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Research article

Solvability and spectral properties of a boundary value problem for differential equations with involution

Kairat Usmanov, Kulzina Nazarova* **and Zhazira Yerkisheva**

Department of Mathematics, Khoja Akhmet Yassawi International Kazakh-Turkish University, Kazakhstan

* **Correspondence:** Email: kulzina.nazarova@ayu.edu.kz; Tel: +77479901964.

Abstract: Differential equations with involution give rise to nonlocal constraints that significantly limit the application of classical methods in the theory of boundary value problems. In this paper, we study a boundary value problem for a system of differential equations with involution. By applying the parameterization method, the original boundary value problem is reduced to an equivalent Cauchy problem and a system of linear algebraic equations with respect to the introduced parameters. Explicit analytical solvability conditions are obtained, and the spectral properties of the problem are investigated. In cases where the solvability conditions are not satisfied, the spectrum of the corresponding boundary value problem is analyzed. The obtained results extend the existing analytical approaches for studying boundary value problems for differential equations with involution.

Keywords: involution; differential equations; boundary value problem; parameterization method; parameter; Cauchy problem; solvability; eigenvalues
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1. Introduction

Differential and integro-differential equations with deviating arguments constitute an important class of functional-differential equations arising in the mathematical modeling of nonlocal and hereditary processes. Such equations appear naturally in control theory, mathematical physics, population dynamics, engineering systems, and models with delayed or reflected interactions. In recent years, interest in these equations has increased significantly due to their applications in nonlinear partial differential equations, nonlocal boundary value problems, and integro-differential models with weakly singular kernels [1]. Moreover, equations with deviating arguments play an essential role in the development of modern analytical and numerical methods, including compact finite-difference schemes, operator-theoretic approaches, and spectral analysis of nonlocal problems.

The practical significance of differential and integro-differential equations with deviating arguments is underscored by their role in modeling complex systems. Recent advances include the development of high-order compact Crank–Nicolson alternating direction implicit (CN-ADI) on graded meshes for nonlinear problems and numerical methods for equations with Riemann–Liouville fractional kernels. The reliable implementation of such approaches relies heavily on a rigorous understanding of solvability and stability, as discussed in [2–4]. Although the present study focuses on the analytical investigation of equations with classical derivatives, the established spectral properties provide an essential theoretical framework for ensuring the consistency of more sophisticated numerical models.

Certain classes of equations with deviating arguments possess special structural properties arising from involutive transformations. Differential equations containing terms of the form $u(\sigma(x))$ with the mapping $\sigma(\sigma(x)) = x$ are commonly referred to as differential equations with involution or equations with Carleman shifts. On the interval $[0, T]$, the involution is often defined by the transformation $\sigma(x) = T - x$. Such equations arise in various problems involving symmetric structures, reflection phenomena, and nonlocal interactions. The presence of involution substantially changes the qualitative behavior of solutions and gives rise to operators with specific spectral and analytical properties. As a result, the study of equations with involution has become an active area of research in the theory of functional-differential equations and nonlocal boundary value problems.

The foundations of the theory of differential equations with involution were established in the classical works of Carleman [5], Przeworska-Rolewicz [6], and Wiener [7], where equations with transformed arguments and Carleman shifts were systematically investigated. Karapetiants and Samko [8] further developed the operator-theoretic approach to involutive equations. Cabada and Tojo [9] obtained important results on Green's functions, solvability, and boundary value problems for differential equations with involution.

In recent years, considerable attention has been devoted to the study of boundary value problems and spectral properties of differential equations with involution. Solvability issues for second-order differential equations with involution were investigated in [10], and boundary value problems for fractional differential equations with involution perturbations were studied in [11]. Spectral properties of nonlocal operators with multiple involutions were investigated in [12], and eigenvalue problems for differential operators with involution were analyzed in [13]. Nonlinear parabolic equations with involution and mixed problems for equations with involutive transformations were considered in [14, 15]. In [16], the method of deviating argument was employed to solve a singularly perturbed Cauchy problem for a second-order ordinary differential equation with variable coefficients. Asymptotic properties, spectral structure, and well-posedness issues for involution equations were investigated in [17–19]. Methods for solving ill-posed and nonlocal problems associated with differential operators were developed in [20], and inverse and fractional evolution problems involving involution perturbations were studied in [21]. These investigations demonstrate the continuing development of the theory of involution differential equations and highlight this theory's important role in the analysis of nonlocal and functional-differential models.

Various analytical and numerical approaches have been developed for the investigation of boundary value problems for differential and integro-differential equations. Among them, the parameterization method proposed by Dzhumabaev [22] has proven effective in the analysis of nonlocal boundary value problems. This method is based on reducing the original boundary value problem to an equivalent Cauchy problem together with a system of algebraic equations in terms of the introduced parameters.

The parameterization method has been successfully applied to various classes of differential and integro-differential equations [23–25], where explicit solvability conditions and constructive solution representations were obtained. These results demonstrate the efficiency of the parameterization method in the analysis of nonlocal problems and motivate its application to differential equations with involution.

Despite the significant progress achieved in the theory of differential equations with involution, solvability and spectral issues for boundary value problems associated with such equations have not been fully addressed, especially within the framework of the parameterization method. In particular, the derivation of explicit solvability criteria and the investigation of spectral properties for systems with involutive transformations require further analysis.

The results obtained extend existing analytical approaches to the study of differential equations with involution and provide new solvability criteria for nonlocal boundary value problems.

On the interval $[0, T]$, we consider a boundary value problem for a system of differential equations with involution:

$$\frac{dx(t)}{dt} = \mathbf{A}_0(t)x(t) + \mathbf{A}_1(t)x(T-t) + f(t), \tag{1.1}$$

$$\mathbf{C}_0 x(0) + \mathbf{C}_1 x(T) = d, \ d \in \mathbb{R}^n. \tag{1.2}$$

Here, $x(t) \in \mathbb{R}^n$, $\mathbf{A}_0(t)$, $\mathbf{A}_1(t) \in \mathbf{C}([0,T], \mathbb{R}^{n\times n})$, $\mathbf{C}_0$, and $\mathbf{C}_1$ are constant matrices, and $f(t) \in C([0,T], \mathbb{R}^n)$. We seek classical solutions $x \in C([0,T]) \cap C^1((0,T))$. Therefore, the value $x(T/2)$ is well-defined.

2. Parameterization method

Introducing the notation $\mu = x(T/2)$ and using the substitution $x(t) = u(t) + \mu$, we reduce the boundary value problem (1.1) and (1.2) to the following equivalent problem:

$$\frac{du(t)}{dt} = \mathbf{A}_0(t)u(t) + \mathbf{A}_1(t)u(T-t) + f(t) + (\mathbf{A}_0(t) + \mathbf{A}_1(t))\mu, \tag{2.1}$$

$$u\left(\frac{T}{2}\right) = 0, \tag{2.2}$$

$$\mathbf{C}_0 u(0) + \mathbf{C}_1 u(T) + (\mathbf{C}_0 + \mathbf{C}_1)\mu = d. \tag{2.3}$$

Lemma 1. *Let* $\mathbf{A}_0(\cdot), \mathbf{A}_1(\cdot), f(\cdot) \in C([0,T])$. *Then, between the solutions of the boundary value problem* (1.1)-(1.2) *and the solutions of the boundary value problem with parameter* (2.1)–(2.3)*, there exists a one-to-one correspondence, defined by the substitution* $x(t) = u(t) + \mu$*, where* $\mu = x(T/2)$.

Proof. Let $x(t)$ be a solution of the boundary value problem (1.1)-(1.2). Then, the function defined as $u(t) = x(t) - \mu$ satisfies (2.1)–(2.3), where $\mu = x(T/2)$.

Indeed, $du(t)/dt = dx(t)/dt$. Then,

$$\frac{du(t)}{dt} - \mathbf{A}_0(t)u(t) - \mathbf{A}_1(t)u(T-t) - f(t) - (\mathbf{A}_0(t) + \mathbf{A}_1(t))\mu$$

$$= \frac{dx(t)}{dt} - \mathbf{A}_0(t)x(t) - \mathbf{A}_1(t)x(T-t) - f(t) \ = 0.$$

Consequently, $u(t) = x(t) - \mu$ satisfies (2.1).

Because $\mu = x(T/2)$, we obtain

$$u\left(\frac{T}{2}\right) = x\left(\frac{T}{2}\right) - \mu = \mu - \mu = 0.$$

Condition (2.3) follows from

$$\mathbf{C}_0 u(0) + \mathbf{C}_1 u(T) + (\mathbf{C}_0 + \mathbf{C}_1)\mu - d = \mathbf{C}_0 x(0) + \mathbf{C}_1 x(T) - d = 0.$$

Conversely, let the pair $u(t)$, μ be a solution to system (2.1)–(2.3). Then, the function defined as $x(t) = u(t) + \mu$ satisfies (1.1), (1.2). Indeed, $dx(t)/dt = du(t)/dt$. Then,

$$\frac{dx(t)}{dt} - \mathbf{A}_0(t)x(t) - \mathbf{A}_1(t)x(T-t) - f(t)$$

$$= \frac{du(t)}{dt} - \mathbf{A}_0(t)u(t) - \mathbf{A}_1(t)u(T-t) - f(t) - (\mathbf{A}_0(t) + \mathbf{A}_1(t))\mu = 0.$$

Consequently, solution (2.1)–(2.3) satisfies (1.1).

Let us prove that $x(t)$ defined with the equality $x(t) = u(t) + \mu$ also satisfies the boundary condition (1.2):

$$\mathbf{C}_0 x(0) + \mathbf{C}_1 x(T) - d = \mathbf{C}_0 u(0) + \mathbf{C}_1 u(T) + (\mathbf{C}_0 + \mathbf{C}_1)\mu - d = 0.$$

The lemma is proven. □

As can be seen from (2.1)–(2.3), the boundary value problem is formally divided into two parts, that is, the Cauchy problem (2.1)-(2.2) for the original equation and the linear algebraic equation (2.3) for determining the value of the parameter μ.

3. Solution of the Cauchy problem

For a fixed parameter μ, Problem (2.1) is a linear Cauchy problem with a known inhomogeneous term.

Let us consider Eq (2.1) at points $t^* = T - t$:

$$\frac{du(T-t)}{dt} = \mathbf{A}_0(T-t)u(T-t) + \mathbf{A}_1(T-t)u(t) + f(T-t) + (\mathbf{A}_0(T-t) + \mathbf{A}_1(T-t))\mu. \tag{3.1}$$

Let us assume that the matrices $\mathbf{A}_0(t)$ and $\mathbf{A}_1(t)$ are symmetric with respect to the point $T/2$, that is,

$$\mathbf{A}_i(t) = \mathbf{A}_i(T-t),\ i = 0, 1,\ t \in [0, T].$$

Let us introduce the notation $v(t) = u(t) + u(T-t)$, $w(t) = u(t) - u(T-t)$. Then, we obtain the following Cauchy problem for systems of differential equations:

$$\frac{dw(t)}{dt} = (\mathbf{A}_0(t) + \mathbf{A}_1(t))\, v(t) + f_+^*(t), \tag{3.2}$$

$$\frac{dv(t)}{dt} = (\mathbf{A}_0(t) - \mathbf{A}_1(t))\, w(t) + f_-^*(t), \tag{3.3}$$

$$w\left(\frac{T}{2}\right) = 0, \tag{3.4}$$

$$v\left(\frac{T}{2}\right) = 0, \tag{3.5}$$

where $f_+^*(t) = f(t) + f(T-t) + 2(\mathbf{A}_0(t) + \mathbf{A}_1(t))\,\mu,\ \ f_-^*(t) = f(t) - f(T-t)$.

The solution of the Cauchy problem (3.2)–(3.5) is equivalent to the following integral equations:

$$w(t) = \int_{\frac{T}{2}}^{t} \mathbf{A}_+(\xi)\, v(\xi) d\xi + \int_{\frac{T}{2}}^{t} f_+^*(\xi) d\xi,$$

$$v(t) = \int_{\frac{T}{2}}^{t} \mathbf{A}_-(\xi)\, w(\xi) d\xi + \int_{\frac{T}{2}}^{t} f_-^*(\xi) d\xi,$$

where $\mathbf{A}_+(t) = \mathbf{A}_0(t) + \mathbf{A}_1(t)$ and $\mathbf{A}_-(t) = \mathbf{A}_0(t) - \mathbf{A}_1(t)$.

$$w(t) = \int_{\frac{T}{2}}^{t} \mathbf{A}_+(\xi) \left(\int_{\frac{T}{2}}^{\xi} \mathbf{A}_-(\tau)\, w(\tau) d\tau \right) d\xi + \int_{\frac{T}{2}}^{t} \mathbf{A}_+(\xi) \left(\int_{\frac{T}{2}}^{\xi} f_-^*(\tau)\, d\tau \right) d\xi + \int_{\frac{T}{2}}^{t} f_+^*(\xi) d\xi, \tag{3.6}$$

$$v(t) = \int_{\frac{T}{2}}^{t} \mathbf{A}_-(\xi) \left(\int_{\frac{T}{2}}^{\xi} \mathbf{A}_+(\tau)\, v(\tau) d\tau \right) d\xi + \int_{\frac{T}{2}}^{t} \mathbf{A}_-(\xi) \left(\int_{\frac{T}{2}}^{\xi} f_+^*(\tau)\, d\tau \right) d\xi + \int_{\frac{T}{2}}^{t} f_-^*(\xi) d\xi. \tag{3.7}$$

Let us introduce the following denomination:

$$\mathbf{K}_\pm(t,\tau) = \left(\int_{\tau}^{t} \mathbf{A}_+(\xi)\, d\xi \right) \mathbf{A}_-(\tau),\ \ \mathbf{K}_\mp(t,\tau) = \left(\int_{\tau}^{t} \mathbf{A}_-(\xi)\, d\xi \right) \mathbf{A}_+(\tau),$$

$$g_+(t) = \int_{\frac{T}{2}}^{t} \mathbf{A}_+(\xi) \left(\int_{\frac{T}{2}}^{\xi} f_-^*(\tau)\, d\tau \right) d\xi + \int_{\frac{T}{2}}^{t} f_+^*(\xi) d\xi,\ g_-(t) = \int_{\frac{T}{2}}^{t} \mathbf{A}_-(\xi) \left(\int_{\frac{T}{2}}^{\xi} f_+^*(\tau)\, d\tau \right) d\xi + \int_{\frac{T}{2}}^{t} f_-^*(\xi) d\xi.$$

Then, the integral equations (3.6) and (3.7) can be written in the following form:

$$w(t) = \int_{\frac{T}{2}}^{t} \mathbf{K}_\pm(t,\tau)\, w(\tau) d\tau + g_+(t), \tag{3.8}$$

$$v(t) = \int_{\frac{T}{2}}^{t} \mathbf{K}_\mp(t,\tau)\, v(\tau) d\tau + g_-(t). \tag{3.9}$$

Because the kernels $\mathbf{K}_{\pm}(t,\tau), \mathbf{K}_{\mp}(t,\tau)$ are continuous on the domain $\{(t,\tau) : T/2 \le \tau \le t \le T\}$, and the functions $g_{+}(t), g_{-}(t)$ are continuous on $[T/2, T]$, Eqs (3.8) and (3.9) are Volterra equations of the second kind. Consequently, they have unique solutions in the class of continuous functions on $[T/2, T]$.

Lemma 2. *The solutions to the Cauchy problem* (3.2)–(3.5) *satisfy the conditions*

$$w(t) = -w(T-t), \ v(t) = v(T-t)$$

on the interval $[0, T]$.

Proof. Consider the functions $w(T-t), \quad v(T-t)$ as the solution to the Cauchy problem (3.2)–(3.5) on $[0, T]$ due to the symmetry of the coefficients

$$\mathbf{A}_i(t) = \mathbf{A}_i(T-t), \ i = 0, 1, \ t \in [0, T].$$

The definitions $\mathbf{A}_{\pm}(t) = \mathbf{A}_0(t) \pm \mathbf{A}_1(t)$ of the Cauchy problem (3.2)–(3.5) are invariant under the replacement $T-t$. Therefore, the functions $w(T-t), \quad v(T-t)$ also satisfy the Cauchy problem (3.2)–(3.5) on $[0, T]$.

From (3.8), we have

$$w(T-t) = \int_{\frac{T}{2}}^{T-t} \mathbf{K}_{\pm}(T-t,\tau)\, w(\tau) d\tau + g_{+}(T-t).$$

Applying the change of variables $\tau = T - \tau_1$, we obtain

$$w(T-t) = -\int_{\frac{T}{2}}^{t} \mathbf{K}_{\pm}(T-t, T-\tau_1)\, w(T-\tau_1) d\tau_1 + g_{+}(T-t).$$

Consider the kernel $\mathbf{K}_{\pm}(T-t, T-\tau_1)$ and perform the substitution $\xi = T - \xi_1$:

$$\mathbf{K}_{\pm}(T-t, T-\tau_1) = \left(\int_{T-\tau_1}^{T-t} \mathbf{A}_{+}(\xi)\, d\xi \right) \mathbf{A}_{-}(T-\tau_1) = -\left(\int_{\tau_1}^{t} \mathbf{A}_{+}(\xi_1)\, d\xi_1 \right) \mathbf{A}_{-}(\tau_1) = -\mathbf{K}_{\pm}(t, \tau_1),$$

where $\mathbf{A}_{\pm}(T-t) = \mathbf{A}_0(T-t) \pm \mathbf{A}_1(T-t) = \mathbf{A}_{\pm}(t)$.

Using

$$g_{+}(t) = \int_{\frac{T}{2}}^{t} \mathbf{A}_{+}(\xi) \left(\int_{\frac{T}{2}}^{\xi} f_{-}^{*}(\tau)\, d\tau \right) d\xi + \int_{\frac{T}{2}}^{t} f_{+}^{*}(\xi) d\xi,$$

we therefore obtain

$$g_{+}(T-t) = \int_{\frac{T}{2}}^{T-t} \mathbf{A}_{+}(\xi) \left(\int_{\frac{T}{2}}^{\xi} f_{-}^{*}(\tau)\, d\tau \right) d\xi + \int_{\frac{T}{2}}^{T-t} f_{+}^{*}(\xi) d\xi.$$

Using the symmetry relations

$$\mathbf{A}_{\pm}(T-t)=\mathbf{A}_{\pm}(t), \qquad f_{+}^{*}(T-t)=f_{+}^{*}(t), \qquad f_{-}^{*}(T-t)=-f_{-}^{*}(t),$$

and applying the changes of variables

$$\xi = T-\xi_1, \qquad \tau = T-\tau_1,$$

we obtain

$$g_{+}(T-t)=-g_{+}(t).$$

Considering the function

$$z_1\left(t\right)=w\left(t\right)+w\left(T-t\right),$$

we obtain that $z_1\left(t\right)$ satisfies the homogeneous integral Volterra equation of the second kind:

$$z_1\left(t\right)=\int\limits_{\frac{T}{2}}^{t}\mathbf{K}_{\pm}\left(t,\tau\right)z_1(\tau)d\tau, \quad t\in\left[\frac{T}{2},T\right].$$

The Grönwall–Bellman lemma [26] implies $z_1(t)=0$, that is, $w(t)=-w(T-t)$. Because the substitution $T-t$ converts $[0,T/2]$ to $[T/2,T]$, the resulting equalities extend to the entire interval $[0,T]$.

The proof of $v\left(t\right)=v\left(T-t\right)$ is similar. The lemma is proved. □

Lemma 3. *Let* $w\left(t\right)$, $v\left(t\right)$ *the solutions of Cauchy problem* (3.2)–(3.5) *defined in the form* (3.6)*,* (3.7)*. Then,* $u\left(t\right)=1/2\left(w\left(t\right)+v\left(t\right)\right)$ *is the unique solution of the Cauchy problem* (2.1)*,* (2.2)*.*

Proof. Let $w\left(t\right)$ and $v\left(t\right)$ be the solutions of problems (3.2)–(3.5), respectively. Then,

$$\frac{du(t)}{dt}=\frac{1}{2}\left(\frac{dw(t)}{dt}+\frac{dv(t)}{dt}\right)$$

and

$$\begin{aligned}&\frac{du(t)}{dt}-\mathbf{A}_0(t)u(t)-\mathbf{A}_1(t)u\left(T-t\right)-f(t)-\left(\mathbf{A}_0(t)+\mathbf{A}_1(t)\right)\mu\\
&=\frac{1}{2}\left(\frac{dw(t)}{dt}+\frac{dv(t)}{dt}\right)-\frac{\mathbf{A}_0(t)}{2}\left(w\left(t\right)+v\left(t\right)\right)-\frac{\mathbf{A}_1(t)}{2}\left(w\left(T-t\right)+v\left(T-t\right)\right)-f(t)-\left(\mathbf{A}_0(t)+\mathbf{A}_1(t)\right)\mu\\
&=\frac{1}{2}\left[\frac{dw(t)}{dt}-\left(\mathbf{A}_0\left(t\right)+\mathbf{A}_1\left(t\right)\right)v(t)-f_{+}^{*}\left(t\right)\right]+\frac{1}{2}\left[\frac{dv(t)}{dt}-\left(\mathbf{A}_0\left(t\right)-\mathbf{A}_1\left(t\right)\right)w(t)-f_{-}^{*}\left(t\right)\right]=0.\end{aligned}$$

The lemma is proven. □

Resolvent representation of the solutions

Because Eqs (3.8) and (3.9) are Volterra integral equations of the second kind, their solutions admit the following resolvent representation:

$$w\left(t\right)=g_{+}\left(t\right)+\int\limits_{\frac{T}{2}}^{t}\mathbf{R}_{\pm}\left(t,\tau\right)g_{+}\left(\tau\right)d\tau, \tag{3.10}$$

$$v(t)=g_-(t)+\int\limits_{\frac{T}{2}}^{t}\mathbf{R}_{\mp}(t,\tau)\,g_-(\tau)\,d\tau, \tag{3.11}$$

where $\mathbf{R}_{\pm}(t,\tau)$, $\mathbf{R}_{\mp}(t,\tau)$ are the resolvent kernels of the Volterra integral equations (3.8) and (3.9) with kernels $\mathbf{K}_{\pm}(t,\tau)$ and $\mathbf{K}_{\mp}(t,\tau)$ respectively.

Based on Lemma 3, the solution to the Cauchy problem (2.1), (2.2) can be written as

$$u(t)=\frac{1}{2}\left(g_+(t)+g_-(t)\right)+\frac{1}{2}\int\limits_{\frac{T}{2}}^{t}\left(\mathbf{R}_{\pm}(t,\tau)\,g_+(\tau)+\mathbf{R}_{\mp}(t,\tau)\,g_-(\tau)\right)d\tau. \tag{3.12}$$

Considering the obtained representations, we define

$$\begin{aligned}
g_+(t)&=\int\limits_{\frac{T}{2}}^{t}\mathbf{A}_+(\xi)\left(\int\limits_{\frac{T}{2}}^{\xi}f_-^*(\tau)\,d\tau\right)d\xi+\int\limits_{\frac{T}{2}}^{t}f_+^*(\xi)\,d\xi,\\
g_-(t)&=\int\limits_{\frac{T}{2}}^{t}\mathbf{A}_-(\xi)\left(\int\limits_{\frac{T}{2}}^{\xi}f_+^*(\tau)\,d\tau\right)d\xi+\int\limits_{\frac{T}{2}}^{t}f_-^*(\xi)\,d\xi,\\
f_+^*(t)&=f(t)+f(T-t)+2\mathbf{A}_+(t)\mu,\\
f_-^*(t)&=f(t)-f(T-t).
\end{aligned}$$

Then, $g_{\pm}(t)=\mathbf{G}_{\pm}^{0}(t)+2\mathbf{G}_{\pm}^{1}(t)\,\mu$, where

$$\begin{aligned}
\mathbf{G}_+^0(t)&=\int\limits_{\frac{T}{2}}^{t}\mathbf{A}_+(\xi)\left(\int\limits_{\frac{T}{2}}^{\xi}f_-(\tau)\,d\tau\right)d\xi+\int\limits_{\frac{T}{2}}^{t}f_+(\xi)d\xi,\\
\mathbf{G}_+^1(t)&=\int\limits_{\frac{T}{2}}^{t}\mathbf{A}_+(\xi)d\xi,\\
\mathbf{G}_-^0(t)&=\int\limits_{\frac{T}{2}}^{t}\mathbf{A}_-(\xi)\left(\int\limits_{\frac{T}{2}}^{\xi}f_+(\tau)\,d\tau\right)d\xi+\int\limits_{\frac{T}{2}}^{t}f_-(\xi)d\xi,\\
\mathbf{G}_-^1(t)&=\int\limits_{\frac{T}{2}}^{t}\mathbf{A}_-(\xi)\left(\int\limits_{\frac{T}{2}}^{\xi}\mathbf{A}_+(\tau)d\tau\right)d\xi.
\end{aligned}$$

Then, the solution to the Cauchy problem (3.12) can be written in the following form:

$$u(t) = \frac{1}{2}\left(\mathbf{G}_{+}^{0}(t) + \mathbf{G}_{-}^{0}(t)\right) + \frac{1}{2}\int\limits_{\frac{T}{2}}^{t}\left(\mathbf{R}_{\pm}(t,\tau)\,\mathbf{G}_{+}^{0}(\tau) + \mathbf{R}_{\mp}(t,\tau)\,\mathbf{G}_{-}^{0}(\tau)\right)d\tau$$

$$+\left(\mathbf{G}_{+}^{1}(t) + \mathbf{G}_{-}^{1}(t) + \int\limits_{\frac{T}{2}}^{t}\left(\mathbf{R}_{\pm}(t,\tau)\,\mathbf{G}_{+}^{1}(\tau)\,d\tau + \mathbf{R}_{\mp}(t,\tau)\,\mathbf{G}_{-}^{1}(\tau)\right)d\tau\right)\mu$$

or

$$u(t) = u_0(t) + u_1(t)\mu, \tag{3.13}$$

where

$$u_0(t) = \frac{1}{2}\left(\mathbf{G}_{+}^{0}(t) + \mathbf{G}_{-}^{0}(t) + \int\limits_{\frac{T}{2}}^{t}\left(\mathbf{R}_{\pm}(t,\tau)\,\mathbf{G}_{+}^{0}(\tau) + \mathbf{R}_{\mp}(t,\tau)\,\mathbf{G}_{-}^{0}(\tau)\right)d\tau\right),$$

$$u_1(t) = \mathbf{G}_{+}^{1}(t) + \mathbf{G}_{-}^{1}(t) + \int\limits_{\frac{T}{2}}^{t}\left(\mathbf{R}_{\pm}(t,\tau)\,\mathbf{G}_{+}^{1}(\tau)\,d\tau + \mathbf{R}_{\mp}(t,\tau)\,\mathbf{G}_{-}^{1}(\tau)\right)d\tau.$$

Let us substitute the obtained solution of the Cauchy problem (2.1)-(2.2) into the boundary condition (2.3):

$$\mathbf{C}_0 u_0(0) + \mathbf{C}_0 u_1(0)\mu + \mathbf{C}_1 u_0(T) + \mathbf{C}_1 u_1(T)\mu + (\mathbf{C}_0 + \mathbf{C}_1)\,\mu = d.$$

The expression corresponding to the parameter μ is kept on the left side, and the remaining expression is moved to the right side; then,

$$(\mathbf{C}_0 u_1(0) + \mathbf{C}_1 u_1(T) + (\mathbf{C}_0 + \mathbf{C}_1))\,\mu = d - \mathbf{C}_0 u_0(0) - \mathbf{C}_1 u_0(T). \tag{3.14}$$

Let us denote the matrix corresponding to the system of linear equations by $\mathbf{Q}$, and the vector function on the right side by $\mathbf{b}$. Then, system (3.14) can be written as

$$\mathbf{Q}\mu = \mathbf{b}. \tag{3.15}$$

4. Solvability and spectral properties

In this section, we establish solvability conditions for the boundary value problem under consideration. Furthermore, an illustrative example is presented, and the spectral properties of the corresponding problem are investigated.

Theorem 1. *Let*

$$\mathbf{A}_0(\cdot), \mathbf{A}_1(\cdot), f(\cdot) \in C([0,T])$$

and

$$\mathbf{A}_i(t) = \mathbf{A}_i(T-t), \quad i = 0,1, \quad t \in [0,T].$$

Then, the boundary value problem (1.1)-(1.2) *has a unique solution at* $[0, T]$ *if and only if the matrix* $\mathbf{Q}$ *is invertible.*

Proof. Sufficiency. Let the matrix $\mathbf{Q}$ be invertible. Then, system (3.15)

$$\mathbf{Q}\mu = \mathbf{b}$$

has a unique solution μ. Because the Cauchy problem (2.1)-(2.2) has a unique solution, the solution to the boundary value problem (1.1)-(1.2) is determined by the formula

$$x(t) = u_0(t) + u_1(t)\mu.$$

Therefore, the boundary value problem has a unique solution.

Necessity. Let the boundary value problem (1.1)-(1.2) have a unique solution. Let us assume that the matrix $\mathbf{Q}$ is noninvertible. Then, system (3.15) has at least two solutions, μ_1 and μ_2. By Lemma 1, boundary value problems (1.1)-(1.2) and (2.1)–(2.3) are equivalent. Therefore, Problem (2.1)–(2.3) has two distinct solutions,

$$u_0(t) + u_1(t)\mu_1, \qquad u_0(t) + u_1(t)\mu_2,$$

which contradicts the uniqueness of the solution. Therefore, the matrix $\mathbf{Q}$ is invertible. The theorem is proved. □

Corollary 1. *If* $\det(\mathbf{Q}) \neq 0$, *then the solution to the boundary value problem* (1.1)-(1.2) *has the form*

$$x(t) = u_0(t) + u_1(t)\left(\mathbf{Q}^{-1}\mathbf{b}\right).$$

Theorem 2. *If the matrix* $\mathbf{Q}$ *is noninvertible, then for the boundary value problem* (1.1)-(1.2) *to be solvable, it is necessary and sufficient that* $\mathbf{b} \perp \ker(\mathbf{Q}^*)$.

Proof. The Cauchy problem (2.1)-(2.2) has a unique solution. The solution to the boundary value problem is defined as $x(t) = u(t) + \mu$, where μ is determined from the system of linear equations $\mathbf{Q}\mu = \mathbf{b}$. From the theory of systems of linear equations, it is known that the system $\mathbf{Q}\mu = \mathbf{b}$ is solvable if and only if $\mathbf{b} \perp \ker(\mathbf{Q}^*)$. The theorem is proven. □

Let us consider a model example that allows us to investigate the spectral properties of the boundary value problem under consideration.

Example 1. Let us consider the boundary value problem on the interval $[0, T]$

$$y'(x) = b_0 y(x) + b_1 y(1 - x) + f(x), \tag{4.1}$$

$$c_0 y(0) + c_1 y(1) = d, \tag{4.2}$$

where $b_0,\ b_1,\ c_0,\ c_1 \in R,\ \ f(x) \in C[0, 1]$.

Let us introduce the notation $\mu = y(1/2)$ and make the substitution $y(x) = u(x) + \mu$ from the boundary value problem (4.1)-(4.2) and move on to the following equivalent boundary value problem with the parameter

$$u'(x) = b_0 u(x) + b_1 u(1 - x) + f^*(x), \tag{4.3}$$

$$u\left(\frac{1}{2}\right) = 0, \tag{4.4}$$

$$c_0 u(0) + c_1 u(1) + (c_0 + c_1)\mu = d, \tag{4.5}$$

where $f^*(x) = f(x) + (b_0 + b_1)\mu$.

Let us consider Eq (4.3) for $x^* = 1 - x$. Then,

$$u'(1-x) = b_0 u(1-x) + b_1 u(x) + f^*(1-x). \tag{4.6}$$

Let us add and subtract Eqs (4.3) and (4.6). Let us denote $v(x) = u(x) + u(1-x)$, $w(x) = u(x) - u(1-x)$, and then, we obtain the Cauchy problem for the system of differential equations

$$w'(x) = (b_0 + b_1) v(x) + f_+^*(x), \tag{4.7}$$

$$v'(x) = (b_0 - b_1) w(x) + f_-^*(x), \tag{4.8}$$

$$w\left(\frac{1}{2}\right) = 0, \tag{4.9}$$

$$v\left(\frac{1}{2}\right) = 0, \tag{4.10}$$

where $f_+^*(x) = f^*(x) + f^*(1-x)$, $f_+^*(x) = f^*(x) - f^*(1-x)$.

The solution to the Cauchy problem (4.7)–(4.10) is equivalent to the following system of integral equations:

$$w(x) = \left(b_0^2 - b_1^2\right) \int_{\frac{1}{2}}^{x} (x-\xi) w(\xi) d\xi + (b_0 + b_1) \int_{\frac{1}{2}}^{x} (x-\xi) f_-^*(\xi) d\xi + \int_{\frac{1}{2}}^{x} f_+^*(\xi) d\xi, \tag{4.11}$$

$$v(x) = \left(b_0^2 - b_1^2\right) \int_{\frac{1}{2}}^{x} (x-\xi) v(\xi) d\xi + (b_0 - b_1) \int_{\frac{1}{2}}^{x} (x-\xi) f_+^*(\xi) d\xi + \int_{\frac{1}{2}}^{x} f_-^*(\xi) d\xi. \tag{4.12}$$

Let us introduce the notations $\lambda = b_0^2 - b_1^2$, $\lambda_1 = b_0 + b_1$. Consider Eq (4.11) for $x = t + 1/2$. Then,

$$w\left(t + \frac{1}{2}\right) = \lambda \int_{\frac{1}{2}}^{t+\frac{1}{2}} \left(t + \frac{1}{2} - \xi\right) w(\xi) d\xi + \lambda_1 \int_{\frac{1}{2}}^{t+\frac{1}{2}} \left(t + \frac{1}{2} - \xi\right) f_-^*(\xi) d\xi + \int_{\frac{1}{2}}^{t+\frac{1}{2}} f_+^*(\xi) d\xi.$$

After the substitution $s = \xi - 1/2$, we obtain

$$W(t) = \lambda \int_0^t (t-s) W(s) ds + \lambda_1 \int_0^t (t-s) F_-^*(s) ds + \int_0^t F_+^*(s) ds, \tag{4.13}$$

where $W(t) = w(t + 1/2)$, $F_-^*(t) = f_-^*(t + 1/2)$, $F_+^*(t) = f_+^*(t + 1/2)$.

Let us apply the Laplace transform to Eq (4.13). Using the properties

$$\mathcal{L}\left\{\int_0^t (t-s)\,g(s)\,ds\right\} = \frac{1}{p^2}\tilde{g}(p),$$

$$\mathcal{L}\left\{\int_0^t g(s)\,ds\right\} = \frac{1}{p}\tilde{g}(p),$$

then

$$\tilde{W}(p) = \lambda\frac{1}{p^2}\tilde{W}(p) + \lambda_1\frac{1}{p^2}\tilde{F}_-(p) + \frac{1}{p}\tilde{F}_+(p), \tag{4.14}$$

where $\tilde{W}(p) = \mathcal{L}\{W(t)\}$, $\tilde{F}_\pm(p) = \mathcal{L}\{F_\pm^*(t)\}$.

From (4.14), we obtain

$$\tilde{W}(p) = \frac{\lambda_1\tilde{F}_-(p) + p\tilde{F}_+(p)}{p^2-\lambda}.$$

From the Laplace transform table at $\lambda > 0$, we have

$$\mathcal{L}^{-1}\left\{\frac{1}{p^2-\lambda}\right\} = \frac{1}{\sqrt{\lambda}}\sinh\left(\sqrt{\lambda}t\right), \quad \mathcal{L}^{-1}\left\{\frac{p}{p^2-\lambda}\right\} = \cosh\left(\sqrt{\lambda}t\right);$$

at $\lambda < 0$:

$$\mathcal{L}^{-1}\left\{\frac{1}{p^2-\lambda}\right\} = \frac{1}{\sqrt{-\lambda}}\sin\left(\sqrt{-\lambda}t\right), \quad \mathcal{L}^{-1}\left\{\frac{p}{p^2-\lambda}\right\} = \cos\left(\sqrt{-\lambda}t\right).$$

Then, at $\lambda > 0$:

$$W(t) = \frac{\lambda_1}{\sqrt{\lambda}}\int_0^t \sinh\left(\sqrt{\lambda}(t-s)\right)F_-^*(s)\,ds + \int_0^t \cosh\left(\sqrt{\lambda}(t-s)\right)F_+^*(s)\,ds,$$

at $\lambda < 0$:

$$W(t) = \frac{\lambda_1}{\sqrt{\lambda}}\int_0^t \sin\left(\sqrt{-\lambda}(t-s)\right)F_-^*(s)\,ds + \int_0^t \cos\left(\sqrt{-\lambda}(t-s)\right)F_+^*(s)\,ds.$$

Returning to the variables $x = t + 1/2$ at $\lambda > 0$, we obtain,

$$w(x) = \frac{\lambda_1}{\sqrt{\lambda}}\int_{\frac{1}{2}}^x \sinh\left(\sqrt{\lambda}(x-s)\right)f_-^*(s)\,ds + \int_{\frac{1}{2}}^x \cosh\left(\sqrt{\lambda}(x-s)\right)f_+^*(s)\,ds,$$

at $\lambda < 0$:

$$w(x) = \frac{\lambda_1}{\sqrt{-\lambda}}\int_{\frac{1}{2}}^x \sin\left(\sqrt{-\lambda}(x-s)\right)f_-^*(s)\,ds + \int_{\frac{1}{2}}^x \cos\left(\sqrt{-\lambda}(x-s)\right)f_+^*(s)\,ds.$$

Similarly, at $\lambda > 0$, we find

$$v(x) = \frac{\lambda_2}{\sqrt{\lambda}} \int\limits_{\frac{1}{2}}^{x} \sinh\left(\sqrt{\lambda}(x-s)\right) f_+^*(s)\,ds + \int\limits_{\frac{1}{2}}^{x} \cosh\left(\sqrt{\lambda}(x-s)\right) f_-^*(s)\,ds,$$

at $\lambda < 0$:

$$v(x) = \frac{\lambda_2}{\sqrt{-\lambda}} \int\limits_{\frac{1}{2}}^{x} \sin\left(\sqrt{-\lambda}(x-s)\right) f_+^*(s)\,ds + \int\limits_{\frac{1}{2}}^{x} \cos\left(\sqrt{-\lambda}(x-s)\right) f_-^*(s)\,ds,$$

where $\lambda_2 = b_0 - b_1$.

At $\lambda = 0$:

$$w(x) = \lambda_1 \int\limits_{\frac{1}{2}}^{x} (x-\xi) f_-^*(\xi)d\xi + \int\limits_{\frac{1}{2}}^{x} f_+^*(\xi)d\xi,$$

$$v(x) = \lambda_2 \int_{\frac{1}{2}}^{x} (x-\xi) f_+^*(\xi)d\xi + \int_{\frac{1}{2}}^{x} f_-^*(\xi)d\xi.$$

The solution to the Cauchy problem (4.3)-(4.4) at $\lambda > 0$ can be written in the following form:

$$u(x) = \frac{b_0}{\sqrt{\lambda}} \int\limits_{\frac{1}{2}}^{x} \sinh\left(\sqrt{\lambda}(x-s)\right) f^*(s)\,ds - \frac{b_1}{\sqrt{\lambda}} \int\limits_{\frac{1}{2}}^{x} \sinh\left(\sqrt{\lambda}(x-s)\right) f^*(1-s)\,ds$$

$$+ \int_{\frac{1}{2}}^{x} \cosh\left(\sqrt{\lambda}(x-s)\right) f^*(s)\,ds,$$

at $\lambda < 0$:

$$u(x) = \frac{b_0}{\sqrt{-\lambda}} \int\limits_{\frac{1}{2}}^{x} \sin\left(\sqrt{-\lambda}(x-s)\right) f^*(s)\,ds - \frac{b_1}{\sqrt{-\lambda}} \int\limits_{\frac{1}{2}}^{x} \sin\left(\sqrt{-\lambda}(x-s)\right) f^*(1-s)\,ds$$

$$+ \int\limits_{\frac{1}{2}}^{x} \cos\left(\sqrt{-\lambda}(x-s)\right) f^*(s)\,ds;$$

at $\lambda = 0$:

$$u(x) = b_0 \int\limits_{\frac{1}{2}}^{x} (x-\xi) f^*(\xi)d\xi - b_1 \int\limits_{\frac{1}{2}}^{x} (x-\xi) f^*(1-\xi)d\xi + \int\limits_{\frac{1}{2}}^{x} f^*(\xi)d\xi.$$

Let $\lambda = b_0^2 - b_1^2 = 0$. Because $f^*(x) = f(x) + (b_0 + b_1)\mu$,

$$b_0 \int\limits_{\frac{1}{2}}^{x} (x-\xi)(b_0+b_1)\mu d\xi - b_1 \int\limits_{\frac{1}{2}}^{x} (x-\xi)(b_0+b_1)\mu d\xi = \left(b_0^2 - b_1^2\right)\mu \int\limits_{\frac{1}{2}}^{x} (x-\xi)\,d\xi = 0,$$

we obtain

$$u(x) = b_0 \int_{\frac{1}{2}}^{x} (x-\xi) f(\xi) d\xi - b_1 \int_{\frac{1}{2}}^{x} (x-\xi) f(1-\xi) d\xi + \int_{\frac{1}{2}}^{x} f(\xi) d\xi + \mu (b_0 + b_1) \left(x - \frac{1}{2} \right). \quad (4.15)$$

Substituting (4.15) into condition (4.5), we obtain

$$\left(\frac{1}{2} (c_1 - c_0)(b_0 + b_1) + (c_1 + c_0) \right) \mu$$

$$= d - (c_0 b_0 - c_1 b_1) \int_0^{\frac{1}{2}} \xi f(\xi) d\xi + (c_0 b_1 - c_1 b_0) \int_{\frac{1}{2}}^{1} (1-\xi) f(\xi) d\xi + c_0 \int_0^{\frac{1}{2}} f(\xi) d\xi - c_1 \int_{\frac{1}{2}}^{1} f(\xi) d\xi. \quad (4.16)$$

From (4.16), it follows that

(1) for $b_0 = -b_1$, the solution to the boundary value problem (4.1)-(4.2) is unique if $q_1 = c_0 + c_1 \neq 0$;
(2) for $b_0 = b_1$, the solution to the boundary value problem (4.1)-(4.2) is unique if

$$q_2 = ((c_1 - c_0) b_0 + (c_1 + c_0)) \neq 0.$$

If $\lambda > 0$, then

$$u(x) = \frac{b_0}{\sqrt{\lambda}} \int_{\frac{1}{2}}^{x} \sinh\left(\sqrt{\lambda}\, (x-s) \right) f(s)\, ds - \frac{b_1}{\sqrt{\lambda}} \int_{\frac{1}{2}}^{x} \sinh\left(\sqrt{\lambda}\, (x-s) \right) f(1-s)\, ds$$

$$+ \int_{\frac{1}{2}}^{x} \cosh\left(\sqrt{\lambda}\, (x-s) \right) f(s)\, ds - \mu + \mu \cosh \sqrt{\lambda} \left(x - \frac{1}{2} \right) + \frac{(b_0 + b_1)\mu}{\sqrt{\lambda}} \sinh \sqrt{\lambda} \left(x - \frac{1}{2} \right).$$

Substituting the obtained solution into condition (4.5), we obtain

$$\mu \left\{ (c_0 + c_1) \cosh\left(\frac{\sqrt{\lambda}}{2} \right) + (c_1 - c_0) \frac{(b_0 + b_1)}{\sqrt{\lambda}} \sinh\left(\frac{\sqrt{\lambda}}{2} \right) \right\} = d - \frac{c_0 b_0 - c_1 b_1}{\sqrt{\lambda}} \int_0^{\frac{1}{2}} \sinh\left(\sqrt{\lambda} s \right) f(s)\, ds$$

$$- \frac{c_0 b_1 - c_1 b_0}{\sqrt{\lambda}} \int_{\frac{1}{2}}^{1} \sinh\left(\sqrt{\lambda} (1-s) \right) f(s)\, ds - (c_0 - c_1) \int_0^{1} \cosh\left(\sqrt{\lambda} (1-s) \right) f(s)\, ds. \quad (4.17)$$

Therefore, for $\lambda > 0$, the solution to the boundary value problem (4.1)-(4.2) is unique if and only if

$$q_3 = \left\{ (c_0 + c_1) \cosh\left(\frac{\sqrt{\lambda}}{2} \right) + (c_1 - c_0) \frac{(b_0 + b_1)}{\sqrt{\lambda}} \sinh\left(\frac{\sqrt{\lambda}}{2} \right) \right\} \neq 0.$$

If $q_3 = 0,\ c_0 \neq c_1$, then

$$(c_0+c_1)\cosh\left(\frac{\sqrt{\lambda}}{2}\right)+(c_1-c_0)\frac{(b_0+b_1)}{\sqrt{\lambda}}\sinh\left(\frac{\sqrt{\lambda}}{2}\right)=0,$$

i.e.,

$$\tanh\left(\frac{\sqrt{\lambda}}{2}\right)=\frac{\sqrt{\lambda}\,(c_0+c_1)}{(c_0-c_1)\left(b_0+b_1\right)}. \tag{4.18}$$

Because $\lambda>0$, (4.18) has a solution when the condition is met:

$$0<\frac{\sqrt{\lambda}\,(c_0+c_1)}{(c_0-c_1)\left(b_0+b_1\right)}<1. \tag{4.19}$$

Because the function

$$\frac{\tanh\left(\sqrt{\lambda}\right)}{\sqrt{\lambda}}$$

is strictly decreasing on $(0,\infty)$, Eq (4.18) can have at most one positive eigenvalue.

If $c_0=c_1$, then $q_3\neq 0$. If $c_0=-c_1$, then $q_3\neq 0$, since, at $\lambda>0$, $\sinh\left(\sqrt{\lambda}/2\right)\neq 0$.

Consequently, for $\lambda>0$ and $q_3=0$, then the boundary value problem (4.1)-(4.2) has a solution if and only if condition (4.19) is satisfied.

If $\lambda<0$, then

$$u(x)=\frac{b_0}{\sqrt{-\lambda}}\int_{\frac{1}{2}}^{x}\sin\left(\sqrt{-\lambda}\,(x-s)\right)f(s)\,ds-\frac{b_1}{\sqrt{-\lambda}}\int_{\frac{1}{2}}^{x}\sin\left(\sqrt{-\lambda}\,(x-s)\right)f(1-s)\,ds$$

$$+\int_{\frac{1}{2}}^{x}\cos\left(\sqrt{-\lambda}\,(x-s)\right)f(s)\,ds-\mu+\mu\cos\left(\sqrt{-\lambda}\left(x-\frac{1}{2}\right)\right)+\frac{(b_0+b_1)\mu}{\sqrt{-\lambda}}\sin\left(\sqrt{-\lambda}\left(x-\frac{1}{2}\right)\right).$$

Substituting the obtained expression into the boundary condition (4.5), we obtain

$$\mu\left\{(c_0+c_1)\cos\left(\frac{\sqrt{-\lambda}}{2}\right)+(c_1-c_0)\frac{(b_0+b_1)}{\sqrt{-\lambda}}\sin\left(\frac{\sqrt{-\lambda}}{2}\right)\right\}=d+\frac{c_0b_0-c_1b_1}{\sqrt{-\lambda}}\int_0^{\frac{1}{2}}\sin\left(\sqrt{-\lambda}s\right)f(s)\,ds$$

$$-\frac{c_0b_1-c_1b_0}{\sqrt{-\lambda}}\int_{\frac{1}{2}}^{1}\sin\left(\sqrt{-\lambda}\,(1-s)\right)f(s)\,ds+c_0\int_0^{\frac{1}{2}}\cos\left(\sqrt{-\lambda}s\right)f(s)\,ds$$

$$-c_1\int_{\frac{1}{2}}^{1}\cos\left(\sqrt{-\lambda}(1-s)\right)f(s)\,ds. \tag{4.20}$$

Therefore, at $\lambda<0$, the solution to the boundary value problem (4.1)-(4.2) is unique if and only if

$$q_4=\left\{(c_0+c_1)\cos\left(\frac{\sqrt{-\lambda}}{2}\right)+(c_1-c_0)\frac{(b_0+b_1)}{\sqrt{-\lambda}}\sin\left(\frac{\sqrt{-\lambda}}{2}\right)\right\}\neq 0.$$

Theorem 3. *Let* $\lambda < 0$. *If* $q_4 = 0$, *and*

(1) $c_1 = c_0$, *then* $\lambda_k = -(2k+1)^2\pi^2$;
(2) $c_1 = -c_0$, *then* $\lambda_k = -4k^2\pi^2$;
(3) $c_1 \neq \pm c_0$, $2(c_0+c_1)/\left((c_0-c_1)\left(b_0+b_1\right)\right) > 1$, *then* $\nu_k \in (k\pi, (2k+1)\pi/2)$, $k = 1, 2, \ldots$, $\lambda_k = -4\nu_k^2$;
(4) $c_1 \neq \pm c_0$, $0 < 2(c_0+c_1)/\left((c_0-c_1)\left(b_0+b_1\right)\right) < 1$ *then* $\nu_k \in (k\pi, (2k+1)\pi/2\pi)$, $k = 1, 2, \ldots$, *and* $\nu_0 \in \left(0, \frac{\pi}{2}\right)$, $\lambda_0 = -4\nu_0^2$;
(5) $c_1 \neq \pm c_0$, $2(c_0+c_1)/\left((c_0-c_1)\left(b_0+b_1\right)\right) < 0$, *then* $\nu_k \in ((2k-1)\pi/2, k\pi)$, $k = 1, 2, \ldots$, $\lambda_k = -4\nu_k^2$.

Proof. Let $q_4 = 0$. Then,

$$q_4 = (c_0+c_1)\cos\left(\frac{\sqrt{-\lambda}}{2}\right) + (c_1-c_0)\frac{(b_0+b_1)}{\sqrt{-\lambda}}\sin\left(\frac{\sqrt{-\lambda}}{2}\right) = 0.$$

If $c_1 = c_0$, then $c_1 \neq -c_0$. Indeed, if $c_1 = c_0$, and $c_1 = -c_0$ simultaneously, then $c_1 = c_0 = 0$, which contradicts the well-posedness of the boundary condition (4.2). Therefore,

$$\cos\left(\frac{\sqrt{-\lambda}}{2}\right) = 0,$$

which implies

$$\frac{\sqrt{-\lambda_k}}{2} = \frac{(2k+1)\pi}{2}, \quad k = 0, 1, 2, \ldots.$$

Hence, the eigenvalues are given by $\lambda_k = -(2k+1)^2\pi^2$, $k = 0, 1, 2, \ldots$.

Similarly, if $c_1 = -c_0$, then $c_1 \neq c_0$. Considering that $\lambda = b_0^2 - b_1^2 < 0$, we get

$$\sin\left(\frac{\sqrt{-\lambda}}{2}\right) = 0.$$

Hence,

$$\frac{\sqrt{-\lambda_k}}{2} = k\pi, \qquad k = 1, 2, \ldots.$$

Therefore,

$$\lambda_k = -4k^2\pi^2, \qquad k = 1, 2, \ldots.$$

Let us assume that $c_1 \neq \pm c_0$. Then,

$$\tan\left(\frac{\sqrt{-\lambda}}{2}\right) = \frac{\sqrt{-\lambda}\,(c_0+c_1)}{(c_0-c_1)\left(b_0+b_1\right)}.$$

Introduce the notation $\nu = \sqrt{-\lambda}/2$. Then, the characteristic equation takes the form

$$\tan(\nu) = \frac{2(c_0+c_1)\,\nu}{(c_0-c_1)\left(b_0+b_1\right)}.$$

Because the function $\tan(\nu)$ is monotone on each interval between its singular points, the location of the roots follows from the sign of the coefficient on the right-hand side.

If $2(c_0+c_1)/\left((c_0-c_1)\left(b_0+b_1\right)\right) > 1$, then $\nu_k \in (k\pi, (2k+1)\pi/2)$, $k = 1, 2, \ldots$, $\lambda_k = -4\nu_k^2$.

If $0 < 2(c_0+c_1)/\left((c_0-c_1)\left(b_0+b_1\right)\right) < 1$, the spectrum contains one additional eigenvalue $\nu_0 \in (0, \pi/2)$, $\lambda_0 = -4\nu_0^2$.

If $2(c_0+c_1)/\left((c_0-c_1)\left(b_0+b_1\right)\right) < 0$, then $\nu_k \in ((2k-1)\pi/2, k\pi)$, $k = 1, 2, \ldots$, $\lambda_k = -4\nu_k^2$.

The theorem is proven. □

Remark. The spectral proporties are investigated under the solvability conditions of the corresponding equations. If the operator on the left-hand side degenerates, then the right-hand side must satisfy the compatibility conditions of the system.

5. Conclusions

In this paper, the parameterization method proposed by D. Dzhumabaev is applied to the study of the boundary value problem for a system of differential equations with involutive transformation of the argument.

By introducing a parameter defined as the value of the solution at the midpoint of the interval, the original problem is reduced to an equivalent system. This system consists of a Cauchy problem for a system of differential equations with involution and a linear algebraic equation for the introduced parameter.

Explicit necessary and sufficient conditions for the solvability of the boundary value problem are obtained.

Using a model example, the spectral properties of the corresponding boundary value problem are investigated.

The results obtained can be used in studying the solvability of initial-boundary value problems for parabolic and pseudoparabolic equations with involution.

Furthermore, the proposed approach extends the applicability of the parameterization method to systems of differential equations with involution and provide a constructive framework for the analysis of solvability and spectral properties of nonlocal boundary value problems. The proposed approach opens directions for further investigations for further investigations of more general classes of functional-differential equations with involution, including nonlinear and partial differential models.

Author contributions

Kairat Usmanov: writing–original draft, investigation, methodology. Kulzina Nazarova: writing – review, investigation. Zhazira Yerkisheva: writing – review and editing, investigation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

AI was used only for text editing and linguistic quality improvement of the manuscript. No AI tools were used for scientific analysis, mathematical calculations, research results, interpretations, or conclusions. The authors bear full responsibility for the content, interpretation, and conclusions of this work.

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Conflict of interest

The authors declare no conflicts of interestin this paper.

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