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Research article

Global existence of solutions to a non-isentropic compressible diffuse interface model

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Abstract: In this paper, we investigate the existence of solutions to a class of compressible Navier–Stokes/Allen–Cahn systems describing the motion of a mixture of two viscous compressible fluids. Using the energy method and refined interpolation inequalities, we establish the global existence of solutions in the Sobolev spaces H^i for $i = 2, 4$. The system features a temperature-dependent heat conductivity $k(\theta) = \theta^\beta$ with $\beta > 0$, which characterizes the flow of a two-phase immiscible, thermally viscous, compressible fluid mixture. The main technical difficulty of this work lies in the complicated estimates induced by high-order partial derivatives in the proof of solution regularity.

Keywords: Navier–Stokes/Allen–Cahn system; global existence; uniqueness; non-isentropic; compressible diffuse interface
Mathematics Subject Classification: 35Q30, 36C20, 76T30

1. Introduction

Fluid-particle interaction models are significant in sedimentation analysis of disperse suspensions of particles in fluids, and are applied to biotechnology, medicine, engineering, mineral processing, and more. In this paper, we are interested in the non-isentropic compressible Navier–Stokes/Allen–Cahn system (in Lagrange form) which can be written as

$$\begin{cases} v_t = u_x, \\ u_t + (\frac{\theta}{v})_x = (\frac{u_x}{v})_x - \frac{\delta}{2}(\frac{\phi_x^2}{v^2})_x, \\ \phi_t = -v\mu, \\ \mu = (\phi^3 - \phi) - (\frac{\phi_x}{v})_x, \\ \theta_t + (\frac{\theta}{v})u_x - (\frac{\theta^\beta \theta_x}{v})_x = c_v(\frac{u_x^2}{v}) + v\mu^2. \end{cases} \tag{1.1}$$

Here, ρ, u, ϕ, θ represent the total density, the mean velocity of the fluid mixture, the difference of the two components for the fluid mixture, and the absolute temperature, respectively. $c_v > 0$ represents viscosity coefficients. $\delta > 0$ is the thickness of the diffuse interface. The constant β is $\beta > 0$. We consider a typical initial boundary value problem for (1.1) in the reference domain $\{(x,t) : 0 < x < 1, t \geq 0\}$ under the initial conditions and boundary conditions

$$(v, u, \theta, \phi)(x, 0) = (v_0, u_0, \theta_0, \phi_0), \quad x \in (0, 1), \tag{1.2}$$

$$(u, \theta_x, \phi_x)(0, t) = (u, \theta_x, \phi_x)(1, t) = 0, \quad t \geq 0, \tag{1.3}$$

and the compatibility conditions

$$u_0(0) = u_0(1) = 0, \quad \theta_x(0) = \theta_x(1) = 0, \quad \phi_x(0) = \phi_x(1) = 0, \tag{1.4}$$

where $v = \frac{1}{\rho}$ represents specific volume. Without loss of generality, we assume that

$$c_v = \delta = 1, \quad \int_0^1 v dx = \int_0^1 v_0 dx, \quad \bar{\theta(t)} = \int_0^1 \theta(x, t) dt. \tag{1.5}$$

Before introducing our main result, we will give a brief review on some related works.

For the isentropic compressible Navier-Stokes/Allen-Cahn model with constant viscosity: The one-dimensional case, Ding [1] and Chen [2] established the global existence and uniqueness of classical solutions without vacuum and with initial vacuum, respectively. Under boundary conditions different from those in [1, 2], Ding [3] obtained the global existence and uniqueness of strong solutions to the one-dimensional coupled system with a free boundary. For large initial density perturbations and large initial velocity data, Chen [4] derived the global existence and large-time behavior of strong solutions under L-periodic boundary conditions $\Omega = R$ and mixed boundary conditions $\Omega = [0, L]$. Using the method of characteristics, Li [5] constructed weak solutions to the diffuse interface model with fixed boundary values $\rho(0, \tau) = \rho(1, \tau) = 0$. For density-dependent viscosity $v(\varrho) > 0$ satisfying $1 < \tilde{v} \leq v(\rho)$, Su [6] proved the global existence and uniqueness of strong and classical solutions to the initial-boundary value problem with initial vacuum. Yin [13] investigated the large-time behavior of solutions to the inflow problem in the half-space. For higher-dimensional problems, Chen [14, 15] constructed global weak solutions to the compressible Navier-Stokes/Allen-Cahn system with possibly large initial data in three dimensions, provided finite initial energy and the adiabatic exponent $\gamma > 2$. For the viscosity coefficient $\eta(\varrho, \chi) = 1 + \varrho^\alpha \chi^\beta$, Chen [16] established blow-up criteria for strong solutions to the initial-boundary value problem. Ding [17] obtained a blow-up criterion for local strong solutions to the three-dimensional compressible fluid-particle interaction model with vacuum. Li [18] studied a coupled Navier-Stokes/Allen-Cahn system modeling the two-phase flow of viscous incompressible fluids with different densities in a bounded domain $\Omega \subset R^N (N = 2, 3)$. Based on the renormalized weak solution framework, Feireisl [11] proved the global existence of weak solutions. Grasselli [19] showed the existence of trajectory attractors for incompressible Navier-Stokes/Allen-Cahn and Navier-Stokes/Cahn-Hilliard systems, and derived convergence rate estimates in the phase-space metric. Xu [20] discussed the global regularity of axisymmetric solutions in R^3 for large viscosity and small initial data. Under no-slip and pure-slip boundary conditions, Kotschote [21] proved the existence and uniqueness of local strong solutions for arbitrary initial data. Assuming that the external force is a smooth potential $f_{\text{ext}}(\rho) = -\nabla\phi(\rho)$ with $\phi \in C^2(R_+)$, Kotschote [22] studied the stability of traveling wave solutions. Zhao [23] investigated the vanishing viscosity limit and showed that solutions of the Navier-Stokes/Allen-Cahn system converge to those of the Euler/Allen-Cahn system on a sufficiently small time interval. Luo [24] proved the stability of rarefaction waves in R^n. Under spherical symmetry and without initial vacuum, Song [25] obtained the existence and uniqueness of local classical solutions. Song [26] studied time-periodic solutions to the compressible Navier-Stokes/Allen-Cahn system. Chen [27] considered immiscible compressible two-phase flows governed by a compressible Navier-Stokes system coupled with a modified Allen-Cahn equation.

For the non-isentropic model, Chen [7] established the global existence and uniqueness of strong solutions to system (1.1). Assuming a phase-dependent viscosity $\eta(\chi) = \chi^\alpha$ and a temperature-dependent heat conductivity $k(\theta) = \theta^\beta (\beta > 0)$, Yan [8] proved the global existence of strong solutions. Chen [9] obtained the global existence and uniqueness of strong solutions to the Cauchy problem for system (1.1)-(1.3). Recently, Kotschote [10] proved the existence and uniqueness of local strong solutions in bounded domains. Luo [12] studied the large-time behavior of solutions for compressible viscous gases in the half-space.

From the above review, most existing results focus on the existence and uniqueness of solutions and their partial regularity in $H^1([0, 1])$. Regularity properties in $H^2([0, 1])$ and $H^4([0, 1])$ have not been addressed. Motivated by these observations, we investigate the higher-order regularity of global solutions in $H^2([0, 1])$ and $H^4([0, 1])$. To establish higher regularity, we face complicated estimates induced by high-order partial derivatives, which is the main technical difficulty of this work. To overcome it, we employ the Sobolev embedding theorem and refined interpolation inequalities to derive sharp a priori estimates.

2. Main results

We shall use the following notation: For $p \geq 1$, $L^p = L^p(0, 1)$ denotes the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$. For $k \geq 1$ and $p \geq 1$, $W^{k,p} = W^{k,p}(0, 1)$ denotes the Sobolev space with the norm $\|\cdot\|_{W^{k,p}}$ and $H^k(0, 1) = W^{k,2}(0, 1)$. Now, we give our main results as the following.

Theorem 2.1. *Suppose that $(v_0, u_0, \theta_0) \in (H^2([0, 1]))^3$, $\phi_0 \in H^3([0, 1])$, and the compatibility conditions (1.4) hold. Then, there exist unique global solutions (v, u, θ, ϕ) to the system (1.1)-(1.3) for any $T > 0$, satisfying*

$$\begin{aligned} &\|v - 1\|_{H^2}^2 + \|u\|_{H^2}^2 + \|\theta - \bar{\theta}\|_{H^2}^2 + \|\phi_x\|_{H^2}^2 + \|u_t\|^2 + \|\theta_t\|^2 + \|\phi_t\|_{H^1}^2 \\ &+ \int_0^t (\|v - 1\|_{H^2}^2 + \|u\|_{H^3}^2 + \|\theta - \bar{\theta}\|_{H^3}^2 + \|\phi_x\|_{H^3}^2 + \|u_t\|_{H^1}^2 + \|\theta_t\|_{H^1}^2 + \|\phi_t\|_{H^2}^2 + \|\phi_{tt}\|^2)(\tau)d\tau \leq C_2. \end{aligned} \quad (2.1)$$

Here a positive constant C_2 depends on the $H^2([0,1])$ norm of the initial data (v_0, u_0, θ_0), the $H^3([0,1])$ norm of the initial data ϕ_0, $\min_{x\in[0,1]} v_0$, $\min_{x\in[0,1]} \theta_0$, $\min_{x\in[0,1]} \phi_0$ and T.

Theorem 2.2. *Suppose that* $(v_0, u_0, \theta_0) \in (H^4([0,1]))^3$, $\phi_0 \in H^5([0,1])$*, and the compatibility conditions (1.4) hold. Then, there exist unique global solutions* (v, u, θ, ϕ) *to the system* $(1.1)-(1.3)$ *for any* $T > 0$*, satisfying*

$$\|v-1\|_{H^4}^2 + \|u\|_{H^4}^2 + \|\theta-\bar{\theta}\|_{H^4}^2 + \|\phi\|_{H^4}^2 + \|u_t\|_{H^2}^2 + \|u_{tt}\|^2 + \|\theta_t\|_{H^2}^2 + \|\theta_{tt}\|^2 + \|\phi_t\|_{H^2}^2 + \|\phi_{tt}\|_{H^1}^2$$
$$+ \int_0^t (\|v-1\|_{H^4}^2 + \|u\|_{H^5}^2 + \|\theta-\bar{\theta}\|_{H^5}^2 + \|\phi\|_{H^5}^2 + \|u_t\|_{H^3}^2 + +\|u_{tt}\|_{H^1}^2 + \|\theta_t\|_{H^3}^2 + +\|\theta_{tt}\|_{H^1}^2$$
$$+\|\phi_t\|_{H^4}^2 + \|\phi_{tt}\|_{H^2}^2)(\tau)d\tau \le C_4. \tag{2.2}$$

Here, a positive constant C_4 depends on the $H^4([0,1])$ norm of the initial data (v_0, u_0, θ_0), the $H^5([0,1])$ norm of the initial data ϕ_0, $\min_{x\in[0,1]} v_0$, $\min_{x\in[0,1]} \theta_0$, $\min_{x\in[0,1]} \phi_0$, and T.

3. Proof of Theorem 1.1 and a priori estimates

In this section, Theorem 1.1 can be achieved on the following series of priori estimates.

Lemma 3.1. *Assume that* $(v_0, \theta_0) \in H^1(0,1)$*,* $\phi_0 \in H^2(0,1)$*,* $u_0 \in H_0^1(0,1)$*, and*

$$\inf_{x\in(0,1)} v_0(x) > 0, \inf_{x\in(0,1)} \theta_0(x) > 0, \phi_0(x) \in [-1,1].$$

Then, the initial boundary value problem $(1.1)-(1.3)$ *has a unique strong solutions* (v, u, θ, ϕ) *for any* $T > 0$*, satisfying*

$$0 < C_1^{-1} \le v(x,t) \le C_1, 0 < C_1^{-1} \le \theta(x,t) \le C_1, |\phi(x,t)| \le C_1, \qquad (x,t) \in [0,1] \times [0,\propto)$$

$$\sup_{0\le t\le T} \int_0^1 (\frac{u^2}{2} + \frac{1}{4\delta}(\phi^2-1)^2 + \frac{\delta}{2}\frac{\phi_x^2}{v} + (v - \ln v) + (\theta - \ln\theta))dx + \int_0^T V(t)dt \le M_0, \tag{3.1}$$

where

$$M_0 := \int_0^1 (\frac{u_0^2}{2} + \frac{1}{4\delta}(\phi_0^2-1)^2 + \frac{\delta}{2}\frac{\phi_{0x}^2}{v_0} + (v_0 - \ln v_0) + (\theta_0 - \ln\theta_0))dx,$$

$$V(t) := \int_0^1 (\frac{\theta^\beta \theta_x^2}{v\theta^2} + \frac{u_x^2}{v\theta} + \frac{v\mu^2}{\theta})dx,$$

$$\sup_{0\le t\le T} (\|v_x\|^2 + \|u_x\|^2 + \|\theta\|_{H^1}^2 + \|\phi_x\|_{H^1}^2 + \|\phi_t\|^2) + \int_0^T (\|v_x\|_{H^1}^2 + \|u_x\|_{H^1}^2 + \|\theta_x\|_{H^1}^2$$
$$+ \|\phi_x\|_{H^2}^2 + \|\phi_t\|_{H^1}^2 + \|v\mu\|^2 + \|u_t\|^2 + \|\theta_t\|^2)(t)dt \le C_1, \tag{3.2}$$

$$\int_0^T \max_{x\in[0,1]} (u_x^4 + \mu^4 + \theta^2)(t)dt \le C_1. \tag{3.3}$$

Here, a positive constant C_1 depends on the initial data $(v_0, u_0, \theta_0, \phi_0)$, $\min_{x\in[0,1]} v_0$, $\min_{x\in[0,1]} \theta_0$, $\min_{x\in[0,1]} \phi_0$ and is dependent on the variable T.
Proof. See, e.g. [7].

Lemma 3.2. *Let* (v, u, θ, ϕ) *be a smooth solution of* (1.1) − (1.3) *on* $[0, 1] \times [0, T]$. *Then, it holds that for* $\forall(x, t) \in [0, 1] \times [0, T]$,

$$\|u_t\|^2 + \int_0^t \|u_{xt}\|^2(\tau)d\tau \le C_2. \tag{3.4}$$

Proof. Differentiating $(1.1)_2$ with respect to t leads to

$$u_{tt} + (\frac{\theta}{v})_{xt} = (\frac{u_x}{v})_{xt} - \frac{1}{2}(\frac{\phi_x^2}{v^2})_{xt}. \tag{3.5}$$

Multiplying the above equation by u_t and integrating over $[0, 1]$, using integration by parts and Young's inequality, one obtains

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_t\|^2 + \int_0^1 \frac{u_{xt}^2}{v}dx = -\int_0^1 (-\frac{\theta}{v} + \frac{u_x}{v} - \frac{\phi_x^2}{2v^2})_t u_{xt}dx\\
&= \int_0^1 (-\frac{\theta_t}{v} - \frac{u_x}{v} + \frac{u_x^2}{v} + \frac{\phi_x\phi_{xt}}{v^2} - \frac{\phi^2 u_x}{v^3})u_{xt}dx\\
&\le \frac{1}{2}\int_0^1 \frac{u_{xt}^2}{v}dx + C_1(\|\theta_t\|^2 + \|u_x\|_{L^\infty}^2\|u_x\|^2 + \|\phi_x\|_{L^\infty}^2\|\phi_{xt}\|^2 + \|\phi_x\|_{L^\infty}^2\|u_x\|^2),
\end{aligned} \tag{3.6}$$

which together Lemma 3.1, yields

$$\|u_t\|^2 + \int_0^t \|u_{xt}\|^2 d\tau \le C_2, \quad \forall t > 0.$$

Lemma 3.3. *Let* (v, u, θ, ϕ) *be a smooth solution of* (1.1) − (1.3) *on* $[0, 1] \times [0, T]$. *Then, it holds that for* $\forall(x, t) \in [0, 1] \times [0, T]$,

$$\sup_{0\le t\le T} \|((\frac{\phi_x}{v})_t)\|^2 + \int_0^t \|((\frac{\phi_t}{v})_t, (\frac{\phi_x}{v})_{xt})\|^2(\tau)d\tau \le C_2. \tag{3.7}$$

Proof. Rewrite $(1.1)_{3,4}$ as

$$\frac{\phi_t}{v} = (\frac{\phi_x}{v})_x - (\phi^3 - \phi). \tag{3.8}$$

Differentiating (3.8) with respect to t, one has

$$(\frac{\phi_t}{v})_t = (\frac{\phi_x}{v})_{tx} - (3\phi^2 - 1)\phi_t. \tag{3.9}$$

Multiplying it by $(\frac{\phi_t}{v})_t$ and integrating the result over [0,1] yields

$$\int_0^1 (\frac{\phi_t}{v})_t^2 dx + \frac{1}{2}\frac{d}{dt}\int_0^1 (\frac{\phi_x}{v})_t^2 dx = -\int_0^1 (\frac{\phi_x}{v})_t(\frac{\phi_x v_t}{v^2} - \frac{\phi_t v_x}{v^2})_t dx - \int_0^1 (3\phi^2 - 1)\phi_t(\frac{\phi_t}{v})_t dx, \tag{3.10}$$

where we have used the fact

$$(\frac{\phi_t}{v})_x = (\frac{\phi_x}{v})_t + (\frac{\phi_x v_t}{v^2} - \frac{\phi_t v_x}{v^2}).$$

Thus, using Lemmas 3.1-3.2 and Young's inequality, Cauchy inequality, it follows from (3.10) that

$$\begin{aligned}
&\int_0^1 (\frac{\phi_t}{v})_t^2 dx + \frac{1}{2}\frac{d}{dt}\int_0^1 (\frac{\phi_x}{v})_t^2 dx \\
&= -\int_0^1 (\frac{\phi_x}{v})_t [\frac{\phi_{xt}u_x + \phi_x u_{xt}}{v^2} - \frac{2\phi_x u_x^2}{v^3} - (\frac{\phi_t}{v})_t \frac{v_x}{v} - (\frac{\phi_t}{v})\frac{u_{xx}}{v} + (\frac{\phi_t}{v})\frac{v_x u_x}{v^2}]dx \\
&\quad - \int_0^1 (3\phi^2 - 1)\phi_t (\frac{\phi_t}{v})_t dx. \\
&\le \frac{1}{4}\int_0^1 (\frac{\phi_t}{v})_t^2 dx + C_1 \int_0^1 (v_x (\frac{\phi_x}{v})_t)^2 dx + C_1(\|\phi_t\|_{L^\infty}^2 + \|u_x\|_{L^\infty}^2 + \|\phi_x\|_{L^\infty}^2)\|(\frac{\phi_x}{v})_t\|^2 \\
&\quad + C_1(\|\phi_t\|^2 + \|\phi_{xt}\|^2 + \|u_{xt}\|^2 + \|u_x\|_{L^\infty}^2\|u_x\|^2 + \|u_{xx}\|^2) \\
&\le \frac{1}{4}\int_0^1 (\frac{\phi_t}{v})_t^2 dx + C_1\|(\frac{\phi_x}{v})_t\|_{L^\infty}^2 + C_1(\|\phi_t\|_{H^1}^2 + \|u_x\|_{H^1}^2 + \|\phi_x\|_{H^1}^2)\|(\frac{\phi_x}{v})_t\|^2 \\
&\quad + C_1(\|\phi_t\|_{H^1}^2 + \|u_{xt}\|^2 + \|u_{xx}\|^2) \\
&\le \frac{1}{4}\|(\frac{\phi_t}{v})_t\|^2 + C_1\|(\frac{\phi_x}{v})_t\|^2 + C_2\|(\frac{\phi_x}{v})_{xt})\|^2 + C_1(\|\phi_t\|_{H^1}^2 + \|u_{xt}\|^2 + \|u_{xx}\|^2) \\
&\le \frac{1}{4}\|(\frac{\phi_t}{v})_t\|^2 + C_1\|(\frac{\phi_x}{v})_t\|^2 + C_2\|(\frac{\phi_t}{v} + (\phi^3 - \phi))_t\|^2 + C_1(\|\phi_t\|_{H^1}^2 + \|u_{xt}\|^2 + \|u_{xx}\|^2) \\
&\le \frac{1}{2}\|(\frac{\phi_t}{v})_t\|^2 + C_2\|(\frac{\phi_x}{v})_t\|^2 + C_2(\|\phi_t\|_{H^1}^2 + \|u_{xt}\|^2 + \|u_{xx}\|^2). \qquad (3.11)
\end{aligned}$$

Using Grönwall's inequality, we obtain

$$\|(\frac{\phi_x}{v})_t\|^2 + \int_0^t \|(\frac{\phi_t}{v})_t\|^2 d\tau \le C_2. \qquad (3.12)$$

Finally, we get from (3.9) that

$$\int_0^t \|(\frac{\phi_x}{v})_{tx}\|^2 d\tau \le C_2 \int_0^t [\|(\frac{\phi_t}{v})_t\|^2 + \|\phi_t\|^2]d\tau \le C_2. \qquad (3.13)$$

This, together with (3.12)-(3.13), leads to Lemma 3.3.

Lemma 3.4. *Let* (v, u, θ, ϕ) *be a smooth solution of* (1.1) − (1.3) *on* $[0, 1] \times [0, T]$. *Then, it holds that for* $\forall (x, t) \in [0, 1] \times [0, T]$,

$$\|\theta_t\|^2 + \int_0^t \|\theta_{xt}\|^2(\tau)d\tau \le C_2. \qquad (3.14)$$

Proof. Differentiating $(1.1)_4$ with respect to t, by $(1.1)_1$, we get

$$\theta_{tt} = (-\frac{\theta_t u_x}{v} - \frac{\theta u_{xt}}{v} + \frac{\theta u_x^2}{v^2} + \frac{2u_x u_{xt}}{v} - \frac{u_x^3}{v^2}) + (\frac{\beta\theta^{\beta-1}\theta_t\theta_x}{v} + \frac{\theta^\beta\theta_{xt}}{v} - \frac{\theta^\beta\theta_x u_x}{v^2})_x + (u_x\mu^2 + 2v\mu\mu_t). \qquad (3.15)$$

Multiplying the resulting equation (3.15) by θ_t and integrating over $[0,1]$, using integration by parts and Young's inequality, Lemma 3.1, one obtains

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\theta_t\|^2+\int_0^1\frac{\theta^\beta\theta_{xt}^2}{v}dx=\int_0^1(-\frac{\theta_t u_x}{v}-\frac{\theta u_{xt}}{v}+\frac{\theta u_x^2}{v^2}+\frac{2u_x u_{xt}}{v}-\frac{u_x^3}{v^2})\theta_t dx\\
&-\int_0^1(\frac{\beta\theta^{\beta-1}\theta_t\theta_x}{v}-\frac{\theta^\beta\theta_x u_x}{v^2})\theta_{xt}dx+\int_0^1(u_x\mu^2+2v\mu\mu_t)\theta_t dx\\
&\le\frac{1}{2}\int_0^1\frac{\theta^\beta\theta_{xt}^2}{v}dx+C_1\int_0^1(\theta_x^2u_x^2+\theta_x^2\theta_t^2)dx+C_1\int_0^1(u_x\mu^2+2v\mu\mu_t)\theta_t dx\\
&+C_1\int_0^1(\theta_t^2|u_x|+|u_{xt}\theta_t|+|u_x^2\theta_t|+|u_xu_{xt}\theta_t|+|u_x^3\theta_t|)dx\\
&\le\frac{1}{2}\int_0^1\frac{\theta^\beta\theta_{xt}^2}{v}dx+C_1\sup_{t\in[0,T]}\|\theta_x\|^2\int_0^1(u_x^2+\theta_t^2)dx+C_1[\max_{x\in[0,1]}\mu^4\int_0^1u_x^2dx+\int_0^1(v\mu)^2\mu_t^2dx]\\
&+C_1(\|u_x\|_{L^\infty}\|\theta_t\|^2+\|u_{xt}\|^2+\|u_x\|^4+\|u_x\|_{L^\infty}^2\|u_{xt}\|^2+\|u_x\|_{L^\infty}^2\|u_x\|^4)+C_1\int_0^1\theta_t^2dx\\
&\le\frac{1}{2}\int_0^1\frac{\theta^\beta\theta_{xt}^2}{v}dx+C_2(\|u_x\|^2+\|\theta_t\|^2+\max_{x\in[0,1]}\mu^4+\|v\mu\|^2\|\mu_t\|^2+\|u_{xt}\|^2+\|u_x\|^4)+C_1\int_0^1\theta_t^2dx\\
&\le\frac{1}{2}\int_0^1\frac{\theta^\beta\theta_{xt}^2}{v}dx++C_2[\|u_x\|^2+\|\theta_t\|^2+\max_{x\in[0,1]}\mu^4+\|u_{xt}\|^2+\|u_x\|^4\\
&+\|v\mu\|^2(\sup_{0\le t\le T}\|\phi_t\|^2\sup_{0\le t\le T}\|\phi^2-1\|^2+\sup_{0\le t\le T}\|(\frac{\phi_x}{v})_{xt}\|^2)].
\end{aligned}
\tag{3.16}
$$

Integrating the above equality (3.16) over $[0,t]$, using Lemmas 3.1 − 3.3, we get

$$
\|\theta_t\|^2+\int_0^t\|\theta_{xt}\|d\tau\le C_2\int_0^t[\max_{x\in[0,1]}(\mu^4+\|u_x\|^4)+\|v\mu\|^2+\|u_{xt}\|^2+\|\theta_t\|^2]d\tau\le C_2.
$$

The proof of (3.14) is completed.

Lemma 3.5. *Let* (v,u,θ,ϕ) *be a smooth solution of* (1.1) − (1.3) *on* $[0,1]\times[0,T]$. *Then, it holds that for* $\forall(x,t)\in[0,1]\times[0,T]$,

$$
\|v_{xx}\|^2+\|u_{xx}\|^2+\|\theta_{xx}\|^2+\int_0^t(\|v_{xx}\|^2+\|u_{xxx}\|^2+\|\theta_{xxx}\|^2)(\tau)d\tau\le C_2. \tag{3.17}
$$

Proof. Using $(1.1)_2$ and Lemma 3.1, we obtain

$$
\begin{aligned}
\|u_{xx}\|&\le C_1(\|u_x\|_{L^\infty}\|v_x\|+\|v_x\|+\|\theta_x\|+\|\phi_x\|_{H^1}+\|u_x\|)\\
&\le C_1(\|v_x\|+\|\theta_x\|+\|\phi_x\|_{H^1}+\|u_t\|),
\end{aligned}
\tag{3.18}
$$

$$
\begin{aligned}
\|\theta_{xx}\|&\le C_2(\|u_x\|+\|\phi_t\|_{L^\infty}\|\phi_x\|_{L^\infty}\|\phi_{xx}\|+\|\theta_x\|+\|\theta_t\|)\\
&\le C_2(\|u_x\|+\|\phi_x\|_{H^1}+\|\theta_x\|+\|\theta_t\|),
\end{aligned}
\tag{3.19}
$$

$$
\|\phi_{xx}\|\le C_2(\|\phi^2-1\|+\|v_x\|+\|\phi_t\|). \tag{3.20}
$$

Differentiating $(1.1)_2$ with respect to x, using $v_{txx} = u_{xxx}$, we get

$$(\frac{v_{xx}}{v})_t + \frac{\theta}{v}(\frac{v_{xx}}{v}) = u_{xt} + E(x,t), \tag{3.21}$$

where

$$E(x,t) = \frac{\theta_{xx}}{v} - \frac{2\theta_x v_x}{v^2} + \frac{2\theta_x v_x^2}{v^3} + \frac{2u_{xx}v_x^2}{v^2} - \frac{2u_x v_x^2}{v^3} + \frac{1}{2}(\frac{\phi_x^2}{v^2})_{xx}.$$

Multiplying the resulting equation (3.21) by $\frac{v_{xx}}{v}$ and integrating over $[0,1]$, along with integration by parts and Young's inequality, using Lemmas 3.1 − 3.2, one obtains

$$\begin{aligned}\frac{d}{dt}\|\frac{v_{xx}}{v}\|^2 + C_1^{-1}\|\frac{v_{xx}}{v}\|^2 \le \frac{1}{2C_1}\|\frac{v_{xx}}{v}\|^2 + C_1(\|\theta_{xx}\|^2 + \|\theta_x\|_{L^\infty}^2\|v_x\|^2 + \|\theta_x\|_{L^\infty}^2\|v_x\|^4 \\ + \|u_{xt}\|^2 + \|v_x\|_{L^\infty}^2\|u_{xx}\|^2 + \|u_x\|_{L^\infty}^2\|v_x\|^4 + \|(\frac{\phi_x^2}{v^2})_{xx}\|^2),\end{aligned} \tag{3.22}$$

which implies

$$\|v_{xx}\|^2 + \int_0^t \|v_{xx}\|^2(\tau)d\tau \le C_2. \tag{3.23}$$

Differentiating $(1.1)_2, (1.1)_3, (1.1)_4$ with respect to x, respectively, using Lemmas 3.1 − 3.2, we get

$$\|u_{xt}\| \le C_2(\|u_x\|_{H^2} + \|\theta_x\|_{H^1} + \|v_x\|_{H^1} + \|\phi_x\|_{H^2}), \tag{3.24}$$

$$\|\phi_{xt}\| \le C_2(\|\phi_x\|_{L^\infty}\|v_{xx}\| + \|v_x\|_{L^\infty}\|\phi_{xx}\| + \|\phi_x\|_{H^2}) \le C_2(\|v_x\|_{H^1} + \|\phi_x\|_{H^2}), \tag{3.25}$$

$$\|\theta_{xt}\| \le C_2(\|u_x\|_{H^1} + \|\theta_x\|_{H^2} + \|v_x\|_{H^1} + \|\phi_x\|_{H^2}), \tag{3.26}$$

or

$$\|u_{xxx}\| \le C_2(\|u_x\|_{H^1} + \|\theta_x\|_{H^1} + \|v_x\|_{H^1} + \|\phi_x\|_{H^2} + \|u_{xt}\|), \tag{3.27}$$

$$\|\phi_{xxx}\| \le C_2(\|v_x\|_{H^1} + \|\phi_x\|_{H^1} + \|\phi_{xt}\|), \tag{3.28}$$

$$\|\theta_{xxx}\| \le C_2(\|u_x\|_{H^1} + \|\theta_x\|_{H^1} + \|v_x\|_{H^1} + \|\phi_x\|_{H^2} + \|\theta_{xt}\|). \tag{3.29}$$

By (3.18) − (3.20), (3.27) − (3.29), (3.23) and Lemmas 3.1 − 3.4, we get (3.17).
Proof of Theorem 1.1. By Lemmas 3.1-3.5, we complete the proof of Theorem 1.1.

4. Proof of Theorem 1.2 and a priori estimates

In this section, we give the proof of Theorem 1.2 by a series of a priori estimates.

Lemma 4.1. *Let* (v, u, θ, ϕ) *be a smooth solution of* (1.1) − (1.3) *on* $[0, 1] \times [0, T]$. *Then, it holds that for* $\forall (x, t) \in [0, 1] \times [0, T]$,

$$\|\phi_{xt}\|^2 + \|u_{xt}\|^2 + \|\theta_{xt}\|^2 + \int_0^t (\|\phi_{tt}\|^2 + \|u_{tt}\|^2 + \|\theta_{tt}\|^2)(\tau)d\tau \le C_4. \tag{4.1}$$

Proof. By virtue of (3.8), we obtain

$$\phi_t = -v(\phi^3 - \phi) + \phi_{xx} - \frac{\phi_x v_x}{v}. \tag{4.2}$$

Differentiating (4.2) with respect to t, multiplying the resulting equation by ϕ_{tt} in $L^2[0, 1]$, and using Lemma 3.1, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\phi_{xt}\|^2 + \int_0^1 \phi_{tt}^2 dx = -\int_0^1 u_x(\phi^3 - \phi)\phi_{tt}dx - \int_0^1 v(3\phi^2 - 1)\phi_t\phi_{tt}dx \\
&- \int_0^1 \frac{v_x}{v}\phi_{xt}\phi_{tt}dx - \int_0^1 \frac{u_{xx}}{v}\phi_x\phi_{tt}dx + \int_0^1 \frac{v_x u_x}{v^2}\phi_x\phi_{tt}dx \\
&\le C_1\|\phi_{tt}\|(\|\phi^2 - 1\|_{L^\infty}\|u_x\| + \|\phi^2 - 1\|_{L^\infty}\|\phi_t\| + \|v_x\|_{L^\infty}\|\phi_{xt}\| + \|\phi_x\|_{L^\infty}\|u_{xx}\| + \|\phi_x\|_{L^\infty}\|u_x\|_{L^\infty}\|v_x\|) \\
&\le \varepsilon\|\phi_{tt}\|^2 + C_1(\varepsilon)(\|u_x\|^2 + \|\phi_t\|^2 + \|\phi_{xt}\|^2 + \|u_{xx}\|^2 + \|v_x\|^2).
\end{aligned} \tag{4.3}$$

Choosing $\varepsilon \in (0, 1)$ small enough, we have

$$\|\phi_{xt}\|^2 + \int_0^t \|\phi_{tt}\|^2 d\tau \le \varepsilon \int_0^t \|\phi_{tt}\|^2 d\tau + C_1(\varepsilon)\int_0^t (\|u_x\|^2 + \|\phi_t\|^2 + \|\phi_{xt}\|^2 + \|u_{xx}\|^2 + \|v_x\|^2)d\tau \le C_4. \tag{4.4}$$

Multiplying (3.5) by u_{tt} and integrating over $[0, 1]$, using integration by parts, we obtain

$$\begin{aligned}
&\|u_{tt}\|^2 + \frac{1}{2}\frac{d}{dt}\int_0^1 \frac{u_{xt}^2}{v}dx = -\frac{1}{2}\int_0^1 \frac{u_x u_{xt}^2}{v^2}dx + \int_0^1 \frac{u_x^2 u_{xtt}}{v^2}dx + \int_0^1 (-\frac{\theta}{v})_{xt}u_{tt}dx - \frac{1}{2}\int_0^1 (\frac{\phi_x^2}{v^2})_{xt}u_{tt}dx \\
&= \sum_{i=1}^4 A_i.
\end{aligned} \tag{4.5}$$

Now, we focus on the estimates of the terms on the right-hand side of (4.5). First, due to Young's inequality and Lemmas 3.1-3.5, we obtain

$$A_1 \le C_1\|u_x\|_{L^\infty}\|u_{xt}\|^2 \le C_2\|u_{xt}\|^2, \tag{4.6}$$

$$A_2 = \frac{d}{dt}\int_0^1 \frac{u_x^2}{v^2}u_{tx}dx + \int_0^1 \frac{2u_x^3}{v^3}u_{tx}dx - \int_0^1 \frac{2u_x}{v^2}u_{tx}^2 dx$$

$$\leq \frac{d}{dt}\int_0^1 \frac{u_x^2}{v^2}u_{tx}dx + C_1\|u_x\|_{L^\infty}^2 \int_0^1 u_x^4 u_{tx}^2 dx + C_1\|u_x\|_{L^\infty}\|u_{tx}\|^2$$
$$\leq \frac{d}{dt}\int_0^1 \frac{u_x^2}{v^2}u_{tx}dx + C_2 \int_0^1 u_x^4 u_{tx}^2 dx + C_2\|u_{tx}\|^2, \tag{4.7}$$

$$A_3 \leq C_1(\varepsilon)(\|\theta_{xt}\|^2 + \|u_x\|_{L^\infty}^2\|\theta_x\|^2 + \|v_x\|_{L^\infty}^2\|\theta_t\|^2 + \|u_{xx}\|^2 + \|u_x\|_{L^\infty}^2\|v_x\|^2) + \frac{\varepsilon}{2}\|u_{tt}\|^2$$
$$\leq C_1(\varepsilon)(\|\theta_{xt}\|^2 + \|\theta_x\|^2 + \|\theta_t\|^2 + \|u_{xx}\|^2 + \|v_x\|^2) + \frac{\varepsilon}{2}\|u_{tt}\|^2. \tag{4.8}$$

Furthermore, A_4 is estimated as

$$A_4 \leq C_1(\varepsilon)(\|(\frac{\phi_x}{v})_t\|^2\|(\frac{\phi_x}{v})_x\|^2 + \|(\frac{\phi_x}{v})_{xt}\|^2\|(\frac{\phi_x}{v})\|^2) + \frac{\varepsilon}{2}\|u_{tt}\|^2. \tag{4.9}$$

Substituting the above estimates into (4.5), we obtain

$$\|u_{tt}\|^2 + \frac{1}{2}\frac{d}{dt}\int_0^1 \frac{u_{xt}^2}{v}dx \leq C_2\|u_{xt}\|^2 + C_1(\varepsilon)(\|\theta_{xt}\|^2 + \|\theta_x\|^2 + \|\theta_t\|^2 + \|u_{xx}\|^2 + \|v_x\|^2)$$
$$+C_2 \max_{x\in[0,1]} u_x^4 \int_0^1 u_{xt}^2 dx + \frac{d}{dt}\int_0^1 \frac{u_x^2}{v^2}u_{tx}dx + C_1(\varepsilon)(\|(\frac{\phi_x}{v})_t\|^2\|(\frac{\phi_x}{v})_x\|^2 + \|(\frac{\phi_x}{v})_{xt}\|^2\|(\frac{\phi_x}{v})\|^2) + \varepsilon\|u_{tt}\|^2. \tag{4.10}$$

Integrating (4.10) over $[0, t]$, using Grönwall's inequality, we show that

$$\|u_{xt}\|^2 + \int_0^t \|u_{tt}\|^2 d\tau \leq C_1(\varepsilon)\int_0^t (\|\theta_{xt}\|^2 + \|\theta_x\|^2 + \|\theta_t\|^2 + \|u_{xx}\|^2 + \|v_x\|^2)d\tau$$
$$+C_2 \int_0^t \max_{x\in[0,1]} u_x^4 d\tau \int_0^t \|u_{xt}\|^2 d\tau + C_1(\varepsilon)(\sup_{0\leq t\leq T}\|(\frac{\phi_x}{v})_t\|^2 \int_0^t \|(\frac{\phi_x}{v})_x\|^2 d\tau + \int_0^t \|(\frac{\phi_x}{v})_{xt}\|^2 d\tau). \tag{4.11}$$

Using Lemmas 3.1-3.5, we obtain

$$\|u_{xt}\|^2 + \int_0^t \|u_{tt}\|^2 d\tau \leq C_4. \tag{4.12}$$

Due to (3.15), we get

$$\theta_{tt} = (-\frac{\theta_t u_x}{v} - \frac{\theta u_{xt}}{v} + \frac{\theta u_x^2}{v^2} + \frac{2u_x u_{xt}}{v} - \frac{u_x^3}{v^2}) + (\frac{\theta^\beta}{v})_{xt}\theta_x + (\frac{\theta^\beta}{v})_x\theta_{xt} + (\frac{\theta^\beta}{v})_t\theta_{xx} + (\frac{\theta^\beta}{v})\theta_{xxt}$$
$$+u_x\mu^2 + 2\phi_t(\frac{\phi_x}{v})_{xt} - 2\phi_t^2(3\phi^2 - 1). \tag{4.13}$$

Multiplying (4.13) by θ_{tt} and integrating over $[0, 1]$, using integration by parts, we obtain

$$\int_0^1 \theta_{tt}^2 dx == \int_0^1 (-\frac{\theta_t u_x}{v} - \frac{\theta u_{xt}}{v} + \frac{\theta u_x^2}{v^2} + \frac{2u_x u_{xt}}{v} - \frac{u_x^3}{v^2})\theta_{tt}dx + \int_0^1 [(\frac{\theta^\beta}{v})_{xt}\theta_x + (\frac{\theta^\beta}{v})_t\theta_{xx}]\theta_{tt}dx$$
$$-\frac{1}{2}\frac{d}{dt}\int_0^1 (\frac{\theta^\beta}{v})\theta_{xt}^2 dx + \frac{1}{2}\int_0^1 (\frac{\theta^\beta}{v})_t\theta_{xt}^2 dx + \int_0^1 [u_x\mu^2 + 2\phi_t(\frac{\phi_x}{v})_{xt} - 2\phi_t^2(3\phi^2 - 1)]\theta_{tt}dx. \tag{4.14}$$

Then, we obtain

$$\int_0^1 \theta_{tt}^2 dx + \frac{1}{2}\frac{d}{dt}\int_0^1 (\frac{\theta^\beta}{v})\theta_{xt}^2 dx$$
$$\leq C_2[\|\theta_t\|^2 + \|u_{xt}\|^2 + \|u_x\|^4 + \|\theta_{xt}\|^2 + \|\theta_x\|_{H^1}^2 + \|u_x\|_{H^1}^2] + C_2(\|\theta_t\|_{L^\infty}\|\theta_{xt}\|^2$$
$$+ \int_0^1 \mu^4 dx + \|(\frac{\phi_x}{v})_{xt}\|^2 + \|\phi_t\|^2\|) + \varepsilon\|\theta_{tt}\|^2. \quad (4.15)$$

Using Lemmas 3.1-3.5, Young's inequality and Nirenberg's interpolation inequality, we obtain

$$\|\theta_{xt}\|^2 + \int_0^t \|\theta_{tt}\|^2 d\tau \leq C_2 \int_0^t (\|\theta_t\|^2 + \|u_{xt}\|^2 + \|\theta_{xt}\|^2 + \|\theta_x\|_{H^1}^2 + \|u_x\|_{H^1}^2)d\tau + \int_0^t \max_{x\in[0,1]} u_x^4 d\tau$$
$$+C_2(\int_0^t \max_{x\in[0,1]} \mu^4 d\tau + \int_0^t \|(\frac{\phi_x}{v})_{xt}\|^2 d\tau + \int_0^t \|\phi_t\|^2 d\tau) \leq C_4. \quad (4.16)$$

The proof of Lemma 4.1 is completed.

Lemma 4.2. *Let* (v, u, θ, ϕ) *be a smooth solution of* $(1.1) - (1.3)$ *on* $[0, 1] \times [0, T]$. *Then, it holds that for* $\forall(x, t) \in [0, 1] \times [0, T]$,

$$\|\phi_{xt}\|^2 + \|\theta_{xt}\|^2 + \int_0^t (\|\phi_{txx}\|^2 + \|u_{txx}\|^2 + \|\theta_{txx}\|^2)(\tau)d\tau \leq C_4. \quad (4.17)$$

Proof. Differentiating $(1.1)_3$ with respect to x, t respectively, using (3.8), we obtain

$$\phi_{ttx} = -(u_x\mu + v\mu_t)_x = [u_x\frac{\phi_t}{v} + v((\frac{\phi_x}{v})_{xt} - (3\phi^2 - 1)\phi_t)]_x. \quad (4.18)$$

Multiplying the resulting equation (4.18) by ϕ_{tx} in $L^2[0, 1]$, and integrating by parts, using Lemma 3.1 and Young's inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\phi_{tx}\|^2 = -\int_0^1 u_x\frac{\phi_t}{v}\phi_{txx}dx - \int_0^1 v[(\frac{\phi_x}{v})_{xt} - (3\phi^2 - 1)\phi_t]\phi_{txx}dx$$
$$\leq -C_1\|\phi_{txx}\|^2 + C_1(\varepsilon)(\|\frac{\phi_t}{v}\|^2\|u_x\|_{L^\infty}^2 + \|u_x\|_{L^\infty}^2\|\phi_{xx}\|^2 + \|v_x\|_{L^\infty}^2\|\phi_{tx}\|^2$$
$$+\|\phi_x\|_{L^\infty}^2\|u_{xx}\|^2 + \|v_x\|_{L^\infty}^2\|u_x\|_{L^\infty}^2\|\phi_x\|^2 + \|\phi_t\|_{L^\infty}^2\|\phi^2 - 1\|^2)$$
$$\leq -C_1\|\phi_{txx}\|^2 + C_1(\varepsilon)(\|\frac{\phi_t}{v}\|^2 + \|\phi_x\|_{H^1}^2 + \|\phi_{tx}\|^2 + \|u_{xx}\|^2 + \|\phi^2 - 1\|^2). \quad (4.19)$$

Thus,

$$\|\phi_{tx}\|^2 + \int_0^t \|\phi_{txx}\|^2 d\tau$$
$$\leq C_2 + C_2(\varepsilon)\int_0^t (\|\frac{\phi_t}{v}\|^2 + \|\phi_x\|_{H^1}^2 + \|\phi_{tx}\|^2 + \|u_{xx}\|^2 + \|\phi^2 - 1\|^2)d\tau \leq C_4. \quad (4.20)$$

By virtue of (3.15), we obtain

$$\theta_{tt} = (-\frac{\theta}{v} + \frac{u_x}{v})_t u_x + (-\frac{\theta}{v} + \frac{u_x}{v})u_{xt} + (\frac{\theta^\beta\theta_x}{v})_{xt} + (v\mu^2)_t. \quad (4.21)$$

Similarly, we derive from (4.21) that

$$\frac{1}{2}\frac{d}{dt}\|\theta_{tx}\|^2 = \int_0^1 (-\frac{\theta}{v}u_x + \frac{u_x^2}{v})_{tx}\theta_{tx}dx + \int_0^1 (\frac{\theta^\beta\theta_x}{v})_{xxt}\theta_{tx}dx + \int_0^1 (v\mu^2)_{tx}\theta_{tx}dx$$
$$= \sum_i^3 J_i. \tag{4.22}$$

Using Lemmas 3.1-3.5-4.1 and the interpolation inequality, we have

$$\begin{aligned} J_1 &\le C_1\|\theta_{tx}\|(\|\theta_t\|_{L^\infty} + \|v_x\|_{L^\infty}\|\theta_t\| + \|u_{xx}\| + \|u_x\|_{L^\infty}\|\theta_x\| \\ &+ \|u_{txx}\| + \|u_x\|_{L^\infty}\|v_x\| + \|v_x\|_{L^\infty}\|u_{tx}\| + \|v_x\|_{L^\infty}\|u_x\|^2) \\ &\le \varepsilon\|u_{txx}\|^2 + C_1(\varepsilon)(\|\theta_{tx}\|^2 + \|u_x\|_{H^1}^2 + \|u_{tx}\|^2), \end{aligned} \tag{4.23}$$

$$\begin{aligned} J_2 &\le -C_1\|\theta_{txx}\|^2 + C_1\|\theta_{txx}\|(\|\theta_t\|\|\theta_x\|_{L^\infty}^2 + \|\theta_{tx}\|\|\theta_x\|_{L^\infty} + \|u_x\|\|\theta_x\|_{L^\infty}^2 + \|u_x\|_{L^\infty}\|\theta_x\|_{L^\infty}\|\theta_t\| \\ &+ \|v_x\|_{L^\infty}\|\theta_{tx}\| + \|\theta_x\|_{L^\infty}\|u_{xx}\| + \|v_x\|_{L^\infty}\|\theta_x\|_{L^\infty}\|u_x\| + \|\theta_t\|_{L^\infty}\|\theta_{xx}\| + \|u_x\|_{L^\infty}\|\theta_{xx}\|) \\ &\le -C_2(\varepsilon)\|\theta_{txx}\|^2 + C_2\varepsilon(\|\theta_t\|^2 + \|\theta_{tx}\|^2 + \|u_x\|_{H^1}^2 + \|\theta_{xx}\|^2). \end{aligned} \tag{4.24}$$

By virtue of (3.9), we obtain

$$\begin{aligned} J_3 &= -\int_0^1 (v\mu^2)_t\theta_{txx}dx = -\int_0^1 [\mu^2 u_x + 2\phi_t(\frac{\phi_t}{v})_t]\theta_{txx}dx \\ &\le \varepsilon\|\theta_{txx}\|^2 + C_1(\varepsilon)[\|u_x\|_{L^\infty}^2\int_0^1 \mu^4 dx + \|\phi_t\|_{L^\infty}^2(\|(\frac{\phi_x}{v})_{xt}\|^2 + \|\phi_t\|^2)] \\ &\le \varepsilon\|\theta_{txx}\|^2 + C_2(\varepsilon)[\int_0^1 \mu^4 dx + \|(\frac{\phi_x}{v})_{xt}\|^2 + \|\phi_t\|^2]. \end{aligned} \tag{4.25}$$

Differentiating $(1.1)_2$ with respect to t, using Lemmas 3.1-3.5-4.1, we have

$$\|u_{xxt}\| \le C_1\|u_{tt}\| + C_2(\|u_x\|_{H^1} + \|u_{xt}\| + \|\theta_{xt}\| + \|\theta_t\| + \|\phi_{xt}\| + \|\phi_{xxt}\| + \|\phi_x\|_{H^1}). \tag{4.26}$$

Moreover, using Lemma 3.1, (4.12), (4.20), we get

$$\begin{aligned} \int_0^t \|u_{xxt}\|^2 d\tau &\le C_1\int_0^t \|u_{tt}\|^2 d\tau + C_2\int_0^t (\|u_x\|_{H^1}^2 + \|u_t\|_{H^1}^2 + \|\theta_t\|_{H^1}^2)d\tau \\ &+C_2\int_0^t (\|\phi_{xt}\|^2 + \|\phi_{xxt}\|^2 + \|\phi_x\|_{H^1}^2)d\tau \le C_4, \end{aligned} \tag{4.27}$$

which, combined with (4.22)-(4.27) and Lemmas 3.1-4.2, gives that for $\varepsilon \in (0,1)$ small enough

$$\begin{aligned} &\|\theta_{tx}\|^2 + \int_0^t \|\theta_{txx}\|^2 d\tau \le C_2(\varepsilon) + C_2(\varepsilon)[\int_0^t \max_{x\in[0,1]}\mu^4 d\tau + \int_0^t (\|(\frac{\phi_x}{v})_{xt}\|^2 + \|\phi_t\|^2 + \|u_{txx}\|^2)d\tau] \\ &\le C_4(\varepsilon) + C_2(\varepsilon)\int_0^t \|u_{txx}\|^2 d\tau \le C_4(\varepsilon). \end{aligned}$$

The proof of Lemma 4.2 is completed.

Lemma 4.3. *Let* (v, u, θ, ϕ) *be a smooth solution of* (1.1) − (1.3) *on* $[0, 1] \times [0, T]$. *Then, it holds that for* $\forall(x, t) \in [0, 1] \times [0, T]$,

$$\|\phi_{tt}\|^2 + \|u_{tt}\|^2 + \int_0^t (\|\phi_{ttx}\|^2 + \|u_{ttx}\|^2)(\tau)d\tau \leq C_4, \tag{4.28}$$

$$\|\theta_{tt}\|^2 + \int_0^t \|\theta_{ttx}\|^2(\tau)d\tau \leq C_4 + C_4(\varepsilon)\int_0^t \|(\frac{\phi_x}{v})_{xtt}\|^2(\tau)d\tau. \tag{4.29}$$

Proof. Differentiating (4.2) with respect to t twice, multiplying the resulting equation by ϕ_{tt} in $L^2[0, 1]$, and integrating by parts, using Lemmas 3.1 − 4.1 and Young's inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\phi_{tt}\|^2 + \int_0^1 \phi_{ttx}^2 dx \\
&\leq C_1\|\phi_{tt}\|(\|u_{xt}\|\|\phi^2 - 1\| + \|\phi^2 - 1\|\|u_x\|_{L^\infty}\|\phi_t\| + \|\phi_t\|^4) + C_2\|\phi_{tt}\|^2 \\
&+ C_1\|\phi_{tt}\|(\|v_x\|_{L^\infty}\|\phi_{xtt}\| + \|u_{xx}\|\|\phi_{xt}\| + \|u_x\|_{L^\infty}\|\phi_{xt}\|\|v_x\|_{L^\infty} + \|\phi_x\|_{L^\infty}\|u_{xxt}\|) \\
&+ \|u_x\|_{L^\infty}\|\phi_x\|_{L^\infty}\|u_{xx}\| + \|v_x\|_{L^\infty}\|\phi_x\|_{L^\infty}\|u_{xt}\| + \|v_x\|_{L^\infty}\|\phi_x\|_{L^\infty}\|u_x\|^2 \\
&\leq C_2(\|u_{xt}\|^2 + \|\phi_t\|_{H^1}^2 + \|u_{xx}\|^2 + \|u_x\|^4 + \|u_{xxt}\|^2) + \varepsilon\|\phi_{xtt}\|^2 + C_4(\varepsilon)\|\phi_{tt}\|^2.
\end{aligned} \tag{4.30}$$

Choosing $\varepsilon \in (0, 1)$ small enough, by Lemmas 3.1-4.2, we have

$$\|\phi_{tt}\|^2 + \int_0^t \|\phi_{ttx}\|^2 d\tau \leq C_2\int_0^t (\|u_{xt}\|^2 + \|\phi_t\|_{H^1}^2 + \|u_{xx}\|^2 + \max_{x\in[0,1]} u_x^4 + \|u_{xxt}\|^2)d\tau + C_4(\varepsilon)\int_0^t \|\phi_{tt}\|^2 d\tau \leq C_4. \tag{4.31}$$

Differentiate $(1.1)_2, (1.1)_3, (1.1)_4$ with respect to x twice, respectively, using Lemmas 3.1-3.5 to get

$$\|u_{xxt}\| \leq C_2(\|u_x\|_{H^3} + \|\theta_x\|_{H^2} + \|v_x\|_{H^2} + \|\phi_x\|_{H^3}), \tag{4.32}$$

$$\|\phi_{xxt}\| \leq C_2(\|v_x\|_{H^2} + \|\phi_x\|_{H^3}), \tag{4.33}$$

$$\|\theta_{xxt}\| \leq C_2(\|u_x\|_{H^2} + \|\theta_x\|_{H^3} + \|v_x\|_{H^2} + \|\phi_x\|_{H^3}), \tag{4.34}$$

or

$$\|u_{xxxx}\| \leq C_2(\|u_x\|_{H^2} + \|\theta_x\|_{H^2} + \|v_x\|_{H^2} + \|\phi_x\|_{H^3} + \|u_{xxt}\|), \tag{4.35}$$

$$\|\phi_{xxxx}\| \leq C_2(\|v_x\|_{H^2} + \|\phi_x\|_{H^2} + \|\phi_{xxt}\|), \tag{4.36}$$

$$\|\theta_{xxxx}\| \leq C_2(\|u_x\|_{H^2} + \|\theta_x\|_{H^2} + \|v_x\|_{H^2} + \|\phi_x\|_{H^3} + \|\theta_{xxt}\|). \tag{4.37}$$

Differentiating $(1.1)_2$ with respect to t twice, multiplying the resulting equation by in $L^2[0, 1]$, and performing an integration by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_{tt}\|^2 + \int_0^1 (\frac{u_x}{v})_{tt}u_{ttx}dx = \int_0^1 (\frac{\theta}{v})_{tt}u_{ttx}dx + \frac{1}{2}\int_0^1 (\frac{\phi_x^2}{v^2})_{tt}u_{ttx}dx. \tag{4.38}$$

Using Theorem 1.1, we then get

$$\frac{d}{dt}\|u_{tt}\|^2+\int_0^1\frac{u_{ttx}^2}{v}dx\le\varepsilon\int_0^1\frac{u_{ttx}^2}{v}dx+C_4(\varepsilon)(\|u_x\|_{L^\infty}\|u_{xt}\|^2+\|u_x\|_{L^6}^6+\|\theta_{tt}\|^2$$
$$+\|u_x\|_{L^\infty}^2\|\theta_t\|^2+\|u_{xt}\|^2+\|u_x\|_{L^\infty}^4+\|\phi_x\|_{L^\infty}^2\|\phi_{ttx}\|^2+\|\phi_{tx}\|^4$$
$$+\|u_x\|_{L^\infty}^2\|\phi_{xt}\|^2\|\phi_x\|_{L^\infty}^2+\|\phi_x\|_{L^\infty}^4\|u_{xt}\|^2+\|\phi_x\|_{L^\infty}^4\|u_x\|^4)$$
$$\le\varepsilon\int_0^1\frac{u_{ttx}^2}{v}dx+C_4(\varepsilon)(\|u_{xt}\|^2+\|u_x\|_{H^1}^2+\|\theta_t\|^2+\|\theta_{tt}\|^2+\|\phi_{xtt}\|^2+\|\phi_{tx}\|_{H^1}^2).\tag{4.39}$$

Thus, by Theorem 1.1, Lemmas 3.1-3.5, and (4.31), we obtain

$$\|u_{tt}\|^2+\int_0^t\|u_{ttx}\|^2d\tau\le C_4+C_4(\varepsilon)\int_0^t(\|\phi_{xtt}\|^2+\|\theta_{tt}\|^2)d\tau\le C_4.\tag{4.40}$$

Differentiating (3.15) with respect to t, and multiplying the resulting equation by θ_{tt} in $L^2[0,1]$, we have

$$\frac{1}{2}\frac{d}{dt}\|\theta_{tt}\|^2=\int_0^1(-\frac{\theta}{v}+\frac{u_x}{v})_{tt}u_x\theta_{tt}dx+2\int_0^1(-\frac{\theta}{v}+\frac{u_x}{v})_tu_{xt}\theta_{tt}dx+\int_0^1(-\frac{\theta}{v}+\frac{u_x}{v})u_{xtt}\theta_{tt}dx$$
$$+\int_0^1(\frac{\theta^\beta\theta_x}{v})_{xtt}\theta_{tt}dx+\int_0^1(v\mu^2)_{tt}\theta_{tt}dx=\sum_{i=1}^5B_i.\tag{4.41}$$

We employ Lemmas 3.1-3.5-4.1-4.2, the interpolation inequality and Young's inequality to get

$$B_1\le C_2\|u_x\|_{L^\infty}\|\theta_{tt}\|(\|u_x\|_{L^\infty}\|\theta_t\|+\|u_{xt}\|+\|u_x\|_{L^\infty}\|u_x\|+\|\theta_{tt}\|+\|u_{ttx}\|+\|u_x\|_{L^\infty}\|u_{xt}\|+\|u_x\|_{H^1})$$
$$\le C_2\|\theta_{tt}\|(\|\theta_t\|+\|u_{xt}\|+\|\theta_{tt}\|+\|u_{ttx}\|+\|u_x\|_{H^1})$$
$$\le\varepsilon\|u_{ttx}\|^2+C_2(\varepsilon)(\|\theta_t\|^2+\|u_{xt}\|^2+\|\theta_{tt}\|^2+\|u_x\|_{H^1}^2),\tag{4.42}$$

$$B_2\le C_2(\|u_{tx}\|^2\|\theta_t\|^2+\|u_{tx}\|^2\|\theta_{tt}\|^2+\|u_x\|_{L^\infty}^2\|u_{tx}\|^2+\|\theta_{tt}\|^2+\|u_x\|_{L^\infty}^4\|u_{tx}\|^2)$$
$$\le C_2(\|u_{tx}\|^2+\|\theta_{tt}\|^2),\tag{4.43}$$

$$B_3\le\varepsilon\|u_{ttx}\|^2+C_2(\varepsilon)\|\theta_{tt}\|^2,\tag{4.44}$$

$$B_4\le-C_1^{-1}\|\theta_{ttx}\|^2+C_2(\|\theta_{xt}\|^2\|\theta_t\|^2+\|u_x\|_{L^\infty}^2\|\theta_{xt}\|^2+\|\theta_x\|_{L^\infty}^2\|\theta_t\|^4+\|\theta_x\|_{L^\infty}^2\|\theta_{tt}\|^2$$
$$+\|\theta_x\|_{L^\infty}^2\|\theta_t\|^2\|u_x\|^2+\|\theta_x\|_{L^\infty}^2\|u_{xt}\|^2+\|\theta_x\|_{L^\infty}^2\|u_x\|^4)$$
$$\le-C_1^{-1}\|\theta_{ttx}\|^2+C_2(\|\theta_{xt}\|^2+\|\theta_{tt}\|^2+\|u_{xt}\|^2+\|u_x\|_{H^1}^2),\tag{4.45}$$

$$B_5\le C_2[\|\phi_t\|_{L^\infty}^2(\|u_{xt}\|+\|u_x\|)+\|\phi_t\|_{L^\infty}\|u_x\|_{L^\infty}\|(\frac{\phi_x}{v})_{xt}\|+\|\phi_t\|_{L^\infty}\|\phi_{tt}\|$$
$$+\|\phi_{tt}\|\|(\frac{\phi_x}{v})_{xt}\|+\|\phi_t\|_{L^\infty}^2\|\phi_t\|+\|\phi_t\|_{L^\infty}\|(\frac{\phi_x}{v})_{xtt}\|]\|\theta_{tt}\|$$

$$\leq C_2\varepsilon[\|u_{xt}\|^2 + \|u_x\|^2 + \|(\frac{\phi_x}{v})_{xt}\|^2 + \|\phi_{tt}\|^2 + \|\phi_t\|^2 + \|(\frac{\phi_x}{v})_{xtt}\|^2] + C_2(\varepsilon)\|\theta_{tt}\|^2. \tag{4.46}$$

Thus, we get from (4.42)-(4.46), and Lemma 4.1 that for $\varepsilon \in (0,1)$ small enough,

$$\begin{aligned}\|\theta_{tt}\|^2 + \int_0^t \|\theta_{ttx}\|^2(\tau)d\tau &\leq C_4 + C_4(\varepsilon)\int_0^t (\|\theta_{tt}\|^2 + \|\phi_{tt}\|^2 + \|(\frac{\phi_x}{v})_{xtt}\|^2)(\tau)d\tau \\ &\leq C_4 + C_4(\varepsilon)\int_0^t \|(\frac{\phi_x}{v})_{xtt}\|^2(\tau)d\tau. \end{aligned}\tag{4.47}$$

The proof of Lemma 4.3 is completed.

Lemma 4.4. *Let* (v,u,θ,ϕ) *be a smooth solution of* $(1.1)-(1.3)$ *on* $[0,1]\times[0,T]$. *Then, it holds that for* $\forall(x,t)\in[0,1]\times[0,T]$,

$$\|(\frac{\phi_x}{v})_{tt}\|^2 + \int_0^t \|(\frac{\phi_x}{v})_{xtt}\|^2(\tau)d\tau \leq C_4. \tag{4.48}$$

Proof. Differentiating (3.8) with respect to x, we obtain

$$(\frac{\phi_t}{v})_x = (\frac{\phi_x}{v})_{xx} - (\phi^3-\phi)_x. \tag{4.49}$$

Thus

$$(\frac{\phi_x}{v})_t - (\frac{\phi_x}{v})_{xx} = -(\phi^3-\phi)_x + \frac{\phi_t v_x}{v^2} - \frac{\phi_x u_x}{v^2}. \tag{4.50}$$

Differentiating (4.50) with respect to t twice, multiplying the resulting equation by $(\frac{\phi_x}{v})_{tt}$ in $L^2[0,1]$, and integrating by parts, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(\frac{\phi_x}{v})_{tt}\|^2 + \int_0^1 (\frac{\phi_x}{v})^2_{xtt}dx = -\int_0^1(\phi^3-\phi)_{xtt}(\frac{\phi_x}{v})_{tt}dx + \int_0^1(\frac{\phi_t v_x}{v^2})_{tt}(\frac{\phi_x}{v})_{tt}dx - \int_0^1(\frac{\phi_x u_x}{v^2})_{tt}(\frac{\phi_x}{v})_{tt}dx \\ &= \sum_{i=1}^3 M_i. \end{aligned}\tag{4.51}$$

By virtue of Lemmas 3.1-3.5, we deduce

$$\begin{aligned}M_1 &\leq \frac{1}{8}\int_0^1 (\frac{\phi_x}{v})^2_{xtt}dx + C_1(\|\phi_x\|^2_{L^\infty}\|\phi_t\|^2_{L^\infty}\|\phi_t\|^2 + \|\phi_t\|^2_{L^\infty}\|\phi_{tx}\|^2 + \|\phi_x\|^2_{L^\infty}\|\phi_{tt}\|^2 + \|\phi^2-1\|^2_{L^\infty}\|\phi_{ttx}\|^2) \\ &\leq \frac{1}{8}\int_0^1 (\frac{\phi_x}{v})^2_{xtt}dx + C_2(\|\phi_t\|^2 + \|\phi_{tx}\|^2 + \|\phi_{tt}\|^2 + \|\phi_{ttx}\|^2). \end{aligned}\tag{4.52}$$

Now, we will estimate M_2.
Using (4.50), we have

$$\begin{aligned}(\frac{\phi_t v_x}{v^2})_{tt} &= (\frac{\phi_t}{v})_{tt}(\frac{v_x}{v}) + 2(\frac{\phi_t}{v})_t(\frac{v_x}{v})_t + (\frac{\phi_t}{v})(\frac{v_x}{v})_{tt} \\ &= [(\frac{\phi_x}{v})_{ttx} - 6\phi\phi_t^2 - (3\phi^2-1)\phi_{tt}](\frac{v_x}{v}) + 2(\frac{\phi_{tt}}{v} - \frac{\phi_t u_x}{v^2})(\frac{u_{xx}}{v} - \frac{u_x v_x}{v^2}) \end{aligned}$$

$$+(\frac{\phi_t}{v})(\frac{u_{txx}}{v}-\frac{2u_{xx}u_x}{v^2}-\frac{u_{xt}v_x}{v^2}+\frac{2u_xv_x^2}{v^3}). \quad (4.53)$$

Using Young's inequality, Poincaré's inequality, and Lemmas 3.1-3.5, we get

$$\begin{aligned} M_2 \le& \frac{1}{8}\int_0^1(\frac{\phi_x}{v})_{xtt}^2dx + C_1(\|\phi_t\|_{L^\infty}^2\|v_x\|_{L^\infty}^2\|\phi_t\|^2 + \|\phi^2-1\|_{L^\infty}^2\|v_x\|_{L^\infty}^2\|\phi_{tt}\|^2 + \|\phi_{tt}\|^2\|u_{xx}\|^2 \\ &+ \|u_x\|_{L^\infty}^2\|v_x\|_{L^\infty}^2\|\phi_{tt}\|^2 + \|\phi_t\|_{L^\infty}^2\|u_x\|_{L^\infty}^2\|u_{xx}\|^2 + \|\phi_t\|_{L^\infty}^2\|v_x\|_{L^\infty}^2\|u_x\|^4) \\ &+ C_1(\|\phi_t\|_{L^\infty}^2\|u_{txx}\|^2 + \|\phi_t\|_{L^\infty}^2\|v_x\|_{L^\infty}^2\|u_{xt}\|^2 + \|\phi_t\|^2\|u_x\|_{L^\infty}^2\|v_x\|_{L^\infty}^4) \\ \le& \frac{1}{8}\int_0^1(\frac{\phi_x}{v})_{xtt}^2dx + C_2(\|\phi_t\|^2+\|\phi_{tt}\|^2+\|u_{xx}\|^2+\|u_x\|^4+\|u_{txx}\|^2+\|u_{tx}\|^2). \end{aligned} \quad (4.54)$$

Analogously, we get

$$\begin{aligned} M_3 \le& \frac{1}{8}\int_0^1(\frac{\phi_x}{v})_{xtt}^2dx + C_1[\|\phi_{ttx}\|^2 + \|\phi_{xt}\|^2\|u_x\|_{L^\infty}^2 + \|u_{xt}\|^2\|\phi_x\|_{L^\infty}^2 + \|u_x\|^4\|\phi_x\|_{L^\infty}^2 \\ &+ \|(\frac{\phi_x}{v})_t\|_{L^\infty}^2(\|u_{xt}\|^2+\|u_x\|^4) + \|\phi_x\|_{L^\infty}^2\|u_{ttx}\|^2 + \|\phi_x\|_{L^\infty}^2\|u_x\|_{L^\infty}^2\|u_{tx}\|^2 + \|\phi_x\|_{L^\infty}^2\|u_x\|_{L^\infty}^2\|u_x\|^4] \\ \le& \frac{1}{8}\int_0^1(\frac{\phi_x}{v})_{xtt}^2dx + C_2[\|\phi_{ttx}\|^2+\|u_{xt}\|^2+\|\phi_{xt}\|^2+\|u_x\|^4+\|u_{ttx}\|^2], \end{aligned} \quad (4.55)$$

which, combined with (4.52)-(4.55), yields

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|(\frac{\phi_x}{v})_{tt}\|^2 + \int_0^1(\frac{\phi_x}{v})_{xtt}^2dx \le& \frac{3}{8}\int_0^1(\frac{\phi_x}{v})_{xtt}^2dx + C_2(\|\phi_t\|_{H^1}^2+\|\phi_{tt}\|_{H^1}^2 \\ &+\|u_{xx}\|^2+\|u_x\|^4+\|u_{tx}\|_{H^1}^2+\|u_{ttx}\|^2). \end{aligned} \quad (4.56)$$

Using Lemmas 3.1-4.2 and (4.28), we obtain

$$\begin{aligned} \|(\frac{\phi_x}{v})_{tt}\|^2 + \int_0^t\|(\frac{\phi_x}{v})_{xtt}\|^2d\tau \le& C_2\int_0^t(\|\phi_t\|_{H^1}^2+\|\phi_{tt}\|_{H^1}^2 \\ &+\|u_{xx}\|^2+\|u_x\|^4+\|u_{tx}\|_{H^1}^2+\|u_{ttx}\|^2)d\tau \le C_4. \end{aligned}$$

The proof of Lemma 4.4 is completed. By Lemma 4.4, we obtain Lemma 4.3.

Lemma 4.5. *Let* (v,u,θ,ϕ) *be a smooth solution of* (1.1) − (1.3) *on* $[0,1]\times[0,T]$*. Then, it holds that for* $\forall(x,t)\in[0,1]\times[0,T]$*,*

$$\begin{aligned} &\|v_{xxx}\|_{H^1}^2+\|u_{xxx}\|_{H^1}^2+\|\theta_{xxx}\|_{H^1}^2+\|\phi_{xxx}\|_{H^1}^2+\|u_{xxt}\|^2+\|\phi_{xxt}\|^2+\|\theta_{xxt}\|^2 \\ &+\int_0^t(\|v_{xxx}\|_{H^1}^2+\|u_{xxx}\|_{H^1}^2+\|\theta_{xxx}\|_{H^1}^2+\|\phi_{xxx}\|_{H^1}^2+\|u_{xxt}\|^2+\|\phi_{xxt}\|^2+\|\theta_{xxt}\|^2)(\tau)d\tau \le C_4, \end{aligned} \quad (4.57)$$

$$\int_0^t(\|v_{xxxx}\|^2+\|u_{xxxxx}\|^2+\|\theta_{xxxxx}\|^2+\|\phi_{xxxxx}\|^2+\|u_{xxxt}\|^2+\|\phi_{xxxt}\|_{H^1}^2+\|\theta_{xxxt}\|^2)(\tau)d\tau \le C_4. \quad (4.58)$$

Proof. Differentiating (3.21) with respect to x, we obtain

$$(\frac{v_{xxx}}{v})_t + \frac{\theta}{v}(\frac{v_{xxx}}{v}) = u_{xxt} + E_1(x,t), \tag{4.59}$$

where

$$E_1(x,t) = (\frac{v_{xx}v_x}{v^2})_t - (\frac{\theta}{v})_x\frac{v_{xx}}{v} + \frac{\theta v_{xx}v_x}{v^3} + E_x.$$

Differentiating (4.49) with respect to x, we obtain

$$(\frac{\phi_t}{v})_{xx} = (\frac{\phi_x}{v})_{xxx} - (\phi^3 - \phi)_{xx}. \tag{4.60}$$

Hence,

$$\begin{aligned} \|(\frac{\phi_x}{v})_{xxx}\|^2 &\le C_1(\|(\frac{\phi_t}{v})_{xx}\|^2 + \|\phi_x\|^2_{L^\infty}\|\phi_x\|^2 + \|\phi^2 - 1\|^2_{L^\infty}\|\phi_{xx}\|^2) \\ &\le C_2(\|\phi_x\|^2 + \|\phi_{txx}\|^2 + \|\phi_{tx}\|^2 + \|\phi_{xx}\|^2 + \|v_{xx}\|^2). \end{aligned} \tag{4.61}$$

We can infer from Lemmas 3.1-4.2, and (4.61) that

$$\begin{aligned} \|E_1(x,t)\| &\le C_2(\|u_x\|_{H^2} + \|v_x\|_{H^1} + \|\theta_x\|_{H^2} + \|\phi_x\|_{H^3}) \\ &\le C_2(\|u_x\|^2_{H^2} + \|v_x\|^2_{H^1} + \|\theta_x\|^2_{H^2} + \|\phi_x\|^2_{H^1} + \|\phi_{txx}\|^2 + \|\phi_{tx}\|^2). \end{aligned} \tag{4.62}$$

This leads to

$$\int_0^t \|E_1(x,\tau)\|^2 d\tau \le C_4, \quad \forall t > 0. \tag{4.63}$$

Multiplying (4.59) by $\frac{v_{xxx}}{v}$ in $[0,1]\times[0,t]$, and integrating by parts, using Lemmas 3.1-4.4, (4.63), and Young's inequality, we obtain

$$\|\frac{v_{xxx}}{v}\|^2 + \int_0^t \|\frac{v_{xxx}}{v}\|^2 d\tau \le C_2\int_0^t (\|u_{xxt}\|^2 + \|E_1(x,\tau)\|^2)d\tau \le C_4. \tag{4.64}$$

By virtue of (3.27)-(3.29), (4.35)-(4.37), and Lemmas 3.1-4.4, we have

$$\|u_{xxx}\|^2 + \|\theta_{xxx}\|^2 + \|\phi_{xxx}\|^2 + \int_0^t (\|u_{xxx}\|^2_{H^1} + \|\theta_{xxx}\|^2_{H^1} + \|\phi_{xxx}\|^2_{H^1})d\tau \le C_4. \tag{4.65}$$

Differentiating $(1.1)_4, (1.1)_5$ with respect to t, using Lemmas 3.1-4.3, and (4.28), we have

$$\|\phi_{xxt}\| \le C_1\|\phi_{tt}\| + C_2(\|v_x\| + \|u_x\|_{H^1} + \|\phi_{tx}\| + \|\phi_x\|), \tag{4.66}$$

$$\|\theta_{xxt}\| \le C_1(\|\theta_{tt}\| + \|\phi_{tt}\|) + C_2(\|\theta_x\|_{H^1} + \|u_x\|_{H^1} + \|u_{xt}\| + \|\theta_{xt}\|), \tag{4.67}$$

which, combined with (4.35)-(4.37), implies

$$\|u_{xxxx}\|^2 + \|\theta_{xxxx}\|^2 + \|\phi_{xxxx}\|^2 + \|u_{xxt}\|^2 + \|\theta_{xxt}\|^2 + \|\phi_{xxt}\|^2$$

$$+\int_0^t(\|u_{xxxx}\|^2+\|\theta_{xxxx}\|^2+\|\phi_{xxxx}\|^2+\|u_{xxt}\|^2+\|\theta_{xxt}\|^2+\|\phi_{xxt}\|^2)d\tau\le C_4. \tag{4.68}$$

Differentiating (4.59) with respect to x, we obtain

$$(\frac{v_{xxxx}}{v})_t+\frac{\theta}{v}(\frac{v_{xxxx}}{v})=u_{xxxt}+E_2(x,t), \tag{4.69}$$

where

$$E_2(x,t)=(\frac{v_{xxx}v_{xx}}{v^2})_t-(\frac{\theta}{v})_x\frac{v_{xxx}}{v}+\frac{\theta v_{xxx}v_x}{v^3}+E_{1x}.$$

Using the embedding theorem, Lemmas 4.1-4.4, and (4.66)-(4.69), we obtain

$$\|E_{xx}\|\le C_2(\|u_x\|_{H^3}+\|v_x\|_{H^2}+\|\theta_x\|_{H^3}+\|\phi_x\|_{H^4}), \tag{4.70}$$

$$\|E_{1x}\|\le C_2(\|u_x\|_{H^3}+\|v_x\|_{H^2}+\|\theta_x\|_{H^1}+\|E_{xx}\|). \tag{4.71}$$

Hence,

$$\|E_2\|\le C_2(\|u_x\|_{H^3}+\|v_x\|_{H^2}+\|E_{1x}\|)\le C_2(\|u_x\|_{H^3}+\|v_x\|_{H^2}+\|\theta_x\|_{H^3}+\|\phi_x\|_{H^4}). \tag{4.72}$$

Differentiating $(1.1)_2$ with respect to x,t, using Theorem 1.1 and Lemmas 4.1-4.3, we obtain

$$\begin{aligned}\|u_{txxx}\|&\le C_2(\|u_{ttx}\|+\|u_x\|_{H^2}+\|\theta_x\|_{H^1}+\|v_x\|_{H^1}+\|\theta_t\|_{H^2}+\|u_t\|_{H^2}+\|(\frac{\phi_x}{v})_t\|_{H^2}+\|\phi_x\|_{H^1})\\&\le C_2(\|u_{ttx}\|+\|u_x\|_{H^2}+\|\theta_x\|_{H^1}+\|v_x\|_{H^1}+\|\theta_t\|_{H^2}+\|u_t\|_{H^2}+\|\phi_{ttx}\|+\|\phi_t\|_{H^1}+\|\phi_{tt}\|+\|\phi_x\|_{H^1}).\end{aligned} \tag{4.73}$$

Hence,

$$\int_0^t\|u_{txxx}\|^2d\tau\le C_2\int_0^t(\|u_{ttx}\|^2+\|\theta_{txx}\|^2+\|u_t\|_{H^2}^2+\|\phi_{ttx}\|^2+\|\phi_{tt}\|^2)d\tau+C_2\le C_4. \tag{4.74}$$

Multiplying (4.69) by $\frac{v_{xxxx}}{v}$ in $L^2(0,1)$, we derive

$$\frac{d}{dt}\|\frac{v_{xxxx}}{v}\|^2+C_1\|\frac{v_{xxxx}}{v}\|^2\le C_1(\|u_{xxxt}\|^2+\|E_2(x,t)\|^2). \tag{4.75}$$

Whence by (4.75), we obtain

$$\begin{aligned}\|\frac{v_{xxxx}}{v}\|^2+\int_0^t\|\frac{v_{xxxx}}{v}\|^2d\tau&\le C_4+C_1\int_0^t\|E_2(x,t)\|^2d\tau\\ \le C_4+C_2\int_0^t(\|u_x\|_{H^3}^2+\|v_x\|_{H^2}^2+\|\theta_x\|_{H^3}^2+\|\phi_x\|_{H^4}^2)d\tau&\le C_4+C_2\int_0^t\|\phi_x\|_{H^4}^2d\tau.\end{aligned} \tag{4.76}$$

Differentiating $(1.1)_4$ with respect to x,t, using Lemma 3.1, we get

$$\|\phi_{txxx}\|\le C_2(\|\phi_{ttx}\|+\|\phi_{txx}\|+\|\phi_x\|_{H^1}+\|u_x\|_{H^2}+\|v_x\|_{H^1}+\|\phi_t\|_{H^1}). \tag{4.77}$$

Differentiating $(1.1)_4$ with respect to x three times, using Lemmas 2.1-4.3, (4.77) and Young inequality, we get

$$\|\phi_{xxxxx}\|^2 \le C_1\|\phi_{txxx}\|^2 + \varepsilon\|v_x\|_{H^3}^2 + C_2\|\phi_x\|_{H^3}^2 \le \varepsilon\|v_x\|_{H^3}^2 + C_2(\|\phi_x\|_{H^3}^2 + \|\phi_{ttx}\|^2 + \|\phi_t\|_{H^2}^2 + \|u_x\|_{H^2}^2). \tag{4.78}$$

Then,

$$\int_0^t \|\phi_{xxxxx}\|^2 d\tau \le \varepsilon \int_0^t \|v_x\|_{H^3}^2 d\tau + C_2 \int_0^t (\|\phi_x\|_{H^3}^2 + \|\phi_{ttx}\|^2 + \|\phi_t\|_{H^2}^2 + \|u_x\|_{H^2}^2) d\tau \le C_4 + \varepsilon \int_0^t \|v_x\|_{H^3}^2 d\tau. \tag{4.79}$$

Choose ε enough small and combining with (4.79), we get

$$\|v_{xxxx}\|^2 + \int_0^t \|v_{xxxx}\|^2 d\tau \le C_4. \tag{4.80}$$

Differentiating $(1.1)_5$ with respect to x, t , using Lemmas 3.1-4.4, we obtain

$$\|\theta_{txxx}\| \le C_2(\|\theta_{ttx}\| + \|\theta_t\|_{H^2} + \|u_t\|_{H^2} + \|u_x\|_{H^1} + \|\theta_x\|_{H^1} + \|\phi_{tt}\|_{H^1} + \|(\frac{\phi_t}{v})_{xt}\|). \tag{4.81}$$

Thus,

$$\begin{aligned}\int_0^t (\|\theta_{txxx}\|^2 + \|\phi_{txxx}\|^2) d\tau \le C_2 \int_0^t (\|u_{ttx}\|^2 + \|\theta_{ttx}\|^2 + \|\theta_t\|_{H^2}^2 + \|u_t\|_{H^2}^2 + \|\phi_t\|_{H^2}^2 \\ + \|v_x\|_{H^1}^2 + \|u_x\|_{H^2}^2 + \|\theta_x\|_{H^1}^2 + \|\phi_x\|_{H^1}^2 + \|\phi_{tt}\|_{H^1}^2 + \|(\frac{\phi_t}{v})_{xt}\|^2) d\tau \le C_4.\end{aligned} \tag{4.82}$$

Differentiating $(1.1)_2, (1.1)_5$ with respect to x three times, respectively, using Lemmas 3.1-4.4, we get

$$\|u_{xxxxx}\| \le C_2(\|\theta_x\|_{H^3} + \|u_x\|_{H^3} + \|v_x\|_{H^3} + \|\phi_x\|_{H^4} + \|u_{xxxt}\|), \tag{4.83}$$

$$\|\theta_{xxxxx}\| \le C_2(\|\theta_x\|_{H^3} + \|u_x\|_{H^3} + \|v_x\|_{H^3} + \|\phi_t\|_{H^3} + \|\theta_{xxxt}\|). \tag{4.84}$$

By (4.79) and (4.80), (4.82), and (4.65), we get

$$\begin{aligned}\int_0^t (\|u_{xxxxx}\|^2 + \|\theta_{xxxxx}\|^2) d\tau \le C_2 \int_0^t (\|\theta_x\|_{H^3}^2 + \|u_x\|_{H^3}^2 + \|v_x\|_{H^3}^2 + \|\phi_x\|_{H^4}^2 \\ + \|u_{xxxt}\|^2 + \|\phi_t\|_{H^3}^2 + \|\theta_{xxxt}\|^2) d\tau \le C_4.\end{aligned} \tag{4.85}$$

Differentiating $(1.1)_4$ with respect to x twice and differentiating $(1.1)_4$ with respect to t, using Lemmas 3.1-4.4, (4.64)- (4.65), we obtain

$$\|\phi_{txxxx}\| \le C_2(\|\phi_{ttxx}\| + \|\phi_t\|_{H^3} + \|\phi_x\|_{H^2} + \|u_x\|_{H^3} + \|v_x\|_{H^2}). \tag{4.86}$$

Then,

$$\int_0^t \|\phi_{txxxx}\|^2 d\tau \le C_2 \int_0^t (\|\phi_{ttxx}\|^2 + \|\phi_t\|_{H^3}^2 + \|\phi_x\|_{H^2}^2 + \|u_x\|_{H^3}^2 + \|v_x\|_{H^2}^2) d\tau \le C_4.$$

Proof of Theorem 1.2. By Lemmas 4.1-4.5, we thus complete the proof of Theorem 1.2.

Author contributions

Chunxiang Kong: Writing original draft and Editing; Xiaofang Feng: Reviewing documents. The authors jointly contributed to the research and manuscript preparation.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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