Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(6): 17039–17061.
DOI:10.3934/math.2026698
Received: 26 March 2026
Revised: 16 May 2026
Accepted: 27 May 2026
Published: 12 June 2026

Research article

The $\omega^{\sharp}$-operator in ideal topological spaces and its associated topology

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Abstract: In this paper, we introduced a new set-theoretic operator $(\cdot)^{\sharp}_{\omega}$ in the framework of ideal topological spaces and investigated its fundamental properties, including its connections with the classical $\sharp$-operator and the ω-local function. Using this operator, we defined a closure-type operator $\mathrm{Cl}^{\sharp}_{\omega}$ and showed that it satisfies the Kuratowski closure axioms. Consequently, a topology $\mathcal{T}^{\sharp}_{\omega}$ was obtained, which is strictly finer than the topology induced by the $\sharp$-operator. Furthermore, the structural relationships among these topologies were examined, and some applications of the $\omega^{\sharp}$-operator were presented. Finally, we introduced the notions of ω^{*}-continuity and $\omega^{\sharp}$-continuity, investigated their relationship, and established a new decomposition of continuity. We also compared these notions with related concepts such as ω-continuity and $\sharp$-continuity.

Keywords: ideal topological space; ω-open set; $(\cdot)^{\sharp}_{\omega}$-operator; $\mathrm{Cl}^{\sharp}_{\omega}$- operator; $\mathcal{T}^{\sharp}_{\omega}$-topology; $\mathcal{T}^{*}_{\omega}$-topology; ω^{*}-continuity; $\omega^{\sharp}$-continuity
Mathematics Subject Classification: 54A05, 54A10, 54C10

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1. Introduction and preliminaries

The interaction between algebraic structures and topology such as ideals [1], filters [2], primals [3], and grills [4] has played an important role in general topology. Among these structures, ideals provide a convenient framework for formalizing notions of smallness.

Let $(\mathbb{X}, \mathcal{T})$ be a topological space. A nonempty family $\mathfrak{I} \subseteq 2^{\mathbb{X}}$ is called an ideal if it is hereditary and closed under finite unions. The triple $(\mathbb{X}, \mathcal{T}, \mathfrak{I})$ is called an ideal topological space (briefly, an ideal top-space), with no separation axioms assumed. For an ideal top-space, the associated local function (or $*$-operator) is defined by

$$H^* = \{x \in \mathbb{X} : \forall U \in \mathcal{T}(x),\ U \cap H \notin \Im\},$$

where $\mathcal{T}(x) = \{U \in \mathcal{T} : x \in U\}$ and $H \subseteq \mathbb{X}$. This operator, originating in Kuratowski's work [5], was further developed by Vaidyanathaswamy [6]. Moreover, the operator

$\mathrm{Cl}^*(H) = H \cup H^*$. Defines a Kuratowski closure operator on $2^{\mathbb{X}}$. Later contributions include the complementary operator Ψ introduced by Natkaniec [7] and the systematic study of ideal topological spaces by Janković and Hamlett [8]. Islam and Modak provided an in-depth study of the $*$ and Ψ operators and their properties [9].

Recent developments in the theory of ideal topological spaces have emphasized the interaction between ideals and various generalized topological notions. In particular, investigations have addressed weakened forms of separation axioms as well as extensions of closedness concepts within this framework [10, 11]. Moreover, approaches based on nano-topology, especially those employing covering-generated neighborhood systems in the presence of multiple ideals, have further enriched the structure of such spaces [12].

Earlier contributions have also examined decompositions of continuity in the setting of ideal topologies and $\Im$-Alexandroff spaces, providing a deeper understanding of how classical continuity notions can be refined through the incorporation of ideal-related constraints [13, 14].

More recently, attention has shifted toward the formulation of new operators and localized functions in ideal topological spaces, such as sharp-type operators and aura-based local functions. These constructions serve as effective tools for producing finer or alternative topological structures, thereby extending the scope of operator-driven methodologies [15–18]. Collectively, these advancements reflect a growing interest in operator-oriented frameworks and their role in shaping modern generalizations of ideal topological spaces.

Throughout this paper, a pair $(\mathbb{X}, \mathcal{T})$, or simply $\mathbb{X}$ when the topology is clear, denotes a topological space (top-space) without assuming separation axioms. For any $H \subseteq \mathbb{X}$, $\mathrm{Cl}(H)$ and $\mathrm{Int}(H)$ represent the closure and interior of H relative to $\mathcal{T}$, respectively. Let $(\mathbb{X}, \mathcal{T})$ be a top-space and $H \subseteq \mathbb{X}$. A point $x \in \mathbb{X}$ is called a *condensation point* of H if

$$\forall U \in \mathcal{T}\ (x \in U \Rightarrow U \cap H \text{ is uncountable}).$$

The set H is said to be ω*-closed* [19] if it contains all of its condensation points. A subset $W \subseteq \mathbb{X}$ is ω-open if $\mathbb{X} \setminus W$ is ω-closed. Equivalently, W is ω-open if and only if

$$\forall x \in W\ \exists U \in \mathcal{T} \text{ such that } x \in U \text{ and } U \setminus W \text{ is countable.}$$

The family of all ω-open subsets of $\mathbb{X}$ is denoted by $\mathcal{T}_\omega$ and forms a topology on $\mathbb{X}$ finer than $\mathcal{T}$. The ω-closure and ω-interior, which can be defined in the same way as $\mathrm{Cl}(H)$ and $\mathrm{Int}(H)$, respectively, for $H \subseteq \mathbb{X}$, is denoted by $\mathrm{Cl}_\omega(H)$ and $\mathrm{Int}_\omega(H)$, respectively.

Motivated by the concept of ω-open sets, Al-Omari and Al-Saadi [20] introduced the notion of ω-local functions with respect to a topology $\mathcal{T}$ and an ideal $\Im$ on a set $\mathbb{X}$. For any subset $H \subseteq \mathbb{X}$, define $H^*_\omega(\mathcal{T}, \Im) = \{x \in \mathbb{X} : W \cap H \notin \Im \text{ for every } W \in \mathcal{T}_\omega(x)\}$, where $\mathcal{T}_\omega(x) = \{W \in \mathcal{T}_\omega : x \in W\}$. For brevity, we write H^*_ω instead of $H^*_\omega(\mathcal{T}, \Im)$. Moreover, for an ideal $\Im$ on $\mathbb{X}$, the operator $\mathrm{Cl}^*_\omega(H) = H \cup H^*_\omega$ defines a Kuratowski closure operator on $2^{\mathbb{X}}$. The topology generated by the operator Cl^*_ω is $\mathcal{T}^*_\omega(\mathbb{X}, \mathcal{T}) = \{H \subseteq \mathbb{X} : \mathrm{Cl}^*_\omega(\mathbb{X} \setminus H) = \mathbb{X} \setminus H\}$. This topology is called the ω^*-topology, and it is finer than $\mathcal{T}_\omega$.

Recently, Issaka and Özköc [21] developed a framework based on extremal ideals and investigated their structural behavior within ideal-induced topologies. Let $\mathbb{X} \neq \emptyset$ and let $\Im \subsetneq 2^{\mathbb{X}}$ be an ideal. The ideal $\Im$ is said to be *maximal* if

$$\forall \mathcal{K} \, (\Im \subseteq \mathcal{K} \subseteq 2^{\mathbb{X}} \;\Rightarrow\; (\mathcal{K} = \Im \;\vee\; \mathcal{K} = 2^{\mathbb{X}})).$$

Dually, assuming $\Im \neq \{\emptyset\}$, the ideal $\Im$ is called minimal whenever

$$\forall \mathcal{K} \, (\{\emptyset\} \subseteq \mathcal{K} \subseteq \Im \;\Rightarrow\; (\mathcal{K} = \Im \;\vee\; \mathcal{K} = \{\emptyset\})).$$

Let $\mathcal{T}$ be a topology on $\mathbb{X}$. For any $H \subseteq \mathbb{X}$, define the operator

$$H^{\sharp}(\Im, \mathcal{T}) = \{x \in \mathbb{X} : \forall\, U \in \mathcal{T},\ x \in U \Rightarrow \exists\, J \in \Im \setminus \{\emptyset\} \text{ such that } J \subseteq U \cap H\}.$$

This operator is referred to as the *sharp transform* of H relative to $(\Im, \mathcal{T})$. It is briefly denoted by $H^{\sharp}$. Moreover, define the dual operator by $\Psi^{\sharp}(H) = \mathbb{X} \setminus (\mathbb{X} \setminus H)^{\sharp}$. Using $H^{\sharp}$, one obtains a Kuratowski-type closure operator $\mathrm{Cl}^{\sharp}(H) = H \cup H^{\sharp},\ H \subseteq \mathbb{X}$. The family of open sets induced by $\mathrm{Cl}^{\sharp}$ is given by $\mathcal{T}^{\sharp} = \{U \subseteq \mathbb{X} : \mathrm{Cl}^{\sharp}(\mathbb{X} \setminus U) = \mathbb{X} \setminus U\}$, which defines a topology on $\mathbb{X}$ satisfying $\mathcal{T} \subseteq \mathcal{T}^{\sharp}$.

In Section 2, we introduce a new operator $(\cdot)^{\sharp}_{\omega}$, which we denote by $\omega^{\sharp}$ when no confusion arises, and investigate its principal properties, emphasizing its connections with the classical sharp operator and the ω-local function.

In Section 3, using this operator, we define a Kuratowski-type closure $\mathrm{Cl}^{\sharp}_{\omega}$ and analyze the topology it induces, proving that this topology is strictly finer than $\mathcal{T}^{\sharp}$. Furthermore, we examine the relationships between $\mathcal{T}^{\sharp}_{\omega}$ and other related topologies, highlighting both implications and independence results among them.

In Subsection 3.1, we present some applications of the operator $\omega^{\sharp}$.

Finally, in Section 4, we define the notions of ω^{*}-continuity and $\omega^{\sharp}$-continuity for functions; and examine their relationships with other established forms of continuity, such as ω-continuity and $\sharp$-continuity.

The $\omega^{\sharp}$-operator modifies the classical sharp operator by replacing open neighborhoods with the broader class of ω-open neighborhoods, while preserving the ideal-based inclusion condition. Consequently, the $\omega^{\sharp}$-structure extends the classical sharp structure. Although it is not intrinsically equivalent to the ω-local function, it admits a representation in terms of the annihilator ideal.

In this study, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the sets of integers, rational numbers, and real numbers, respectively.

Definition 1.1. *[21] Let $\mathbb{X}$ be a nonempty set and $H \subseteq \mathbb{X}$ with $H \neq \emptyset$. The* ideal generated by H *is defined as* $\Im(H) = \{ J \subseteq \mathbb{X} \mid J \subseteq H \}$.

Theorem 1.2. *[22] Let $(\mathbb{X}, \mathcal{T}, \Im)$ be an ideal top-space and let $H, G \subseteq \mathbb{X}$. Then the following hold.*

(1) $(H^{*}_{\omega})^{*}_{\omega} \subseteq H^{*}_{\omega}$,
(2) $(H \cup G)^{*}_{\omega} = H^{*}_{\omega} \cup G^{*}_{\omega}$,
(3) $H^{*}_{\omega} \subseteq \mathrm{Cl}_{\omega}(H)$,
(4) $H^{*}_{\omega} \setminus G^{*}_{\omega} \subseteq (H \setminus G)^{*}_{\omega}$.

Lemma 1.3. *[21] Let $\Im$ be an ideal on $\mathbb{X}$, and let $\mathcal{K} \subseteq 2^{\mathbb{X}}$. Define* $(\Im : \mathcal{K}) = \{H \subseteq \mathbb{X} \mid (H \cap K \in \Im)(\ \forall K \in \mathcal{K})\}$*. Then $(\Im : \mathcal{K})$ is itself an ideal on $\mathbb{X}$.*

Definition 1.4. *[21] (Ideal quotient and annihilator) Let $\mathbb{X} \neq \emptyset$ and let $\Im$ be an ideal on $\mathbb{X}$. Suppose $\mathcal{K} \subseteq 2^{\mathbb{X}}$.*

(1) The quotient of the ideal $\Im$ by $\mathcal{K}$ is the collection

$$(\Im : \mathcal{K}) = \{H \subseteq \mathbb{X} \mid H \cap K \in \Im \text{ for every } K \in \mathcal{K}\}.$$

(2) In the special case where $\Im = \{\emptyset\}$, the quotient $(\{\emptyset\} : \mathcal{K})$ is called the annihilator of $\mathcal{K}$ relative to $\mathbb{X}$, denoted by $\mathcal{ANN}(\mathcal{K}) = (\{\emptyset\} : \mathcal{K})$.
(3) For a singleton subset $H \subseteq \mathbb{X}$, we write $\mathcal{ANN}_H = \mathcal{ANN}(\{H\})$ for brevity.

Definition 1.5. *[21] Let $\mathbb{X} \neq \emptyset$ and $\Im \subseteq 2^{\mathbb{X}}$. The ideal $\Im$ is called* faithful *if $\mathcal{ANN}(\Im) = \{\emptyset\}$.*

Lemma 1.6. *[21] Let $\Im$ be an ideal on $\mathbb{X}$. Then $\Im \cap \mathcal{ANN}(\Im) = \{\emptyset\}$.*

Theorem 1.7. *[21] A proper ideal $\Im$ on a non-empty set $\mathbb{X}$ is maximal if and only if for every $H \subseteq \mathbb{X}$, either $H \in \Im$ or $\mathbb{X} \setminus H \in \Im$.*

Corollary 1.8. *[21] Let $\mathbb{X} \neq \emptyset$ with an ideal $\Im$. Then the following hold.*

(1) For every $H \subseteq \mathbb{X}$, one has $\mathcal{ANN}_H = \mathcal{ANN}(\Im(H))$.
(2) If $\Im$ is a minimal ideal, then $\mathcal{ANN}(\Im)$ is a maximal ideal.

2. Basic properties of the $\omega^\sharp$-operator

Definition 2.1. *Let $(\mathbb{X}, \mathcal{T}, \Im)$ be an ideal top-space. For any subset $H \subseteq \mathbb{X}$, define the operator* $(\cdot)^\sharp_\omega : 2^{\mathbb{X}} \longrightarrow 2^{\mathbb{X}}$ *by* $H^\sharp_\omega(\Im, \mathcal{T}) = \left\{x \in \mathbb{X} \;\middle|\; \forall W \in \mathcal{T}_\omega(x),\ \exists J \in \Im \setminus \{\emptyset\}\ (J \subseteq W \cap H)\right\}$*, where* $\mathcal{T}_\omega(x) = \{W \in \mathcal{T}_\omega \mid x \in W\}$*. The set $H^\sharp_\omega(\Im, \mathcal{T})$, also denoted by $H^\sharp_\omega$, is called the $\omega^\sharp$-operator of H relative to the ideal $\Im$ and the topology $\mathcal{T}$. When the context is clear, we simply write $H^\sharp_\omega$.*

Corollary 2.2. *Let $(\mathbb{X}, \mathcal{T}, \Im)$ be an ideal top-space and let $H \subseteq \mathbb{X}$. Then $H^\sharp_\omega \subseteq H^\sharp$.*

Proof. Since every $\mathcal{T}$-open is also $\mathcal{T}_\omega$-open, the inclusion $H^\sharp_\omega \subseteq H^\sharp$ follows immediately by definition. □

The next example illustrates that, in general, the containment $H^\sharp \subseteq H^\sharp_\omega$ fails to be valid.

Example 2.3. *Let $\mathbb{R}$ be equipped with the indiscrete topology $\mathcal{T}_{\text{ind}} = \{\emptyset, \mathbb{R}\}$, and set $\mathcal{T} = \mathcal{T}_{\text{ind}}$.*

Then a subset $W \subseteq \mathbb{R}$ is ω-open if and only if $\mathbb{R} \setminus W$ is countable. Hence,

$$\mathcal{T}_\omega = \{W \subseteq \mathbb{R} : \mathbb{R} \setminus W \text{ is countable}\} \cup \{\emptyset\}.$$

Define an ideal on $\mathbb{R}$ by $\Im = \Im(\{0, \sqrt{2}\}) = \{\emptyset, \{0\}, \{\sqrt{2}\}, \{0, \sqrt{2}\}\}$. Let $H = \mathbb{P}$ be the set of irrational numbers.

Step 1: Computation of $H^\sharp$. Since the only nonempty open set is $\mathbb{R}$, for every $x \in \mathbb{R}$ and every nonempty open neighborhood U of x, we have $U = \mathbb{R}$. Moreover, $\{\sqrt{2}\} \subset \mathbb{R} \cap H$, thus every point of

$\mathbb{R}$ satisfies the sharp condition, and consequently $H^\sharp = \mathbb{R}$.
Step 2: Computation of $H^\sharp_\omega$. Let $x \in \mathbb{R}$. We analyze whether $x \in H^\sharp_\omega$. Indeed, observe that the set $\mathbb{R} \setminus \{\sqrt{2}\}$ is ω-open. Moreover, for every $J \in \Im \setminus \{\emptyset\}$, we have

$$[(\mathbb{R} \setminus \{\sqrt{2}\}) \cap H] \cap J = \emptyset \quad \text{or equivalently} \quad (\mathbb{R} \setminus \{\sqrt{2}\}) \cap H \in \mathcal{ANN}(\Im).$$

Consequently, for each $x \in \mathbb{R} \setminus \{\sqrt{2}\}$, it follows that $x \notin H^\sharp_\omega$. In particular, for $x = \sqrt{2}$ and any $W \in \mathcal{T}_\omega$ containing $\sqrt{2}$, we obtain $\{\sqrt{2}\} \subseteq \mathbb{P} \cap W$. Hence, $H^\sharp_\omega = \{\sqrt{2}\}$.
Step 3: Comparison. We conclude that $H^\sharp = \mathbb{R}$ and $H^\sharp_\omega = \{\sqrt{2}\}$; so that $H^\sharp \nsubseteq H^\sharp_\omega$.

2.1. Remarks

Remark 2.4. *From Example 2.3, we have $H = \mathbb{P}$, the set of irrational numbers, and $H^\sharp_\omega = \{\sqrt{2}\}$. Hence, $H \nsubseteq H^\sharp_\omega$.*

Next, let $(\mathbb{R}, \mathcal{T})$ be as defined in Example 2.3, let $\Im = \Im(\mathbb{Z})$ be the ideal on $\mathbb{R}$ generated by $\mathbb{Z}$, and consider the set $H = \mathbb{R} \setminus \mathbb{Z}$. For every $W \in \mathcal{T}_\omega$, we obtain ($H \cap W \in \mathcal{ANN}(\Im)$, which implies that $H^\sharp_\omega = \emptyset$. Consequently, $H \nsubseteq H^\sharp_\omega$.

Finally, consider the ideal $\mathcal{K} = \{H \subseteq \mathbb{R} : 1 \notin H\}$. Then let $H = \mathbb{R} \setminus \{1\}$. It follows that
$H^\sharp_\omega = \mathbb{R}$, since

$$\forall x \in \mathbb{R},\ \forall W \in \mathcal{T}_\omega(x),\ \exists K \in \mathcal{K} \setminus \{\emptyset\} \text{ such that } K \subseteq W \cap (\mathbb{R} \setminus \{1\}),$$

and hence, $H^\sharp_\omega \nsubseteq H$. Therefore, in general, there is no inclusion relationship between a set H and its associated set $H^\sharp_\omega$.

Remark 2.5. *Consider a topological space $(\mathbb{X}, \mathcal{T})$ and two ideals $\mathcal{K}$ and $\Im$ on $\mathbb{X}$. Assume that $\mathcal{K} \subseteq \Im$. Then, for any $H \subseteq \mathbb{X}$, the following implication holds:*

$$\mathcal{K} \subseteq \Im \implies H^\sharp_\omega(\mathcal{K}, \mathcal{T}) \subseteq H^\sharp_\omega(\Im, \mathcal{T}).$$

Remark 2.6. *In general, there is no intrinsic correspondence between the ω-local function and the $\omega^\sharp$-operator. This lack of dependence can be demonstrated through the following construction.*

Let $(\mathbb{X}, \mathcal{U})$ be the usual topological space where $\mathbb{X} = \mathbb{R}$. Let $\Im = \Im(\mathbb{Q})$ be the ideal on $\mathbb{R}$, and consider the set $F = \mathbb{R} \setminus \mathbb{Q}$. Then $F \cap W \in \mathcal{ANN}(\Im)$ for every $W \in \mathcal{T}_\omega$.

In this setting, one observes that

$$F^\sharp_\omega(\Im, \mathcal{T}) = \emptyset \quad \text{while} \quad F^*_\omega(\Im, \mathcal{T}) = \mathbb{R}.$$

Conversely, since $F \in \mathcal{ANN}(\Im)$, the behavior of the corresponding operators is reversed.

Indeed,

$$(\mathbb{X} \setminus F)^\sharp_\omega(\Im, \mathcal{T}) = \mathbb{Q} \quad \text{and} \quad (\mathbb{X} \setminus F)^*_\omega(\Im, \mathcal{T}) = \emptyset.$$

This example illustrates that the two operators behave independently.

Based on Theorem 1.7, we obtain the following corollary.

Corollary 2.7. *Let $(\mathbb{X}, \mathcal{T}, \Im)$ be an ideal top-space. If the ideal $\Im$ is maximal, then for every subset $H \subseteq \mathbb{X}$, we have either $H^*_\omega = \emptyset$ or $(\mathbb{X} \setminus H)^*_\omega = \emptyset$.*

Theorem 2.8. *Let $(\mathbb{X}, \mathcal{T}, \Im)$ be an ideal top-space and let $H \subseteq \mathbb{X}$. If $\Im$ is a maximal ideal, then H is either $\mathcal{T}_\omega^*$-closed or $\mathcal{T}_\omega^*$-open.*

Proof. Let $\Im$ be a maximal ideal on $\mathbb{X}$, and consider an arbitrary subset $H \subseteq \mathbb{X}$. Because $\Im$ is maximal, it follows from Theorem 1.7, that

$$H \in \Im \quad \vee \quad \mathbb{X} \setminus H \in \Im.$$

Case 1. If $H \in \Im$, then $H_\omega^* = \emptyset$, $\mathrm{Cl}_\omega^*(H) = H$. Thus, H is $\mathcal{T}_\omega^*$-closed, and hence, $\mathbb{X} \setminus H$ is $\mathcal{T}_\omega^*$-open.
Case 2. If $\mathbb{X} \setminus H \in \Im$, then $(\mathbb{X} \setminus H)_\omega^* = \emptyset$, $\mathrm{Cl}_\omega^*(\mathbb{X} \setminus H) = \mathbb{X} \setminus H$. Thus, $\mathbb{X} \setminus H$ is $\mathcal{T}_\omega^*$-closed, and hence, H is $\mathcal{T}_\omega^*$-open. Therefore, H is either $\mathcal{T}_\omega^*$-closed or H is $\mathcal{T}_\omega^*$-open. □

Theorem 2.9. *Let $(\mathbb{X}, \mathcal{T}, \Im)$ be an ideal top-space. If the ideal $\Im$ is maximal, then the space $(\mathbb{X}, \mathcal{T}_\omega^*)$ satisfies the T_0 separation axiom.*

Proof. Let $\alpha, \beta \in \mathbb{X}$ with $\alpha \neq \beta$. Define $H = \{\alpha\}$. Since the ideal $\Im$ is maximal, we have
$H \in \Im \ \vee \ (\mathbb{X} \setminus H) \in \Im$.
Case 1: $\mathbb{X} \setminus H \in \Im$. By Theorem 2.8, $\mathbb{X} \setminus H \in \Im \ \Rightarrow \ H \in \mathcal{T}_\omega^*$. Hence, $H \in \mathcal{T}_\omega^* \ \wedge \ (\alpha \in H \wedge \beta \notin H)$.
Case 2: $H \in \Im$. Then, $H \in \Im \ \Rightarrow \ \mathbb{X} \setminus H \in \mathcal{T}_\omega^*$. Hence, $(\mathbb{X} \setminus H) \in \mathcal{T}_\omega^* \ \wedge \ (\beta \in \mathbb{X} \setminus H \wedge \alpha \notin \mathbb{X} \setminus H)$.

Thus in all cases, $\exists U \in \mathcal{T}_\omega^* \ [(\alpha \in U \wedge \beta \notin U) \ \vee \ (\beta \in U \wedge \alpha \notin U)]$. Hence, $(\mathbb{X}, \mathcal{T}_\omega^*)$ satisfies the T_0 separation axiom. □

The following theorem establishes a connection between the ω-localized function and the $\omega^\sharp$-operator.

Theorem 2.10. *For any ideal top-space $(\mathbb{X}, \mathcal{T}, \Im)$ and any $H \subseteq X$, we have*

$$H_\omega^\sharp(\Im, \mathcal{T}) = H_\omega^*(\mathcal{ANN}(\Im), \mathcal{T}).$$

Proof. Let $H \subseteq X$. Then, for any $x \in \mathbb{X}$,

$$\begin{aligned} x \in H_\omega^\sharp(\Im, \mathcal{T}) &\Longleftrightarrow \forall W \in \mathcal{T}_\omega(x),\ \exists J \in \Im \setminus \{\emptyset\},\ J \subseteq W \cap H \\ &\Longleftrightarrow \forall W \in \mathcal{T}_\omega(x),\ \exists J \in \Im \setminus \{\emptyset\},\ J \cap (W \cap H) \neq \emptyset \\ &\Longleftrightarrow \forall W \in \mathcal{T}_\omega(x),\ W \cap H \notin \mathcal{ANN}(\Im) \\ &\Longleftrightarrow x \in H_\omega^*(\mathcal{ANN}(\Im), \mathcal{T}). \end{aligned}$$

Hence, we conclude; $H_\omega^\sharp(\Im, \mathcal{T}) = H_\omega^*(\mathcal{ANN}(\Im), \mathcal{T})$. □

The following lemma is essential for proving part (10) of the next theorem.

Lemma 2.11. *Let $(\mathbb{X}, \mathcal{T}, \Im)$ be an ideal top-space and let $H \subseteq \mathbb{X}$. If H is ω-closed, then $H_\omega^\sharp \subseteq H$.*

Proof. Let H be ω-closed and let $x \in H_\omega^\sharp$. Suppose that $x \notin H$. Since H is ω-closed, $\mathbb{X} \setminus H \in \mathcal{T}_\omega(x)$. Hence, taking $W = \mathbb{X} \setminus H$, there exists $J \in \Im \setminus \{\emptyset\}$ such that $J \subseteq H \cap (\mathbb{X} \setminus H)$. But $H \cap (\mathbb{X} \setminus H) = \emptyset$, hence, $J = \emptyset$, a contradiction since $J \neq \emptyset$. This contradicts the definition of $x \in H_\omega^\sharp$. Therefore, $x \in H$, and hence, $H_\omega^\sharp \subseteq H$. □

Theorem 2.12. *Consider an ideal top-space $(\mathbb{X}, \mathcal{T}, \Im)$. For any subsets H and G of $\mathbb{X}$, the operator $\omega^\sharp$ satisfies the following properties:*

(1) $H \subseteq G \Rightarrow H^\sharp_\omega \subseteq G^\sharp_\omega$;
(2) $H^\sharp_\omega = \mathrm{Cl}_\omega(H^\sharp_\omega) \subseteq \mathrm{Cl}_\omega(H)$ and $H^\sharp_\omega$ is ω-closed in $(\mathbb{X}, \mathcal{T})$;
(3) $(H \cap G)^\sharp_\omega \subseteq H^\sharp_\omega \cap G^\sharp_\omega$;
(4) $(H \cup G)^\sharp_\omega = H^\sharp_\omega \cup G^\sharp_\omega$;
(5) $H^\sharp_\omega \setminus G^\sharp_\omega \subseteq (H \setminus G)^\sharp_\omega$;
(6) If $H \in \mathcal{ANN}(\Im)$, then $H^\sharp_\omega = \emptyset$;
(7) If $H \in \mathcal{ANN}(\Im)$, then $(H \cup G)^\sharp_\omega = (G \setminus H)^\sharp_\omega$;
(8) If $\Im$ is faithful, then $H^\sharp_\omega = \mathrm{Cl}_\omega(H)$;
(9) If $H \in \mathcal{T}_\omega$, then $H \cap G^\sharp_\omega \subseteq (H \cap G)^\sharp_\omega$;
(10) $(H^\sharp_\omega)^\sharp_\omega \subseteq H^\sharp_\omega$.

Proof. We verify the properties of the operator $\omega^\sharp$ as follows:
(1) Let $x \in H^\sharp_\omega$

$$\begin{aligned} x \in H^\sharp_\omega &\Rightarrow \forall\, W \in \mathcal{T}_\omega(x),\ \exists\, J \in \Im,\ J \neq \emptyset,\ \text{satisfying } J \subseteq W \cap H \\ &\Rightarrow H \subseteq G \text{ implies } J \subseteq W \cap G \\ &\Rightarrow x \in G^\sharp_\omega \\ &\Rightarrow H^\sharp_\omega \subseteq G^\sharp_\omega. \end{aligned}$$

(2) Clearly, $H^\sharp_\omega \subseteq \mathrm{Cl}_\omega(H^\sharp_\omega)$. To prove the reverse inclusion $\mathrm{Cl}_\omega(H^\sharp_\omega) \subseteq H^\sharp_\omega$, let $x \in \mathrm{Cl}_\omega(H^\sharp_\omega)$. Then, for every $W \in \mathcal{T}_\omega(x)$, $W \cap H^\sharp_\omega \neq \emptyset$, so there exists $z \in W \cap H^\sharp_\omega$. Since $z \in H^\sharp_\omega$, by definition, $(\forall V \in \mathcal{T}_\omega(z))\,(\exists J_V \in \Im \setminus \{\emptyset\})\quad$ such that $\quad J_V \subseteq V \cap H$.

Now, fix $W \in \mathcal{T}_\omega(x)$ and choose $z \in W \cap H^\sharp_\omega$. Since $z \in W$ and W is a $\mathcal{T}_\omega$-neighborhood of z, we may take $V := W$ in the defining condition of $z \in H^\sharp_\omega$. Hence, there exists an ideal element

$$J_W \in \Im \setminus \{\emptyset\} \quad \text{such that} \quad J_W \subseteq W \cap H.$$

Therefore,

$$(\forall W \in \mathcal{T}_\omega(x))\,(\exists J_W \in \Im \setminus \{\emptyset\}) \quad \text{such that} \quad J_W \subseteq W \cap H,$$

which implies $x \in H^\sharp_\omega$. Hence, $\mathrm{Cl}_\omega(H^\sharp_\omega) \subseteq H^\sharp_\omega$. By combining both inclusions, it follows that $\mathrm{Cl}_\omega(H^\sharp_\omega) = H^\sharp_\omega$.

Next, let $x \notin \mathrm{Cl}_\omega(H)$. Then there exists $W \in \mathcal{T}_\omega(x)$ such that $W \cap H = \emptyset$. Hence, $W \cap H \in \mathcal{ANN}(\Im)$, which implies $x \notin H^\sharp_\omega$. Therefore, $H^\sharp_\omega = \mathrm{Cl}_\omega(H^\sharp_\omega) \subseteq \mathrm{Cl}_\omega(H)$.
(3) This follows directly from (1).
(4) From (1), we immediately obtain $(H \cup G)^\sharp_\omega \supseteq H^\sharp_\omega \cup G^\sharp_\omega$. We now prove the reverse inclusion.

Let $x \notin H^\sharp_\omega \cup G^\sharp_\omega$. Then $x \notin H^\sharp_\omega$ and $x \notin G^\sharp_\omega$. Hence, there exist $W_1, W_2 \in \mathcal{T}_\omega(x)$ such that

$$W_1 \cap H \in \mathcal{ANN}(\Im), \quad W_2 \cap G \in \mathcal{ANN}(\Im).$$

Since $\mathcal{ANN}(\Im)$ is an additive ideal, we obtain:

$$(W_1 \cap H) \cup (W_2 \cap G) \in \mathcal{ANN}(\Im).$$

Moreover,

$$\begin{aligned}(W_1 \cap H) \cup (W_2 \cap G) &= [(W_1 \cap H) \cup W_2] \cap [(W_1 \cap H) \cup G] \\ &= (W_1 \cup W_2) \cap (H \cup W_2) \cap (W_1 \cup G) \cap (H \cup G) \\ &\supseteq (W_1 \cap W_2) \cap (H \cup G).\end{aligned}$$

By the hereditary property of $\mathcal{ANN}(\Im)$, it follows that $(W_1 \cap W_2) \cap (H \cup G) \in \mathcal{ANN}(\Im)$.

Thus, $[(W_1 \cap W_2) \cap (H \cup G)] \cap J = \emptyset, \quad \forall J \in \Im$. Since $W_1 \cap W_2 \in \mathcal{T}_\omega(x)$, we conclude that $x \notin (H \cup G)^\sharp_\omega$. Hence, $(H \cup G)^\sharp_\omega \subseteq H^\sharp_\omega \cup G^\sharp_\omega$. Therefore, $(H \cup G)^\sharp_\omega = H^\sharp_\omega \cup G^\sharp_\omega$.

(5) Since $H = (H \setminus G) \cup (H \cap G)$, we have

$$\begin{aligned} H^\sharp_\omega(\Im, \mathcal{T}) &= (H \setminus G)^\sharp_\omega(\Im, \mathcal{T}) \ \cup \ (H \cap G)^\sharp_\omega(\Im, \mathcal{T}) && \text{by (4)}, \\ &\subseteq (H \setminus G)^\sharp_\omega(\Im, \mathcal{T}) \ \cup \ G^\sharp_\omega(\Im, \mathcal{T}) && \text{by (1)}, \\ \Longrightarrow \quad H^\sharp_\omega(\Im, \mathcal{T}) \setminus G^\sharp_\omega(\Im, \mathcal{T}) &\subseteq (H \setminus G)^\sharp_\omega(\Im, \mathcal{T}) \setminus G^\sharp_\omega(\Im, \mathcal{T}). \end{aligned} \tag{2.1}$$

$$\begin{aligned} &\text{By axiom (1),} \ \ (H \setminus G)^\sharp_\omega(\Im, \mathcal{T}) \subseteq H^\sharp_\omega(\Im, \mathcal{T}), \\ \Longrightarrow \quad &(H \setminus G)^\sharp_\omega(\Im, \mathcal{T}) \setminus G^\sharp_\omega(\Im, \mathcal{T}) \subseteq H^\sharp_\omega(\Im, \mathcal{T}) \setminus G^\sharp_\omega(\Im, \mathcal{T}). \end{aligned} \tag{2.2}$$

Combining (2.1) and (2.2), we get,

$$\begin{aligned} H^\sharp_\omega(\Im, \mathcal{T}) \setminus G^\sharp_\omega(\Im, \mathcal{T}) &= (H \setminus G)^\sharp_\omega(\Im, \mathcal{T}) \setminus G^\sharp_\omega(\Im, \mathcal{T}) \\ &\subseteq (H \setminus G)^\sharp_\omega(\Im, \mathcal{T}). \end{aligned}$$

Equivalent expression via Theorems 1.2 (4) and 2.10. Using the canonical forms, we also have

$$\begin{aligned} H^\sharp_\omega(\Im, \mathcal{T}) \setminus G^\sharp_\omega(\Im, \mathcal{T}) &= H^*_\omega(\mathcal{ANN}(\Im), \mathcal{T}) \setminus G^*_\omega(\mathcal{ANN}(\Im), \mathcal{T}) \quad \text{(Thm. 2.10)} \\ &\subseteq (H \setminus G)^*_\omega(\mathcal{ANN}(\Im), \mathcal{T}) \quad \text{(Thm. 1.2 (4))} \\ &= (H \setminus G)^\sharp_\omega(\Im, \mathcal{T}). \end{aligned}$$

This completes the proof.

(6) It follows directly from the definition of $\mathcal{ANN}(\Im)$.

(7)

$$\begin{aligned} (H \cup G)^\sharp_\omega &= [(H \setminus G) \cup (H \cap G) \cup (G \setminus H)]^\sharp_\omega \\ &\overset{by(4)}{=} (H \setminus G)^\sharp_\omega \cup (H \cap G)^\sharp_\omega \cup (G \setminus H)^\sharp_\omega. \end{aligned}$$

Since $H \in \mathcal{ANN}(\Im)$, by (6), we get

$$= (H \cup G)^\sharp_\omega = \emptyset \cup \emptyset \cup (G \setminus H)^\sharp_\omega = (G \setminus H)^\sharp_\omega.$$

(8) Since $\Im$ is assumed to be faithful, it follows that $\mathcal{ANN}(\Im) = \{\emptyset\}$. From Theorem 2.10, we deduce that

$$H^\sharp_\omega(\Im, \mathcal{T}) = H^*_\omega(\{\emptyset\}, \mathcal{T}) = \{x \in \mathbb{X} \mid (\forall W \in \mathcal{T}_\omega(x)) \, (W \cap H \neq \emptyset)\} = \mathrm{Cl}_\omega(H).$$

(9) Let $x \in H \cap G^{\sharp}_{\omega}$. Then,

$$x \in H \cap G^{\sharp}_{\omega} \Rightarrow \begin{cases} x \in H, \\ x \in G^{\sharp}_{\omega} \Rightarrow (\forall W \in \mathcal{T}_{\omega}(x))(\exists J \in \Im \setminus \{\emptyset\})\,(J \subseteq W \cap G). \end{cases}$$

$$\begin{aligned} H \in \mathcal{T}_{\omega}(x) &\Rightarrow W \cap H \in \mathcal{T}_{\omega}(x), \exists I \in \Im \setminus \{\emptyset\}, \\ &\Rightarrow I \subseteq (W \cap H) \cap G = W \cap (H \cap G), \\ &\Rightarrow x \in (H \cap G)^{\sharp}_{\omega}, \\ &\Rightarrow H \cap G^{\sharp}_{\omega} \subseteq (H \cap G)^{\sharp}_{\omega}. \end{aligned}$$

(10) Let $H \subseteq \mathbb{X}$.

$$H \subseteq \mathbb{X} \overset{(2)}{\Rightarrow} H^{\sharp}_{\omega} \text{ is } \omega\text{-closed} \overset{\text{Lemma 2.11}}{\Rightarrow} (H^{\sharp}_{\omega})^{\sharp}_{\omega} \subseteq H^{\sharp}_{\omega}.$$

□

Example 2.13. *Let $\mathbb{R}$ be equipped with the usual topology $\mathcal{U}$, and let $\Im = \{\emptyset, \{0\}\}$ be an ideal on $\mathbb{R}$. Consider the operator $\omega^{\sharp}$.*

For any $x \in \mathbb{R}$ with $x \neq 0$, there exists $\varepsilon > 0$ such that

$$(x - \varepsilon, x + \varepsilon) \in \mathcal{U}_{\omega}(x) \quad \text{and} \quad 0 \notin (x - \varepsilon, x + \varepsilon).$$

Hence,

$$[\mathbb{R} \cap (x - \varepsilon, x + \varepsilon)] \cap \{0\} = \emptyset,$$

which implies $x \notin \mathbb{R}^{\sharp}_{\omega}$. Therefore, $\mathbb{R}^{\sharp}_{\omega} = \{0\}$. This shows that $\omega^{\sharp}$ is not extensive, since $\mathbb{R}^{\sharp}_{\omega} \not\supseteq \mathbb{R}$.

Lemma 2.14. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space and let $\Im, \mathcal{K} \subseteq 2^{\mathbb{X}}$ be ideals on $\mathbb{X}$. Then, for any subset $H \subseteq X$, $H^{*}_{\omega}(\Im \cap \mathcal{K}, \mathcal{T}) = H^{*}_{\omega}(\Im, \mathcal{T}) \cup H^{*}_{\omega}(\mathcal{K}, \mathcal{T})$.*

Proof. ($\subseteq$) Let $x \in H^{*}_{\omega}(\Im \cap \mathcal{K}, \mathcal{T})$. If $x \notin H^{*}_{\omega}(\Im, \mathcal{T})$ and $x \notin H^{*}_{\omega}(\mathcal{K}, \mathcal{T})$, then there exist $E, F \in \mathcal{T}_{\omega}(x)$ such that $E \cap H \in \Im$ and $F \cap H \in \mathcal{K}$. Hence, $(E \cap F) \cap H \in \Im \cap \mathcal{K}$, a contradiction. Thus, $x \in H^{*}_{\omega}(\Im, \mathcal{T}) \cup H^{*}_{\omega}(\mathcal{K}, \mathcal{T})$.

($\supseteq$) Let $x \in H^{*}_{\omega}(\Im, \mathcal{T}) \cup H^{*}_{\omega}(\mathcal{K}, \mathcal{T})$. Without loss of generality, assume $x \in H^{*}_{\omega}(\Im, \mathcal{T})$. Then for every $W \in \mathcal{T}_{\omega}(x)$, $W \cap H \notin \Im$. Since $\Im \cap \mathcal{K} \subseteq \Im$, it follows that $W \cap H \notin \Im \cap \mathcal{K}$. Thus, $x \in H^{*}_{\omega}(\Im \cap \mathcal{K}, \mathcal{T})$. The case $x \in H^{*}_{\omega}(\mathcal{K}, \mathcal{T})$ is similar. Hence, $H^{*}_{\omega}(\Im \cap \mathcal{K}, \mathcal{T}) = H^{*}_{\omega}(\Im, \mathcal{T}) \cup H^{*}_{\omega}(\mathcal{K}, \mathcal{T})$. □

Theorem 2.15. *In an ideal top-space $(\mathbb{X}, \mathcal{T}, \Im)$, the ω-closure of any subset $H \subseteq \mathbb{X}$ decomposes as the union of the sets $H^{\sharp}_{\omega}$ and H^{*}_{ω}. Explicitly,*

$$\mathrm{Cl}_{\omega}(H) = H^{\sharp}_{\omega}(\Im, \mathcal{T}) \cup H^{*}_{\omega}(\Im, \mathcal{T}).$$

Proof. By Theorem 2.10,

$$H^{\sharp}_{\omega}(\Im, \mathcal{T}) \cup H^{*}_{\omega}(\Im, \mathcal{T}) = H^{*}_{\omega}(\mathcal{ANN}(\Im), \mathcal{T}) \cup H^{*}_{\omega}(\Im, \mathcal{T}).$$

By Lemma 2.14,

$$\begin{aligned} &= H^*_\omega(\mathcal{ANN}(\mathfrak{I}) \cap \mathfrak{I}, \mathcal{T}), \text{ since } \mathcal{ANN}(\mathfrak{I}) \cap \mathfrak{I} = \emptyset \text{ (Lemma 1.6)},\\ &= H^*_\omega(\{\emptyset\}, \mathcal{T}) = \{x \in \mathbb{X} : W \cap H \neq \emptyset \text{ for every } W \in \mathcal{T}_\omega(x)\} = \mathrm{Cl}_\omega(H). \end{aligned}$$

□

The following example illustrates the equality stated in Theorem 2.15.

Example 2.16. *Referring to Example 2.13, we observe that* $\mathrm{Cl}_\omega(\mathbb{Q}) = \mathbb{Q}$*, since* $\mathbb{R} \setminus \mathbb{Q} \in \mathcal{T}_\omega$*.*

Furthermore, we have $\mathbb{Q}^\sharp_\omega = \{0\}$. In addition, for every $x \in \mathbb{Q}$ and each $W \in \mathcal{T}_\omega(x)$, it holds that $\mathbb{Q} \cap W \notin \mathfrak{I}$. On the other hand, since $\mathbb{R} \setminus \mathbb{Q} \in \mathcal{T}_\omega$ and $\mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q}) = \emptyset \in \mathfrak{I}$, it follows that $\mathbb{Q}^*_\omega = \mathbb{Q}$.

Consequently, by Theorem 2.15, we conclude that

$$\mathrm{Cl}_\omega(\mathbb{Q}) = \mathbb{Q}^\sharp_\omega \cup \mathbb{Q}^*_\omega = \{0\} \cup \mathbb{Q} = \mathbb{Q}.$$

Corollary 2.17. *Let* $(\mathbb{X}, \mathcal{T}, \mathfrak{I})$ *be an ideal top-space and let* $H \subseteq \mathbb{X}$*. Then,*

(1) $H \in \mathfrak{I} \implies H^\sharp_\omega = \mathrm{Cl}_\omega(H)$.
(2) $H \in \mathcal{ANN}(\mathfrak{I}) \implies H^*_\omega = \mathrm{Cl}_\omega(H)$.

3. On the $\omega^\sharp$-closure operator and the associated sharp topology

Definition 3.1. *Let* $(\mathbb{X}, \mathcal{T}, \mathfrak{I})$ *be an ideal top-space. The* $\omega^\sharp$-closure *operator,* $\mathrm{Cl}^\sharp_\omega : 2^{\mathbb{X}} \to 2^{\mathbb{X}}$ *is defined by* $\forall H \subseteq \mathbb{X},\ \mathrm{Cl}^\sharp_\omega(H) = H \cup H^\sharp_\omega$*.*

Theorem 3.2. *Let* $(\mathbb{X}, \mathcal{T}, \mathfrak{I})$ *be an ideal top-space, and* $H, G \subseteq \mathbb{X}$*. Then,* $\mathrm{Cl}^\sharp_\omega$ *satisfies.*

(1) $\mathrm{Cl}^\sharp_\omega(\emptyset) = \emptyset$.
(2) $\mathrm{Cl}^\sharp_\omega(\mathbb{X}) = \mathbb{X}$.
(3) $H \subseteq \mathrm{Cl}^\sharp_\omega(H)$.
(4) $H \subseteq G \Rightarrow \mathrm{Cl}^\sharp_\omega(H) \subseteq \mathrm{Cl}^\sharp_\omega(G)$.
(5) $\mathrm{Cl}^\sharp_\omega(H \cup G) = \mathrm{Cl}^\sharp_\omega(H) \cup \mathrm{Cl}^\sharp_\omega(G)$.
(6) $\mathrm{Cl}^\sharp_\omega(\mathrm{Cl}^\sharp_\omega(H)) = \mathrm{Cl}^\sharp_\omega(H)$.
(7) If $H \in \mathcal{T}_\omega$, then $H \cap \mathrm{Cl}^\sharp_\omega(G) \subseteq \mathrm{Cl}^\sharp_\omega(H \cap G)$.

Proof. We verify the properties of $\mathrm{Cl}^\sharp_\omega$ as follows:

(1) By definition, $\emptyset^\sharp_\omega = \emptyset$, so

$$\mathrm{Cl}^\sharp_\omega(\emptyset) = \emptyset \cup \emptyset^\sharp_\omega = \emptyset.$$

(2) Since $\mathbb{X}^\sharp_\omega \subseteq \mathbb{X}$,

$$\mathrm{Cl}^\sharp_\omega(\mathbb{X}) = \mathbb{X} \cup \mathbb{X}^\sharp_\omega = \mathbb{X}.$$

(3) Clearly, $H \subseteq H \cup H^\sharp_\omega = \mathrm{Cl}^\sharp_\omega(H)$.
(4) If $H \subseteq G$, then $H^\sharp_\omega \subseteq G^\sharp_\omega$ (Theorem 2.12(1)), so

$$\mathrm{Cl}^\sharp_\omega(H) = H \cup H^\sharp_\omega \subseteq G \cup G^\sharp_\omega = \mathrm{Cl}^\sharp_\omega(G).$$

(5) For unions, using Theorem 2.12(4),

$$\begin{aligned}\mathrm{Cl}^{\sharp}_{\omega}(H \cup G) &= (H \cup G) \cup (H \cup G)^{\sharp}_{\omega} \\ &= (H \cup G) \cup (H^{\sharp}_{\omega} \cup G^{\sharp}_{\omega}) \\ &= (H \cup H^{\sharp}_{\omega}) \cup (G \cup G^{\sharp}_{\omega}) \\ &= \mathrm{Cl}^{\sharp}_{\omega}(H) \cup \mathrm{Cl}^{\sharp}_{\omega}(G).\end{aligned}$$

(6) Using the definition of $\mathrm{Cl}^{\sharp}_{\omega}$, we obtain

$$\begin{aligned}\mathrm{Cl}^{\sharp}_{\omega}(\mathrm{Cl}^{\sharp}_{\omega}(H)) &= \mathrm{Cl}^{\sharp}_{\omega}(H \cup H^{\sharp}_{\omega}) = (H \cup H^{\sharp}_{\omega}) \cup (H \cup H^{\sharp}_{\omega})^{\sharp}_{\omega} \\ &= (H \cup H^{\sharp}_{\omega}) \cup \left(H^{\sharp}_{\omega} \cup (H^{\sharp}_{\omega})^{\sharp}_{\omega}\right) && \text{(by Theorem 2.12(4))} \\ &= (H \cup H^{\sharp}_{\omega}) \cup H^{\sharp}_{\omega} && \text{(since } (H^{\sharp}_{\omega})^{\sharp}_{\omega} \subseteq H^{\sharp}_{\omega} \text{ by Theorem 2.12(10))} \\ &= H \cup H^{\sharp}_{\omega} \\ &= \mathrm{Cl}^{\sharp}_{\omega}(H).\end{aligned}$$

(7) If $H \in \mathcal{T}_{\omega}$, then by Theorem 2.12(9),

$$H \cap \mathrm{Cl}^{\sharp}_{\omega}(G) = (H \cap G) \cup (H \cap G^{\sharp}_{\omega}) \subseteq (H \cap G) \cup (H \cap G)^{\sharp}_{\omega} = \mathrm{Cl}^{\sharp}_{\omega}(H \cap G).$$

□

Proposition 3.3. *Let* $(\mathbb{X}, \mathcal{T}, \Im)$ *be an ideal top-space and* $H \subseteq \mathbb{X}$*. If* $H \subseteq H^{\sharp}_{\omega}$*, then the following properties hold.*

(1) $\mathrm{Cl}_{\omega}(H) = \mathrm{Cl}^{\sharp}_{\omega}(H)$;
(2) $\mathrm{Int}_{\omega}(\mathbb{X} \setminus H) = \mathrm{Int}^{\sharp}_{\omega}(\mathbb{X} \setminus H)$.

Proof. (1) By Theorem 2.12(2), $H^{\sharp}_{\omega} = \mathrm{Cl}_{\omega}(H^{\sharp}_{\omega}) \subseteq \mathrm{Cl}_{\omega}(H)$. Since $H \subseteq H^{\sharp}_{\omega}$, it follows that $\mathrm{Cl}_{\omega}(H) \subseteq \mathrm{Cl}_{\omega}(H^{\sharp}_{\omega})$. Therefore, $\mathrm{Cl}_{\omega}(H) = \mathrm{Cl}_{\omega}(H^{\sharp}_{\omega}) = \mathrm{Cl}^{\sharp}_{\omega}(H)$.

(2) From (1), we obtain $\mathbb{X} \setminus \mathrm{Cl}_{\omega}(H) = \mathbb{X} \setminus \mathrm{Cl}^{\sharp}_{\omega}(H) \Leftrightarrow \mathrm{Int}_{\omega}(\mathbb{X} \setminus H) = \mathrm{Int}^{\sharp}_{\omega}(\mathbb{X} \setminus H)$. Hence, the result follows. □

Corollary 3.4. *If* $(\mathbb{X}, \mathcal{T}, \Im)$ *is an ideal top-space, then for every* $H \subseteq \mathbb{X}$*,*

$$\mathrm{Cl}^{\sharp}_{\omega}(H) = H \cup H^{*}_{\omega}(\mathcal{ANN}(\Im), \mathcal{T}).$$

Remark 3.5. *By Theorem 3.2, the operator* $\mathrm{Cl}^{\sharp}_{\omega}(H) = H \cup H^{\sharp}_{\omega}$ *is a Kuratowski closure. Accordingly,*

$$\mathcal{T}^{\sharp}_{\omega} = \{ W \subseteq \mathbb{X} \mid \mathrm{Cl}^{\sharp}_{\omega}(\mathbb{X} \setminus W) = \mathbb{X} \setminus W \};$$

defines a topology on $\mathbb{X}$, called the $\omega^{\sharp}$-topology. When needed, we write $\mathcal{T}^{\sharp}_{\omega}(\Im)$.

Theorem 3.6. *Let* $(\mathbb{X}, \mathcal{T}, \Im)$ *be an ideal top-space. Then the inclusions* $\mathcal{T}_{\omega} \subseteq \mathcal{T}^{\sharp}_{\omega}$ *and* $\mathcal{T}^{\sharp} \subseteq \mathcal{T}^{\sharp}_{\omega}$ *hold.*

Proof. To show that $\mathcal{T}_\omega \subseteq \mathcal{T}^\sharp_\omega$, let $W \in \mathcal{T}_\omega$. Then, $(\forall x \in W)(\forall J \in \mathfrak{I} \setminus \{\varnothing\})\ [\, J \cap (W \cap (\mathbb{X} \setminus W)) = \emptyset\,]$. Hence $W \cap (\mathbb{X} \setminus W) \in \mathcal{ANN}(\mathfrak{I})$, it follows that

$$\Big[(\forall x \in W)\ (x \notin (\mathbb{X} \setminus W)^\sharp_\omega\Big] \Longrightarrow (\mathbb{X} \setminus W)^\sharp_\omega \subseteq \mathbb{X} \setminus W.$$

By the definition of the closure operator $\mathrm{Cl}^\sharp_\omega$, $\mathrm{Cl}^\sharp_\omega(\mathbb{X} \setminus W) = (\mathbb{X} \setminus W) \cup (\mathbb{X} \setminus W)^\sharp_\omega = \mathbb{X} \setminus W$. Therefore, $\mathbb{X} \setminus W$ is $\mathcal{T}^\sharp_\omega$-closed, and consequently $W \in \mathcal{T}^\sharp_\omega$. Hence, $\mathcal{T}_\omega \subseteq \mathcal{T}^\sharp_\omega$.

To show that $\mathcal{T}^\sharp \subseteq \mathcal{T}^\sharp_\omega$, let $W \in \mathcal{T}^\sharp$. Then $\mathbb{X} \setminus W$ is $\mathcal{T}^\sharp$-closed, so that $(\mathbb{X} \setminus W)^\sharp \subseteq \mathbb{X} \setminus W$.
By Corollary 2.2, $(\mathbb{X} \setminus W)^\sharp_\omega \subseteq (\mathbb{X} \setminus W)^\sharp \subseteq \mathbb{X} \setminus W$. Thus, $\mathrm{Cl}^\sharp_\omega(\mathbb{X} \setminus W) = (\mathbb{X} \setminus W) \cup (\mathbb{X} \setminus W)^\sharp_\omega = \mathbb{X} \setminus W$.
Therefore, $\mathbb{X} \setminus W$ is $\mathcal{T}^\sharp_\omega$-closed, and hence $W \in \mathcal{T}^\sharp_\omega$. It follows that $\mathcal{T}^\sharp \subseteq \mathcal{T}^\sharp_\omega$. □

Corollary 3.7. *Let* $(\mathbb{X}, \mathcal{T}, \mathfrak{I})$ *be an ideal top-space. Then,*

$$(\forall H \subseteq \mathbb{X})\ \Big(\mathrm{Cl}^\sharp_\omega(H) \subseteq \mathrm{Cl}^\sharp(H)\ \wedge\ \mathrm{Cl}^\sharp_\omega(H) \subseteq \mathrm{Cl}_\omega(H)\Big).$$

Remark 3.8. *The relationships induced by the definitions of the $\sharp$-topology and the $\omega^\sharp$-topology are summarized in the diagram below. The example that follows shows that these implications do not admit converses. Moreover, the notions of $\mathcal{T}^\sharp_\omega$-open sets and $\mathcal{T}^*_\omega$-open sets are independent, as illustrated in the next example.*

The implications and independence relations among the classes of $\mathcal{T}$-open, $\mathcal{T}_\omega$-open, $\mathcal{T}^\sharp$-open, $\mathcal{T}^\sharp_\omega$-open, and $\mathcal{T}^*_\omega$-open sets are summarized in Figure 1.

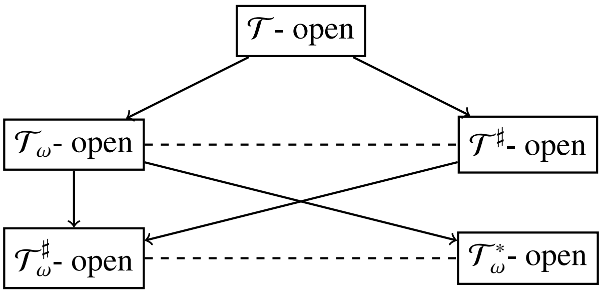

Figure 1. Diagram illustrating the relationships among various classes of open sets; dashed line denotes independence.

Example 3.9. *Let $\mathcal{T}$ be the indiscrete topology on $\mathbb{R}$; and let $\mathfrak{I} = \mathfrak{I}(\mathbb{Q})$ denote the ideal generated by $\mathbb{Q}$. From Example 2.3, it is evident that $W \subseteq \mathbb{R}$ is ω-open if and only if $\mathbb{R} \setminus W$ is a countable set.*

Since

$$\mathbb{R} \setminus \mathbb{Q} \in \mathcal{ANN}(\mathfrak{I}) \implies (\mathbb{R} \setminus \mathbb{Q})^\sharp = \emptyset,$$

it follows that $\mathbb{Q} \in \mathcal{T}^\sharp\ \wedge\ \mathbb{Q} \notin \mathcal{T}_\omega$. Moreover, since $\mathcal{T}^\sharp \subseteq \mathcal{T}^\sharp_\omega$, we obtain

$$\mathbb{Q} \in \mathcal{T}^\sharp_\omega\ \wedge\ \mathbb{Q} \notin \mathcal{T}_\omega.$$

On the other hand, since

$$\mathbb{R} \setminus \mathbb{Q} \in \mathcal{T}_\omega\ \wedge\ \mathbb{Q}^\sharp = \mathbb{R},$$

it follows that

$$\mathbb{R} \setminus \mathbb{Q} \notin \mathcal{T}^{\sharp}.$$

Hence, $\mathcal{T}_{\omega}$-open sets and $\mathcal{T}^{\sharp}$-open sets are independent concepts. Furthermore, since $\mathbb{R} \setminus \mathbb{Q} \in \mathcal{T}_{\omega} \ \wedge \ \mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q}) = \emptyset$, we obtain $\mathbb{Q}^{\sharp}_{\omega} \subseteq \mathbb{Q}$. Consequently,

$$\mathbb{R} \setminus \mathbb{Q} \in \mathcal{T}^{\sharp}_{\omega} \ \wedge \ \mathbb{R} \setminus \mathbb{Q} \notin \mathcal{T}^{\sharp}.$$

Example 3.10. *Let* $(\mathbb{R}, \mathcal{T}_{ind})$ *be an indiscrete topological space, and let* $\Im = \Im((0, \infty))$ *denote the ideal generated by the interval* $(0, \infty)$*. First, consider the set* $H = (0, \infty)$*. Then,*

$$\mathbb{R} \setminus H = (-\infty, 0] \in \mathcal{ANN}(\Im).$$

By Corollary 2.17(2), we obtain $(\mathbb{R} \setminus H)^{\sharp}_{\omega} = \emptyset$ and $(\mathbb{R} \setminus H)^{*}_{\omega} = \mathrm{Cl}_{\omega}(\mathbb{R} \setminus H) = \mathbb{R}$. Therefore, $H \in \mathcal{T}^{\sharp}_{\omega}$, while $H \notin \mathcal{T}^{*}_{\omega}$. Next, let $G = (-\infty, 0]$. Then,

$$\mathbb{R} \setminus G = (0, \infty) \in \Im.$$

Hence, by Corollary 2.17(1), $(\mathbb{R} \setminus G)^{*}_{\omega} = \emptyset$ and $(\mathbb{R} \setminus G)^{\sharp}_{\omega} = \mathrm{Cl}_{\omega}(\mathbb{R} \setminus G) = \mathbb{R}$. Thus, $G \in \mathcal{T}^{*}_{\omega}$, whereas $G \notin \mathcal{T}^{\sharp}_{\omega}$.

Definition 3.11. *Let* $(\mathbb{X}, \mathcal{T}, \Im)$ *be an ideal top-space, and let* $H \subseteq \mathbb{X}$*. The following operators are defined:*

(1) The Ψ^{*}_{ω}-operator is defined by $\Psi^{*}_{\omega}(H) = \mathbb{X} \setminus (\mathbb{X} \setminus H)^{*}_{\omega}$.
(2) The $\Psi^{\sharp}_{\omega}$-operator is defined by $\Psi^{\sharp}_{\omega}(H) = \mathbb{X} \setminus (\mathbb{X} \setminus H)^{\sharp}_{\omega}$.

Remark 3.12. *Let* $(\mathbb{X}, \mathcal{T}, \Im)$ *be an ideal top-space, and let* $H \subseteq \mathbb{X}$*. According to the definition of the* $\Psi^{\sharp}_{\omega}$*-operator together with Theorem 2.10, we have* $\Psi^{*}_{\omega}(H(\mathcal{ANN}(\Im), \mathcal{T})) = \Psi^{\sharp}_{\omega}(H(\Im, \mathcal{T}))$.

Theorem 3.13. *Let* $(\mathbb{X}, \mathcal{T}, \Im)$ *be an ideal top-space, and let* $H \subseteq \mathbb{X}$*. Then,* $H \in \mathcal{T}^{\sharp}_{\omega} \iff H \subseteq \Psi^{\sharp}_{\omega}(H)$.

Proof. Let $H \subseteq \mathbb{X}$. Then,

$$\begin{aligned} H \in \mathcal{T}^{\sharp}_{\omega} &\iff \mathbb{X} \setminus H \text{ is } \mathcal{T}^{\sharp}_{\omega}\text{-closed} \\ &\iff \mathrm{Cl}^{\sharp}_{\omega}(\mathbb{X} \setminus H) = \mathbb{X} \setminus H \\ &\iff (\mathbb{X} \setminus H)^{\sharp}_{\omega} \subseteq \mathbb{X} \setminus H \\ &\iff \mathbb{X} \setminus (\mathbb{X} \setminus H)^{\sharp}_{\omega} \supseteq H \\ &\iff \Psi^{\sharp}_{\omega}(H) \supseteq H. \end{aligned}$$

Hence,

$$H \in \mathcal{T}^{\sharp}_{\omega} \iff H \subseteq \Psi^{\sharp}_{\omega}(H).$$

□

Theorem 3.14. *Let* $(\mathbb{X}, \mathcal{T}, \Im)$ *be an ideal top-space and* $H \subseteq \mathbb{X}$*. Then,*

$$\Psi^{\sharp}_{\omega}(H) \cap \Psi^{*}_{\omega}(H) = \mathrm{Int}_{\omega}(H).$$

Proof. Consider any subset $H \subseteq \mathbb{X}$. By Theorem 2.15, we have

$$\begin{aligned}
\mathrm{Cl}_\omega(\mathbb{X} \setminus H) &= (\mathbb{X} \setminus H)^\sharp_\omega(\Im, \mathcal{T}) \cup (\mathbb{X} \setminus H)^*_\omega(\Im, \mathcal{T}) \\
&\Leftrightarrow \mathbb{X} \setminus \mathrm{Cl}_\omega(\mathbb{X} \setminus H) = \mathbb{X} \setminus \left[(\mathbb{X} \setminus H)^\sharp_\omega(\Im, \mathcal{T}) \cup (\mathbb{X} \setminus H)^*_\omega(\Im, \mathcal{T})\right] \\
&\Leftrightarrow \mathrm{Int}_\omega(H) = \mathbb{X} \setminus (\mathbb{X} \setminus H)^\sharp_\omega(\Im, \mathcal{T}) \ \cap \ \mathbb{X} \setminus (\mathbb{X} \setminus H)^*_\omega(\Im, \mathcal{T}) \\
&\Leftrightarrow \mathrm{Int}_\omega(H) = \Psi^\sharp_\omega(H) \cap \Psi^*_\omega(H).
\end{aligned}$$

□

From Theorem 3.14, we obtain the following.

Corollary 3.15. *For an ideal top-space* $(\mathbb{X}, \mathcal{T}, \Im)$ *and a subset* $H \subseteq \mathbb{X}$*:*

(1) If $\mathbb{X} \setminus H \in \Im$, then it follows that $\Psi^\sharp_\omega(H) = \mathrm{Int}_\omega(H)$.
(2) If $\mathbb{X} \setminus H \in \mathcal{ANN}(\Im)$, then, $\Psi^*_\omega(H) = \mathrm{Int}_\omega(H)$.

The following theorem plays a key role in establishing the next result.

Theorem 3.16. *Let* $(\mathbb{X}, \mathcal{T})$ *be a top-space, and let* $\Im, \mathcal{K} \subseteq 2^{\mathbb{X}}$ *be ideals on* $\mathbb{X}$*. Then,*

$$\mathcal{T}^*_\omega(\Im \cap \mathcal{K}, \mathcal{T}) = \mathcal{T}^*_\omega(\Im, \mathcal{T}) \cap \mathcal{T}^*_\omega(\mathcal{K}, \mathcal{T}).$$

Proof. Let $W \subseteq \mathbb{X}$. Then,

$$\begin{aligned}
W \in \mathcal{T}^*_\omega(\Im \cap \mathcal{K}, \mathcal{T}) &\Leftrightarrow (\mathbb{X} \setminus W)^*_\omega(\Im \cap \mathcal{K}, \mathcal{T}) \subseteq \mathbb{X} \setminus W \\
&\Leftrightarrow [(\mathbb{X} \setminus W)^*_\omega(\Im, \mathcal{T}) \cup (\mathbb{X} \setminus W)^*_\omega(\mathcal{K}, \mathcal{T})] \subseteq \mathbb{X} \setminus W \quad \text{(by Lemma 2.14)} \\
&\Leftrightarrow \begin{cases} (\mathbb{X} \setminus W)^*_\omega(\Im, \mathcal{T}) \subseteq \mathbb{X} \setminus W, \\ (\mathbb{X} \setminus W)^*_\omega(\mathcal{K}, \mathcal{T}) \subseteq \mathbb{X} \setminus W \end{cases} \\
&\Leftrightarrow W \in \mathcal{T}^*_\omega(\Im, \mathcal{T}) \cap \mathcal{T}^*_\omega(\mathcal{K}, \mathcal{T}).
\end{aligned}$$

Hence,

$$\mathcal{T}^*_\omega(\Im \cap \mathcal{K}, \mathcal{T}) = \mathcal{T}^*_\omega(\Im, \mathcal{T}) \cap \mathcal{T}^*_\omega(\mathcal{K}, \mathcal{T}),$$

which is equivalent to the two inclusions:

$$\mathcal{T}^*_\omega(\Im \cap \mathcal{K}, \mathcal{T}) \subseteq \mathcal{T}^*_\omega(\Im, \mathcal{T}) \cap \mathcal{T}^*_\omega(\mathcal{K}, \mathcal{T}),$$

and

$$\mathcal{T}^*_\omega(\Im, \mathcal{T}) \cap \mathcal{T}^*_\omega(\mathcal{K}, \mathcal{T}) \subseteq \mathcal{T}^*_\omega(\Im \cap \mathcal{K}, \mathcal{T}).$$

□

Theorem 3.17. *For any ideal top-space* $(\mathbb{X}, \mathcal{T}, \Im)$, $\mathcal{T}_\omega = \mathcal{T}^*_\omega(\Im, \mathcal{T}) \cap \mathcal{T}^\sharp_\omega(\Im, \mathcal{T})$.

Proof. Obviously, $\mathcal{T}_\omega \subseteq \mathcal{T}^*_\omega(\Im,\mathcal{T}) \cap \mathcal{T}^\sharp_\omega(\Im,\mathcal{T})$. Let $W \in \mathcal{T}^*_\omega(\Im,\mathcal{T}) \cap \mathcal{T}^\sharp_\omega(\Im,\mathcal{T})$. Then, by Theorem 2.10, we obtain

$$W \in \mathcal{T}^*_\omega(\Im,\mathcal{T}) \cap\ \mathcal{T}^*_\omega(\mathcal{ANN}(\Im),\mathcal{T}) \overset{\text{Theorem 3.16}}{\Rightarrow} W \in \mathcal{T}^*_\omega(\Im \cap \mathcal{ANN}(\Im),\mathcal{T}).$$

By Lemma 1.6, $\Im \cap \mathcal{ANN}(\Im) = \{\emptyset\}$, hence, $W \in \mathcal{T}^*_\omega(\{\emptyset\},\mathcal{T})$, implying $W \in \mathcal{T}_\omega$. Hence,

$$\mathcal{T}^*_\omega(\Im,\mathcal{T}) \cap \mathcal{T}^\sharp_\omega(\Im,\mathcal{T}) \subseteq \mathcal{T}_\omega.$$

Therefore,

$$\mathcal{T}_\omega = \mathcal{T}^*_\omega(\Im,\mathcal{T}) \cap \mathcal{T}^\sharp_\omega(\Im,\mathcal{T}).$$

□

Theorem 3.18. *Let* $(\mathbb{X},\mathcal{T},\Im)$ *be an ideal top-space with* $\Im$ *as proper minimal ideal. Then, for every* $H \subseteq \mathbb{X}$,

$$H^\sharp_\omega = \emptyset \quad or \quad (\mathbb{X} \setminus H)^\sharp_\omega = \emptyset.$$

Proof. By the duality between ideals and their annihilators (Corollary 1.8(2)), $\Im$ minimal $\Leftrightarrow$ $\mathcal{ANN}(\Im)$ is maximal. We conclude that

$$\forall H \subseteq \mathbb{X},\ (H \in \mathcal{ANN}(\Im)\ \vee\ \mathbb{X} \setminus H \in \mathcal{ANN}(\Im)).$$

From Theorem 2.12(6), we obtain

$$\Big(H \in \mathcal{ANN}(\Im) \Rightarrow H^\sharp_\omega = \emptyset\Big) \vee \Big(\mathbb{X} \setminus H \in \mathcal{ANN}(\Im) \Rightarrow (\mathbb{X} \setminus H)^\sharp_\omega = \emptyset\Big).$$

Therefore,

$$\forall H \subseteq \mathbb{X},\ (H^\sharp_\omega = \emptyset\ \vee\ (\mathbb{X} \setminus H)^\sharp_\omega = \emptyset).$$

□

Corollary 3.19. *Let* $(\mathbb{X},\mathcal{T},\Im)$ *be an ideal top-space with* $\Im$ *minimal. Then,*

$$\forall H \subseteq \mathbb{X},\ H \in \mathcal{T}^\sharp_\omega\text{-closed}\ \vee\ H \in \mathcal{T}^\sharp_\omega\text{-open}.$$

Proof. The proof proceeds analogously to that of Theorem 2.8. □

Corollary 3.20. *Let* $(\mathbb{X},\mathcal{T},\Im)$ *be an ideal top-space. If* $\Im$ *is minimal, then the space* $(\mathbb{X},\mathcal{T}^\sharp_\omega)$ *satisfies the* T_0 *separation axiom.*

Theorem 3.21. *Assume that* $(\mathbb{X},\mathcal{T},\Im)$ *is an ideal top-space such that* $\Im$ *is a maximal ideal on* $\mathbb{X}$. *Then, for any subset* $H \subseteq \mathbb{X}$,

$$H^\sharp_\omega = \mathrm{Cl}_\omega(H)\ \vee\ \Psi^\sharp_\omega(H) = \mathrm{Int}_\omega(H).$$

Proof. Let $H \subseteq \mathbb{X}$. Since $\Im$ is a maximal ideal on $\mathbb{X}$, it follows that

$$H \in \Im\ \vee\ \mathbb{X} \setminus H \in \Im.$$

Consequently,

$$H^*_\omega = \emptyset\ \vee\ (\mathbb{X} \setminus H)^*_\omega = \emptyset.$$

From Theorem 2.15, we have

$$\mathrm{Cl}_\omega(H) = H^*_\omega \cup H^\sharp_\omega \quad \forall\, H \subseteq \mathbb{X}.$$

Hence,

$$(H^*_\omega = \emptyset \Rightarrow \mathrm{Cl}_\omega(H) = H^\sharp_{\omega)}) \ \vee \ ((\mathbb{X} \setminus H)^*_\omega = \emptyset \Rightarrow \mathrm{Cl}_\omega(\mathbb{X} \setminus H) = (\mathbb{X} \setminus H)^\sharp_{\omega)}).$$

Using the identity

$$\mathrm{Cl}_\omega(\mathbb{X} \setminus H) = \mathbb{X} \setminus \mathrm{Int}_\omega(H),$$

we deduce

$$\Psi^\sharp_\omega(H) = \mathbb{X} \setminus (\mathbb{X} \setminus H)^\sharp_\omega = \mathrm{Int}_\omega(H).$$

Therefore,

$$\mathrm{Cl}_\omega(H) = H^\sharp_\omega \ \vee \ \Psi^\sharp_\omega(H) = \mathrm{Int}_\omega(H),$$

which completes the proof. □

Corollary 3.22. *Suppose that* $(\mathbb{X}, \mathcal{T}, \Im)$ *is an ideal top-space such that* $\Im$ *is a proper minimal ideal on* $\mathbb{X}$*. Then, for every subset* $H \subseteq \mathbb{X}$*,*

$$H^*_\omega = \mathrm{Cl}_\omega(H) \ \vee \ \Psi^*_\omega(H) = \mathrm{Int}_\omega(H).$$

Proof. Applying Theorems 2.15 and 3.18, the statement is established. □

3.1. Some applications of $\omega^\sharp$-operator

Theorem 3.23. *[21] Let* $(\mathbb{X}, \mathcal{T})$ *be a topological space and let* H *be a subset of* $\mathbb{X}$*. Then* H *is dense in* $\mathbb{X}$ *if and only if the ideal topological space* $(\mathbb{X}, \mathcal{T}, \mathcal{ANN}_H)$ *is a Hayashi–Samuel space.*

Remark 3.24. *Let* $\mathcal{U}$ *be the usual topology on* $\mathbb{R}$*, and let the annihilator ideal of the rationals be*

$$\mathcal{ANN}(\{\mathbb{Q}\}) = 2^{\mathbb{P}}, \quad \mathbb{P} = \mathbb{R} \setminus \mathbb{Q}.$$

The associated ω-topology is

$$\mathcal{U}_\omega = \left\{ W \subseteq \mathbb{R} \,\middle|\, \forall x \in W,\ \exists U \in \mathcal{U} \text{ with } x \in U \text{ and } U \setminus W \text{ countable} \right\}.$$

Since $\mathbb{R} \setminus \mathbb{Q} \in \mathcal{U}_\omega \cap \mathcal{ANN}(\{\mathbb{Q}\}) \neq \emptyset$, we can write symbolically that the space
$(\mathbb{R}, \mathcal{U}_\omega, \mathcal{ANN}(\{\mathbb{Q}\}))$ fails to be a Hayashi–Samuel space as $\mathcal{U}_\omega \cap \mathcal{ANN}(\{\mathbb{Q}\}) \neq \emptyset$. Furthermore, applying the $\mathrm{Cl}^\sharp_\omega$ operator, we have $\mathrm{Cl}^\sharp_\omega(\mathbb{Q}) = \mathbb{Q} \cup \mathbb{Q}^\sharp_\omega = \mathbb{Q}$, because $\mathbb{Q}^\sharp_\omega \subseteq \mathbb{Q}$.

Remark 3.25. *In [21], Issaka and Özköc established that the set of rational numbers* $\mathbb{Q}$ *is* $\mathcal{U}^\sharp$*-dense in the ideal topological space or dense in* $(\mathbb{R}, \mathcal{U}^\sharp)$ $(\mathbb{R}, \mathcal{U}, \Im(\mathbb{Q}))$*, where* $\mathcal{U}$ *denotes the usual topology on* $\mathbb{R}$*.*

Proposition 3.26. *There exists a Hausdorff topology on* $\mathbb{R}$ *such that* $\mathbb{Q}$ *is not dense in* $\mathbb{R}$ *with respect to the ideal* $\Im(\mathbb{Q})$*.*

The following example illustrates Proposition 3.26.

Example 3.27. *Let $\mathcal{U}$ be the usual topology on $\mathbb{R}$ and let $\Im = \Im(\mathbb{Q})$. Since $\mathbb{Q} \in \Im$, it follows from Corollary 2.17(1) that $\mathbb{Q}^*_\omega = \emptyset$ and $\mathbb{Q}^\sharp_\omega = \mathrm{Cl}_\omega(\mathbb{Q}) = \mathbb{Q}$, since $\mathbb{R} \setminus \mathbb{Q}$ is ω-open.*

Thus,

$$\mathrm{Cl}^\sharp_\omega(\mathbb{Q}) = \mathbb{Q} \cup \mathbb{Q}^\sharp_\omega = \mathbb{Q}.$$

Hence, $\mathbb{Q}$ is not dense in $(\mathbb{R}, \mathcal{U}^\sharp_\omega)$.

Finally, since $\mathcal{U} \subseteq \mathcal{U}^\sharp_\omega$ and $(\mathbb{R}, \mathcal{U})$ is Hausdorff, it follows that $(\mathbb{R}, \mathcal{U}^\sharp_\omega)$ is also Hausdorff.

Theorem 3.28. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space. Assume that there is a set $F \in \mathcal{T}^c_\omega \setminus \{\emptyset, \mathbb{X}\}$. Then the ideal top-space $(\mathbb{X}, \mathcal{T}, \Im(F))$ is $\mathcal{T}^\sharp_\omega$-disconnected.*

Proof. Let $F \in \mathcal{T}^c_\omega \setminus \{\varnothing, \mathbb{X}\}$. Then $\mathbb{X} \setminus F \in \mathcal{T}_\omega \setminus \{\emptyset, \mathbb{X}\}$. Since $\mathcal{T}_\omega \subseteq \mathcal{T}^\sharp_\omega$, it follows that $\mathbb{X} \setminus F \in \mathcal{T}^\sharp_\omega \setminus \{\emptyset, \mathbb{X}\}$. Moreover, $\mathbb{X} \setminus F \in \mathcal{ANN}_F = \mathcal{ANN}(\Im(F))$, and hence, by Theorem 2.12 (6), $(\mathbb{X} \setminus F)^\sharp_\omega = \emptyset$. By the definition of $\mathrm{Cl}^\sharp_\omega$,

$$\begin{aligned} \mathrm{Cl}^\sharp_\omega(\mathbb{X} \setminus F) &= (\mathbb{X} \setminus F)^\sharp_\omega \cup (\mathbb{X} \setminus F) = \mathbb{X} \setminus F \\ &\implies \mathbb{X} \setminus F \in (\mathcal{T}^\sharp_\omega)^c \setminus \{\emptyset, \mathbb{X}\} \\ &\implies F \in \mathcal{T}^\sharp_\omega \cap (\mathcal{T}^\sharp_\omega)^c \\ &\implies (\mathbb{X}, \mathcal{T}, \Im(F)) \text{ is } \mathcal{T}^\sharp_\omega\text{-disconnected.} \end{aligned}$$

□

Corollary 3.29. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space. If $F \in \mathcal{T}^c \setminus \{\emptyset, \mathbb{X}\}$, then $(\mathbb{X}, \mathcal{T}, \Im(F))$ is $\mathcal{T}^\sharp_\omega$-disconnected.*

Example 3.30. *Consider the real line $\mathbb{R}$ endowed with its usual topology $\mathcal{U}$. Let $\Im = \Im(F)$ be the ideal generated by the finite set $F = \{0, 1\}$. The set F is ω-closed in the topological space $(\mathbb{R}, \mathcal{U})$. By Theorem 3.28, the corresponding ideal topological space $(\mathbb{R}, \mathcal{U}, \Im(F))$ is $\mathcal{T}^\sharp_\omega$-disconnected.*

4. Decomposition results for continuity

Definition 4.1. *A function $f : (\mathbb{X}, \mathcal{T}, \Im) \to (\mathbb{Y}, \Gamma)$ is said to be ω-*continuous *[23] (resp. $\sharp$-*continuous *[21]) if for every $U \in \Gamma$, $f^{-1}(U) \in \mathcal{T}_\omega$ (resp. $f^{-1}(U) \in \mathcal{T}^\sharp$).*

Definition 4.2. *Let $(\mathbb{X}, \mathcal{T}, \Im)$ be an ideal top-space, and $(\mathbb{Y}, \Gamma)$ a topological space. A function $f : \mathbb{X} \to \mathbb{Y}$ is called ω^*-*continuous *(respectively, $\omega^\sharp$-*continuous*) if and only if*

$$f^{-1}(U) \in \mathcal{T}^*_\omega \quad (\text{respectively, } f^{-1}(U) \in \mathcal{T}^\sharp_\omega) \quad \text{for every } U \in \Gamma.$$

The following corollaries are immediate consequences of the definitions. However, to show that the converses are not valid in general, we provide appropriate counterexamples.

Corollary 4.3. *Let $f : (\mathbb{X}, \mathcal{T}, \Im) \to (\mathbb{Y}, \Gamma)$ be a function. Then, f is ω^*-continuous if and only if $f : (\mathbb{X}, \mathcal{T}^*_\omega) \to (\mathbb{Y}, \Gamma)$ is continuous, and f is $\omega^\sharp$-continuous if and only if $f : (\mathbb{X}, \mathcal{T}^\sharp_\omega) \to (\mathbb{Y}, \Gamma)$ is continuous.*

Corollary 4.4. *Let $f : (\mathbb{X}, \mathcal{T}, \Im) \to (\mathbb{Y}, \Gamma)$ be a function. If f is continuous, then it is also $\omega^\sharp$-continuous and ω^*-continuous.*

Corollary 4.5. *Let $f : (\mathbb{X}, \mathcal{T}, \Im) \to (\mathbb{Y}, \Gamma)$ be a function. Then the following hold.*

1. If f is ω-continuous, it is also $\omega^\sharp$-continuous.
2. If f is $\sharp$-continuous, it is also $\omega^\sharp$-continuous.
3. If f is ω-continuous, it is also ω^*-continuous.

The relationships illustrated in the following figure follow directly from the definitions and Corollaries 4.4 and 4.5.

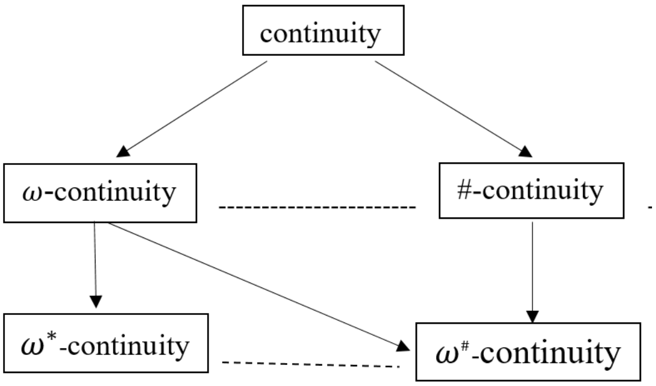

Figure 2. Relationships among different generalized forms of continuity induced by various open-set structures; dashed line denotes independence.

The converses of the implications in Figure 2 do not hold in general, as demonstrated by the following examples.

First, note that in [21], Issaka and Özkoç proved that every continuous function $f : (\mathbb{X}, \mathcal{T}, \Im) \to (\mathbb{Y}, \Gamma)$ is $\sharp$-continuous. However, the converse need not always be true, as shown in [21].

Example 4.6. *Let $\mathcal{U}$ be the standard topology on $\mathbb{R}$, and let $\Im = \{J \subseteq \mathbb{R} : J \text{ is countable}\}$ be the ideal of countable sets. Define the co-countable topology*

$$\mathcal{T}_c = \{U \subseteq \mathbb{R} : \mathbb{R} \setminus U \text{ is countable}\} \cup \{\emptyset\}.$$

Consider the identity function $f : (\mathbb{R}, \mathcal{U}, \Im) \longrightarrow (\mathbb{R}, \mathcal{T}_c),\ f(x) = x$.

For any $U \in \mathcal{T}_c \setminus \{\emptyset\}$, $f^{-1}(U) = U$. Since $\mathbb{R} \setminus U$ is countable, for any $W \in \mathcal{U}_\omega$,

$$(\mathbb{R} \setminus U) \cap W \in \Im \quad \Rightarrow \quad (\mathbb{R} \setminus U)^*_\omega = \emptyset,$$

so $U \in \mathcal{U}^*_\omega$. Therefore, f is ω^*-continuous. However, f is not continuous because $\mathbb{R} \setminus \mathbb{Q} \in \mathcal{T}_c$ but $f^{-1}(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R} \setminus \mathbb{Q} \notin \mathcal{U}$. Similarly, using the $\omega^\sharp$-operator, we have $\mathbb{R} \setminus (\mathbb{R} \setminus U) \in \mathcal{U}_\omega$ and

$$(\mathbb{R} \setminus U) \cap \left[\mathbb{R} \setminus (\mathbb{R} \setminus U)\right] \in \mathcal{ANN}(\Im) \quad \Rightarrow \quad (\mathbb{R} \setminus U)^\sharp_\omega \subseteq \mathbb{R} \setminus U,$$

so $\mathrm{Cl}^\sharp_\omega(\mathbb{R} \setminus U) = \mathbb{R} \setminus U$, and $U \in \mathcal{U}^\sharp_\omega$. Hence, f is $\omega^\sharp$-continuous as well.

Example 4.7. *Let $(\mathbb{R}, \mathcal{U}, \Im)$ be an ideal top-space, where $\mathcal{U}$ is the usual topology on $\mathbb{R}$ and $\Im = \Im(\mathbb{Q})$. Define a function $f : (\mathbb{R}, \mathcal{U}, \Im) \to (\mathbb{R}, \mathcal{U})$ by*

$$f(x)=\begin{cases}1, & x\in\mathbb{R}\setminus\mathbb{Q},\\ 0, & x\in\mathbb{Q}.\end{cases}$$

For any $U\in\mathcal{U}$,

$$f^{-1}(U)=\begin{cases}\emptyset, & 0\notin U,\ 1\notin U,\\ \mathbb{Q}, & 0\in U,\ 1\notin U,\\ \mathbb{R}\setminus\mathbb{Q}, & 0\notin U,\ 1\in U,\\ \mathbb{R}, & 0\in U,\ 1\in U.\end{cases}$$

Since $\mathbb{R}\setminus\mathbb{Q}\in\mathcal{ANN}(\Im)$, Theorem 2.12 (6) yields $(\mathbb{R}\setminus\mathbb{Q})^{\sharp}_{\omega}=\emptyset$. Hence,

$$\mathbb{Q}\subseteq\Psi^{\sharp}_{\omega}(\mathbb{Q})=\mathbb{R}\setminus(\mathbb{R}\setminus\mathbb{Q})^{\sharp}_{\omega}=\mathbb{R},$$

which implies $\mathbb{Q}\in\mathcal{U}^{\sharp}_{\omega}$ by Theorem 3.13. Similarly, $\mathbb{R}\setminus\mathbb{Q}\in\mathcal{U}^{\sharp}_{\omega}$ by Theorem 3.13. Indeed, since $\mathbb{Q}\cap(\mathbb{R}\setminus\mathbb{Q})=\emptyset$ and $\mathbb{R}\setminus\mathbb{Q}\in\mathcal{U}_{\omega}$, it follows that $\mathbb{Q}^{\sharp}_{\omega}\subseteq\mathbb{Q}$. Consequently, $\Psi^{\sharp}_{\omega}(\mathbb{R}\setminus\mathbb{Q})=\mathbb{R}\setminus\mathbb{Q}$, and therefore, $\mathbb{R}\setminus\mathbb{Q}\in\mathcal{U}^{\sharp}_{\omega}$. Therefore, $f^{-1}(U)\in\mathcal{U}^{\sharp}_{\omega}$ for every $U\in\mathcal{U}$, and thus f is $\omega^{\sharp}$-continuous. However, f is not ω-continuous since $\mathbb{Q}\notin\mathcal{U}_{\omega}$.

Remark 4.8. *Following Issaka and Özköc [21], for any $H\in\Im$, we have $H^{\sharp}=\mathrm{Cl}(H)$. In Example 4.6, $\mathbb{Q}\in\Im$ yields $\mathbb{Q}^{\sharp}=\mathrm{Cl}(\mathbb{Q})=\mathbb{R}$. Hence, $\mathbb{R}\setminus\mathbb{Q}\notin\mathcal{U}^{\sharp}$. Let $U=\mathbb{R}\setminus\mathbb{Q}$. Then,*

$$U\in\mathcal{T}_c \quad\textit{and}\quad f^{-1}(U)=U\notin\mathcal{U}^{\sharp}.$$

Therefore, the function $f:(\mathbb{R},\mathcal{U},\Im)\to(\mathbb{R},\mathcal{T}_c)$ is not $\sharp$-continuous. On the other hand, Example 4.6 shows that f is $\omega^{\sharp}$-continuous.

Remark 4.9. *The example below demonstrates that ω-continuity and $\sharp$-continuity are independent notions.*

Example 4.10. *Let $(\mathbb{R},\mathcal{U},\Im)$ and $(\mathbb{R},\mathcal{U},\mathcal{K})$ be ideal top-spaces, where $\mathcal{U}$ is the usual topology on $\mathbb{R}$, $\Im=\Im(\mathbb{Q})$, and $\mathcal{K}=\{\emptyset,\{0\}\}$. Consider $\mathbb{Y}=\{0,1\}$ equipped with the topology $\Gamma=\{\emptyset,\mathbb{Y},\{0\}\}$.*

(1) Define

$$f:(\mathbb{R},\mathcal{U},\mathcal{K})\longrightarrow(\mathbb{Y},\Gamma),\quad f(x)=\begin{cases}1, & x\in\mathbb{R}\setminus\mathbb{Q},\\ 0, & x\in\mathbb{Q}.\end{cases}$$

Note that $f^{-1}(\{0\})=\mathbb{Q}$. Since $\mathbb{R}\setminus\mathbb{Q}\in\mathcal{ANN}(\mathcal{K})$, by Theorem 2.12, we have $(\mathbb{R}\setminus\mathbb{Q})^{\sharp}=\emptyset$, which implies $\mathbb{Q}\in\mathcal{U}^{\sharp}$. Therefore, for each $U\in\Gamma$, $f^{-1}(U)\in\mathcal{U}^{\sharp}$. Hence, f is $\sharp$-continuous but not ω-continuous.

(2) Define

$$f:(\mathbb{R},\mathcal{U},\Im)\longrightarrow(\mathbb{Y},\Gamma),\quad f(x)=\begin{cases}0, & x\in\mathbb{R}\setminus\mathbb{Q},\\ 1, & x\in\mathbb{Q}.\end{cases}$$

In this case, f is ω-continuous but not $\sharp$-continuous.

Remark 4.11. *The concepts of ω^{*}-continuity and $\omega^{\sharp}$-continuity are independent as illustrated by the following examples. Let $\mathcal{T}_{ind}$ denote the indiscrete topology on $\mathbb{R}$; and set $\mathcal{T}=\mathcal{T}_{ind}$.*

We now consider the following ideals. $\Im = \{\emptyset, \{1\}\}$, $\mathcal{K} = \mathcal{K}((0,\infty))$ on $\mathbb{R}$. Define the codomain $\mathbb{Y} = \{0,1\}$ with topologies

$$\Gamma_1 = \{\emptyset, \mathbb{Y}, \{0\}\}, \quad \Gamma_2 = \{\emptyset, \mathbb{Y}, \{1\}\}. \text{ Let } f : \mathbb{R} \longrightarrow \mathbb{Y}, \quad f(x) = \begin{cases} 0, & x > 0, \\ 1, & x \leq 0. \end{cases}$$

(1) When f is considered as $f : (\mathbb{R}, \mathcal{T}, \Im) \longrightarrow (\mathbb{Y}, \Gamma_1)$, it is $\omega^\sharp$-continuous but not ω^*-continuous. Observe that $f^{-1}(\{0\}) = (0,\infty) \in \mathcal{T}^\sharp_\omega$. Since $(-\infty, 0] \in \mathcal{ANN}(\Im)$, Corollary 2.17(2) implies $(-\infty,0]^\sharp_\omega = \emptyset$, $\quad (-\infty,0]^*_\omega = \mathrm{Cl}_\omega((-\infty,0]) = \mathbb{R}$. Therefore, $(0,\infty) \notin \mathcal{T}^*_\omega$, so f is not ω^*-continuous.

(2) When f is considered as $f : (\mathbb{R}, \mathcal{T}, \mathcal{K}) \longrightarrow (\mathbb{Y}, \Gamma_2)$, it is ω^*-continuous but not $\omega^\sharp$-continuous. Observe that $f^{-1}(\{1\}) = (-\infty, 0] \in \mathcal{T}^*_\omega$. Since $(0,\infty) \in \mathcal{K}$, Corollary 2.17(1) gives $(0,\infty)^*_\omega = \emptyset$, $\quad (0,\infty)^\sharp_\omega = \mathrm{Cl}_\omega((0,\infty)) = \mathbb{R}$. Therefore, $(-\infty,0] \notin \mathcal{T}^\sharp_\omega$, so f is not $\omega^\sharp$-continuous.

(3) From statement (2), the function $f : (\mathbb{R}, \mathcal{T}, \mathcal{K}) \longrightarrow (\mathbb{Y}, \Gamma_2)$ is ω^*-continuous. However, it fails to be ω-continuous. Indeed, we observe that $f^{-1}(\{1\}) = (-\infty, 0] \notin \mathcal{T}_\omega$. This follows from the fact that for every $x \in (-\infty, 0]$, $\mathbb{R} \setminus (-\infty, 0]$ is not countable.

Theorem 4.12. *A function* $f : (\mathbb{X}, \mathcal{T}, \Im) \to (\mathbb{Y}, \Gamma)$ *is* ω*-continuous if and only if it is both* ω^**-continuous and* $\omega^\sharp$*-continuous.*

Proof. The proof follows directly from Theorem 3.17. □

5. Conclusions

In this paper, we introduced and developed the operator $(\cdot)^\sharp_\omega$ within the setting of ideal topological spaces, and we demonstrated that it provides a coherent and effective framework for refining topological structures and analyzing generalized notions of openness and continuity. The results establish that this operator is not merely an extension of existing constructions, but a structurally meaningful tool that interacts in a nontrivial way with both ω-local functions and classical sharp operators.

The main findings and contributions of this work are as follows:

(1) Definition and structural characterization of the $\omega^\sharp$-operator: We introduced the $\omega^\sharp$-operator and carried out a rigorous structural analysis of its fundamental properties. In particular, we established its monotonicity, its stability with respect to standard set-theoretic operations, and a form of weak idempotence. A key contribution is the characterization $H^\sharp_\omega(\Im, \mathcal{T}) = H^*_\omega(\mathcal{ANN}(\Im), \mathcal{T})$, which links the $\omega^\sharp$-operator to the annihilator of the ideal (Theorem 2.10). Moreover, the decomposition theorem (Theorem 2.15) shows that $\mathrm{Cl}_\omega(H) = H^\sharp_\omega \cup H^*_\omega$. This result provides a precise structural link between the $\omega^\sharp$-operator and the ω-local function, offering a new perspective on ω-closure.

(2) Generation of a strictly finer topology: We proved that the operator $\mathrm{Cl}^\sharp_\omega(H) = H \cup H^\sharp_\omega$ satisfies the Kuratowski closure axioms and hence induces a topology $\mathcal{T}^\sharp_\omega$. We further showed that this topology is strictly finer than both $\mathcal{T}_\omega$ and $\mathcal{T}^\sharp$, and we characterized the precise relationships among the associated topologies. In particular, the identity $\mathcal{T}_\omega = \mathcal{T}^*_\omega \cap \mathcal{T}^\sharp_\omega$ reveals an intrinsic structural decomposition of the ω-topology (Theorem 3.17) .

(3) Sharpness via examples and independence results: A series of carefully constructed examples and counterexamples demonstrates that the introduced concepts are genuinely new and independent. In particular, we showed that:

- $H^\sharp \not\subseteq H^\sharp_\omega$ in general,
- $\mathcal{T}^\sharp_\omega$-openness and $\mathcal{T}^*_\omega$-openness are independent, and
- $\omega^\sharp$-continuity is independent of both ω- and $\sharp$-continuity.

These results confirm that the $\omega^\sharp$-framework is not reducible to previously known notions.

(4) A decomposition theory of continuity: We introduced ω^*- and $\omega^\sharp$-continuity and established that a function is ω-continuous if and only if it is both ω^*-continuous and $\omega^\sharp$-continuous. This decomposition provides a finer analytical tool for studying continuity in ideal topological spaces and clarifies the internal structure of ω-continuity.

The framework developed here suggests several directions for further investigation:

- Studying separation axioms in $(\mathbb{X}, \mathcal{T}^\sharp_\omega)$, particularly higher separation properties such as regularity and normality.
- Investigating deeper properties of $\omega^\sharp$-continuous functions, including their stability under composition and their interaction with compactness and connectedness.
- Exploring potential applications in algebraic topology, especially in constructing invariants based on refined notions of smallness induced by ideals.

Overall, the $(\cdot)^\sharp_\omega$-operator introduces a flexible and structurally rich mechanism for refining topology via ideals. The combination of theoretical results, structural decompositions, and independence examples shows that this framework is both robust and nontrivial. It is expected that the ideas developed in this work will contribute to ongoing research in ideal topology and stimulate further developments in generalized topological structures.

Author contributions

Abdo Qahis and Mohd Salmi Md Noorani contributed equally to the conception, analysis, and preparation of the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors are thankful to the Deanship of Graduate Studies and Scientific Research at Najran University for funding this work under the Consortium Funding Program grant code (NU/CPI/SERC/14/3066-1).

Conflict of interest

The authors declare no conflicts of interest

References

1. T. R. Hamlett, D. Jankovic, Ideals in topological spaces and the set operator Ψ, *Boll. Unione Mat. Ital.*, **4** (1990), 863–874.
2. K. Kuratowski, *Topology, Volume I*, New York: Academic Press, 1966.
3. S. Acharyee, M. Ã-zkoç, F. Y. Issaka, Primal topological spaces, *Boletim Da Sociedade Paranaense De Matemática*, **43** (2025), 1–9. https://doi.org/10.5269/bspm.66792
4. G. Choquet, Sur les notions de filtre et grille, *Paris, Ser. A, C. R. Acad. Sci.*, **224** (1947), 171–173.
5. K. Kuratowski, *Topologie I*, Warszawa: Instytut Matematyczny Polskiej Akademii Nauk, 1933.
6. R. Vaidyanathaswamy, The localization theory in set-topology, *Proc. Indian Acad. Sci.*, **20** (1944), 51–61. https://doi.org/10.1007/BF03048958
7. T. Natkaniec, On $\mathcal{I}$-continuity and $\mathcal{I}$-semicontinuity points, *Math. Slovaca*, **36** (1986), 297–312.
8. D. Jankovic, T. R. Hamlett, New topologies from old via ideals, *The American Mathematical Monthly*, **97** (1990), 295–310. https://doi.org/10.2307/2324512
9. M. M. Islam, S. Modak, Operator associated with the * and Ψ operators, *J. Taibah Univ. Sci.*, **12** (2018), 444–449. https://doi.org/10.1080/16583655.2018.1489585
10. F. G. Arenas, J. Dontchev, M. L. Puertas, Idealization of some weak separation axioms, *Acta Math. Hung.*, **89** (2000), 47–53. https://doi.org/10.1023/A:1026773308067
11. M. Navaneethakrishnan, J. P. Joseph, g-Closed sets in ideal topological spaces, *Acta Math. Hung.*, **119** (2008), 365–371. https://doi.org/10.1007/s10474-007-7050-1
12. K. Kaur, A. Gupta, T. M. Al-shami, M. Hosny, A new multi-ideal nano-topological model via neighborhoods for diagnosis and cure of dengue, *Comp. Appl. Math.*, **43** (2024), 400. https://doi.org/10.1007/s40314-024-02910-4
13. E. Hatır, On decompositions of continuity and complete continuity in ideal topological spaces, *Eur. J. Pure Appl. Math.*, **6** (2017), 352–362.
14. E. Ekici, On I-Alexandroff and Ig-Alexandroff ideal topological spaces, *Filomat*, **25** (2011), 99–108. https://doi.org/10.2298/FIL1104099E
15. A. Acikgoz, M. Ozkoc, Ideal-aura topological spaces, new local functions, and generalized open sets, 2026, arXiv:2602.07692.
16. M. Matejdes, On topologies induced by ideals, primals, filters and grills, *Axioms*, **13** (2024), 698. https://doi.org/10.3390/axioms13100698
17. R. Roshdey, A. A. Nasef, N. Youns, M. Badr, Some issues on topological operators via ideals, *Journal of Contemporary Technology and Applied Engineering*, **3** (2024), 25–31.
18. A. N. Tunç, S. O. Yıldırım, Characterization of two specific cases with new operators in ideal topological spaces, *Journal of New Theory*, **46** (2024), 51–70. https://doi.org/10.53570/jnt.1418949

19. H. Z. Hdeib, ω-Closed mappings, *Revista Colombiana de Matematicas*, **16** (1982), 65–80.

20. A. Al-Omari, H. Al-Saadi, A topology via ω-local functions in ideal spaces, *Mathematica*, **60** (2018), 103–110. https://doi.org/10.24193/mathcluj.2018.2.01

21. F. Y. Issaka, M. Özköc, New topologies derived from the old one via operators, *Demonstr. Math.*, **58** (2025), 20240094. https://doi.org/10.1515/dema-2024-0094

22. E. Hatir, A. Al-Omari, S. Jafari, δ-Local function and its properties in ideal topological spaces, *Fasc. Math.*, **53** (2014), 53–64.

23. H. Hdeib, ω-Continuous functions, *Dirasat*, **16** (1996), 136–142.

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