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Research article

Existence and multiplicity of solutions to p-Kirchhoff-Choquard-type equations

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Abstract: In this paper, we investigate the existence and multiplicity of solutions for a p-Kirchhoff-Choquard-type equation involving a critical exponent. First, we employ the concentration-compactness principle to overcome the lack of compactness arising from the critical exponent. Then, under appropriate assumptions, we obtain the existence of solutions by applying the mountain pass theorem, the symmetric mountain pass theorem, the dual fountain theorem, and various necessary analytical techniques.

Keywords: Kirchhoff-Choquard equation; critical exponent; mountain pass theorem; symmetric mountain pass theorem; dual fountain theorem
Mathematics Subject Classification: 31A30, 35J60, 35J75

1. Introduction

This paper is devoted to the study of the existence and multiplicity of solutions for the following p-Kirchhoff-Choquard equation involving a critical exponent:

$$K(u)\left(-\Delta_p u + V(x)|u|^{p-2}u\right) = \beta f(x,u) + \lambda\left(\int_{\mathbb{R}^N} \frac{|u(y)|^{p_\mu^*}}{|x-y|^\mu} dy\right)|u(x)|^{p_\mu^*-2}u, x \in \mathbb{R}^N, \tag{1.1}$$

where $K(u) = M\left(\int_{\mathbb{R}^N} (|\nabla u|^p + V(x)|u|^p)\, dx\right)$, $M(\cdot)$ is a Kirchhoff-type function, $V(x)$, $f(x,u)$ are continuous functions, $\mu \in (0,N)$, $1 < p < p_\mu^*$, $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is the p-Laplacian operator, and $p_\mu^* = \frac{2Np-\mu p}{2N-2p}$ is the critical exponent with respect to the Hardy-Littlewood-Sobolev inequality.

Problem (1.1) is intrinsically characterized by the interplay of two distinct nonlinear effects: the Kirchhoff-type nonlocal term and the Choquard-type nonlocal term. Such a coupling arises naturally in physical contexts, particularly in quantum mechanics, where it models long-range interactions in many-body systems, and in condensed matter physics, where it describes the self-trapping of electrons in nonlinear media. Additionally, analogous structures emerge in wave packet dynamics in nonlinear optics, reflecting the nonlocal nature of photon interactions. Recently, the study of Choquard-type problems, particularly the existence, multiplicity, and qualitative properties of their solutions, has become a central topic in nonlinear analysis. The intricate balance between local and nonlocal nonlinearities in such problems presents rich mathematical challenges, driving advances in variational methods, critical point theory, and regularity analysis. Consider the following Choquard problem:

$$-\Delta u + V(x)u = (I_\alpha * F(u))\, f(u),\ x \in \mathbb{R}^N, \tag{1.2}$$

where $\alpha \in (0, N)$, $u : \mathbb{R}^N \to \mathbb{R}$, f serves as the primitive function of F, and $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential, defined as follows:

$$I_\alpha = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right) 2^\alpha \pi^{\frac{N}{2}} |x|^{N-\alpha}},$$

where Γ is the gamma function. When the potential function $F(u) = |u|^{p-2} u$ is considered, problem (1.2) simplifies to the following form:

$$-\Delta u + V(x)\, u = (I_\alpha * |u|^p)\, |u|^{p-2} u,\ x \in \mathbb{R}^N. \tag{1.3}$$

Existing results on Choquard-type problems (1.3) are largely governed by the admissible range of the exponent p, which critically influences the existence, regularity, and asymptotic behavior of solutions. In the case where $V(x) = 1$ and $p = 2$, the equation was first introduced by Pekar [18] in the context of modeling the quantum theory of static polarons. Subsequently, Penrose [19] employed this equation as a self-gravitational model to describe the motion of a single particle in its own gravitational field while studying the phenomenon of quantum state collapse. Advances in nonlinear functional analysis techniques have led to substantial progress in understanding solution properties for Choquard-type problems, particularly when the exponent p reaches the Hardy-Littlewood-Sobolev critical threshold. When $V(x) = 0$, $p = 2^*_\mu = \frac{2N-\mu}{N-2}$, and certain perturbation terms are added to the right-hand side of equation (1.3). Lan et al. [11] demonstrated the existence of multiple solutions to problem (1.3) by employing the Nehari manifold. Liu et al. [17] utilized the Pohožaev identity, the Nehari manifold, and the mountain pass theorem to prove both the existence and non-existence of solutions to problem (1.3) on bounded domains. For more detailed results concerning the Choquard-type problems, we refer to [20–23] and the references therein.

On the other hand, Kirchhoff-type nonlocal problems have attracted considerable and sustained research interest due to their distinctive mathematical structure and wide-ranging physical applications. Specifically, Kirchhoff in [9] extended the classical D'Alembert wave equation by incorporating the dependence of string tension on vibration amplitude and established a foundational model described by the following equation:

$$\rho \frac{\partial^2 u}{\partial t^2} - M\left(\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.4}$$

where $M\left(\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right) = \frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx$ and L, h, E, ρ, and ρ_0 are constants. The presence of the nonlocal term endows it with significant applications across various fields, such as elasticity, geophysics, acoustics, and fluid dynamics. Since then, many mathematicians began to investigate the following Kirchhoff-Dirichlet problem on a bounded domain $\Omega \subset \mathbb{R}^N$:

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2\,dx\right)\Delta u = f(x,u),\ x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.5}$$

where $a, b > 0$ are constants and $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$. For recent results about Kirchhoff problems, see for example [1, 5, 6] and the references cited there. Cheng et al. [4] investigated the following p-Kirchhoff equation:

$$\left(a + \lambda M\left(\int_{\mathbb{R}^N} (|\nabla u|^p + b|u|^p)dx\right)\right)\left(-\Delta_p u + b|u|^{p-2}u\right) = f(u), x \in \mathbb{R}^N, \tag{1.6}$$

where a, $b > 0$, $\lambda \geq 0$, $\mu \in \mathbb{R}$, $1 < p < N$, functions M, $f \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\mathbb{R}^+ = [0, \infty)$, and $\Delta_p u = div\left(|\nabla u|^{p-2}\nabla u\right)$ is the p-Laplacian operator. The authors proved the existence of at least a positive ground state solution via variational methods, monotonicity methods, cut-off functional techniques, and a priori estimates techniques. When $f(u) = |u|^{m-2}u + \mu|u|^{q-2}u$, Chen et al. [3] obtained a positive ground state solution to problem (1.6) using the Nehari manifold. By employing variational methods, Li et al. [12] established the existence of at least one positive solution to problem (1.6) with $p = 2$ and $M(t) = t$. Moreover, in the case where $f(u) = f(x,u) + g(x)|u|^{p-2}u$, Fan et al. [7] proved that the problem (1.6) admits at least two positive and two negative solutions under different conditions by using the variational methods. Liu [13] considered the following p-Kirchhoff equation:

$$\left[M\left(\int_\Omega (|\nabla u|^p + \lambda(x)|u|^p)\,dx\right)\right]^{p-1}\left(-\Delta_p u + \lambda(x)|u|^{p-2}\right) = f(x,u),\ x \in \Omega,$$

where Ω is a smooth bounded domain, $\Delta_p u = div\left(|\nabla u|^{p-2}\nabla u\right)$ is the p-Laplacian with $1 < p < N$, and $\lambda(x) \in L^\infty(\Omega)$ satisfies $\operatorname{essinf}_{x\in\bar{\Omega}}\lambda(x) > 0$. The authors employed the fountain theorem and the dual fountain theorem to demonstrate the existence of multiple solutions to this problem.

Although the individual theories of Kirchhoff-type and Choquard-type problems have seen substantial development, their nonlinear coupling through the p-Kirchhoff-Choquard framework presents new analytical challenges that require systematic study. In present paper, we are interested in the existence and multiplicity of solutions for the p-Kirchhoff-Choquard equation involving critical exponents. Due to the energy functional associated to (1.1) involving the non-local terms, specifically $\int_{\mathbb{R}^N}|\nabla u|^p\,dx$ and $\left(\int_{\mathbb{R}^N}\frac{|u(y)|^{p_\mu^*}}{|x-y|^\mu}dy\right)|u(x)|^{p_\mu^*}$, it presents some difficulties in analyzing the validity of the Palais-Smale condition and discussing the mountain pass geometry on the energy functional. Inspired by [14], we shall apply the concentration-compactness principle to ensure the necessary compactness. Clearly, problem (1.1) is different from the problem studied in the literature, since (1.1) deals with p-Kirchhoff equations with Choquard type and critical exponent p_μ^*. In this paper, we will prove the multiplicity of nontrivial solutions for problem (1.1).

Before presenting our main results, we first introduce the following fundamental assumptions on the Kirchhoff term $M(\cdot)$, the function $V(\cdot)$, and the nonlinearity term f:
(M_1) $M \in C(\mathbb{R}_0^+, \mathbb{R}_0^+)$, and for $\theta \in \left[1, \frac{2N-\mu}{N-p}\right)$, there exist $0 < m_1 < m_2$ such that

$$m_1 t^{\theta-1} \leq M(t) \leq m_2 t^{\theta-1}\ for\ \ all\ t \geq 0.$$

(V_1) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ and $V(x) \geq \min_{x \in \mathbb{R}^N} V(x) = 0$.
(V_2) There exists a constant $R > 0$ such that $|\{x \in B_{\mathbb{R}}(y) : V(x) \leq c\}| \to 0$ *as* $y \to \infty$ for any $c > 0$, where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^N$.
(F_1) $f(x, 0) = 0, f(x, t) \in C^1\left(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}\right)$.
(F_2) There exist constants r_i with $1 < r_1 < r_2 < \cdots < r_m < p$ and functions $w_i(x) \in L^{\vartheta_i}\left(\mathbb{R}^N, \mathbb{R}^+\right)$, $\vartheta_i \in \left[\frac{p^*}{p^*-r_i}, \frac{1}{1-r_i}\right]$, $i = 1, 2, ..., m$, such that $|f(x, t)| \leq \sum_{i=1}^m w_i(x) |t|^{r_i - 1}$ for all $(x, t) \in \left(\mathbb{R}^N, \mathbb{R}\right)$.
(F_3) There exist constants $a > 0$, $q \in \left(\theta p, p_\mu^*\right)$ such that $F(x, t) := \int_0^t f(x, s)\, ds \geq a |t|^q$, for a.e. $x \in \mathbb{R}^N$, and $t \in \mathbb{R}$.
(F_4) There exists a constant $\sigma \in (\theta p, 2p_\mu^*)$ such that $0 < \sigma F(x, t) \leq f(x, t)t$, for every $x \in \mathbb{R}^N$ and $t \neq 0$.

Our main results are as follows.

Theorem 1.1. *Suppose that the conditions* (M_1)*,* (V_1)*–*(V_2)*, and* (F_1)*–*(F_4) *hold. Then for* $\beta = 1$*:*

(i) There exists $\lambda_0 > 0$ such that, for any $\lambda \in (0, \lambda_0)$, problem (1.1) admits at least a nontrivial solution.

(ii) If $f(x, t)$ is odd with respect to t, then for any $m \in \mathbb{N}$, there exist $\lambda_{00} > 0$ such that problem (1.1) has at least m pairs solutions $u_{\lambda,i}$ and $u_{\lambda,-i}$ (i=1,2,...,m) for any $\lambda \in (0, \lambda_{00})$.

Theorem 1.2. *Suppose that* λ*,* $\beta > 0$ *and conditions* (M_1)*,* (V_1)*,* (F_1)*,* (F_2)*, and* (F_4) *hold. Then problem* (1.1) *has a sequence of negative energy solutions* $\{u_n\}$ *with* $J_{\lambda,\beta}(u_n) < 0$ *and* $J'_{\lambda,\beta}(u_n) \to 0$ *as* $n \to \infty$*, where* $J_{\lambda,\beta}(u_n)$ *is given in* (4.1)*.*

Remark 1.3. *To establish the aforementioned results, we employ the mountain pass theorem and the symmetric mountain pass theorem, motivated by the work of Liang et al. [14] and Yang et al. [25], in combination with the dual fountain theorem [24]. However, in the proof process, we will encounter several technical challenges. First, due to the presence of two nonlocal terms, we must verify whether the nonlocal part of the energy functional is weakly continuous. To address this issue, we shall utilize the concentration-compactness principle to ensure the necessary compactness. Second, the existence of the Hardy-Littlewood-Sobolev critical exponents makes it difficult to verify the compactness of* $(PS)_c$ *sequences. To overcome this obstacle, we determine an appropriate critical level c and constrain the energy below this threshold to restore compactness. Moreover, the involvement of dual parameters further complicates the problem, particularly in proving that the energy functional satisfies the mountain pass geometry. Additionally, we must tackle several technical difficulties, including deriving estimates for the nonlocal terms and establishing the strong convergence of certain sequences. Finally, it is worth emphasizing that the results obtained in this work exhibit significant differences from those in previous studies, highlighting the novelty of our approach.*

Throughout this paper, Section 2 is devoted to presenting the preliminary framework, including some definitions, notations, and functional-analytic tools required for the analysis of problem (1.1). In Section 3, we prove Theorem 1.1 by applying both the mountain pass theorem and the symmetric mountain pass theorem. In Section 4, Theorem 1.2 is established by using the dual fountain theorem.

2. Preliminaries

In this section, we begin by introducing the functional framework for analyzing problem (1.1). We then recall the Hardy-Littlewood-Sobolev inequality and the concentration-compactness principle, which serve as essential analytical tools. Finally, we define the corresponding energy functional associated with problem (1.1).

Let us start by recalling the definition of the Sobolev space:

$$W^{1,p}(\mathbb{R}^N) := \left\{ u \in L^p\left(\mathbb{R}^N\right) : \int_{\mathbb{R}^N} |\nabla u|^p \, dx < \infty \right\},$$

with the norm

$$\|u\|_{W^{1,p}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p) \, dx \right)^{\frac{1}{p}}.$$

We work within the following space framework:

$$W_V^{1,p}\left(\mathbb{R}^N\right) := \left\{ u \in W^{1,p}\left(\mathbb{R}^N\right) : \int_{\mathbb{R}^N} V(x) \, |u|^p \, dx < \infty \right\},$$

and

$$\|u\| := \left(\int_{\mathbb{R}^N} (|\nabla u|^p + V(x) \, |u|^p) \, dx \right)^{\frac{1}{p}}.$$

Here, we still denote the norm of $W_V^{1,p}\left(\mathbb{R}^N\right)$ by $\|\cdot\|$, and the scalar product is defined as

$$(u, v) := \int_{\mathbb{R}^N} |\nabla u|^{p-2} \, \nabla u \nabla v + V(x) \, |u|^{p-2} \, uv dx.$$

Proposition 2.1. (Hardy-Littlewood-Sobolev inequality [15]). Let $1 < k, l < \infty$, $0 < \mu < N$, and

$$\frac{1}{k} + \frac{1}{l} + \frac{\mu}{N} = 2.$$

Then, there exists a constant $C(N, \mu, k, l) > 0$ such that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)| \, |u(y)|}{|x - y|^\mu} dxdy \le C(N, \mu, k, l) \, \|u\|_{L^k(\mathbb{R}^N)} \|u\|_{L^l(\mathbb{R}^N)}.$$

The best constant for the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ is defined as

$$S := \inf_{u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}, \|u\|_{p^*} = 1} \int_{\mathbb{R}^N} |\nabla u|^p dx. \tag{2.1}$$

Consequently, we define

$$S_\mu := \inf_{u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p dx}{\left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{p_\mu^*} |u(y)|^{p_\mu^*}}{|x-y|^\mu} dxdy \right)^{\frac{p}{2p_\mu^*}}}. \tag{2.2}$$

To prove that the Palais-Smale condition satisfies for our settings, we shall first revisit the concentration-compactness principle. Following [14, 16], the concentration-compactness principle for the p-Laplacian setting can be stated as follows.

Lemma 2.2. Let $\{u_n\}$ be a bounded sequence in $W^{1,p}\left(\mathbb{R}^N\right)$ converging weakly to $u \in W^{1,p}\left(\mathbb{R}^N\right)$ such that $|\nabla u|^p \rightharpoonup \omega$ and $|u_n|^{p^*} \rightharpoonup \xi$ in the sense of measure. Assume that

$$\left(\int_{\mathbb{R}^N} \frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu} dy\right) |u_n(x)|^{p_\mu^*} dx \rightharpoonup \nu$$

converges weakly in the sense of measures, where ν is a bounded positive measure on $\mathbb{R}^N$. Then, there exist families of distinct points $\left\{\nu_j : j \in \mathcal{I}\right\}$, $\left\{\omega_j : j \in \mathcal{I}\right\}$, and $\left\{\xi_j : j \in \mathcal{I}\right\}$ in $\mathbb{R}^N$ satisfying for most countable set $\mathcal{I}$ such that

$$\nu = \left(\int_{\mathbb{R}^N} \frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu} dy\right) |u_n(x)|^{p_\mu^*} dx + \sum_{j \in \mathcal{I}} \nu_j \delta_{z_j}, \sum_{j \in \mathcal{I}} \nu_j^{\frac{1}{p_\mu^*}} < \infty,$$

$$\omega \geq |\nabla u_n|^p dx + \sum_{j \in \mathcal{I}} \omega_j \delta_{z_j}, \omega_j > 0,$$

$$\xi \geq |u_n|^{p^*} dx + \sum_{j \in \mathcal{I}} \xi_j \delta_{z_j}, \xi_j > 0, \quad S \xi_j^{\frac{p}{p^*}} \leq \omega_j,$$

where δ_{z_j} is the Dirac function of mass 1 concentrated at $z \in \mathbb{R}^N$ and ν, ω, and ξ are nonnegative measurements and bounded on $\mathbb{R}^N$. For all $j \in \mathcal{I}$, there holds

$$S_\mu \nu_j^{\frac{p}{2p_\mu^*}} \leq \omega_j, \; and \; \nu^{\frac{N}{2N-\mu}} \leq C(N,\mu)^{\frac{N}{2N-\mu}} \xi_j.$$

Lemma 2.3. Let $\{u_n\} \in W^{1,p}\left(\mathbb{R}^N\right)$ be a sequence as in Lemma 2.2, and defined as

$$\omega_\infty = \lim_{R\to\infty} \lim_{n\to\infty} \int_{|x|>R} |\nabla u_n|^p dx, \; \xi_\infty = \lim_{R\to\infty} \lim_{n\to\infty} \int_{|x|>R} |u_n|^{p^*} dx.$$

Then, there hold

$$S \xi_\infty^{\frac{p}{p^*}} \leq \omega_\infty,$$

and

$$\lim_{R\to\infty} \sup \int_{|x|>R} |\nabla u_n|^p dx = \omega_\infty + \int_{\mathbb{R}^N} d\omega, \; \lim_{R\to\infty} \sup \int_{|x|>R} |u_n|^{p^*} dx = \xi_\infty + \int_{\mathbb{R}^N} d\xi.$$

Lemma 2.4. Let $\{u_n\}$ be a bounded sequence in $W^{1,p}\left(\mathbb{R}^N\right)$ converging weakly to some $u \in W^{1,p}\left(\mathbb{R}^N\right)$. Let ν, ω, and ξ be non-negative measurements and bounded on $\mathbb{R}^N$, while ω_∞ and ξ_∞ be the numbers given as in Lemma 2.3. Assume that

$$\nu_\infty = \lim_{R\to\infty} \lim_{n\to\infty} sup \int_{|x|>R} \left(\int_{\mathbb{R}^N} \frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu} dy\right) |u_n(x)|^{p_\mu^*} dx.$$

Then, there hold

$$\limsup_{R\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^p dx = \omega_\infty + \int_{\mathbb{R}^N} d\omega, \; \limsup_{R\to\infty} \int_{\mathbb{R}^N} |u_n|^{p^*} dx = \xi_\infty + \int_{\mathbb{R}^N} d\xi,$$

$$\limsup_{R\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu}dy\,|u_n(x)|^{p_\mu^*}dx=\nu_\infty+\int_{\mathbb{R}^N}d\nu,$$

and

$$C(N,\mu)^{-\frac{2N}{2N-\mu}}\nu_\infty^{\frac{2N}{2N-\mu}}=\xi_\infty\left(\xi_\infty+\int_{\mathbb{R}^N}d\xi\right),$$

$$S^pC(N,\mu)^{-\frac{p}{p_\mu^*}}\nu_\infty^{\frac{p}{p_\mu^*}}=\omega_\infty\left(\omega_\infty+\int_{\mathbb{R}^N}d\omega\right).$$

3. Proof of Theorem 1.1

In this section, by applying both the mountain pass theorem and the symmetric mountain pass theorem, we shall establish the existence and multiplicity of solutions for problem (1.1) when $\beta = 1$.

We rewrite problem (1.1) as follows:

$$K(u)\left(-\Delta_p u+V(x)|u|^{p-2}u\right)=f(x,u)+\lambda\left(\int_{\mathbb{R}^N}\frac{|u(y)|^{p_\mu^*}}{|x-y|^\mu}dy\right)|u(x)|^{p_\mu^*-2}u,\ x\in\mathbb{R}^N, \tag{3.1}$$

with the associated functional $J_\lambda : W_V^{1,p}(\mathbb{R}^N)\to\mathbb{R}$:

$$J_\lambda(u)=\frac{1}{p}\hat{M}(\|u\|^p)-\frac{\lambda}{2p_\mu^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{p_\mu^*}|u(y)|^{p_\mu^*}}{|x-y|^\mu}dxdy-\int_{\mathbb{R}^N}F(x,u)dx, \tag{3.2}$$

where $\hat{M}(t):=\int_0^t M(s)ds$.

Definition 3.1. We say that $u\in W_V^{1,p}(\mathbb{R}^N)$ is a weak solution of problem (3.1), if, for all $v\in W_V^{1,p}\left(\mathbb{R}^N\right)$, there holds

$$\begin{aligned}\langle J_\lambda'(u),v\rangle&=M(\|u\|^p)\int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\nabla v+V(x)|u|^{p-2}uvdx\\&\quad-\lambda\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{p_\mu^*}|u(y)|^{p_\mu^*-2}uv}{|x-y|^\mu}dxdy-\int_{\mathbb{R}^N}f(x,u)vdx=0.\end{aligned}$$

Lemma 3.2. Assume that $c\geq 0$ and conditions (M_1), (V_1), (F_1), and (F_4) hold. Then, any Palais-Smale sequence $\{u_n\}$ for the energy functional J_λ at level c is bounded in $W_V^{1,p}\left(\mathbb{R}^N\right)$.

Proof. Let $\{u_n\}$ be a Palais-Smale sequence for the energy functional J_λ at the level c in $W_V^{1,p}\left(\mathbb{R}^N\right)$, then we have

$$c+o_n(1)=\frac{1}{p}\hat{M}(\|u_n\|^p)-\frac{\lambda}{2p_\mu^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^{p_\mu^*}|u_n(y)|^{p_\mu^*}}{|x-y|^\mu}dxdy-\int_{\mathbb{R}^N}F(x,u_n)dx, \tag{3.3}$$

and

$$\begin{aligned}\langle J_\lambda'(u_n),v\rangle&=M(\|u_n\|^p)\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u\nabla v+V(x)|u_n|^{p-2}u_nvdx\\&\quad-\lambda\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^{p_\mu^*}|u_n(y)|^{p_\mu^*-2}u_n(y)v(y)}{|x-y|^\mu}dxdy-\int_{\mathbb{R}^N}f(x,u_n)vdx,\end{aligned} \tag{3.4}$$

for all $v \in W_V^{1,p}\left(\mathbb{R}^N\right)$.

Consequently, it follows from (3.3) and (3.4) that there holds

$$\begin{aligned} c + o_n(1)\|u_n\| &= J_\lambda(u_n) - \frac{1}{\sigma}\langle J'_\lambda(u_n), u_n\rangle \\ &\geq \left(\frac{1}{\theta p} - \frac{1}{\sigma}\right) m_1 \|u\|^{\theta p} + \lambda\left(\frac{1}{\sigma} - \frac{1}{2p_\mu^*}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^{p_\mu^*}|u_n(y)|^{p_\mu^*}}{|x-y|^\mu}dxdy \\ &\quad + \int_{\mathbb{R}^N}\left(\frac{1}{\sigma}f(x,u_n)u_n - F(x,u_n)\right)dx. \end{aligned}$$

Condition (F_4) implies $\theta p < \sigma < 2p_\mu^*$, thus, we know that any Palais-Smale sequence for J_λ is bounded in $W_V^{1,p}\left(\mathbb{R}^N\right)$. The proof is completed. □

Lemma 3.3. Assume that (M_1), (V_1), (F_1), and (F_2) hold. Then, the functional J_λ satisfies the $(PS)_c$ condition for all $c \in \left(0, a\left(\frac{1}{p} - \frac{1}{\sigma}\right)\alpha_1 \lambda^{\frac{\theta p}{\theta p - 2p_\mu^*}}\right)$, where

$$\alpha_1 = \min\left\{\left(m_1 S_\mu^{\frac{2p_\mu^*}{p}}\right)^{\frac{\theta p}{2p_\mu^* - p}}, \left(m_1 \hat{C}^{-1} S^{\frac{p_\mu^*}{p}}\right)^{\frac{\theta p}{p_\mu^* - p}}\right\},$$

and $\hat{C}$ is a fixed constant that depends only on the parameters N and μ.

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence for J_λ, namely,

$$J_\lambda(u_n) \to c \ \ and \ J'_\lambda(u_n) \to 0, \ as\ n \to \infty.$$

By Lemma 3.2, $\{u_n\}$ is bounded, and consequently, there exist a function $u \in W_V^{1,p}\left(\mathbb{R}^N\right)$ and subsequence still denoted by $u_n \rightharpoonup u$ such that

$$u_n \rightharpoonup u \ \ weakly\ in\ W_V^{1,p}(\mathbb{R}^N), \ \ u_n \to u \ \ in \ \ L^t_{loc}\ for\ t \in [1,\ p^*),\ and\ u_n \to u \ \ a.e.\ \ in\ \mathbb{R}^N.$$

Furthermore, according to [8], there exist the bounded nonnegative measures ω, ξ, and ν such that as $n \to \infty$, there hold

$$|\nabla u_n|^p\,dx \overset{*}{\rightharpoonup} \omega, \quad |u_n|^{p^*} \overset{*}{\rightharpoonup} \xi,\ and\ \left(\int_{\mathbb{R}^N}\frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu}dy\right)|u_n(x)|^{p_\mu^*}\,dx \overset{*}{\rightharpoonup} \nu.$$

By Lemma 2.2, there exist at most countable set $\mathcal{I}$, families of positive numbers $\left\{\omega_j : j \in \mathcal{I}\right\}$, $\left\{\xi_j : j \in \mathcal{I}\right\}$, and $\left\{\nu_j : j \in \mathcal{I}\right\}$, and sequence of points $\left\{x_j\right\}_{j\in\mathcal{I}} \subset \mathbb{R}^N$ such that

$$\nu = \left(\int_{\mathbb{R}^N}\frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu}dy\right)|u_n|^{p_\mu^*}\,dx + \sum_{j\in\mathcal{I}}\nu_j\delta_{x_j},$$

$$\omega \geq |\nabla u_n|^p\,dx + \sum_{j\in\mathcal{I}}\omega_j\delta_{x_j}, \quad \xi \geq |u_n|^{p^*}\,dx + \sum_{j\in\mathcal{I}}\xi_j\delta_{x_j},$$

$$\nu_j \le S_\mu^{-\frac{2p_\mu^*}{p}} \omega_j^{\frac{2p_\mu^*}{p}}, \ and \ \nu_j^{\frac{N}{2N-\mu}} \le C(N,\mu)^{\frac{N}{2N-\mu}} \xi, \tag{3.5}$$

where δ_x is Dirac-mass of mass 1 concentrated at $x \in \mathbb{R}^N$.

Case 1: To verify the $(PS)_c$ condition, we construct a smooth cut-off function $\varphi \in C_0^\infty\left(\mathbb{R}^N\right)$ satisfying

$$0 \le \varphi \le 1, \ \ \varphi \equiv 1 \ in \ B_1(0), \ \varphi \equiv 0 \ in \ \mathbb{R}^N \backslash B_2(0), \ \ and \ |\nabla\varphi| \le 2 \ in \ \mathbb{R}^N.$$

For $\epsilon > 0$, define the rescaled function

$$\varphi_{\epsilon,j} = \varphi\left(\frac{x - x_j}{\epsilon}\right).$$

By Hölder's inequality, the following inequality holds:

$$\begin{aligned} \lim_{\epsilon\to0}\lim_{n\to\infty}\int_{\mathbb{R}^N}\left|u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi_{\epsilon,j}\right|dx &\le \lim_{\epsilon\to0}\lim_{n\to\infty}\left(\int_{\mathbb{R}^N}|\nabla u_n|^p dx\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^N}\left|u_n\nabla\varphi_{\epsilon,j}\right|^p dx\right)^{\frac{1}{p}} \\ &\le C\lim_{\epsilon\to0}\left(\int_{B_{2\epsilon}(z_j)}|u_n|^{\frac{Np}{N-p}}dx\right)^{\frac{N-p}{pN}}\left(\int_{B_{2\epsilon}(z_j)}\left|\nabla\varphi_{\epsilon,j}\right|^N dx\right)^{\frac{1}{N}} \\ &\le \tilde{C}\lim_{\epsilon\to0}\left(\int_{B_{2\epsilon}(z_j)}|u_n|^{p^*}dx\right)^{\frac{1}{p^*}} = 0, \end{aligned} \tag{3.6}$$

where $C = sup_n\|u_n\|$ and $\tilde{C} = C\left(\int_{B_{2(0)}}\left|\nabla\varphi_{\epsilon,j}\right|^N dx\right)^{\frac{p}{N}}$.

In addition, according to the definition of the truncation function $\varphi_{\epsilon,j}$ and the boundedness of $\{u_n\}$, we can obtain

$$\lim_{\epsilon\to0}\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x,u_n)u_n\varphi_{\epsilon,j}dx = 0. \tag{3.7}$$

An integration of conditions (3.6) and (3.7) yields

$$\begin{aligned} 0 &= \lim_{\epsilon\to0}\lim_{n\to\infty}\left\langle J'_\lambda(u_n), u_n\varphi_{\epsilon,j}\right\rangle \\ &= \lim_{\epsilon\to0}\lim_{n\to\infty} M(\|u_n\|^p)\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u_n\nabla u_n\varphi_{\epsilon,j} + V(x)|u_n|^{p-2}u_nu_n\varphi_{\epsilon,j}dx \\ &\quad - \lambda\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^{p_\mu^*}|u_n(y)|^{p_\mu^*}\varphi_{\epsilon,j}}{|x-y|^\mu}dxdy - \int_{\mathbb{R}^N} f(x,u_n)u_n\varphi_{\epsilon,j}dx \\ &= \lim_{\epsilon\to0}\lim_{n\to\infty} M(\|u_n\|^p)\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u_n\left(\varphi_{\epsilon,j}\nabla u_n + u_n\nabla\varphi_{\epsilon,j}\right) + V(x)|u_n|^p\varphi_{\epsilon,j}dx \\ &\quad - \lambda\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^{p_\mu^*}|u_n(y)|^{p_\mu^*}\varphi_{\epsilon,j}}{|x-y|^\mu}dxdy - \int_{\mathbb{R}^N} f(x,u_n)u_n\varphi_{\epsilon,j}dx \\ &\ge \lim_{\epsilon\to0}\lim_{n\to\infty} M(\|u_n\|^p)\int_{\mathbb{R}^N}|\nabla u_n|^p\varphi_{\epsilon,j} + V(x)|u_n|^p\varphi_{\epsilon,j}dx - \lambda\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^{p_\mu^*}|u_n(y)|^{p_\mu^*}\varphi_{\epsilon,j}}{|x-y|^\mu}dxdy \\ &\ge \lim_{\epsilon\to0}\lim_{n\to\infty} m_1\left(\int_{\mathbb{R}^N}|\nabla u_n|^p\varphi_{\varepsilon,j}dx\right)^\theta - \lambda\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^{p_\mu^*}|u_n(y)|^{p_\mu^*}\varphi_{\epsilon,j}}{|x-y|^\mu}dxdy \\ &\ge m_1\omega_j^\theta - \lambda\nu_j, \end{aligned} \tag{3.8}$$

which implies $m_1\omega_j^{\theta} \leq \lambda\nu_j$. Combining $\nu_j \leq S_\mu^{-\frac{2p_\mu^*}{p}}\omega_j^{\frac{2p_\mu^*}{p}}$, we can obtian

$$either \quad \omega_j \geq \left(m_1\lambda^{-1}S_\mu^{\frac{2p_\mu^*}{p}}\right)^{\frac{p}{2p_\mu^*-\theta p}} \ or\ \omega_j = 0.$$

Assume that case $\omega_j \geq \left(m_1\lambda^{-1}S_\mu^{\frac{2p_\mu^*}{p}}\right)^{\frac{p}{2p_\mu^*-\theta p}}$ holds. Then, we deduce that there exists j_0 such that $\omega_{j_0} \geq \left(m_1\lambda^{-1}S_\mu^{\frac{2p_\mu^*}{p}}\right)^{\frac{p}{2p_\mu^*-\theta p}}$, and we have

$$\begin{aligned}
c &= \lim_{n\to\infty}\left\{J_\lambda(u_n) - \frac{1}{\sigma}\langle J'_\lambda(u_n), u_n\rangle\right\} \\
&\geq \lim_{n\to\infty}\left(\frac{1}{\theta p} - \frac{1}{\sigma}\right)m_1\|u\|^{\theta p} \geq \lim_{n\to\infty} m_1\left(\frac{1}{p} - \frac{1}{\sigma}\right)\left(\int_{\mathbb{R}^N}|\nabla u_n|^p dx\right)^{\theta} \geq m_1\left(\frac{1}{p} - \frac{1}{\sigma}\right)\omega_{j_0}^{\theta} \\
&\geq m_1\left(\frac{1}{\theta p} - \frac{1}{\sigma}\right)\left(m_1\lambda^{-1}S_\mu^{\frac{2p_\mu^*}{p}}\right)^{\frac{\theta p}{2p_\mu^*-\theta p}},
\end{aligned}$$

which is a contradiction with the admissible range of c. Therefore, we deduce that $\omega_j = 0$ holds.
Case 2: To obtain the possible concentration of mass at infinity, we need to take a cut-off function ψ_R in $C^\infty\left(\mathbb{R}^N\right)$ satisfying $\psi_R = 0$ in $B_R(0)$, $\psi_R = 1$ in $\mathbb{R}^N \backslash B_{R+1}(0)$, and $|\nabla\psi_R| \leq 2/R$ in $\mathbb{R}^N$, where R is a positive constant. Furthermore, an application of Lemma 2.4 together with the Hardy-Littlewood-Sobolev and Hölder's inequality yields

$$\begin{aligned}
\nu_\infty &= \lim_{R\to\infty}\lim_{\mathrm{n}\to\infty}\int_{\mathbb{R}^{\mathrm{N}}}\left(\int_{\mathbb{R}^{\mathrm{N}}}\frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu}dy\right)|u_n(x)|^{p_\mu^*}\psi_{\mathbb{R}}(x)\,dx \\
&\leq C(N,\mu)\lim_{R\to\infty}\lim_{n\to\infty}\|u_n\|_{p^*}^{p_\mu^*}\left(\int_{\mathbb{R}^{\mathrm{N}}}|u_n(x)|^{p^*}\psi_{\mathbb{R}}(x)\,dx\right)^{\frac{p_\mu^*}{p^*}} \\
&\leq \hat{C}\xi_\infty^{\frac{p_\mu^*}{p^*}},
\end{aligned} \tag{3.9}$$

where $\hat{C} = sup\|u_n\|_{p^*}^{p_\mu^*}$. It follows from conditions (3.9) and $\left\langle J'_\lambda(u_n), u_n\psi_R\right\rangle = 0$ that

$$\begin{aligned}
0 &= \lim_{R\to\infty}\lim_{n\to\infty}\langle J'_\lambda(u_n), u_n\psi_R\rangle \\
&= \lim_{R\to\infty}\lim_{n\to\infty} M(\|u_n\|^p)\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u_n\nabla u_n\psi_R + V(x)|u_n|^{p-2}u_n u_n\psi_R dx \\
&\quad - \lambda\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^{p_\mu^*}|u_n(y)|^{p_\mu^*}\psi_R(y)}{|x-y|^\mu}dxdy - \int_{\mathbb{R}^N} f(x,u_n)u_n\psi_R dx \\
&\geq \lim_{R\to\infty}\lim_{n\to\infty} m_1\left(\int_{\mathbb{R}^N}|\nabla u_n|^p\psi_R dx\right)^{\theta} - \lambda\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^{p_\mu^*}|u_n(y)|^{p_\mu^*}\psi_R(y)}{|x-y|^\mu}dxdy \\
&\geq m_1\omega_\infty^{\theta} - \hat{C}\lambda\xi_\infty^{\frac{p_\mu^*}{p^*}}.
\end{aligned}$$

Hence, $m_1\omega_\infty^\theta \le \hat{C}\lambda\xi_\infty^{\frac{p_\mu^*}{p^*}}$. Combining Lemma 2.3, we can obtain

$$either \quad \omega_\infty \ge \left(m_1\hat{C}^{-1}\lambda^{-1}S^{\frac{p_\mu^*}{p}}\right)^{\frac{p}{p_\mu^*-\theta p}} \ or\ \omega_\infty = 0.$$

Following the same approach as in Case 1, we conclude

$$c = \lim_{n\to\infty}\left\{J_\lambda(u_n) - \frac{1}{\sigma}\left\langle J_\lambda'(u_n), u_n\right\rangle\right\} \ge m_1\left(\frac{1}{\theta p} - \frac{1}{\sigma}\right)\left(m_1\hat{C}^{-1}\lambda^{-1}S^{\frac{p_\mu^*}{p}}\right)^{\frac{\theta p}{p_\mu^*-\theta p}}.$$

This also leads to a contradiction with the definition of c. Hence, we have $\omega_\infty = 0$.

Based on the foregoing analysis, set

$$\alpha_1 = \min\left\{\left(m_1 S_\mu^{\frac{2p_\mu^*}{p}}\right)^{\frac{\theta p}{2p_\mu^*-\theta p}}, \left(m_1\hat{C}^{-1}S^{\frac{p_\mu^*}{p}}\right)^{\frac{\theta p}{p_\mu^*-\theta p}}\right\}.$$

Thus, for any $0 < c < m_1\left(\frac{1}{p} - \frac{1}{\sigma}\right)\alpha_1\lambda^{\frac{\theta p}{\theta p-2p_\mu^*}}$, we have $\omega_j = 0$ and $\omega_\infty = 0$ for all $j \in \mathcal{I}$. Hence,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^{p_\mu^*}|u_n(y)|^{p_\mu^*}}{|x-y|^\mu}dxdy = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{p_\mu^*}|u(y)|^{p_\mu^*}}{|x-y|^\mu}dxdy,$$

and

$$\int_{\mathbb{R}^N} f(x, u_n - u)(u_n - u)dx \to 0, \quad as\ n \to \infty.$$

Now, we proceed to prove that $\{u_n\}$ strongly converges to u in $W_V^{1,p}\left(\mathbb{R}^N\right)$. First, we need to introduce a continuous linear function $\mathcal{P}(u)$ in $W_V^{1,p}(\mathbb{R}^N)$ defined by

$$\langle\mathcal{P}(u), v\rangle = \int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\nabla v dx + \int_{\mathbb{R}^N}V(x)|u|^{p-2}uvdx.$$

Since $\{u_n\}$ is uniformly bounded and $M(\cdot) \in C(\mathbb{R}^+)$, it easy to obtain that $M(\|u_n\|^p)$ and $M(\|u\|^p)$ are uniformly bounded, which leads to

$$\lim_{n\to\infty}M(\|u_n\|^p)\langle\mathcal{P}(u_n), u_n - u\rangle = \lim_{n\to\infty}M(\|u\|^p)\langle\mathcal{P}(u), u_n - u\rangle = 0.$$

Furthermore, there holds

$$\begin{aligned}M(\|u_n\|^p)\langle\mathcal{P}(u_n), u_n - u\rangle - M(\|u\|^p)\langle\mathcal{P}(u), u_n - u\rangle &= M(\|u_n\|^p)\langle\mathcal{P}(u_n) - \mathcal{P}(u), u_n - u\rangle \\ &\quad - (M(\|u_n\|^p) - M(\|u\|^p))\langle\mathcal{P}(u), u_n - u\rangle \\ &= M(\|u_n\|^p)\langle\mathcal{P}(u_n) - \mathcal{P}(u), u_n - u\rangle.\end{aligned}$$

Clearly, $\langle J_\lambda'(u_n) - J_\lambda'(u), u_n - u\rangle \to 0$ as $n \to \infty$. Hence,

$$\begin{aligned}o_n(1) &= \left\langle J_\lambda'(u_n) - J_\lambda'(u), u_n - u\right\rangle \\ &= M(\|u_n\|^p)\langle\mathcal{P}(u_n), u_n - u\rangle - \lambda\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^{p_\mu^*}|u_n(y)|^{p_\mu^*-2}u_n(u_n-u)}{|x-y|^\mu}dxdy \\ &\quad - M(\|u\|^p)\langle\mathcal{P}(u), u_n - u\rangle - \lambda\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{p_\mu^*}|u(y)|^{p_\mu^*-2}u(u_n-u)}{|x-y|^\mu}dxdy \\ &\quad - \int_{\mathbb{R}^N}f(x, u_n - u)(u_n - u)dx \\ &= M(\|u_n\|^p)\langle\mathcal{P}(u_n) - \mathcal{P}(u), u_n - u\rangle,\end{aligned}$$

and

$$
\begin{aligned}
o_n(1) &= \langle \mathcal{P}(u_n) - \mathcal{P}(u), u_n - u \rangle \\
&= \int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n \nabla (u_n - u) dx + \int_{\mathbb{R}^N} V(x) |u_n|^{p-2} u_n (u_n - u) dx \\
&\quad - \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla (u_n - u) dx - \int_{\mathbb{R}^N} V(x) |u|^{p-2} u (u_n - u) dx \\
&= \int_{\mathbb{R}^N} \left(|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u \right) \nabla (u_n - u) dx \\
&\quad + \int_{\mathbb{R}^N} V(x) \left(|u_n|^{p-2} u_n - |u|^{p-2} u \right) (u_n - u) dx.
\end{aligned}
$$

For the purpose of proof, we recall the well-known Simon inequality. For any $\zeta, \eta \in \mathbb{R}$,

$$
\langle |\zeta|^{p-2} \zeta - |\eta|^{p-2} \eta, \zeta - \eta \rangle \geqslant \begin{cases} C_p |\zeta - \eta|^p, & p \geq 2, \\ C_p \frac{|\zeta - \eta|^2}{(|\zeta| + |\eta|)^{2-p}}, & 1 < p < 2, \end{cases} \tag{3.10}
$$

where C_p is a constant depending only on p. By (3.10), we obtain

$$
\int_{\mathbb{R}^N} |\nabla u_n - \nabla u|^p dx \leq C_p \int_{\mathbb{R}^N} \left(|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u \right) \nabla (u_n - u) dx, \tag{3.11}
$$

and

$$
\int_{\mathbb{R}^N} V(x) |u_n - u|^p dx \leq C_p \int_{\mathbb{R}^N} V(x) \left(|u_n|^{p-2} u_n - |u|^{p-2} u \right) (u_n - u) dx. \tag{3.12}
$$

By combining (3.11) and (3.12), we can deduce that $\|u_n - u\| = o_n(1)$. Thus, $\{u_n\}$ is strongly convergent to u in $W_V^{1,p}\left(\mathbb{R}^N\right)$. The proof is completed. □

Lemma 3.4. Assume that conditions (M_1), (V_1), (F_1), and (F_2) hold. For $\lambda \in (0, \lambda^*)$, there exist $\rho,\ \alpha > 0$ such that $J_\lambda \geq \alpha > 0$ for all $u \in W_V^{1,p}\left(\mathbb{R}^N\right)$ with $\|u\| = \rho$.

Proof. It follows from (F_2) and Hölder's inequality that there hold

$$
|F(x,u)| \leq \int_0^u |f(x,s)| ds \leq \int_0^u \sum_{i=1}^m \omega_i(x) |s|^{r_i - 1} ds \leq C_1 \sum_{i=1}^m \omega_i(x) |u|^{r_i},
$$

and

$$
\int_{\mathbb{R}^N} F(x,u) dx \leq \int_{\mathbb{R}^N} C_1 \sum_{i=1}^m \omega_i(x) |u_n|^{r_i} dx \leq C_1 \sum_{i=1}^m \|\omega_i(x)\|_{\vartheta_i} \|u\|_{\vartheta_i^*}^{r_i} \leq C_2 \sum_{i=1}^m \|u\|^{r_i}. \tag{3.13}
$$

Combining (2.2) with (3.13), assuming $\|u\| = \rho \in (0,1)$ is sufficiently small, there holds

$$\begin{aligned} J_\lambda(u) &= \frac{1}{p}\hat{M}\left(\|u\|^p\right) - \frac{\lambda}{2p_\mu^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{p_\mu^*}|u(y)|^{p_\mu^*}}{|x-y|^\mu}dxdy - \int_{\mathbb{R}^N}F(x,u)dx \\ &\geq \frac{m_1}{\theta p}\|u\|^{\theta p} - \frac{\lambda}{2p_\mu^*}S_\mu^{-\frac{2p_\mu^*}{p}}\|u\|^{2p_\mu^*} - \int_{\mathbb{R}^N}C_1\sum_{i=1}^m w_i(x)|u|^{r_i}dx \\ &\geq \frac{m_1}{\theta p}\rho^{\theta p} - \frac{\lambda}{2p_\mu^*}S_\mu^{-\frac{2p_\mu^*}{p}}\rho^{2p_\mu^*} - C_2\sum_{i=1}^m \rho^{r_i} \\ &\geq \frac{m_1}{\theta p}\rho^{\theta p} - \frac{\lambda}{2p_\mu^*}S_\mu^{-\frac{2p_\mu^*}{p}}\rho^{2p_\mu^*} - C_2 m\rho^{r_1} \\ &= \rho^{r_1}\left(\frac{m_1}{\theta p}\rho^{\theta p - r_1} - \frac{\lambda}{2p_\mu^*}S_\mu^{-\frac{2p_\mu^*}{p}}\rho^{2p_\mu^* - r_1} - C_2 m\right) \\ &= \rho^{r_1}\left(l_1\rho^{\theta p - r_1} - \lambda l_2\rho^{2p_\mu^* - r_1} - C_2 m\right). \end{aligned}$$

Define

$$h(t) = l_1 t^{\theta p - r_1} - \lambda l_2 t^{2p_\mu^* - r_1} - C_2 m, \ for\ all\ \ t > 0,$$

where $l_1,\ l_2$ are some positive constants. Since $0 < r_1 < \theta p < 2p_\mu^*$, the function $h(t)$ attains its maximum at a point t_0 satisfying $h'(t_0) = 0$, which is explicitly given by

$$t_0 = \left(\frac{l_1(\theta p - r_1)}{\lambda l_2\left(2p_\mu^* - r_1\right)}\right)^{\frac{1}{2p_\mu^* - \theta p}},$$

and

$$h(t)_{\max} = h(t_0) = \left(\frac{2p_\mu^* - \theta p}{2p_\mu^* - r_1}\right) l_1^{\frac{2p_\mu^* - r_1}{2p_\mu^* - \theta p}}\left(\frac{1}{l_2}\right)^{\frac{\theta p - r_1}{2p_\mu^* - \theta p}}\left(\frac{\theta p - r_1}{2p_\mu^* - \theta p}\right)^{\frac{\theta p - r_1}{2p_\mu^* - \theta p}}\lambda^{-\frac{\theta p - r_1}{2p_\mu^* - \theta p}} - C_2 m.$$

Take

$$\lambda^* = \left(\frac{\left(\frac{2p_\mu^* - \theta p}{2p_\mu^* - r_1}\right) l_1^{\frac{2p_\mu^* - r_1}{2p_\mu^* - \theta p}}\left(\frac{1}{l_2}\right)^{\frac{\theta p - r_1}{2p_\mu^* - \theta p}}\left(\frac{\theta p - r_1}{2p_\mu^* - \theta p}\right)^{\frac{\theta p - r_1}{2p_\mu^* - \theta p}}}{C_2 m}\right)^{\frac{2p_\mu^* - \theta p}{\theta p - r_1}},$$

then, $h(t_0) > 0$ for any $\lambda \in (0, \lambda^*)$. Therefore, for $\lambda \in (0, \lambda^*)$, there exist $\alpha > 0$ with $J_\lambda(u) \geq \alpha > 0$ while $u \in W_V^{1,p}\left(\mathbb{R}^N\right)$ with $\|u\| = \rho \in (0,1)$ is sufficiently small. This completes the proof. □

Lemma 3.5. Suppose that conditions (M_1), (V_1), and (F_1)–(F_3) are satisfied. Then, there exists an element $e \in W_V^{1,p}\left(\mathbb{R}^N\right)$ with $\|e\| \geq \rho$ such that $J_\lambda(e) < 0$ for all $\lambda > 0$, where ρ is the constant given in Lemma 3.4.

Proof. According to conditions (M_1) and (F_3), for any $t > 0$ and $u \in W_V^{1,p}\left(\mathbb{R}^N\right)$, we have

$$\begin{aligned} J_\lambda(tu) &= \frac{1}{p}\hat{M}(\|tu\|^p) - \frac{\lambda}{2p_\mu^*}t^{2p_\mu^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{p_\mu^*}|u(y)|^{p_\mu^*}}{|x-y|^\mu}dxdy - \int_{\mathbb{R}^N}F(x,tu)dx \\ &\leq \frac{m_2}{\theta p}t^{\theta p}\|u\|^{\theta p} - \frac{\lambda}{2p_\mu^*}t^{2p_\mu^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{p_\mu^*}|u(y)|^{p_\mu^*}}{|x-y|^\mu}dxdy - at^q\int_{\mathbb{R}^N}|u|^q\,dx \\ &\to -\infty \quad as\ t \to +\infty. \end{aligned} \tag{3.14}$$

From the established results, there exists a sufficiently large $t_1 > 0$ such that $J_\lambda(t_1u) < 0$ and $\|t_1u\| > \rho$. Let $e = t_1u$. Then, $\|e\| > \rho$, and $J_\lambda(e) < 0$ for all $\lambda > 0$. This completes the proof. □

In view of Lemma 3.5 and (F_3), we conclude that

$$\begin{aligned} J_\lambda(u) &\leq \frac{m_2}{\theta p}\|u\|^{\theta p} - \frac{\lambda}{2p_\mu^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{p_\mu^*}|u(y)|^{p_\mu^*}}{|x-y|^\mu}dxdy - a\int_{\mathbb{R}^N}|u|^q\,dx \\ &\leq \frac{m_2}{\theta p}\|u\|^{\theta p} - a\int_{\mathbb{R}^N}|u|^q\,dx := J_1(u). \end{aligned} \tag{3.15}$$

Clearly, the functional $J_1(u) : W_V^{1,p}\left(\mathbb{R}^N\right) \to \mathbb{R}$ and $J_\lambda(u) \leq J_1(u)$.

For every $0 < \delta < 1$, we can take $u_\delta \in C_0^\infty(\mathbb{R})$ with $|u_\delta|_q = 1$ and supp $u_\delta \subset B_{r_\delta(0)}$ such that $\|u_\delta\|_{L^p}^p \leq \delta$. Set

$$u = u_\delta\left(\tilde{\lambda}x\right),\ where\ \tilde{\lambda} := \lambda^{-\frac{p}{(\theta p - 2p_\mu^*)N}},$$

and then supp $u \subset B_{\tilde{\lambda}r_\delta}(0)$. Consequently, for $t \geq 0$ and $\lambda \in (0,1)$, we have

$$\begin{aligned} J_1(tu) &= \frac{m_2}{\theta p}t^{\theta p}\|u\|^{\theta p} - at^q\int_{\mathbb{R}^N}|u|^q\,dx \\ &= \frac{m_2}{\theta p}t^{\theta p}\left(\int_{\mathbb{R}^N}|\nabla u|^p + V(x)|u|^p\,dx\right)^\theta - at^q\int_{\mathbb{R}^N}|u|^q\,dx \\ &= \frac{m_1}{\theta p}t^{\theta p}\left(\lambda^{\frac{p}{\theta p - 2p_\mu^*}}\int_{\mathbb{R}^N}|\nabla u_\delta(x)|^p + V\left(\tilde{\lambda}^{-1}x\right)|u_\delta(x)|^p\,dx\right)^\theta - at^q\lambda^{\frac{p}{\theta p - 2p_\mu^*}}\int_{\mathbb{R}^N}|u_\delta(x)|^q dx \\ &= \lambda^{\frac{\theta p}{\theta p - 2p_\mu^*}}\left\{\frac{m_1}{\theta p}t^{\theta p}\left(\int_{\mathbb{R}^N}|\nabla u_\delta(x)|^p + V\left(\tilde{\lambda}^{-1}x\right)|u_\delta(x)|^p\,dx\right)^\theta - at^q\lambda^{\frac{(1-\theta)p}{\theta p - 2p_\mu^*}}\int_{\mathbb{R}^N}|u_\delta(x)|^q dx\right\}. \end{aligned} \tag{3.16}$$

Let

$$\psi_\lambda(tu_\delta) = \frac{m_1}{\theta p}t^{\theta p}\left(\int_{\mathbb{R}^N}|\nabla u_\delta(x)|^p + V\left(\tilde{\lambda}^{-1}x\right)|u_\delta(x)|^p\,dx\right)^\theta - at^q\lambda^{\frac{(1-\theta)p}{\theta p - 2p_\mu^*}}\int_{\mathbb{R}^N}|u_\delta(x)|^q dx, \tag{3.17}$$

which indicates that $\psi_\lambda \in C^1\left(W_V^{1,p}\left(\mathbb{R}^N\right), \mathbb{R}\right)$. According to the definition of ψ_λ, it is evident that there exists $t_2 \geq 0$ satisfying

$$\max_{t_2 \geq 0}\psi_\lambda(t_2u) \leq \frac{m_2}{\theta p}t_2^{\theta p}\left(\int_{\mathbb{R}^N}|\nabla u|^p + V\left(\tilde{\lambda}^{-1}x\right)|u|^p\,dx\right)^\theta.$$

It can be inferred from conditions (V_1)–(V_2), and supp$u_\delta \subset B_{r_\delta(0)}$ that there exists $\lambda_\delta > 0$ such that

$$0 \le V\left(\tilde{\lambda}^{-1}x\right) \le \frac{\delta}{|u_\delta|_p^p},$$

for all $|x| \le r_\delta$ and $\lambda \in (0, \lambda_\delta)$. Combined with the above conclusions, we can get

$$\max_{t_2\ge 0}\psi_\lambda (tu_\delta) \le \frac{m_2}{\theta p} t_2^{\theta p} (2\delta)^\theta .$$

Therefore,

$$\max_{t\ge 0}J_\lambda (tu_\delta) \le \frac{m_2}{\theta p} t_2^{\theta p} (2\delta)^\theta \lambda^{\frac{\theta p}{\theta p - 2p_\mu^*}}, \tag{3.18}$$

for all $\lambda \in (0, \min\{1, \lambda_\delta\})$.

Combining the previous results, we conclude the following result.

Lemma 3.6. Assume that conditions (M_1), (V_1)–(V_2), and (F_1)–(F_3) are satisfied. There exists an element $\hat{e}_\lambda \in W_V^{1,p}\left(\mathbb{R}^N\right)$ with $\|\hat{e}_\lambda\| > \rho$ such that $J_\lambda (\hat{e}_\lambda) < 0$, and

$$\max_{t\in[0,1]} J_\lambda (t\hat{e}_\lambda) \le \left(\frac{1}{p} - \frac{1}{\sigma}\right)\alpha_1 \lambda^{\frac{\theta p}{\theta p - 2p_\mu^*}},$$

where α_1 is defined in Lemma 3.3.

Proof. We have an arbitrarily small $\delta > 0$ satisfying:

$$\frac{m_2}{\theta p} t_2^{\theta p} (2\delta)^\theta \le \left(\frac{1}{p} - \frac{1}{\sigma}\right)\alpha_1.$$

Moreover, we may choose $\hat{t} > 0$ satisfying $\|\hat{t}e_\lambda\| > \rho$ and define $\hat{e}_\lambda = \hat{t}e_\lambda$. It follows that $J_\lambda(\hat{e}_\lambda) < 0$ and

$$\max_{t\in[0,1]} J_\lambda (t\hat{e}_\lambda) \le \left(\frac{1}{p} - \frac{1}{\sigma}\right)\alpha_1 \lambda^{\frac{\theta p}{\theta p - 2p_\mu^*}},$$

for all $t \ge \hat{t}$. This completes the proof. □

For any $m \in \mathbb{N}$, consider functions $u_\delta^i \in C_0^\infty\left(\mathbb{R}^N\right)$ such that supp $u_\delta^i \bigcap$ supp $u_\delta^j = \emptyset$ for all $1 \le i \ne j \le m$, and $\left|u_\delta^i\right|_q = 1$, $\left|u_\delta^i\right|_p^p < \delta$. Let $r_\delta^m > 0$ be such that supp $u_\delta^i \subset B_{r_\delta^m}(0)$ for all $i = 1, 2, ..., m$. Define

$$u^i = u_\delta^i\left(\tilde{\lambda}x\right),$$

and

$$\mathcal{H}_{\lambda,\delta}^m = span\left\{u^1, u^2, ..., u^m\right\}.$$

Put $u = \sum_{i=1}^m c_i u^i \in \mathcal{H}_{\lambda,\delta}^m$. Then for any u, we have

$$J_\lambda (u) \le C^* \sum_{i=1}^m J_\lambda\left(c_i u^i\right),$$

where C^* is some constant. From (3.16), we obtain

$$J_\lambda\left(c_i u^i\right) \leq \lambda^{\frac{\theta p}{\theta p-2p_\mu^*}} \psi_\lambda\left(c_i u^i\right).$$

Set

$$\eta_\delta := \max\left\{\left|u_\delta^i\right|_p^p : i = 1, 2, ..., m\right\},$$

and let $\lambda_{m,\delta} > 0$ be chosen such that for all $|x| \leq r_\delta^m$ and $\lambda \leq \lambda_{m,\delta}$,

$$V\left(\tilde{\lambda}^{-1} x\right) \leq \frac{\delta}{\eta_\delta}.$$

From the preceding results, we deduce that for all $\lambda < \lambda_{m,\delta}$,

$$\max_{u\in\mathcal{H}_\lambda^m} J_\lambda(u) \leq \frac{m_2}{\theta p} t_2^{\theta p} (2\delta)^\theta C^* \lambda^{\frac{\theta p}{\theta p-2p_\mu^*}}, \tag{3.19}$$

which yields the following conclusion.

Lemma 3.7. Assume conditions (M_1), (V_1)–(V_2), and (F_1)–(F_3) hold. Then for any $m \in \mathbb{N}$, there exists $\lambda_{m,\delta}$ such that for all $\lambda < \lambda_{m,\delta}$, we can find an m-dimensional subspace $\mathcal{H}_\lambda^m$ satisfying

$$\max_{u\in\mathcal{H}_\lambda^m} J_\lambda(u) \leq \left(\frac{1}{p} - \frac{1}{\sigma}\right) \alpha_1 \lambda^{\frac{\theta p}{\theta p-2p_\mu^*}}.$$

Proof. We select $\delta > 0$ sufficiently small to ensure

$$\frac{m_2}{\theta p} t_2^{\theta p} (2\delta)^\theta C^* \leq \left(\frac{1}{p} - \frac{1}{\sigma}\right) \alpha_1.$$

Furthermore, combining this with (3.19) and setting $\mathcal{H}_{\lambda,\delta}^m = \mathcal{H}_\lambda^m$, we have

$$\max_{u\in\mathcal{H}_\lambda^m} J_\lambda(u) \leq \left(\frac{1}{p} - \frac{1}{\sigma}\right) \alpha_1 \lambda^{\frac{\theta p}{\theta p-2p_\mu^*}}.$$

This completes the proof. □

Let E be a Banach space. We denote by Σ the family of all closed subsets $A \subset E\backslash\{0\}$ that are symmetric (i.e., $A = -A$).

Definition 3.8 ([10]). For a set $A \in \Sigma$, the Krasnosel'skii genus $\gamma(A)$ is defined as the smallest positive integer k for which there exists an odd continuous mapping $\phi \in C(A, \mathbb{R}^k\backslash 0)$ that is odd with respect to the origin.

Proof of Theorem 1.1. Building upon the preceding analysis, we set $\lambda_0 = \min\{1, \lambda_\delta, \lambda^*\}$. Subsequently, for an arbitrary $\lambda \in (0, \lambda_0)$ and the application of Lemmas 3.2–3.5, there exists a convergent sequence $\{u_n\}$ such that

$$J_\lambda(u_n) \to c \ \ and \ J_\lambda'(u_n) \to 0, \ \ as\ n \to \infty,$$

where

$$c := \inf_{y\in\Gamma_\lambda} \max_{t\in[0,1]} J_\lambda (y (t)),$$

and

$$\Gamma_\lambda = \left\{y \in C\left([0, 1], W_V^{1,p}(\mathbb{R}^N)\right) : \ y(0) = 0 \textit{ and } y(1) = \hat{e}_\lambda\right\}.$$

According to Lemma 3.6, we obtain

$$\alpha \le c \le \left(\frac{1}{p} - \frac{1}{\sigma}\right)\alpha_1 \lambda^{\frac{\theta p}{\theta p-2p_\mu^*}}.$$

By Lemmas 3.2 and 3.3, the functional J_λ satisfies the $(PS)_c$ condition. An application of the mountain pass theorem yields a function $u_0 \in W_V^{1,p}\left(\mathbb{R}^N\right)$ satisfying $J_\lambda(u_0) = c$ and $J'_\lambda(u_0) = 0$. Consequently, u_0 constitutes a solution to problem (3.1).

Let

$$\mathit{\Gamma} := \left\{g \in C\left(W_V^{1,p}\left(\mathbb{R}^N\right), W_V^{1,p}\left(\mathbb{R}^N\right)\right) : g \textit{ is an odd homeomorphism}\right\},$$

where $C(X, Y)$ denotes the space of continuous mappings from X to Y. For every $A \in \mathit{\Sigma}$, we define the quantity

$$j (A) := \min_{g\in\mathit{\Gamma}} \gamma\left(g (A) \cap \partial B_\rho\right),$$

where $\rho > 0$ is the constant given in Lemma 3.3, and $\gamma(\cdot)$ denotes the Krasnosel'skii genus. This construction yields a variant of the Benci pseudo-index [2].

Set

$$c_i := \mathop{inf}_{j(A)\ge i} \mathop{sup}_{u\in A} J_\lambda (u), \quad i = 1, 2, ..., m.$$

Clearly, we have the monotonicity of $\{c_j\}_{j=1}^m$ with $c_1 \le c_2 \le \cdots \le c_m$. We shall prove that the estimates $c_1 \ge \alpha$ and $c_m \le \mathop{sup}_{u\in\mathcal{H}_\lambda^m} J_\lambda (u)$ hold, where α is the constant given in Lemma 3.3. In fact, for any admissible set $A \in \sum$ with $j(A) \ge 1$, it follows that $\gamma\left(g(A) \cap \partial B_\rho\right) \ge 1$, which implies $g(A) \cap \partial B_\rho \ne \emptyset$. From Lemma 3.3, we derive the lower bound

$$J_\lambda(u) \ge \alpha \quad \text{for all } |u| = \rho. \tag{3.20}$$

Consequently, the inequality $\sup_{u\in A} J_\lambda(u) \ge \alpha$ yields $c_1 \ge \alpha$. The Krasnosel'skii genus satisfies Benci's dimension property [2], giving

$$\gamma(g(\mathcal{H}_\lambda^m) \cap \partial B\rho) = \dim \mathcal{H}_\lambda^m = m, \quad \text{for any } g \in \Gamma, \tag{3.21}$$

which implies $j(\mathcal{H}_\lambda^m) = m$. Therefore, we obtain the upper bound $c_m \le \sup u \in \mathcal{H}_\lambda^m J_\lambda(u)$. Combining with Lemma 3.7, there exists $\lambda_{00} = \lambda_{m,\delta}$ such that for all λ satisfying $\lambda < \lambda_{00}$, we establish the chain of inequalities:

$$\alpha \le c_1 \le c_2 \le \cdots \le c_m \le \sup_{u\in\mathcal{H}_\lambda^m} J_\lambda(u) \le \left(\frac{1}{p} - \frac{1}{\sigma}\right)\alpha_1 \lambda^{\frac{\theta p}{\theta p-2p_\mu^*}}.$$

By critical point theory, all critical values c_i $(1 \le i \le m)$ are well-defined. The evenness of J_λ guarantees the existence of at least m pairs of critical points, which correspond to m pairs of solutions for problem (3.1).

4. Proof of Theorem 1.2

In this section, we employ the dual fountain theorem to prove the existence of infinitely many nontrivial solutions to problem (1.1) for any $\lambda,\beta > 0$. Clearly, the functional $J_{\lambda,\beta} : W_V^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ associated with problem (1.1) can be expressd as follows:

$$J_{\lambda,\beta}(u) = \frac{1}{p}\hat{M}\left(\|u\|^p\right) - \frac{\lambda}{2p_\mu^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{p_\mu^*}|u(y)|^{p_\mu^*}}{|x-y|^\mu}dxdy - \beta\int_{\mathbb{R}^N}F(x,u)dx. \tag{4.1}$$

We begin by recalling some relevant theoretical preliminaries. Let X be a reflexive and separable Banach space, then there are $e_j \in X$ and $e_j^* \in X^*$ such that

$$X = \overline{span\left\{e_j \mid j = 1,2,\cdots\right\}}, \quad X^* = \overline{span\left\{e_j^* \mid j = 1,2,\cdots\right\}},$$

and

$$\left\langle e_i^*, e_j\right\rangle = \begin{cases} 1, i = j, \\ 0, i \neq j. \end{cases}$$

To establish our framework, we define the following subspaces: $X_j := span\left\{e_j\right\}$, $Y_k := \oplus_{j=1}^k X_j$, and $Z_k = \overline{\oplus_{j=1}^k X_j}$. Furthermore, we introduce the sets

$$B_k := \{u \in Y_k \mid \|u\|_X \leq \rho_k\} \text{ } and \text{ } N_k := \{u \in Z_k \mid \|u\|_X = Y_k\},$$

where the radii satisfy $\rho_k > \gamma_k > 0$ for all $k \in \mathbb{N}$.

Definition 4.1. Let $\varphi \in C^1(X,\mathbb{R})$ and $c \in \mathbb{R}$. The functional φ satisfies the $(PS)_c^*$ condition (with respect to (Y_n)) if any sequence $\left\{u_{n_j}\right\} \subset X$ such that

$$u_{n_j} \in Y_{n_j},\ \varphi(u_{n_j}) \to c,\ and\ \varphi'\mid_{n_j}(u_{n_j}) \to 0 \ in\ X^*\ as\ n_j \to \infty.$$

Then, φ admits a convergent subsequence.

Theorem 4.2. (Dual fountain theorem [24]). Let $\varphi \in C^1(X,\mathbb{R})$ with $\varphi(u) = \varphi(-u)$. Suppose that for every $k \geq k_1$, there exists $\rho_k > \gamma_k > 0$ such that
(A_1) $a_k := \inf_{\|u\|_X=\rho_k, u\in Z_k} \varphi(u) \geq 0$;
(A_2) $b_k := \max_{\|u\|_X=\gamma_k, u\in Y_k} \varphi(u) < 0$;
(A_3) $d_k := \inf_{\|u\|_X\leq\rho_k, u\in Z_k} \varphi(u) \to 0, as\ k \to \infty$;
(A_4) φ *satisfies the* $(PS)_c^*$ *condition for every* $c \in [d_{k_1}, 0)$.
Then, φ has a sequence of negative critical values converging to 0.

Lemma 4.3. Assume that conditions (V_1), (V_2), and (F_2) hold. Then, for $1 \leq q < p^*$, we have

$$\zeta_k := \sup_{u\in Z_k, \|u\|=1} \|u\|_q \to 0 \ as\ k \to \infty.$$

Proof. It is clear that $0 < \zeta_{k+1} < \zeta_k$, so there exists $\zeta_0 \geq 0$ such that $\zeta_k \to \zeta_0$ as $k \to \infty$. According to the definition of ζ_k, there exists $u_k \in Z_k$ such that $\|u_k\| = 1$, $0 \leq \zeta_0 - |u_k|_q < \frac{1}{k}$ for every $k > 0$. Consequently, there exists a subsequence $\{u_k\}$ (still denoted by the same index) such that $u_k \rightharpoonup u$ in $W_V^{1,p}\left(\mathbb{R}^N\right)$. Moreover, we have

$$\left\langle e_j^*, u\right\rangle = \lim_{k\to\infty}\left\langle e_j^*, u_k\right\rangle = 0, j = 1, 2, \cdots,$$

which yields that $u = 0$. Thus, $u_k \rightharpoonup 0$ weakly in $W_V^{1,p}(\mathbb{R}^N)$. By the compact embedding $W_V^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$, we obtain the strong convergence $u_k \to 0$ in $L^q(\mathbb{R}^N)$. We therefore conclude that $\zeta_0 = 0$. □

Lemma 4.4. Assume that (M_1), (V_1), (F_1), and (F_2) hold. Let $\{u_n\} \subset W_V^{1,p}(\mathbb{R}^N)$ be a Palais-Smale sequence for $J_{\lambda,\beta}$. Then, $J_{\lambda,\beta}$ satisfies the $(PS)_c$ condition for $c < 0$.

Proof. Let $\{u_n\} \subset W_V^{1,p}(\mathbb{R}^N)$ be a $(PS)_c$ sequence for $J_{\lambda,\beta}$. It is clear that $\{u_n\}$ is bounded on $W_V^{1,p}(\mathbb{R}^N)$. Following an argument similar to that in Lemma 3.3, we obtain
Case 1:

$$\begin{aligned}
0 > c &= \lim_{n\to\infty} J_{\lambda,\beta}(u_n) - \frac{1}{\sigma}\left\langle J'_{\lambda,\beta}(u_n), u_n\right\rangle \\
&\geq \lim_{n\to\infty}\left\{\left(\frac{1}{\theta p} - \frac{1}{\sigma}\right) m_1 \|u_n\|^{\theta p} + \beta\int_{\mathbb{R}^N}\left(\frac{1}{\sigma} f(x, u_n) u_n - F(x, u_n)\right) dx\right\} \\
&\geq \left(\frac{1}{\theta p} - \frac{1}{\sigma}\right) m_1 \left(m_1 \lambda^{-1} S_\mu^{\frac{2p_\mu^*}{p}}\right)^{\frac{\theta p}{2p_\mu^* - \theta p}} + \beta\int_{\mathbb{R}^N}\left(\frac{1}{\sigma} f(x, u_n) u_n - F(x, u_n)\right) dx.
\end{aligned}$$

It is clear that for any λ, $\beta > 0$, the right-hand side of the above equation is positive, which are absurd.
Case 2:

$$\begin{aligned}
0 > c &= \lim_{n\to\infty} J_{\lambda,\beta}(u_n) - \frac{1}{\sigma}\left\langle J'_{\lambda,\beta}(u_n), u_n\right\rangle \\
&\geq \left(\frac{1}{\theta p} - \frac{1}{\sigma}\right) m_1 \left(m_1 \hat{C}^{-1} \lambda^{-1} S^{\frac{p_\mu^*}{p}}\right)^{\frac{\theta p}{p_\mu^* - \theta p}} + \beta\int_{\mathbb{R}^N}\left(\frac{1}{\sigma} f(x, u_n) u_n - F(x, u_n)\right) dx.
\end{aligned}$$

For all λ, $\beta > 0$, the right-hand side of the above equation is likewise positive, which are absurd. An analysis of Cases 1 and 2 yields the following: For any λ, $\beta > 0$ and $c < 0$, we have $\omega_i = 0$ for all $i \in \mathcal{I}$ and $\omega_\infty = 0$.

Now, the rest of the proof follows in a similar manner as in the proof of Lemma 3.3. From the above discussion, we obtain the compactness of the $(PS)_c$ sequence with $c < 0$. □

Proof of Theorem 1.2. As established in the proof of Lemma 4.4, condition (A_4) of Theorem 4.2 is satisfied. We proceed to verify conditions (A_1)–(A_3).

To verify condition (A_1), we choose $r > 0$ sufficiently small so that for all $u \in Z_k$ with $\|u\| \leq r$, the following inequality holds:

$$\frac{m_1}{2\theta p}\|u\|^{\theta p} \geq \frac{\lambda}{2p_\mu^*} S_\mu^{-\frac{2p_\mu^*}{p}} \|u\|^{2p_\mu^*}.$$

By combining inequalities (2.2) and (4.1) with assumptions (M_1) and (F_2), we obtain the following estimate:

$$
\begin{aligned}
J_{\lambda,\beta}(u) &= \frac{1}{p}\hat{M}\left(\|u\|^{p}\right) - \frac{\lambda}{2p_{\mu}^{*}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{p_{\mu}^{*}}|u(y)|^{p_{\mu}^{*}}}{|x-y|^{\mu}}dxdy - \beta\int_{\mathbb{R}^N}F(x,u)dx \\
&\geq \frac{m_1}{\theta p}\|u\|^{\theta p} - \frac{\lambda}{2p_{\mu}^{*}}S_{\mu}^{-\frac{2p_{\mu}^{*}}{p}}\|u\|^{2p_{\mu}^{*}} - \beta m C_2\zeta_k^{r_1}\|u\|^{r_1} \\
&\geq \frac{m_1}{2\theta p}\|u\|^{\theta p} - \beta m C_2\zeta_k^{r_1}\|u\|^{r_1}.
\end{aligned}
$$

Set $\rho_k := \left(\frac{2\theta\beta p m C_2\zeta_k^{r_1}}{m_1}\right)^{\frac{1}{\theta p - r_1}}$. It follows from Lemma 4.3 that $\rho_k \to 0$ as $k \to \infty$. Hence, there exists $k_1 \in \mathbb{N}$ such that $\rho_k \leq r$ for all $k \geq k_1$. It follows that for any $k \geq k_1$ and $u \in Z_k$ with $|u| = \rho_k$, we have $J_{\lambda,\beta}(u) \geq 0$. This verifies condition (A_1).

For condition (A_2), for any $u \in Y_k$ and $\|u\| = \gamma_k < 1$ with $0 < \gamma_k < \rho_k$, we have

$$
\begin{aligned}
J_{\lambda,\beta}(u) &= \frac{1}{p}\hat{M}\left(\|u\|^{p}\right) - \frac{\lambda}{2p_{\mu}^{*}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{p_{\mu}^{*}}|u(y)|^{p_{\mu}^{*}}}{|x-y|^{\mu}}dxdy - \beta\int_{\mathbb{R}^N}F(x,u)dx \\
&\leq \frac{m_2}{\theta p}\|u\|^{\theta p} - \beta m C_3\|u\|^{r_m}.
\end{aligned}
$$

Since all norms are equivalent on the finite-dimensional subspace Y_k, and $r_m < \theta p$, for $\beta > 0$, term $\beta m C_3\|u\|^{r_m}$ dominates asymptotically when $\|u\| = \gamma_k$ is sufficiently small, and condition (A_2) follows.

Regarding condition (A_3), for any $k \geq k_1$ and $u \in Z_k$ with $|u| \leq \rho_k$, we obtain the lower bound

$$
J_{\lambda,\beta}(u) \geq -\beta m C_2\zeta_k^{r_1}|u|^{r_1} \geq -\beta m C_2\zeta_k^{r_1}\rho_k^{r_1}.
$$

Lemma 4.3 guarantees that both $\zeta_k \to 0$ and $\rho_k \to 0$ as $k \to \infty$. Consequently, condition (A_3) is satisfied, and we conclude that problem (1.1) admits a sequence of negative energy solutions.

Author contributions

All authors of this article have been contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. R. Alkhal, M. Kratou, K. Saoudi, Combined effects of singular and hardy nonlinearities in fractional Kirchhoff Choquard equation, *J. Appl. Anal. Comput.*, **15** (2025), 76–109. https://doi.org/10.11948/20230429
2. V. Benci, On critical point theory for indefinite functionals in the presence of symmetries, *Trans. Am. Math. Soc.*, **274** (1982), 533–572. https://doi.org/10.2307/1999120
3. C. S. Chen, Q. Zhu, Existence of positive solutions to p-Kirchhoff-type problem without compactness conditions, *Appl. Math. Lett.*, **28** (2014), 82–87. https://doi.org/10.1016/j.aml.2013.10.005
4. X. Y. Cheng, G. W. Dai, Positive solutions for p-Kirchhoff type problems on $\mathbb{R}^N$, *Math. Method. App. Sci.*, **38** (2015), 2650–2662. https://doi.org/10.1002/mma.3396
5. Y. Q. Cai, Y. L. Zhao, Ground state solution for the Logarithmic Schrödinger-Poisson system with critical growth, *Qual. Theory Dyn. Syst.*, **24** (2025), 16. https://doi.org/10.1007/s12346-024-01174-x
6. Y. Q. Cai, Y. L. Zhao, C. L. Luo, Ground state solution for a quasi-linear Schrödinger system with logarithmic perturbation, *Complex Var. Elliptic*, **2025** (2025), 2505226. https://doi.org/10.1080/17476933.2025.2505226
7. H. N. Fan, X. C. Liu, Positive and negative solutions for a class of Kirchhoff type problems on unbounded domain, *Nonlinear Anal.-Theor.*, **114** (2015), 186–196. https://doi.org/10.1016/j.na.2014.07.012
8. I. Fonseca, G. Leoni, *Modern methods in the calculus of variations: L^p spaces*, New York: Springer, 2007. https://doi.org/10.1007/978-0-387-69006-3
9. G. Kirchhoff, *Mechanik*, 1 Eds., Leipzig: Teubner, 1883.
10. M. A. Krasnosel'skiĭ,*Topological methods in the theory of nonlinear integral equations*, 1 Eds., Oxford : Pergamon, 1964.
11. F. Q. Lan, X. M. He, The Nehari manifold for a fractional critical Choquard equation involving sign-changing weight functions, *Nonlinear Anal.*, **180** (2019), 236–263. https://doi.org/10.1016/j.na.2018.10.010
12. Y. H. Li, F. Y. Li, J. P. Shi, Existence of a positive solution to Kirchhoff type problems without compactness conditions, *J. Differ. Equations*, **253** (2012), 2285–2294. https://doi.org/10.1016/j.jde.2012.05.017
13. D.C. Liu, On a p-Kirchhoff equation via fountain theorem and dual fountain theorem, *Nonlinear Anal.* **72**(2010), 302-308. https://doi.org/10.1016/j.na.2009.06.052
14. S. H. Liang, H. Liu, D. L. Zhang, p-Kirchhoff modified Schrödinger equation with critical nonlinearity in $\mathbb{R}^N$, *Results Math.*, **79** (2024), 83. https://doi.org/10.1007/s00025-023-02109-9

15. E. H. Lieb, M. Loss, *Analysis, graduate studies in mathematics*, 2 Eds., Providence: American Mathematical Society, 2001.

16. P. L. Lions, The concentration-compactness principle in the calculus of variations. The limit case, Part 1, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **1** (1984), 109–145. https://doi.org/10.1016/S0294-1449(16)30428-0

17. Z. Y. Liu, Z. Y. Liu, W. H. Xu, On the Choquard equation with double critical Sobolev exponents in $\mathbb{R}^N$, *Complex Var. Elliptic*, **70** (2025), 440–455. https://doi.org/10.1080/17476933.2024.2310228

18. S. I. Pekar, *Untersuchungen über die Elektronentheorie der Kristalle*, 1 Eds., Berlin: De Gruyter, 1954. https://doi.org/10.1515/9783112649305

19. R. Penrose, On gravity's role in quantum state reduction, *Gen. Relat. Gravit.*, **28** (1996), 581–600. https://doi.org/10.1007/BF02105068

20. Q. Y. Ren, Y. L. Zhao, Multiplicity of solutions for a class of critical Choquard double phase problems, *J. Elliptic Parabol. Equ.*, **11** (2025), 1841–1859. https://doi.org/10.1007/s41808-025-00400-0

21. L. Wang, T. Han, J. X. Wang, Infinitely many solutions for Schrödinger-Choquard-Kirchhoff equations involving the fractional p-Laplacian, *Acta Math. Sin.-English Ser.*, **37** (2021), 315–332. https://doi.org/10.1007/s10114-021-0125-z

22. Z. J. Wang, H. R. Sun, J. Liu, Normalized solutions for Kirchhoff equations with Choquard nonlinearity, *Discrete Cont. Dyn.-A*, **45** (2025), 1335–1365. https://doi.org/10.3934/dcds.2024131

23. L. D. Wang, p-Laplacian equations with general Choquard nonlinearity on lattice graphs, *J. Anal.*, **34** (2026), 465–485. https://doi.org/10.1007/s41478-025-00963-0

24. M. Willem, *Minimax theorems*, 1 Eds., Boston: Birkhäuser, 1996. https://doi.org/10.1007/978-1-4612-4146-1

25. M. B. Yang, Y. H. Ding, Existence of semiclassical states for a quasilinear Schrödinger equation with critical exponent in $\mathbb{R}^N$, *Ann. Mat. Pur. Appl.*, **192** (2013), 783–804. https://doi.org/10.1007/S10231-011-0246-6

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