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Research article

Matrix domain of the second-order q-Cesàro matrix in classical sequence spaces

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Abstract: In this paper, we study a second-order q-Cesàro matrix and its matrix domains in the spaces c_0 and c. We establish basic structural results, including inclusion relations, connections with earlier Cesàro-type sequence spaces, the duals of these spaces, and the associated matrix transformations. Our approach extends known work on q-Cesàro operators and provides a starting point for further investigations of higher-order q-averaging methods in summability theory.

Keywords: q-Cesàro matrix; second-order q-Cesàro matrix; matrix domain; sequence spaces
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1. Introduction and preliminaries

Sequence spaces play a central role in functional analysis and summability theory. Let ω denote the linear space of all real or complex sequences $x = (x_s)_{s\in\mathbb{N}_0}$, where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. The classical sequence spaces c_0, c, and ℓ_∞ consist of all null, convergent, and bounded sequences, respectively. For $1 \leq p < \infty$, ℓ_p denotes the space of all absolutely p-summable sequences endowed with the usual norm $\|x\|_p = \left(\sum_{s=0}^{\infty} |x_s|^p\right)^{1/p}$, while the spaces c_0, c, and ℓ_∞ are equipped with the supremum norm $\|x\|_\infty = \sup_{s\in\mathbb{N}} |x_s|$. It is well known that these spaces are Banach sequence spaces (BK-spaces) under their natural coordinatewise structure; see [1–3]. Motivated by the importance of these classical spaces, many authors have introduced new sequence spaces by means of different summability methods. Among these approaches, the matrix-domain technique has proved to be an effective tool for defining and analyzing new sequence spaces.

Cesàro summability and its higher-order extensions have been widely studied due to their close connection with convergence theory [4]. In recent years, the development of q-calculus has led to q-analogues of classical summability methods, including the Cesàro method. While first-order q-Cesàro sequence spaces have been investigated, higher-order cases have not yet been systematically studied within the matrix domain framework.

We recall the notation and basic results on sequence spaces, matrix transformations, and q-calculus that will be used throughout the paper.

An infinite matrix $A = (a_{ms})$ determines a matrix transformation by

$$(Ax)_m = \sum_{s=0}^{\infty} a_{ms} x_s, \qquad m \in \mathbb{N}_0,$$

for all sequences $x = (x_s) \in \omega$ for which the series converges for each $m \in \mathbb{N}_0$. We recall that a matrix $A = (a_{ms})$ is called conservative if it maps c into c and regular if it preserves the limit of each convergent sequence. For a sequence space X, the matrix domain of A in X is defined by

$$X_A = \{x \in \omega : \ Ax \in X\}. \tag{1.1}$$

This concept provides a standard framework for constructing new sequence spaces. Sequence spaces defined via matrix transformations, together with their matrix-domain properties, constitute a central topic in summability theory and functional analysis; see, for example, [5–7].

Let $q \in (0, 1)$. For a positive integer m, the q-integer is defined by

$$[m]_q = 1 + q + \cdots + q^{m-1} = \frac{1 - q^m}{1 - q},$$

which reduces to m as $q \to 1$ [8]. The q-factorial and the Gaussian q-binomial coefficient are given by

$$[m]_q! = [m]_q[m-1]_q \cdots [1]_q, \quad [0]_q! = 1, \qquad \begin{bmatrix} m \\ s \end{bmatrix}_q = \begin{cases} \frac{[m]_q!}{[s]_q!\,[m-s]_q!}, & \text{if } 0 \le s \le m, \\ 0, & \text{otherwise}, \end{cases}$$

and both converge to their classical counterparts as $q \to 1$. The q-shifted factorial $(a; q)_m$ is defined by

$$(a; q)_m = \begin{cases} 1, & m = 0, \\ \prod_{s=0}^{m-1}(1 - aq^s), & m \ge 1. \end{cases}$$

A fundamental example in this setting is the classical Cesàro matrix $\zeta = (\varsigma_{ms})_{m,s\in\mathbb{N}}$, defined by

$$\varsigma_{ms} = \begin{cases} \dfrac{1}{m+1}, & 0 \le s \le m, \\ 0, & s > m. \end{cases}$$

In recent years, this approach has also been combined with q-analogues of classical summability matrices. The theory of q-calculus, also called quantum calculus, originated from the pioneering works of Jackson [9] and was systematically presented by Kac and Cheung [8]. It has provided useful tools for extending many classical objects to their q-analogues. This development has naturally influenced sequence space theory. Aktuğlu and Bekar studied the q-Cesàro matrix and introduced the notion of q-statistical convergence [10]. Later, Demiriz and Şahin introduced q-Cesàro sequence spaces in c_0 and c [11], while Yaying et al. investigated the matrix domain of the q-Cesàro matrix in ℓ_p [12]. Moreover,

q-Fibonacci type sequence spaces were considered by Atabey et al. [13, 14]. More recently, Bilgin Ellidokuzoğlu et al. studied generalized q-difference and q-Schröder type sequence spaces [15–17]. In this direction, Çınar and Et studied q-double Cesàro matrices and q-statistical convergence of double sequences [18]. Related spectral, compactness, and operator-theoretic properties of q-Cesàro matrices and operators were studied in [19–21]. Also, various q-type sequence spaces, difference operators, spectra, operator ideals, and compact operators were studied by Yaying et al. in [22–24]. These studies show that q-analogues are not merely formal extensions of classical matrices, but may also lead to new sequence spaces with different structural, topological, and operator-theoretic features.

Although the above-mentioned studies provide a broad background on q-analogues in sequence space theory, the second-order q-Cesàro matrix has not yet been systematically studied within the matrix-domain framework. Its importance is not limited to the construction of new sequence spaces. The parameter q gives a flexible summability structure, while the second-order Cesàro form provides a higher-order averaging process. Hence, this matrix offers a useful setting for studying convergence, bases, duality, and matrix transformations in a unified way. This motivates the present investigation of the spaces $c_0(\zeta^{(q,2)})$ and $c(\zeta^{(q,2)})$.

More generally, for $\alpha > -1$, the Cesàro matrix of order α, denoted by $\zeta^{\alpha} = (\varsigma_{ms}^{(\alpha)})_{m,s\in\mathbb{N}}$, is defined by

$$\varsigma_{ms}^{(\alpha)} = \begin{cases} \dfrac{\binom{m-s+\alpha-1}{m-s}}{\binom{m+\alpha}{m}}, & 0 \le s \le m, \\ 0, & s > m, \end{cases} \qquad (m, s \in \mathbb{N}).$$

The method of order α extends the classical Cesàro summability and is regular for $\alpha \ge 0$, while it fails to be regular for $-1 < \alpha < 0$.

Recent developments in q-calculus have led to natural q-analogues of Cesàro-type operators. In this context, Aktuğlu and Bekar [10] introduced the q-Cesàro matrix $\zeta^{(q)} = (\varsigma_{ms}^{(q)})_{m,s\in\mathbb{N}}$ defined by

$$\varsigma_{ms}^{(q)} = \begin{cases} \dfrac{q^{m-s}}{[m+1]_q}, & 0 \le s \le m, \\ 0, & s > m. \end{cases}$$

Subsequently, Demiriz and Şahin [11] constructed q-Cesàro sequence spaces generated by this matrix and analyzed their structural properties.

Extending this approach to the higher-order setting, Bekar [10] also introduced the second-order q-Cesàro matrix

$$\zeta^{(q,2)} = (\varsigma_{ms}^{(q,2)})_{m,s\in\mathbb{N}},$$

defined by

$$\varsigma_{ms}^{(q,2)} = \begin{cases} \dfrac{q^{2s}\,[\,m-s+1\,]_q}{\sum\limits_{j=0}^{m} q^{2j}[\,m-j+1\,]_q}, & 0 \le s \le m, \\ 0, & s > m. \end{cases} \tag{1.2}$$

The first-order q-Cesàro matrix and its associated matrix-domain sequence spaces have been widely studied. However, a systematic investigation of the sequence spaces generated by the second-order q-Cesàro matrix $\zeta^{(q,2)}$ appears to be lacking in the literature. Although Bekar [10] defined the matrix in the form given in (1.2), a direct computation of the denominator yields the following equivalent representation:

$$\varsigma_{ms}^{(q,2)} = \begin{cases} \dfrac{q^{2s}\,[\,m-s+1\,]_q}{\begin{bmatrix} m+2 \\ 2 \end{bmatrix}_q}, & 0 \le s \le m, \\ 0, & s > m. \end{cases}$$

Lemma 1.1. *The second-order q-Cesàro matrix $\zeta^{(q,2)}$ has the following properties:*

- *(i) $\zeta^{(q,2)}$ is conservative for every $q > 0$.*
- *(ii) $\zeta^{(q,2)}$ is regular for every $q \ge 1$.*

In the particular case $q = 1$, $\zeta^{(q,2)}$ coincides with the classical second-order Cesàro matrix.

Proof. We verify the Silverman-Toeplitz conditions. Since $\varsigma_{ms}^{(q,2)} \ge 0$ and

$$\sum_{s=0}^{m} q^{2s}[m-s+1]_q = \begin{bmatrix} m+2 \\ 2 \end{bmatrix}_q,$$

we obtain

$$\sum_{s=0}^{\infty} |\varsigma_{ms}^{(q,2)}| = \sum_{s=0}^{m} \varsigma_{ms}^{(q,2)} = 1 \quad (m \in \mathbb{N}),$$

which implies

$$\sup_m \sum_{s=0}^{\infty} |\varsigma_{ms}^{(q,2)}| = 1 \quad \text{and} \quad \lim_{m\to\infty} \sum_{s=0}^{\infty} \varsigma_{ms}^{(q,2)} = 1.$$

Fix $s \in \mathbb{N}$. If $q > 1$, it follows that $\varsigma_{ms}^{(q,2)} \to 0$ as $m \to \infty$. If $q = 1$, the matrix coincides with the classical second order Cesàro matrix, which is known to be regular. Hence, for every $q \ge 1$, the Silverman-Toeplitz conditions are satisfied and $\zeta^{(q,2)}$ is regular. Moreover, since the column limits exist for every $q > 0$, $\zeta^{(q,2)}$ is conservative for every $q > 0$. □

We next determine the inverse of $\zeta^{(q,2)}$, denoted by $(\varsigma^{(q,2)})^{-1} = (\tau_{ms})$,

$$\tau_{ms} = \begin{cases} (-1)^{m-s} q^{-2m+\binom{m-s}{2}} \begin{bmatrix} s+2 \\ 2 \end{bmatrix}_q \begin{bmatrix} 2 \\ m-s \end{bmatrix}_q, & \text{if } m-2 \le s \le m, \\ 0, & \text{otherwise}. \end{cases} \tag{1.3}$$

In this paper, we introduce and investigate the matrix-domain sequence spaces generated by $\zeta^{(q,2)}$. More precisely, we define

$$c(\zeta^{(q,2)}) := \{x \in \omega : \ \zeta^{(q,2)}x \in c\}, \qquad c_0(\zeta^{(q,2)}) := \{x \in \omega : \ \zeta^{(q,2)}x \in c_0\}.$$

We examine the basic topological structure of these spaces, determine their dual spaces, and characterize related matrix transformations.

Beyond being a mere higher-order analogue, the second-order q-Cesàro operator exhibits a substantially different structural pattern, most notably through its three-banded inverse matrix. This structural distinction affects the topology of the generated sequence spaces and the characterization of their duals, thereby enriching the matrix-domain theory in the context of q-calculus.

2. New q-Cesàro sequence spaces of second order

In this section, we introduce new sequence spaces generated by the second-order q-Cesàro matrix $\zeta^{(q,2)}$ within the matrix-domain framework. More precisely, we define

$$c(\zeta^{(q,2)}) = \left\{ x = (x_s) \in \omega : \lim_{m\to\infty} \left(\frac{1}{\begin{bmatrix} m+2 \\ 2 \end{bmatrix}_q} \sum_{s=0}^{m} q^{2s} [m-s+1]_q \, x_s \right) \text{ exists} \right\},$$

$$c_0(\zeta^{(q,2)}) = \left\{ x = (x_s) \in \omega : \lim_{m\to\infty} \left(\frac{1}{\begin{bmatrix} m+2 \\ 2 \end{bmatrix}_q} \sum_{s=0}^{m} q^{2s} [m-s+1]_q \, x_s \right) = 0 \right\}.$$

In view of the notation in (1.1), the spaces $c(\zeta^{(q,2)})$ and $c_0(\zeta^{(q,2)})$ can be written as

$$c(\zeta^{(q,2)}) = \{c\}_{\zeta^{(q,2)}}, \qquad c_0(\zeta^{(q,2)}) = \{c_0\}_{\zeta^{(q,2)}}.$$

We now show that these spaces are BK-spaces under a suitable norm.

Theorem 2.1. *The sequence spaces $c_0(\zeta^{(q,2)})$ and $c(\zeta^{(q,2)})$ are BK-spaces endowed with the norm*

$$\|x\|_{c(\zeta^{(q,2)})} = \sup_{m\in\mathbb{N}} \left| \frac{1}{\begin{bmatrix} m+2 \\ 2 \end{bmatrix}_q} \sum_{s=0}^{m} [m-s+1]_q \, q^{2s} x_s \right|.$$

Proof. By Wilansky's Theorem 4.3.12 [3, p. 63], if X is a BK-space and Ω is a triangle, then the matrix domain X_Ω is also a BK-space equipped with the norm $\|x\|_{X_\Omega} = \|\Omega x\|_X$. Since the spaces c and c_0 are BK-spaces under the supremum norm and $\zeta^{(q,2)}$ is a triangle, it follows that $c(\zeta^{(q,2)})$ and $c_0(\zeta^{(q,2)})$ are BK-spaces endowed with the norm $\|x\|_{c(\zeta^{(q,2)})} = \|\zeta^{(q,2)} x\|_\infty$. □

Remark 2.2. *The spaces $c(\zeta^{(q,2)})$ and $c_0(\zeta^{(q,2)})$ are of non-absolute type. Indeed, consider the sequence $x = (1, -1, 0, 0, \dots)$. Then, we obtain*

$$(\zeta^{(q,2)} x)_m = \frac{1}{\begin{bmatrix} m+2 \\ 2 \end{bmatrix}_q} ([m+1]_q - q^2 [m]_q),$$

whereas for $|x| = (1, 1, 0, 0, \dots)$ we have

$$(\zeta^{(q,2)} |x|)_m = \frac{1}{\begin{bmatrix} m+2 \\ 2 \end{bmatrix}_q} ([m+1]_q + q^2 [m]_q).$$

Hence,

$$\|x\|_{c(\zeta^{(q,2)})} \neq \||x|\|_{c(\zeta^{(q,2)})},$$

and therefore the absolute property fails.

Theorem 2.3. *The sequence spaces $c(\zeta^{(q,2)})$ and $c_0(\zeta^{(q,2)})$ are isometrically isomorphic to c and c_0, respectively.*

Proof. Define the linear operator $T : c(\zeta^{(q,2)}) \to c$ by $T(x) = \zeta^{(q,2)}x$. By definition of the matrix domain, T is well defined and linear. Since $\zeta^{(q,2)}$ is a triangle, its inverse $(\zeta^{(q,2)})^{-1}$ is also triangular and is given in (1.3). Hence, T is bijective with inverse $T^{-1}(y) = (\zeta^{(q,2)})^{-1}y$. Moreover, for every $x \in c(\zeta^{(q,2)})$,

$$\|x\|_{c(\zeta^{(q,2)})} = \|\zeta^{(q,2)}x\|_\infty = \|T(x)\|_\infty,$$

which shows that T is an isometry. Therefore, $c(\zeta^{(q,2)})$ is isometrically isomorphic to c.

The proof for $c_0(\zeta^{(q,2)})$ follows similarly. □

Theorem 2.4. *Let* $0 < q < 1$. *Then, the inclusion*

$$c \subset c(\zeta^{(q,2)})$$

holds and is strict.

Proof. Since $\zeta^{(q,2)}$ is conservative for every $q > 0$, we have $c \subset c(\zeta^{(q,2)})$.

Assume $0 < q < 1$ and consider $x = ([-1]^s)$. Clearly, $x \notin c$. Moreover,

$$(\zeta^{(q,2)}x)_m = \sum_{s=0}^{m} \varsigma_{ms}^{(q,2)}(-1)^s.$$

For each fixed s and $0 < q < 1$,

$$\begin{aligned}
\lim_{m\to\infty} \varsigma_{ms}^{(q,2)} &= \lim_{m\to\infty} \frac{q^{2s}\,[\,m-s+1\,]_q}{\begin{bmatrix} m+2 \\ 2 \end{bmatrix}_q} \\
&= \lim_{m\to\infty} \frac{q^{2s}\,[\,m-s+1\,]_q}{\frac{[\,m+2\,]_q[\,m+1\,]_q}{[\,2\,]_q}} \\
&= \lim_{m\to\infty} \frac{q^{2s}\,[\,m-s+1\,]_q[\,2\,]_q}{[\,m+2\,]_q[\,m+1\,]_q} \\
&= \lim_{m\to\infty} \frac{q^{2s}\,(1-q^{m-s+1})(1+q)(1-q)}{(1-q^{m+2})(1-q^{m+1})} \\
&= q^{2s}(1-q^2).
\end{aligned}$$

Moreover, since $0 < q < 1$, the series is geometric. Hence,

$$\sum_{s=0}^{\infty} q^{2s}(1-q^2) = (1-q^2)\sum_{s=0}^{\infty} q^{2s} = (1-q^2)\frac{1}{1-q^2} = 1.$$

Therefore,

$$\lim_{m\to\infty} (\zeta^{(q,2)}x)_m = \sum_{s=0}^{\infty} q^{2s}(1-q^2)(-1)^s = \frac{1-q^2}{1+q^2},$$

so $\zeta^{(q,2)}x \in c$ and the inclusion is strict. □

Theorem 2.5. *Let* $0 < q < 1$. *Then, the inclusion*

$$c_0 \subset c_0(\zeta^{(q,2)})$$

holds and is strict.

Proof. Since $\zeta^{(q,2)}$ is conservative for every $0 < q < 1$, we have

$$c_0 \subset c_0(\zeta^{(q,2)}).$$

To show that the inclusion is strict, first assume that $0 < q < 1$ and define the sequence $x^{(q)} = (x_s^{(q)})_{s\in\mathbb{N}}$, as

$$x_s^{(q)} = \begin{cases} 1, & s \text{ even}, \\ -\dfrac{1}{q^2}, & s \text{ odd}. \end{cases}$$

Clearly, $x^{(q)} \notin c_0$. Also, $x^{(q)}$ can be written as

$$x_s^{(q)} = \frac{1-q^{-2}}{2} + \frac{1+q^{-2}}{2}(-1)^s.$$

Hence, by the linearity of $\zeta^{(q,2)}$,

$$(\zeta^{(q,2)}x^{(q)})_m = \frac{1-q^{-2}}{2}\,(\zeta^{(q,2)}e)_m + \frac{1+q^{-2}}{2}\,(\zeta^{(q,2)}((-1)^s))_m,$$

where $e = (1,1,1,\dots)$. By Theorem 2.4,

$$(\zeta^{(q,2)}e)_m = 1, \qquad \text{for all } m,$$

and

$$\lim_{m\to\infty} (\zeta^{(q,2)}((-1)^s))_m = \frac{1-q^2}{1+q^2},$$

it follows that

$$\lim_{m\to\infty} (\zeta^{(q,2)}x^{(q)})_m = \frac{1-q^{-2}}{2} + \frac{1+q^{-2}}{2}\cdot\frac{1-q^2}{1+q^2} = 0.$$

Thus, $\zeta^{(q,2)}x^{(q)} \in c_0$. Hence,

$$x^{(q)} \in c_0(\zeta^{(q,2)}) \setminus c_0.$$

Therefore, the inclusion is strict. □

Theorem 2.6. *Let* $0 < q < 1$. *Then,*

$$c_0(\zeta^{(q,2)}) \not\subseteq \ell_\infty, \quad \textit{and} \quad c(\zeta^{(q,2)}) \not\subseteq \ell_\infty.$$

Proof. For $x = (x_s) \in \omega$, let

$$y_m = (\zeta^{(q,2)}x)_m = \frac{[2]_q}{[m+1]_q[m+2]_q}\sum_{s=0}^{m}[m-s+1]_q\, q^{2s}x_s, \qquad m \in \mathbb{N},$$

and define

$$B_m := \frac{[m+1]_q[m+2]_q}{[2]_q}\, y_m.$$

Then,

$$B_m = \sum_{s=0}^{m}[m-s+1]_q q^{2s} x_s.$$

Hence, by the inverse relation for $\zeta^{(q,2)}$ given in (1.3), we obtain

$$x_m = q^{-2m}(B_m - (1+q)B_{m-1} + qB_{m-2}), \qquad m \geq 2.$$

To prove that $c_0(\zeta^{(q,2)}) \not\subseteq \ell_\infty$, choose

$$B_m = q^{3m/2}.$$

Then,

$$y_m = \frac{[2]_q\, q^{3m/2}}{[m+1]_q[m+2]_q}.$$

Since $0 < q < 1$, we have $q^{3m/2} \to 0$. Hence, $y_m \to 0$, that is, $y \in c_0$.

Also,

$$x_m = q^{-m/2}\Big(1 - (1+q)q^{-3/2} + q^{-2}\Big).$$

Therefore, $|x_m| \to \infty$, and so $x \notin \ell_\infty$. As $\zeta^{(q,2)}x = y \in c_0$, it follows that

$$c_0(\zeta^{(q,2)}) \not\subseteq \ell_\infty.$$

Similarly, choosing

$$B_m = 1 + q^{3m/2},$$

we obtain $y \in c$ and $x \notin \ell_\infty$, so that

$$c(\zeta^{(q,2)}) \not\subseteq \ell_\infty.$$

This completes the proof. □

Definition 2.7. *Let X be a normed linear space with norm $\|\cdot\|$. The space X is said to possess a Schauder basis $u = (u^{(s)})$ if for every $w = (w_m) \in X$ there exists a unique scalar sequence $a = (a_s)$ such that*

$$\lim_{j\to\infty}\left\| w - \sum_{s=0}^{j} a_s\, u^{(s)} \right\| = 0.$$

It is known that if T is a triangle, then the matrix domain X_T has a Schauder basis precisely when X has a Schauder basis. As an immediate consequence, we obtain the following result.

Theorem 2.8. *For each $s \in \mathbb{N}$, define the sequence $b^{(s)}(q) = (b^{(s)}_m(q))_{m\in\mathbb{N}}$ by*

$$b^{(s)}_m(q) = \begin{cases} (-1)^{m-s} q^{-2m+\binom{m-s}{2}} \begin{bmatrix} s+2 \\ 2 \end{bmatrix}_q \begin{bmatrix} 2 \\ m-s \end{bmatrix}_q, & \textit{if } m-2 \leq s \leq m, \\ 0, & \textit{otherwise}. \end{cases}$$

Then, each of the following statements holds:

(i) *The set* $\{ b^{(0)}(q), b^{(1)}(q), b^{(2)}(q), \dots \}$ *is a Schauder basis of* $c_0(\zeta^{(q,2)})$*, and every* $x \in c_0(\zeta^{(q,2)})$ *is uniquely represented as*

$$x = \sum_{s=0}^{\infty} t_s \, b^{(s)}(q), \qquad \text{where } t_s = (\zeta^{(q,2)} x)_s \ (s \in \mathbb{N}).$$

(ii) *The set* $\{ b(q), b^{(0)}(q), b^{(1)}(q), b^{(2)}(q), \dots \}$ *is a Schauder basis of* $c(\zeta^{(q,2)})$*, and every* $x \in c(\zeta^{(q,2)})$ *is uniquely represented as*

$$x = \ell \, b(q) + \sum_{s=0}^{\infty} (t_s - \ell) \, b^{(s)}(q),$$

where $\ell = \lim_{m\to\infty} (\zeta^{(q,2)} x)_m$*.*

3. The α-, β-, and γ-duals of the spaces $c_0(\zeta^{(q,2)})$ and $c(\zeta^{(q,2)})$

In this section, we derive the α-, β-, and γ-duals of $c_0(\zeta^{(q,2)})$ and $c(\zeta^{(q,2)})$.

For sequence spaces λ and μ, define the multiplier space

$$\mathcal{M}(\lambda, \mu) = \{ a = (a_s) \in \omega : (a_s x_s) \in \mu \text{ for each } x = (x_s) \in \lambda \} . \tag{3.1}$$

In terms of (3.1), the α-, β- and γ-duals of a sequence space λ are, respectively,

$$\lambda^{\alpha} = \mathcal{M}(\lambda, \ell_1), \qquad \lambda^{\beta} = \mathcal{M}(\lambda, cs), \qquad \lambda^{\gamma} = \mathcal{M}(\lambda, bs),$$

where cs denotes the space of all convergent series and bs denotes the space of all series whose sequence of partial sums is bounded.

For two sequence spaces X and Y, $(X : Y)$ denotes the class of all infinite matrices mapping X into Y.

Before proceeding to the main results, we state an auxiliary lemma due to Stieglitz and Tietz [25], which plays a central role in the proofs of Theorems 3.3–3.5.

Lemma 3.1. *An infinite matrix* $A = (a_{ms})$ *belongs to* $(c_0 : \ell_1) = (c : \ell_1)$ *if and only if*

$$\sup_{M,S \in \mathcal{F}} \left| \sum_{m \in M} \sum_{s \in S} a_{ms} \right| < \infty,$$

where $\mathcal{F}$ *denotes the collection of all finite subsets of* $\mathbb{N}$*.*

Lemma 3.2. *An infinite matrix* $A = (a_{ms})$ *belongs to* $(c : c)$ *if and only if the following conditions hold:*

$$\lim_{m\to\infty} a_{ms} \text{ exists for each } s \in \mathbb{N}_0, \tag{3.2}$$

$$\sup_{m \in \mathbb{N}_0} \sum_{s=0}^{\infty} |a_{ms}| < \infty, \tag{3.3}$$

$$\lim_{m\to\infty} \sum_{s=0}^{\infty} a_{ms} \quad \text{exists.} \tag{3.4}$$

Theorem 3.3. *Define*

$$d_1 = \left\{ a \in \omega : \sup_{M,S \in \mathcal{F}} \left| \sum_{m \in M} \sum_{s \in S} (-1)^{m-s} q^{-2m+\binom{m-s}{2}} \begin{bmatrix} s+2 \\ 2 \end{bmatrix}_q \begin{bmatrix} 2 \\ m-s \end{bmatrix}_q \alpha_m \right| < \infty \right\}.$$

Then,

$$\left\{ c_0(\zeta^{(q,2)}) \right\}^{\alpha} = \left\{ c(\zeta^{(q,2)}) \right\}^{\alpha} = d_1.$$

Proof. Let $\alpha = (\alpha_s) \in \omega$ and define the matrix $B = (b_{ms})$ via the sequence $\alpha = (\alpha_s)$ by

$$b_{ms} = \begin{cases} (-1)^{m-s} q^{-2m+\binom{m-s}{2}} \begin{bmatrix} s+2 \\ 2 \end{bmatrix}_q \begin{bmatrix} 2 \\ m-s \end{bmatrix}_q \alpha_m, & \text{if } m-2 \le s \le m, \\ 0, & \text{otherwise.} \end{cases}$$

Since the inverse matrix $(\zeta^{(q,2)})^{-1} = (\tau_{ms})$ is given by (1.3), we obtain

$$x_m = \sum_{s=m-2}^{m} (-1)^{m-s} q^{-2m+\binom{m-s}{2}} \begin{bmatrix} s+2 \\ 2 \end{bmatrix}_q \begin{bmatrix} 2 \\ m-s \end{bmatrix}_q y_s.$$

We immediately derive that

$$\alpha_m x_m = \sum_{s=m-2}^{m} (-1)^{m-s} q^{-2m+\binom{m-s}{2}} \begin{bmatrix} s+2 \\ 2 \end{bmatrix}_q \begin{bmatrix} 2 \\ m-s \end{bmatrix}_q \alpha_m y_s = (By)_m, (m \in \mathbb{N}).$$

Since Lemma 3.1 characterizes the classes $(c_0 : \ell_1)$ and $(c : \ell_1)$ and the matrix B satisfies the condition of Lemma 3.1, it follows that the α-dual of $c_0(\zeta^{(q,2)})$ and $c(\zeta^{(q,2)})$ is d_1. This completes the proof. □

Theorem 3.4. *Define*

$$d_2 = \left\{ \alpha \in \omega : \sum_{s=0}^{\infty} \left| q^{-2s-3} \begin{bmatrix} s+2 \\ 2 \end{bmatrix}_q \left(q^3 \alpha_s - q[2]_q \alpha_{s+1} + \alpha_{s+2} \right) \right| < \infty \right\},$$

$$d_3 = \left\{ \alpha \in \omega : \sup_{m \in \mathbb{N}} \left| q^{-2m} \begin{bmatrix} m+2 \\ 2 \end{bmatrix}_q \alpha_m \right| < \infty \right\}.$$

Then,

$$\left\{ c_0(\zeta^{(q,2)}) \right\}^{\beta} = d_2 \cap cs \textit{ and } \left\{ c(\zeta^{(q,2)}) \right\}^{\beta} = d_2 \cap d_3.$$

Proof. Since the proof for the space $c_0(\zeta^{(q,2)})$ can be obtained in a similar manner, we present the argument only for the space $c(\zeta^{(q,2)})$.

Consider the equality

$$\sum_{s=0}^{m} \alpha_s x_s = \sum_{s=0}^{m} \alpha_s \left(\sum_{j=s-2}^{s} (-1)^{s-j} q^{-2s+\binom{s-j}{2}} \begin{bmatrix} j+2 \\ 2 \end{bmatrix}_q \begin{bmatrix} 2 \\ s-j \end{bmatrix}_q y_j \right).$$

Rearranging the sums with respect to y_s yields

$$\sum_{s=0}^{m} \alpha_s x_s = \sum_{s=0}^{m-2} q^{-2s-3} \begin{bmatrix} s+2 \\ 2 \end{bmatrix}_q \left(q^3 \alpha_s - q[2]_q \alpha_{s+1} + \alpha_{s+2} \right) y_s$$

$$
\begin{aligned}
&+ q^{-2m}\begin{bmatrix} m+1 \\ 2 \end{bmatrix}_q \left(q^2\alpha_{m-1} - [2]_q\alpha_m\right) y_{m-1} + q^{-2m}\begin{bmatrix} m+2 \\ 2 \end{bmatrix}_q \alpha_m y_m \\
&= (Ty)_m, \qquad (m \in \mathbb{N}),
\end{aligned} \tag{3.5}
$$

where the matrix $T(q) = (t_{ms})$ is defined by

$$
t_{ms} = \begin{cases} q^{-2s-3}\left[\begin{smallmatrix} s+2 \\ 2 \end{smallmatrix}\right]_q \left(q^3\alpha_s - q[2]_q\alpha_{s+1} + \alpha_{s+2}\right), & 0 \le s \le m-2, \\ q^{-2m}\left[\begin{smallmatrix} m+1 \\ 2 \end{smallmatrix}\right]_q \left(q^2\alpha_{m-1} - [2]_q\alpha_m\right), & s = m-1, \\ q^{-2m}\left[\begin{smallmatrix} m+2 \\ 2 \end{smallmatrix}\right]_q \alpha_m, & s = m, \\ 0, & s > m. \end{cases}
$$

Hence,

$$
\sum_{s=0}^{m} \alpha_s x_s = (T(q)y)_m.
$$

The readers can easily verify that

$$
\lim_{m\to\infty} t_{ms} = q^{-2s-3}\left(q^3\alpha_s - q[2]_q\alpha_{s+1} + \alpha_{s+2}\right). \tag{3.6}
$$

Thus, the columns of the matrix $T(q)$ are convergent. Therefore, by Lemma 3.2 and by (3.5), we conclude that $\alpha x = (\alpha_s x_s) \in cs$ for every $x = (x_s) \in c(\zeta^{(q,2)})$ if and only if $T(q)y \in c$ for all $y = (y_s) \in c$. Therefore, from the conditions given in (3.2)–(3.4), and taking in account the Eq (3.6), we obtain

$$
\sum_{s=0}^{\infty} \left| q^{-2s-3}\begin{bmatrix} s+2 \\ 2 \end{bmatrix}_q \left(q^3\alpha_s - q[2]_q\alpha_{s+1} + \alpha_{s+2}\right)\right| < \infty,
$$

and

$$
\alpha = (\alpha_s) \in cs.
$$

This shows that $\left\{c_0(\zeta^{(q,2)})\right\}^{\beta} = d_2 \cap cs$. □

Theorem 3.5. *The γ-dual of the sequence spaces $c_0(\zeta^{(q,2)})$ and $c(\zeta^{(q,2)})$ is given by*

$$
c_0(\zeta^{(q,2)})^{\gamma} = c(\zeta^{(q,2)})^{\gamma} = d_2 \cap d_3.
$$

Proof. Let $\alpha = (\alpha_s) \in \omega$ and put $y = \zeta^{(q,2)}x$. Using the same notation as in the proof of Theorem 3.4, we have

$$
\sum_{s=0}^{m} \alpha_s x_s = (T(q)y)_m.
$$

Hence, $\alpha \in c_0(\zeta^{(q,2)})^{\gamma}$ if and only if $T(q)y \in \ell_\infty$ for every $y \in c_0$, that is,

$$
T(q) \in (c_0 : \ell_\infty).
$$

Similarly, $\alpha \in c(\zeta^{(q,2)})^{\gamma}$ if and only if

$$
T(q) \in (c : \ell_\infty).
$$

By the standard characterizations of these matrix classes and by the definition of the sets d_2 and d_3, both conditions are equivalent to

$$\alpha \in d_2 \cap d_3.$$

Therefore,

$$c_0(\zeta^{(q,2)})^{\gamma} = c(\zeta^{(q,2)})^{\gamma} = d_2 \cap d_3.$$

□

4. Matrix classes

In this section, we introduce certain matrix classes associated with the newly defined q-Cesàro sequence spaces of second order.

We first establish two auxiliary theorems which will be used throughout this section.

Theorem 4.1. *Let μ be any sequence space, and let $\Lambda = (\lambda_{ms})$ be an infinite matrix for all $m, s \in \mathbb{N}_0$. Define the associated matrix $\Upsilon = (\upsilon_{ms})$ by*

$$\upsilon_{ms} := q^{-2s}\begin{bmatrix} s+2 \\ 2 \end{bmatrix}_q \lambda_{ms} - q^{-2s-2}[s+2]_q[s+1]_q\lambda_{m,s+1} + q^{-2s-3}\begin{bmatrix} s+2 \\ 2 \end{bmatrix}_q \lambda_{m,s+2}. \tag{4.1}$$

Then,

(i) $\Lambda \in (c(\zeta^{(q,2)}) : \mu)$ if and only if, for each fixed $m \in \mathbb{N}_0$, the sequence $(\lambda_{ms})_{s=0}^{\infty}$ belongs to $\{c(\zeta^{(q,2)})\}^{\beta}$ and $\Upsilon \in (c : \mu)$.

(ii) $\Lambda \in (c_0(\zeta^{(q,2)}) : \mu)$ if and only if, for each fixed $m \in \mathbb{N}_0$, the sequence $(\lambda_{ms})_{s=0}^{\infty}$ belongs to $\{c_0(\zeta^{(q,2)})\}^{\beta}$ and $\Upsilon \in (c_0 : \mu)$.

Proof. It is sufficient to prove part (i), since part (ii) follows by a similar argument.

Let $x = (x_s) \in c(\zeta^{(q,2)})$, and set

$$y = (y_s) := \zeta^{(q,2)}x.$$

Then, $y \in c$. By Theorem 2.3, the spaces $c(\zeta^{(q,2)})$ and c are linearly isomorphic. Moreover, using the inverse representation of x in terms of y, we have

$$x_s = q^{-2s}\begin{bmatrix} s+2 \\ 2 \end{bmatrix}_q y_s - q^{-2s-2}[s+2]_q[s+1]_q y_{s+1} + q^{-2s-3}\begin{bmatrix} s+2 \\ 2 \end{bmatrix}_q y_{s+2}, \qquad s \in \mathbb{N}_0.$$

Hence, for each fixed $m \in \mathbb{N}_0$,

$$\begin{aligned}(\Lambda x)_m &= \sum_{s=0}^{\infty} \lambda_{ms}x_s = \sum_{s=0}^{\infty} \lambda_{ms}\left(q^{-2s}\begin{bmatrix} s+2 \\ 2 \end{bmatrix}_q y_s - q^{-2s-2}[s+2]_q[s+1]_q y_{s+1} + q^{-2s-3}\begin{bmatrix} s+2 \\ 2 \end{bmatrix}_q y_{s+2}\right) \\ &= \sum_{s=0}^{\infty} \upsilon_{ms}y_s = (\Upsilon y)_m.\end{aligned}$$

Therefore, $\Lambda x = \Upsilon y$.

Assume that $\Lambda \in (c(\zeta^{(q,2)}) : \mu)$. Then, for each fixed $m \in \mathbb{N}_0$, the row $(\lambda_{ms})_{s=0}^{\infty}$ belongs to $\{c(\zeta^{(q,2)})\}^{\beta}$. Also, for every $y \in c$, by Theorem 2.3 there exists $x \in c(\zeta^{(q,2)})$ such that $y = \zeta^{(q,2)}x$. Since $\Lambda x \in \mu$ and $\Lambda x = \Upsilon y$, we obtain $\Upsilon y \in \mu$. Hence, $\Upsilon \in (c : \mu)$.

Conversely, suppose that for each fixed $m \in \mathbb{N}_0$, $(\lambda_{ms})_{s=0}^{\infty} \in \{c(\zeta^{(q,2)})\}^{\beta}$ and that $\Upsilon \in (c : \mu)$. Let $x \in c(\zeta^{(q,2)})$ and set $y = \zeta^{(q,2)}x \in c$. Then, Λx exists, and since $\Upsilon y \in \mu$ and $\Lambda x = \Upsilon y$, we conclude that $\Lambda x \in \mu$. Therefore, $\Lambda \in (c(\zeta^{(q,2)}) : \mu)$. □

Theorem 4.2. *Let μ be a sequence space and let $\Lambda = (\lambda_{ms})$ be an infinite matrix. Define the associated matrix $B = (b_{ms})$ by*

$$b_{ms} := \sum_{k=0}^{m} \varsigma_{mk}^{(q,2)} \lambda_{ks}, \qquad m, s \in \mathbb{N}_0, \tag{4.2}$$

where $\varsigma_{mk}^{(q,2)}$ denotes the (m,k)th entry of the matrix $\zeta^{(q,2)}$. Then,

(i) $\Lambda \in (\mu : c(\zeta^{(q,2)}))$ *if and only if* $B \in (\mu : c)$.

(ii) $\Lambda \in (\mu : c_0(\zeta^{(q,2)}))$ *if and only if* $B \in (\mu : c_0)$.

Proof. It is sufficient to prove part (*i*), since part (*ii*) follows by a similar argument.

Let $z = (z_s) \in \mu$. Then, for each $m \in \mathbb{N}_0$, we have

$$(Bz)_m = \sum_{s=0}^{\infty} b_{ms} z_s = \sum_{s=0}^{\infty} \left(\sum_{k=0}^{m} \varsigma_{mk}^{(q,2)} \lambda_{ks} \right) z_s = \sum_{k=0}^{m} \varsigma_{mk}^{(q,2)} \left(\sum_{s=0}^{\infty} \lambda_{ks} z_s \right) = \sum_{k=0}^{m} \varsigma_{mk}^{(q,2)} (\Lambda z)_k = (\zeta^{(q,2)}(\Lambda z))_m.$$

Hence,

$$Bz = \zeta^{(q,2)}(\Lambda z).$$

By the definition of the matrix domain $c(\zeta^{(q,2)})$, we have $\Lambda z \in c(\zeta^{(q,2)})$ if and only if $\zeta^{(q,2)}(\Lambda z) \in c$. Since $Bz = \zeta^{(q,2)}(\Lambda z)$, it follows that $\Lambda z \in c(\zeta^{(q,2)})$ if and only if $Bz \in c$ for every $z \in \mu$. Therefore, $\Lambda \in (\mu : c(\zeta^{(q,2)}))$ if and only if $B \in (\mu : c)$. □

For an infinite matrix $A = (a_{ms})$, we shall use the following standard conditions due to Stieglitz and Tietz [25]:

$$\sup_{m \in \mathbb{N}_0} \sum_{s} |a_{ms}| < \infty, \tag{4.3}$$

$$\lim_{m \to \infty} a_{ms} \text{ exists} \qquad (\forall s \in \mathbb{N}_0), \tag{4.4}$$

$$\lim_{m \to \infty} \sum_{s} a_{ms} \text{ exists}, \tag{4.5}$$

$$\lim_{m \to \infty} a_{ms} = 0, \tag{4.6}$$

$$\lim_{m \to \infty} \sum_{s} a_{ms} = 0, \tag{4.7}$$

$$\sup_{M, S \in \mathcal{F}} \left| \sum_{m \in M} \sum_{s \in S} a_{ms} \right| < \infty, \tag{4.8}$$

$$\sup_{m, s \in \mathbb{N}_0} |a_{ms}| < \infty, \tag{4.9}$$

$$\lim_{m\to\infty}\sum_s |a_{ms}| = 0, \tag{4.10}$$

$$\lim_{m\to\infty}\sum_s |a_{ms}| = \sum_s \left|\lim_{m\to\infty} a_{ms}\right|, \tag{4.11}$$

where $\mathcal{F}$ denotes the family of all finite subsets of $\mathbb{N}_0$.

Next, we present a table compiled from [25] which characterizes some known matrix classes. By Theorem 4.1 and Table 1, we obtain the following corollaries.

Corollary 4.3. *Let $\Lambda = (\lambda_{ms})$ be an infinite matrix, and let $\Upsilon = (\upsilon_{ms})$ be the matrix associated with Λ in Theorem 4.1. Then,*

(i) $\Lambda \in (c(\zeta^{(q,2)}) : c_0)$ if and only if, for each fixed $m \in \mathbb{N}_0$, $(\lambda_{ms})_{s=0}^{\infty} \in \{c(\zeta^{(q,2)})\}^{\beta}$ and conditions (4.3), (4.6), and (4.7) are satisfied by Υ.

(ii) $\Lambda \in (c(\zeta^{(q,2)}) : c)$ if and only if, for each fixed $m \in \mathbb{N}_0$, $(\lambda_{ms})_{s=0}^{\infty} \in \{c(\zeta^{(q,2)})\}^{\beta}$ and conditions (4.3)–(4.5) are satisfied by Υ.

(iii) $\Lambda \in (c(\zeta^{(q,2)}) : \ell_\infty)$ if and only if, for each fixed $m \in \mathbb{N}_0$, $(\lambda_{ms})_{s=0}^{\infty} \in \{c(\zeta^{(q,2)})\}^{\beta}$ and condition (4.3) is satisfied by Υ.

(iv) $\Lambda \in (c(\zeta^{(q,2)}) : \ell_1)$ if and only if, for each fixed $m \in \mathbb{N}_0$, $(\lambda_{ms})_{s=0}^{\infty} \in \{c(\zeta^{(q,2)})\}^{\beta}$ and condition (4.8) is satisfied by Υ.

Corollary 4.4. *Let $\Lambda = (\lambda_{ms})$ be an infinite matrix, and let $\Upsilon = (\upsilon_{ms})$ be the matrix associated with Λ in Theorem 4.1. Then,*

(i) $\Lambda \in (c_0(\zeta^{(q,2)}) : c_0)$ if and only if, for each fixed $m \in \mathbb{N}_0$, $(\lambda_{ms})_{s=0}^{\infty} \in \{c_0(\zeta^{(q,2)})\}^{\beta}$ and conditions (4.3) and (4.6) are satisfied by Υ.

(ii) $\Lambda \in (c_0(\zeta^{(q,2)}) : c)$ if and only if, for each fixed $m \in \mathbb{N}_0$, $(\lambda_{ms})_{s=0}^{\infty} \in \{c_0(\zeta^{(q,2)})\}^{\beta}$ and conditions (4.3) and (4.4) are satisfied by Υ.

(iii) $\Lambda \in (c_0(\zeta^{(q,2)}) : \ell_\infty)$ if and only if, for each fixed $m \in \mathbb{N}_0$, $(\lambda_{ms})_{s=0}^{\infty} \in \{c_0(\zeta^{(q,2)})\}^{\beta}$ and condition (4.3) is satisfied by Υ.

(iv) $\Lambda \in (c_0(\zeta^{(q,2)}) : \ell_1)$ if and only if, for each fixed $m \in \mathbb{N}_0$, $(\lambda_{ms})_{s=0}^{\infty} \in \{c_0(\zeta^{(q,2)})\}^{\beta}$ and condition (4.8) is satisfied by Υ.

By Theorem 4.2 together with Table 1, we obtain the following corollaries.

Corollary 4.5. *Let $\Lambda = (\lambda_{ms})$ be an infinite matrix, and let $B = (b_{ms})$ be the matrix associated with Λ in Theorem 4.2. Then,*

(i) $\Lambda \in (c_0 : c(\zeta^{(q,2)}))$ if and only if conditions (4.3) and (4.4) are satisfied by B.

(ii) $\Lambda \in (c : c(\zeta^{(q,2)}))$ if and only if conditions (4.3)–(4.5) are satisfied by B.

(iii) $\Lambda \in (\ell_\infty : c(\zeta^{(q,2)}))$ if and only if conditions (4.4) and (4.11) are satisfied by B.

(iv) $\Lambda \in (\ell_1 : c(\zeta^{(q,2)}))$ if and only if conditions (4.4) and (4.9) are satisfied by B.

Corollary 4.6. *Let* $\Lambda = (\lambda_{ms})$ *be an infinite matrix, and let* $B = (b_{ms})$ *be the matrix associated with* Λ *in Theorem 4.2. Then,*

- *(i)* $\Lambda \in (c_0 : c_0(\zeta^{(q,2)}))$ *if and only if conditions (4.3) and (4.6) are satisfied by B.*
- *(ii)* $\Lambda \in (c : c_0(\zeta^{(q,2)}))$ *if and only if conditions (4.3), (4.6), and (4.7) are satisfied by B.*
- *(iii)* $\Lambda \in (\ell_\infty : c_0(\zeta^{(q,2)}))$ *if and only if condition (4.10) is satisfied by B.*
- *(iv)* $\Lambda \in (\ell_1 : c_0(\zeta^{(q,2)}))$ *if and only if conditions (4.6) and (4.9) are satisfied by B.*

Table 1. Characterizations of the matrix classes $(\Lambda : \Upsilon)$, where $\Lambda, \Upsilon \in \{c, c_0, \ell_\infty, \ell_1\}$.

$(\Lambda \downarrow,\ \Upsilon \rightarrow)$	c	c_0	ℓ_∞	ℓ_1
c	(4.3), (4.4), (4.5)	(4.3), (4.6), (4.7)	(4.3)	(4.8)
c_0	(4.3), (4.4)	(4.3), (4.6)	(4.3)	(4.8)
ℓ_∞	(4.4), (4.11)	(4.10)	–	–
ℓ_1	(4.4), (4.9)	(4.6), (4.9)	–	–

The symbol "–" indicates conditions that are not needed in the present study.

5. Conclusions

In this paper, we studied the second-order q-Cesàro matrix together with its matrix domains in the spaces c_0 and c. Within this framework, we established the fundamental structural properties of the newly defined spaces and examined their relationships with classical and previously studied Cesàro-type sequence spaces. In particular, the inclusion results obtained here clarify the position of these spaces within the existing theory of summability and matrix domains. We also considered the associated duals and matrix transformations, thereby placing these spaces in a broader functional-analytic setting.

The results show that the second-order q-Cesàro matrix provides a natural extension of earlier q-Cesàro operators and offers a useful framework for the study of higher-order averaging methods in the q-setting. Thus, the present work contributes to the development of the theory of matrix domains, matrix transformations, and summability methods associated with q-analogues.

As a continuation of this study, it would be natural to investigate the matrix domains of the second-order q-Cesàro matrix in the spaces ℓ_p for $1 \leq p \leq \infty$. Such a study may lead to further results concerning the topological structure of these spaces, their duals, matrix classes, and compact operators. Moreover, the interaction between these spaces and other well-known sequence spaces may offer new insights into the role of q-analogue averaging processes in functional analysis and summability theory.

Author contributions

Hacer Bilgin Ellidokuzoğlu: conceptualization, methodology, formal analysis, investigation, writing-original draft, writing–review & editing, software; Serkan Demiriz: conceptualization, methodology, formal analysis, investigation, writing–original draft. All authors of this article contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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