) = M(n-1) \cup \{\epsilon\} = J_{n-1}^*$, and so $|G(n-1)| = n$. We now have the following.

Lemma 3.8. *Let α, β be elements in $\langle J_{n-1}^* \rangle$. Then $\alpha\beta \in G(n-1)$ if and only if $\alpha, \beta \in G(n-1)$ and $\alpha\beta = \beta$.*

Proof. The proof is similar to that of Lemma 3.5. □

Lemma 3.9. *The rank of $\langle J_{n-1}^* \rangle$ is n.*

Proof. The result follows from Lemma 3.8 and the fact that $|G(n-1)| = |J_{n-1}^*|$. □

The elements of $M(n-2)$ are generally of the form:

$$\alpha_{i,j} = \begin{pmatrix} 2 & \cdots & i & i+1 & \cdots & j-1 & j+1 & \cdots & n \\ 1 & \cdots & i-1 & i+1 & \cdots & j-1 & j+1 & \cdots & n \end{pmatrix} \ (2 \le i < j \le n). \tag{3.2}$$

Notice that $1, j \notin$ Dom $\alpha_{i,j}$ and $i, j \notin$ Im $\alpha_{i,j}$. Also, the element

$$\epsilon_{1,j} = \begin{pmatrix} 2 & \cdots & j-1 & j+1 & \cdots & n \\ 2 & \cdots & j-1 & j+1 & \cdots & n \end{pmatrix} \tag{3.3}$$

is a partial identity in $G(n-2)$ for all $2 \le j \le n$. Notice also that $1, j \notin$ Dom $\epsilon_{1,j}$ and $1, j \notin$ Im $\epsilon_{1,j}$.

The subsequent lemma demonstrates that the requisite elements together with the partial identity $\epsilon_{1,2}$ in $G(n-2)$ are not required to be included in any minimal generating set of $\langle J_{n-2}^* \cup J_{n-1}^* \rangle$. However, the requisite elements in $G(n-1)$ must be included in any generating set of $\langle J_{n-2}^* \cup J_{n-1}^* \rangle$.

Lemma 3.10. $\langle G(n-2)\rangle \subseteq \langle (G(n-2)\setminus (M(n-2)\cup\{\epsilon_{1,2}\}))\cup G(n-1)\rangle = \langle J^*_{n-2}\cup J^*_{n-1}\rangle = \mathcal{SS}'_n$.

Proof. It is enough to show that $M(n-2)\cup\{\epsilon_{1,2}\}\subset\langle G(n-1)\cup(E(J^*_{n-2})\setminus\{\epsilon_{1,2}\})\rangle$. Now let $\alpha_{i,j}\in M(n-2)$ be as expressed in Eq (3.2). So, take the idempotent $\epsilon_{1,j}\in G(n-2)$ and $\alpha_i\in G(n-1)$ as expressed in Eq (3.1) and Eq (3.3), respectively; and observe that for any $i<j$, we have

$$\begin{aligned}\epsilon_{1,j}\alpha_i &= \begin{pmatrix} 2 & \cdots & i & i+1 & \cdots & j-1 & j+1 & \cdots & n \\ 2 & \cdots & i & i+1 & \cdots & j-1 & j+1 & \cdots & n\end{pmatrix}\begin{pmatrix} 2 & 3 & \cdots & i & i+1 & \cdots & n \\ 1 & 2 & \cdots & i-1 & i+1 & \cdots & n\end{pmatrix}\\ &= \begin{pmatrix} 2 & \cdots & i & i+1 & \cdots & j-1 & j+1 & \cdots & n \\ 1 & \cdots & i-1 & i+1 & \cdots & j-1 & j+1 & \cdots & n\end{pmatrix}\\ &= \alpha_{i,j}.\end{aligned}$$

Similarly, observe that if $i=2$, then

$$\alpha_2^2 = \begin{pmatrix} 2 & 3 & \cdots & n \\ 1 & 3 & \cdots & n\end{pmatrix}\begin{pmatrix} 2 & 3 & \cdots & n \\ 1 & 3 & \cdots & n\end{pmatrix} = \begin{pmatrix} 3 & \cdots & n \\ 3 & \cdots & n\end{pmatrix} = \epsilon_{1,2}.$$

The result now follows. □

The following lemma demonstrates that the only idempotent in J^*_{n-2} generated by the elements in J^*_{n-1} is $\epsilon_{1,2}$.

Lemma 3.11. $(E(J^*_{n-2})\setminus\{\epsilon_{1,2}\})\cap\langle J^*_{n-1}\rangle = \emptyset$.

Proof. For $3\le j\le i\le n$, the elements α_i and α_j in J^*_{n-1} in the form of Eq (3.1) are

$$\alpha_i = \begin{pmatrix} 2 & 3 & \cdots & i-1 & i & i+1 & \cdots & n \\ 1 & 2 & \cdots & i-2 & i-1 & i+1 & \cdots & n\end{pmatrix} \text{ and } \alpha_j = \begin{pmatrix} 2 & 3 & \cdots & j-1 & j & j+1 & \cdots & n \\ 1 & 2 & \cdots & j-2 & j-1 & j+1 & \cdots & n\end{pmatrix}.$$

Now observe that

$$\alpha_j\alpha_i = \begin{pmatrix} 3 & 4 & \cdots & j & j+1 & j+2 & \cdots & i & i+1 & \cdots & n \\ 1 & 2 & \cdots & j-2 & j & j+1 & \cdots & i-1 & i+1 & \cdots & n\end{pmatrix},$$

$$\alpha_i\alpha_j = \begin{pmatrix} 3 & 4 & \cdots & j & j+1 & j+2 & \cdots & i-1 & i & i+1 & \cdots & n \\ 1 & 2 & \cdots & j-2 & j-1 & j & \cdots & i-3 & i-1 & i+1 & \cdots & n\end{pmatrix}.$$

It is clear that $\alpha_i\alpha_j$ and $\alpha_j\alpha_i$ are not idempotents. Moreover, for the unique idempotent $\epsilon = \begin{pmatrix} 2 & \cdots & n \\ 2 & \cdots & n\end{pmatrix}$ in J^*_{n-1}, we see that $\epsilon\alpha_i = \alpha\epsilon_i = \alpha_i$. However, if $i=j=2$, then $\alpha_2^2 = \epsilon_{1,2}$ as in the proof of the aforementioned lemma, which is an idempotent. Thus, the products $\epsilon_{1,2}\alpha_i$ and $\alpha_i\epsilon_{1,2}$ are

$$\begin{pmatrix} 3 & 4 & \cdots & i-1 & i & i+1 & \cdots & n \\ 2 & 3 & \cdots & i-2 & i-1 & i+1 & \cdots & n\end{pmatrix} \text{ and } \begin{pmatrix} 4 & \cdots & i-1 & i & i+1 & \cdots & n \\ 3 & \cdots & i-2 & i-1 & i+1 & \cdots & n\end{pmatrix},$$

respectively, none of which is an idempotent also.

This shows that the only idempotent generated by J^*_{n-1} is $\epsilon_{1,2}$, that is, the remaining $2n-4$ idempotents cannot be generated by J^*_{n-1}, as required. □

Finally, we conclude the paper with the following result.

Theorem 3.4. *Let $\mathcal{SS}'_n$ be as defined in Eq* (1.2)*. Then the rank* $\mathcal{SS}'_n = 3n-4$.

Proof. Clearly, $(G(n-2)\setminus(M(n-2)\cup\{\epsilon_{1,2}\}))\cup G(n-1)$ is the minimum generating set of $\langle J^*_{n-2}\cup J^*_{n-1}\rangle = \mathcal{SS}'_n$. Moreover, by Lemma 3.11, the remaining $2n-4$ idempotents in J^*_{n-2} cannot be generated by J^*_{n-1}. Thus, by Lemmas 3.4 and 3.9, we see that

$$\begin{aligned}
\text{rank } \mathcal{SS}'_n &= |G(n-2)| - |M(n-2)| - 1 + |G(n-1)| \\
&= \binom{n-1}{n-3} + \left(\sum_{r=n-2}^{n-1}\binom{n-1}{r}\binom{r-1}{n-3}\right) - \binom{n-1}{n-3} - 1 + n \\
&= \sum_{r=n-2}^{n-1}\binom{n-1}{r}\binom{r-1}{n-3} + n - 1 = 3n-4,
\end{aligned}$$

as stipulated. □

Author contributions

Muhammad Mansur Zubairu: Methodology, Writing original draft; Abdullahi Umar: Conceptualization, Supervision; Fatma Salim Al-Kharousi: Methodology, Writing original draft. All the authors are involved in the proofs and analysis of the main results. All authors have read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The first author is grateful to Bayero University and TETFund (TETF/ES/UNI/KANO/TSAS/2022) for their funding. He also appreciates the hospitality provided by Sultan Qaboos University in Oman during his postdoctoral research visit, which lasted from June to December 2024. Moreover, we thank the anonymous referee for their valuable comments and suggestions, which helped improve the clarity of this paper.

Conflict of interest

The authors affirm that there are no conflicts of interest associated with the publication of this paper.

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