Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(6): 17115–17123.
DOI: 10.3934/math.2026701
Received: 26 March 2026
Revised: 05 May 2026
Accepted: 19 May 2026
Published: 12 June 2026

Research article

Integrability classification and explicit solutions to a class of vc-BKdV equations

Hanze Liu*

Department of General Education, Shandong Xiehe University, Jinan 250109, China

* **Correspondence:** Email: hanze_liu@126.com.

Abstract: The variable-coefficient equations play an important role in mathematical physics and physical applications. However, it is very difficult to study exact solutions and other properties of such equations. In this paper, we investigate the complete integrability classification of a class of the generalized variable-coefficient nonlinear wave equation by the Painlevé method, the integrability and integrable conditions of the variable-coefficient equations are obtained, then the exact solutions are provided by the truncated expansion method. Moreover, some new types of explicit solutions to the nonlinear variable-coefficient equations are investigated by the invariant subspace method (ISM).

Keywords: Painlevé method; integrability; truncated expansion; invariant subspace method; explicit solution
Mathematics Subject Classification: 37K10, 35C05, 35Q53

---

1. Introduction

Integrability of a nonlinear partial differential equation (NLPDE) plays a key role in mathematical physics, nonlinear theory and physical applications. There are different types of integrability actually, such as the Lax integrable, Painlevé integrable and Liouville integrable, and others (see Remark 1.1). For example, if a system is Lax integrable, then it is completely solvable, and possesses soliton solutions, infinitely many conservation laws, etc. On the other hand, the integrability also holds significant importance in applications, e.g., if an equation is integrable, then the system it describes is stable and controllable, such a system is truly practical. However, it is not easy to prove the integrability of an NLPDE, especially for a variable-coefficient NLPDE (vc-NLPDE). In this paper, we investigate the integrability classification of a generalized variable-coefficient nonlinear wave equation as follows:

$$u_t + a(t)u^p u_x + \beta u_{xx} + \gamma u_{xxx} = 0, \tag{1.1}$$

where $u = u(x,t)$ denotes the unknown function of space variable x and time t, $a = a(t) \neq 0$ is an arbitrary analytic function, p is a positive integer, the other parameters β and γ are arbitrary constants. This equation is also called the generalized variable-coefficient Burgers–Korteweg–de Vries (vc-BKdV) equation sometimes. In particular, if $\beta = 0$, then this equation becomes the following generalized vc-Korteweg–de Vries (vc-KdV) equation:

$$u_t + a(t)u^p u_x + \gamma u_{xxx} = 0. \tag{1.2}$$

If $\gamma = 0$, then Eq (1.1) reduces to the generalized vc-Burgers' equation (vc-BE) as follows:

$$u_t + a(t)u^p u_x + \beta u_{xx} = 0. \tag{1.3}$$

Particularly, if $p = 1$, or $p = 2$, then Eqs (1.2) and (1.3) are classical vc-KdV, vc-BE, variable-coefficient modified Korteweg-de Vries (vc-mKdV), and variable-coefficient modified Burgers' equation (vc-mBE), respectively. Such vc-NLPDEs are of great importance in nonlinear wave theory, integrable system, physical applications, etc. [1–5]. Recently, we studied the equivalent transformations, symmetries and exact solutions to some vc-KdV types of equations [6–8], but the integrable classification of the vc-NLPDEs are not involved.

In recent decades, in order to study exact solutions and other properties of NLPDEs, many effective methods have been developed. Generally speaking, there are mainly two methods: One is the dynamical system method, and the other is the integrable method, including the Lie symmetry analysis approach [4–6, 8]. Moreover, the Painlevé test is a method based on complex analysis. It is a critical step in the study of nonlinear wave theory, soliton theory, integrable systems, etc. [1–3], so this Painlevé method can be regarded as an integrable system approach. However, most of the existing studies only give the integrable condition through computer programs; the test procedures are incomplete, e.g., the recursion formulas are not presented, so the results are few and not clear. In the current paper, we investigate the complete integrability classification of a class of generalized vc-NLPDE by the Painlevé method. Summarizing, the contribution and highlight of the current paper are as follows:

► The complete integrable classification of the vc-NLPDE is presented, and all of the integrability cases are provided.

► The explicit solutions to the generalized vc-NLPDEs are constructed.

The rest of the paper is structured as follows: in Section 2, the Painlevé test is performed, the complete Painlevé integrability classification of Eq (1.1) is provided, the integrability, integrable conditions and recursion formulas are provided, and the exact solutions to the equations are obtained. Then the integrability and exact solutions to Eqs (1.2) and (1.3) are given as its special cases. In Section 3, the other types of explicit solutions to vc-NLPDE (1.1) are constructed by the invariant subspace method (ISM) for the first time. Finally, the conclusion and some new findings are given in Section 4.

Remark 1.1. If a nonlinear equation passes the Painlevé test, then it is said to be Painlevé integrable in this paper, so the integrability in the present paper refers to Painlevé integrable. Moreover, the Painlevé integrable is profoundly connected with other integrabilities, such as the Lax integrable, which is beyond the scope of this paper.

2. Integrability classification and exact solutions

First of all, we assume that Eq (1.1) has a Laurent expansion as follows:

$$u = \phi^{-\rho} \sum_{j=0}^{\infty} u_j \phi^j, \tag{2.1}$$

where $u = u(x,t)$ is the unknown function, $\phi = \phi(x,t) = x + \psi(t)$, $u_j = u_j(t)$ $(j = 0, 1, 2, ...)$ are analytic functions in a neighborhood of the noncharacteristic singular manifold, $u_0 \neq 0$, ρ is a positive integer [2,4].

Then, through the leading order analysis, we get the following result:

(I) If $\gamma = 0$, then Eq (1.1) reduces to the following vc-NLPDE:

$$u_t + a(t)u^p u_x + \beta u_{xx} = 0. \tag{2.2}$$

It is the generalized vc-BE (1.3). In this case, by the leading analysis method, we have

$$\rho = \frac{1}{p}. \tag{2.3}$$

In view of p being a positive integer, we have $p = 1$, that is, $\rho = 1$ and $u_0 = 2\beta/a \neq 0$. So, the condition $u_0 \neq 0$ is satisfied.

(II) If $\gamma \neq 0$, through the leading analysis method, we have

$$\rho = \frac{2}{p}. \tag{2.4}$$

In view of p being a positive integer, from (2.4), we get $p = 1$ and $p = 2$, respectively.

When $p = 1$, we have $\rho = 2$ and $u_0 = -12\gamma/a \neq 0$. So, the condition $u_0 \neq 0$ is satisfied.

When $p = 2$, we have $\rho = 1$ and $u_0^2 = -6\gamma/a \neq 0$. So, the condition $u_0 \neq 0$ is satisfied also.

Summarizing, we have the following:

Theorem 2.1 Suppose that $a = a(t) \neq 0$ is an arbitrary analytic function, p is a positive integer, β and γ are arbitrary real numbers. Then Eq (1.1) is non-integrable in the following cases, respectively:

(I) $\gamma = 0$ and $p > 1$;

(II) $\gamma \neq 0$ and $p > 2$. □

In other words, if $\gamma = 0$ and $p = 1$; or $\gamma \neq 0$ and $p = 1, 2$, then Eq (1.1) is a possible integrable. So, we only need to discuss the possible integrable cases in what follows.

2.1. Painlevé analysis of Eq (1.1) in the case $\gamma = 0$

Under this condition, Eq (1.1) becomes Eq (2.2). Furthermore, if this equation is integrable, then $p = 1$, and from (2.3), we get $\rho = 1$. So, (2.1) is $u = \sum_{j=0}^{\infty} u_j \phi^{j-1}$. Substituting it into Eq (2.2), we have

$$j = 0, \quad u_0 = \frac{2\beta}{a}, \tag{2.5a}$$

$$j = 1, \quad u_1 = -\frac{1}{a}\psi', \tag{2.5b}$$

$$j = 2, \quad u_0' = 0, \tag{2.5c}$$

$$j = 3, \quad u_1' + 4\beta u_3 = 0. \tag{2.5d}$$

By (2.5a), (2.5b) and (2.5d), we have u_j $(j = 0, 1, 3)$ in a unique manner. However, from (2.5c), we cannot get u_2 definitely. Generally, we obtain the recursion relation of Eq (2.2) as follows:

$$(j+1)(j-2)\beta u_j = -u_{j-2}' - (j-2)u_{j-1}\psi' - a\sum_{k=1}^{j-1}(j-k-1)u_k u_{j-k}, \tag{2.6}$$

where $j = 2, 3, ...$

In particular, if $j = 2$, referring to (2.6), then we have $0 = 0$, that is, u_2 is arbitrary.

Therefore, in view of the recursion formula (2.6), the other coefficients u_j of (2.1) can be determined successively in a unique manner. This implies that for Eq (2.2) with $p = 1$, there exists a Painlevé series (2.1) with $\rho = 1$ in the expansion function $\phi = x + \psi(t)$ with the coefficients given by (2.6). Thus, we can say that Eq (2.2) possesses the Painlevé property (PP) under the condition (2.5c).

In view of (2.5c), we have that $u_0 = c_1$ is a constant. Furthermore, from (2.5a), we get that $a(t) = \alpha \neq 0$ is a constant also.

In addition, letting $u_2 = u_3 = 0$, by the induction method, we can check $u_j = 0$, for all $j = 4, 5, ...$

Summarizing the above discussion, we get the following:

Theorem 2.2 The vc-NLPDE (2.2) is integrable (in the sense of Painlevé integrability) if and only if $p = 1$ and $a(t) = \alpha \neq 0$ is a constant. □

Moreover, under the condition $u_3 = 0$, and from (2.1), we get that

$$u = u_0\phi^{-1} + u_1 \tag{2.7}$$

is a solution to Eq (2.2), where u_0 and u_1 are given by (2.5a) and (2.5b), $\phi = x + \psi(t)$ is an analytic function.

On the other hand, in view of (2.5d), from $u_3 = 0$, we get $u_1' = 0$, that is, $u_1 = c_1$ is a constant. Hence, from (2.5b), we have $\psi(t) = -\alpha c_1 t + c_2$. Substituting it into (2.7), we obtain the exact solution to Eq (2.2) as follows:

$$u(x,t) = \frac{2\beta}{\alpha(x - \alpha c_1 t + c_2)} + c_1, \tag{2.8}$$

where $\alpha \neq 0$ and c_1, c_2 are arbitrary constants.

2.2. Painlevé test for Eq (1.1) in the case $\gamma \neq 0$

In this case, and in view of (2.4), we only need to discuss the following two subcases respectively.

Case I. $p = 1$. In this case, Eq (1.1) becomes

$$u_t + a(t)uu_x + \beta u_{xx} + \gamma u_{xxx} = 0. \tag{2.9}$$

Referring to (2.4), we get $\rho = 2$. So, (2.1) is $u = \sum_{j=0}^{\infty} u_j\phi^{j-2}$. Substituting it into Eq (2.9), we have

$$j = 0, \quad u_0 = -\frac{12\gamma}{a}, \tag{2.10a}$$

$$j = 1, \quad \beta u_0 + 5\gamma u_1 = 0, \tag{2.10b}$$

$$j=2,\quad 2\psi' u_0 + a u_1^2 + 2 a u_0 u_2 - 2\beta u_1 = 0, \tag{2.10c}$$

$$j=3,\quad u_0' - \psi' u_1 - a u_1 u_2 - a u_0 u_3 = 0, \tag{2.10d}$$

$$j=4,\quad u_1' = 0, \tag{2.10e}$$

$$j=5,\quad u_2' + \psi' u_3 + a u_0 u_5 + a u_1 u_4 + a u_2 u_3 + 2\beta u_4 + 6\gamma u_5 = 0. \tag{2.10f}$$

By (2.10a)–(2.10d) and (2.10f), we can get u_j $(j = 0,1,2,3,5)$ in a unique manner. However, from (2.10e), we cannot get u_4 definitely. Generally, we have the recursion relation of Eq (2.9) as follows:

$$(j+1)(j-4)(j-6)\gamma u_j = -u_{j-3}' - (j-4)u_{j-2}\psi' - a\sum_{k=1}^{j-1}(j-k-2)u_k u_{j-k}$$
$$-\beta(j-3)(j-4)u_{j-1}, \tag{2.11}$$

for $j = 3,4,....$

In terms of the recursion formula (2.11), we can get all the coefficients u_j $(j = 3,4,...)$ of (2.1) except u_4 and u_6. For example, if $j = 3,5$, then the coefficients u_3 and u_5 are given by (2.10d) and (2.10f), respectively. If $j = 4$, and in view of (2.10e), we have

$$j=4,\quad 0u_4 = -u_1' = 0. \tag{2.12}$$

So, u_4 is arbitrary.

Furthermore, if $j = 6$, then from (2.11), we get $0u_6 = -u_3' - 2\psi' u_4 - 2au_1u_5 - 2au_2u_4 - au_3^2 - 6\beta u_5$, i.e.,

$$j=6,\quad u_3' + 2\psi' u_4 + 2au_1u_5 + 2au_2u_4 + au_3^2 + 6\beta u_5 = 0. \tag{2.13}$$

Setting $u_2' = u_3 = 0$ and $u_4 = 0$, then from (2.10f), we have $u_5 = 0$. Substituting these results into (2.13), we get $0 \equiv 0$, so u_6 is arbitrary. Meanwhile, also setting $u_6 = 0$, by the induction method, we can check $u_j = 0$ for $j = 3,4,....$

Therefore, under the condition $u_2' = u_3 = 0$, the Painlevé series (2.1) can be truncated as follows:

$$u = u_0\phi^{-2} + u_1\phi^{-1} + u_2, \tag{2.14}$$

where $\varphi = x + \psi(t)$, u_0, u_1 and u_2 are given by (2.10a), (2.10b) and (2.10c), respectively.

In addition, in view of (2.10b) and (2.10e), we have that $u_1 = \frac{12\beta}{5a}$ is a constant, that is, $a = a(t) = \alpha \neq 0$ is also a constant. Furthermore, from (2.10c), we have $u_2 = \frac{\beta^2}{25a\gamma} - \frac{1}{a}\psi'$.

On the other hand, in view of (2.10c) and $u_2' = 0$, we get $\psi'' = 0$, so we have $\psi(t) = c_1 t + c_2$.

Then, substituting u_0, u_1, u_2 and $u_3 = 0$ into (2.10d), we get $\beta = 0$.

Summarizing the above discussion, we get the following result:

Theorem 2.3. The vc-NLPDE (2.9) is integrable (in the sense of Painlevé integrability) if and only if $\beta = 0$ and $a(t) = \alpha \neq 0$ is a constant. □

Substituting u_0, u_1, u_2 and φ into (2.14), we obtain the exact solution to Eq (2.9) as follows:

$$u(x,t) = \frac{-12\gamma}{\alpha(x + c_1 t + c_2)^2} - \frac{c_1}{\alpha}, \tag{2.15}$$

where $\alpha \neq 0$ and c_1, c_2 are arbitrary constants.

Case II. $p = 2$. In this case, Eq (1.1) becomes

$$u_t + a(t)u^2u_x + \beta u_{xx} + \gamma u_{xxx} = 0. \tag{2.16}$$

Referring to (2.4), we get $\rho = 1$. So (2.1) is $u = \sum_{j=0}^{\infty} u_j\phi^{j-1}$. Substituting it into Eq (2.16), we have

$$j = 0, \quad au_0^2 + 6\gamma = 0, \tag{2.17a}$$

$$j = 1, \quad au_0u_1 - \beta = 0, \tag{2.17b}$$

$$j = 2, \quad \psi' + au_1^2 + au_0u_2 = 0, \tag{2.17c}$$

$$j = 3, \quad u_0' = 0, \tag{2.17d}$$

$$j = 4, \quad u_1' + \psi'u_2 + 2au_0u_1u_3 + au_0u_2^2 + au_1^2u_2 + 2\beta u_3 = 0. \tag{2.17e}$$

By (2.17a)–(2.17c), we can get u_j $(j = 0, 1, 2)$ in a unique manner. However, from (2.17d) and (2.17e), we cannot get u_3 and u_4 definitely. In general, we have the recursion relation of Eq (2.16) as follows:

$$\begin{aligned}(j+1)(j-3)(j-4)\gamma u_j = {}& -u_{j-3}' - (j-3)u_{j-2}\psi' - \beta(j-2)(j-3)u_{j-1} \\ & + a\sum_{k=1}^{j-1} u_0u_ku_{j-k} - a\sum_{k=1}^{j-1}\Big[(j-k-1)\sum_{i=0}^{k} u_iu_{k-i}u_{j-k}\Big],\end{aligned} \tag{2.18}$$

for $j = 3, 4, ...$

In view of the recursion formula (2.18), we can get all the coefficients u_j $(j = 3, 4, 5, ...)$ of (2.1) except u_3 and u_4. For example, if $j = 3$, then we have

$$j = 3, \quad 0u_3 = u_0' = 0. \tag{2.19}$$

If $j = 4$, then we have

$$j = 4, \quad 0u_4 = -u_1' - \psi'u_2 - 2\beta u_3 - 2au_0u_1u_3 - au_0u_2^2 - au_1^2u_2 = 0. \tag{2.20}$$

So, u_3 and u_4 are arbitrary.

Furthermore, if $j = 5$, then we have

$$j = 5, \quad 12\gamma u_5 = -u_2' - 2\psi'u_3 - 6\beta u_4 - 4au_0u_1u_4 - 4au_0u_2u_3 - 2au_1^2u_3 - 2au_1u_2^2 = 0, \tag{2.21}$$

and so on.

Setting $u_2 = u_3 = u_4 = 0$, then from (2.21), we have $u_5 = 0$. Generally, by the induction method, we can check $u_j = 0$ for all $j = 3, 4, ...$

Therefore, under the condition $u_2 = 0$, the Painlevé series (2.1) is truncated as follows:

$$u = u_0\phi^{-1} + u_1, \tag{2.22}$$

where $\varphi = x + \psi(t)$, u_0 and u_1 are given by (2.17), respectively.

In addition, in view of (2.17a) and (2.17d), we get that $a = a(t) = \alpha \neq 0$ is a constant. Furthermore, from (2.17b), we have that $u_1 = \frac{\beta}{\alpha u_0}$ is a constant. So, (2.17e) and (2.20) are satisfied.

On the other hand, substituting u_1 and $u_2 = 0$ into (2.17c), we have $\psi(t) = \frac{\beta^2}{6\gamma}t + c_2$.

Summarizing the above argument, we get the following:

Theorem 2.4. The vc-NLPDE (2.16) is integrable (in the sense of Painlevé integrability) if and only if $a(t) = \alpha \neq 0$ is a constant. □

Substituting u_0, u_1 and φ into (2.22), we obtain the exact solution to Eq (2.16) as follows:

$$u(x,t) = \frac{6\gamma u_0}{6\gamma x + \beta^2 t + c_1} + \frac{\beta}{\alpha u_0}, \tag{2.23}$$

where u_0 is given by (2.17a), $\alpha \neq 0$ and c_1 are arbitrary constants.

3. Explicit solutions to vc-PDEs by the ISM

In this section, we investigate the other types of exact solutions by the invariant subspace method (ISM) [9–11], it is also called the generalized separation of variables method (GSVM) sometimes, this method is a systematic approach to constructing explicit solutions to NLPDEs [10, 12, 13]. Actually, the ISM is a dynamical system approach based on Lie group analysis. Recently, by the ISM, we [8] studied the exact solutions to the nonlocal Timoshenko beam model, which is a PDE system with space-dependent coefficients. Now we employ this method to investigate the exact solutions to the vc-PDEs in the present paper. As an example, we consider the case $p = 1$. In this case, Eq (1.1) is of the form

$$u_t + a(t)uu_x + \beta u_{xx} + \gamma u_{xxx} = 0, \tag{3.1}$$

where $a = a(t)$ is an arbitrary analytic function, β and γ are constants.

First, we rewrite Eq (3.1) as the form $u_t = F[u] = -a(t)uu_x - \beta u_{xx} - \gamma u_{xxx}$. So, Eq (3.1) admits the subspace W_2 defined by the following ordinary differential equation (ODE):

$$L[y] = y'' + a_1 y' + a_0 y = 0, \tag{3.2}$$

where $y' = dy/dx$, $a_1 = a_1(t)$ and $a_0 = a_0(t)$ are functions to be determined.

In this case, the invariance condition takes the form

$$(D^2F + a_1 DF + a_0 F)|_{u \in W_2} = 0. \tag{3.3}$$

By the ISM, we get $W_2 = \{1, x\}$. Therefore, we suppose that Eq (3.1) has the form solution as follows:

$$u = f(t) + g(t)x, \tag{3.4}$$

where the functions $f = f(t)$ and $g = g(t)$ satisfy the following ODE system:

$$\frac{df}{dt} + afg = 0, \quad \frac{dg}{dt} + ag^2 = 0. \tag{3.5}$$

Solving this ODE system, we get $f(t) = c_1 e^{-\int \frac{a(t)}{A(t)+c_2}dt}$, $g(t) = \frac{1}{A(t)+c_2}$. Thus, we obtain the exact solution to Eq (3.1) as follows:

$$u(x,t) = c_1 e^{-\int \frac{a(t)}{A(t)+c_2}dt} + \frac{x}{A(t) + c_2}, \tag{3.6}$$

where $A(t) = \int a(t)dt$, c_1 and c_2 are arbitrary constants.

Similarly, since the other explicit solutions can be constructed by the ISM, the details are omitted.

4. Conclusions and remarks

In the current paper, the Painlevé integrability of a class of generalized variable-coefficient nonlinear evolution equations is investigated by the Painlevé analysis method, the complete integrability classification and recursion formulas of the generalized vc-NLPDE are provided, and the exact explicit solutions are investigated by the truncated expansion and ISM for the first time. Summarizing the above discussion, we have the following result:

Theorem 4.1. If the generalized vc-NLPDE (1.1) is integrable in the sense of Painlevé integrability, then $a(t) = \alpha$ is a constant. □

In other words, if $a = a(t)$ is a function of t, then Eq (1.1) is non-integrable. Generally, for a non-integrable PDE, the exact solution is poorer and hard to study. So far, there is no systematic and effective method to deal with it as far as we know, but the exact solutions to vc-PDE (3.1) can be given by the ISM.

Remark 4.1. In this paper, the Kruskal simplified method is employed to deal with generalized vc-NLPDE (1.1). More generally, if the general Painlevé test [1–3,5,6] is employed to such equations, then the results on integrability are the same as above. However, since some other results such as Bäcklund transformations (BTs) may be obtained (under some conditions), we omit it in the present paper.

Use of Generative-AI tools declaration

This article did not apply any AI and AI-assisted technologies in the writing process.

Conflict of interest

The author declares that he/she has no conflict of interest.

References

1. G. Wei, Y. Gao, W. Hu, C. Zhang, Painlevé analysis, auto-Bäcklund transformation and new analytic solutions for a generalized variable-coefficient Korteweg-de Vries (KdV) equation, *Eur. Phys. J. B*, **53** (2006), 343–350. https://doi.org/10.1140/epjb/e2006-00378-3
2. R. Conte, M. Musette, *The Painlevé handbook*, Dordrecht: Springer, 2008. https://doi.org/10.1007/978-1-4020-8491-1
3. J. Weiss, M. Tabor, G. Carnevale, The Painlevé property for partial differential equations, *J. Math. Phys.*, **24** (1983), 522–526. https://doi.org/10.1063/1.525721
4. H. Liu, C. Yue, Lie symmetries, integrable properties and exact solutions to the variable-coefficient nonlinear evolution equations, *Nonlinear Dyn.*, **89** (2017), 1989–2000. https://doi.org/10.1007/s11071-017-3566-2

5. H. Liu, C. L. Bai, X. Xin, Painlevé analysis, group classification and exact solutions to the nonlinear wave equations, *Eur. Phys. J. B*, **93** (2020), 26. https://doi.org/10.1140/epjb/e2020-100402-6
6. H. Liu, C. L. Bai, X. Xin, Integrable property, Lie symmetry analysis and explicit solutions to the generalized ϕ^4 model, *Appl. Math. Lett.*, **134** (2022), 108316. https://doi.org/10.1016/j.aml.2022.108316
7. H. Liu, Equivalent transformation and integrability of the nonlinear Schrödinger equations with time-dependent coefficients, *Nucl. Phys. B*, **994** (2023), 116303. https://doi.org/10.1016/j.nuclphysb.2023.116303
8. H. Liu, Symmetry analysis and generalized separation of variables method for exact solutions to PDE system with space-dependent coefficients, *Qual. Theor. Dyn. Syst.*, **24** (2025), 189. https://doi.org/10.1007/s12346-025-01349-0
9. N. Liu, X. Zhao, B. Guo, Long-time asymptotic behavior for the matrix modified Korteweg-de Vries equation, *Phys. D*, **443** (2023), 133560. https://doi.org/10.1016/j.physd.2022.133560
10. Y. Li, N. Liu, Riemann-Hilbert approach for discrete mKdV equation with arbitrary-order poles, *J. Geom. Phys.*, **221** (2026), 105739. https://doi.org/10.1016/j.geomphys.2025.105739
11. R. Wang, N. Liu, The Cauchy problem of vector complex modified Korteweg-de Vries equation: large-time asymptotics with decaying initial data, *J. Math. Phys.*, **66** (2025), 053506. https://doi.org/10.1063/5.0249760
12. T. Yin, H. Dong, W. Wang, J. Pang, Rossby waves and their dynamics for an extended mKdV equation with spatiotemporal variable coefficients in geophysical fluid, *Chin. J. Phys.*, **94** (2025), 97–107. https://doi.org/10.1016/j.cjph.2025.01.005
13. T. Yin, F. Chen, Y. Zhang, K. Wu, The interaction and dynamics of multi-Rossby soliton solutions in a variable coefficient inhomogeneous Zakharov-Kuznetsov equation with temporal bottom topography, *Chaos, Soliton. Fract.*, **208** (2026), 118245. https://doi.org/10.1016/j.chaos.2026.118245

AIMS Press