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Research article

Exponentially quasi-mixing ergodicity for symmetric Markov processes

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Abstract: In this paper, we focused on the quasi-mixing limits of symmetric Markov processes. Under mild assumptions, we proved that (intrinsic) ultracontractivity of the transition semigroup implied (uniformly) exponentially quasi-mixing ergodicity of its associated process. As a by-product, we demonstrated that the underlying process exhibited (uniformly) exponentially quasi-ergodicity, (uniformly) exponentially fractional quasi-ergodicity, and (uniformly) mean-ratio quasi-ergodicity being proportional to time.

Keywords: symmetric Markov process; intrinsic ultracontractivity; quasi-mixing limit; quasi-ergodicity; fractional quasi-ergodicity; mean-ratio quasi-ergodicity
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1. Introduction

As fundamental stochastic models in probability theory, Markov processes are characterized by the memoryless property and have been widely adopted to describe random evolutions in physics, engineering, finance, and biology [9, 14]. The long-term asymptotic behavior of Markov processes is a central research topic, where quasi-stationarity and quasi-ergodicity are two essential notions for characterizing the conditional dynamical behavior of absorbed Markov processes [1]. As a generalized and refined concept, quasi-mixing limits encompass quasi-stationarity and quasi-ergodicity, and thus provide a more precise framework for understanding the long-term conditional asymptotic behavior of Markov processes.

Symmetric Markov processes admit rich spectral structures, and ultracontractivity serves as a crucial regularity condition that dominates their asymptotic properties [10]. Although researchers have investigated the quasi-stationarity and various quasi-ergodic behaviors of symmetric Markov processes, the intrinsic connection between ultracontractivity and quasi-mixing limits remains insufficiently explored. Against this background, we focus on the quasi-mixing limits of symmetric Markov processes. Under mild assumptions, we prove that the intrinsic ultracontractivity of the transition semigroup implies the uniformly exponential quasi-mixing ergodicity of the corresponding process. As a by-product, we further establish several types of quasi-ergodic behaviors, including uniformly exponential quasi-ergodicity, uniformly exponential fractional quasi-ergodicity, and uniformly mean-ratio quasi-ergodicity with linear temporal scaling. The results enrich and refine the asymptotic theory of ultracontractive symmetric Markov processes.

Let E be a locally compact separable metric space with a Borel σ-algebra $\mathscr{B}(E)$ and m a positive Radon measure on E with full support. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a regular Dirichet form on $L^2(E; m)$, $(-\mathscr{A}, \mathcal{D}(\mathscr{A}))$ be its generator, and $\{P_t := e^{-\mathscr{A}t}\}_{t\geq 0}$ be its associated semigroup. It is then well-known, cf. Fukushima and Oshima et al. [10, Theorem 7.2.1], that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is associated with an m-symmetric Hunt process $X = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \{X_t\}_{t\geq 0}, \{\mathbb{P}_x\}_{x\in E}, \zeta)$. Here, ζ is the lifetime of the process, i.e., $\zeta = \inf\{t \geq 0 : X_t = \Delta\}$, with Δ being the cemetery state, and $\mathbb{P}_x$ is the initial distribution of X from point $x \in E$. The Dirichlet form provides the analytic framework for constructing the symmetric Markov process, defining its generator $-\mathscr{A}$, and studying its spectral properties. It is the fundamental tool that links the analytic properties of the semigroup $\{P_t\}$ to the probabilistic behavior of the Hunt process X. Notice that such a Hunt process is unique up to a proper exceptional set $\mathcal{N}$. That is, if two Hunt processes are associated with the same Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, then their transition functions coincide on $E \setminus \mathcal{N}$ (see [10, Theorem 4.2.8]). Defining

$$\mathcal{P}_t f(x) := \mathbb{E}_x f(X_t)\mathbb{1}_{\{t<\zeta\}} = \mathbb{E}_x f(X_t),\ t \geq 0,\ x \in E \setminus \mathcal{N},\ f \in \mathscr{B}_b(E),$$

due to [10, Theorem 7.2.1], the relation between $\{P_t\}$ and $\{\mathcal{P}_t\}$ is given by the equation $P_t f = \mathcal{P}_t f$ m-a.e. for all $t > 0$ and $f \in \mathscr{B}_b(E) \cap L^2(E; m)$, where $\mathbb{E}_x$ denotes the expectation corresponding the probability measure $\mathbb{P}_x$, $\mathscr{B}_b(E)$ is the set of all bounded Borel functions on E. Here, we are interested in exploring quasi-mixing limits of X, or alternatively mixing limits of X conditioned to never exit E. There are three useful tools, namely quasi-stationary distribution, fractional quasi-stationary distribution, and mean-ratio quasi-stationary distribution, whose precise definitions are given by Definition 1.1 below (see Chen et al. [3,5], as well as Zhang et al. [16]), for describing quasi-mixing limits.

Definition 1.1. *Let $\mathcal{P}(E)$ be the set of all probability measures on $(E, \mathscr{B}(E))$ and $\mathbb{P}_\rho(\cdot) = \int_E \mathbb{P}_x(\cdot)\rho(dx)$ be the probability taken for X with an initial distribution $\rho \in \mathcal{P}(E)$.*

(i) *$\mu \in \mathcal{P}(E)$ is said to be a quasi-limiting distribution of X if there exists a $\rho \in \mathcal{P}(E)$ such that*

$$\lim_{t\to\infty} \mathbb{P}_\rho(X_t \in B \mid \zeta > t) = \mu(B),\quad B \in \mathscr{B}(E). \tag{1.1}$$

(ii) *For each $0 < a < 1$, $\pi_a \in \mathcal{P}(E)$ is said to be an (a-order) fractional quasi-stationary distribution of X if there exists a $\rho \in \mathcal{P}(E)$ such that*

$$\lim_{t\to\infty} \mathbb{P}_\rho(X_{at} \in B \mid \zeta > t) = \pi_a(B),\quad B \in \mathscr{B}(E). \tag{1.2}$$

(iii) *$\nu \in \mathcal{P}(E)$ is said to be a mean-ratio quasi-stationary distribution of X if there exists a $\rho \in \mathcal{P}(E)$ such that*

$$\lim_{t\to\infty} \mathbb{E}_\rho(L_t(B) \mid \zeta > t) = \nu(B),\quad B \in \mathscr{B}(E), \tag{1.3}$$

where L_t is the empirical distribution of X,

$$L_t(B) = \frac{1}{t}\int_0^t \mathbb{1}_B(X_s)ds, \quad t > 0.$$

A basic and useful property of the quasi-limiting distribution μ is $\mathbb{P}_\mu(X_t \in \cdot \mid \zeta > t) = \mu$ for any $t \geq 0$; therefore, μ is also called the quasi-stationary distribution of X. Provided that π_a in (1.2) are independent of $a \in (0, 1)$, π_a must be a mean-ratio quasi-stationary distribution of X by using Fubini's theorem and the dominated convergence (see Corollary 3.1(iii) below). In (1.1)–(1.3), we also say that ρ is attracted to μ, π_a, and ν, or is in the domain of attraction of μ, π_a, and ν for the conditional evolution. Furthermore, if μ, π_a, and ν attract all initial distributions $\rho \in \mathcal{P}(E)$, we then call that X admits quasi-ergodicity, fractional quasi-ergodicity, and mean-ratio quasi-ergodicity.

Exploring the long-term behavior of Markov processes is of great significance to mathematical and physical sciences (see e.g., [4, 13, 16]). For instance, the existence of quasi-mixing limits implies the existence of the aforementioned three quasi-stationary distributions of the process, which reflects the phase transition phenomena and is beneficial for understanding the large deviation principle of the empirical measure, quasi-ergodic theorem, and conditional functional weak law of large numbers of the process. Besides, Knobloch and Partzsch [12] proved that intrinsic ultracontractivity of the transition semigroup suggests uniform quasi-ergodicity of its associated process; He and Zhang [11] found that the absorbed Markov process fulfilling Champagnat and Villemonais [2, Assumption A] exhibits uniformly and exponentially quasi-mixing ergodicity; Zhang et al. [16] studied the quasi-stationarity and quasi-ergodicity of general Markov processes, but assumed that the underlying measure is finite. It is noteworthy that [11, 12, 16] do not specify a specific value for the exponential convergence rate of the quasi-stationary distribution.

In this paper, we show that (intrinsic) ultracontractivity of the transition semigroup implies (uniformly) exponentially quasi-mixing ergodicity of its associated process. As a corollary, we demonstrate that the process exhibits (uniformly) exponential quasi-ergodicity, (uniformly) exponential fractional quasi-ergodicity, and (uniformly) mean-ratio quasi-ergodicity proportional to time. We emphasize, comparing to [11, 12, 16], that our proof is straightforward, and the exponential convergence rate of quasi-mixing limits is determined by the difference between the higher order eigenvalues (depending on the initial distributions, but it is at least of second order) and the first order eigenvalue of the generator.

The paper is organized as follows: In Section 2, we collect all necessary notations, hypotheses, and technical lemmas. In Section 3, we state the major results (Theorem 3.1 and Corollary 3.1). In Section 4, we provide an illustrative example showing how the abstract theorem applies to a concrete differential operator.

2. Preliminaries

To highlight our results, we first provide an overview of mathematical notations and impose some hypotheses on $\{P_t\}$.

Notation 2.1. *The norm in the usual Banach space $L^p(E; m)$ is written as $\|\cdot\|_{p,m}$, $1 \leq p \leq \infty$; the scalar product and norm in $L^2(E; m)$ are written as $(\cdot, \cdot)_m$, $\|\cdot\|_m$, respectively; $(f, g)_m$ or $m(fg)$ will be used to denote the integration of fg with respect to m if $fg \in L^1(E; m)$; for a fixed $f \in L^1(E; m)$, define a* *signed measure on E by* $(f \cdot m)(A) = m(f\mathbb{1}_A)$, $A \in \mathcal{B}(E)$*; and we occasionally write some symbols by selectively omitting their subscripts when there is no confusion, e.g., writing* $(\cdot,\cdot)_m$ *for* $(\cdot,\cdot)$, $\|\cdot\|_{p,m}$ *for* $\|\cdot\|_p$.

Hypothesis 2.1. *Suppose that for each* $t > 0$, P_t *is a Hilbert-Schmidt operator and has a joint continuous positive integral kernel* $p(t,x,y)$ *on* $(0,\infty) \times E \times E$*, i.e.,*

$$P_t f(x) = \int_E p(t,x,y) f(y) m(dy), \quad \int_E \int_E p^2(t,x,y) m(dx) m(dy) < \infty. \tag{2.1}$$

Denoting the arithmetic square root of $p(t,x,x)$ *by* $b_t(x)$*, i.e.,* $b_t(x) := \sqrt{p(t,x,x)}$*. We also assume the following:*

(A1) *For any* $t > 0$, $b_t \in L^1(E;m) \cap L^\infty(E;m)$.

(A2) *For any* $t > 0$*, there exists a positive* $c_t < \infty$ *so that* $p(t,x,y) \geq c_t b_t(x) b_t(y)$ *for all* $x, y \in E$.

Remark 2.1.
- P_t *is ultracontractive if and only if* $b_t \in L^\infty(E;m)$ *for every* $t > 0$.
- *If* $m(E) < \infty$*, then* $b_t \in L^\infty(E;m)$ *implies that* P_t *is a Hilbert–Schmidt operator and, moreover,* $b_t \in L^p(E;m)$ *for all* $p \in [1,\infty]$.
- *Condition (A2) implies (A1). In fact, (A2) is equivalent to intrinsic ultracontractivity; see Davies and Simon [6, Theorem 3.2] and [7, Theorem 2.1].*

A fundamental consequence of Hypothesis 2.1 is the following Lemmas 2.1–2.3, among which Lemma 2.1(i)–(iv), Lemma 2.2(i), and Lemma 2.3(i) may be found in Davies [8, Theorem 7.2.5] and [6, Section 2], Zhang et al. [17, Lemmas 2.1 and 2.2], respectively.

Lemma 2.1. *Suppose that for each* $t > 0$, P_t *is a Hilbert-Schmidt operator and has a joint continuous positive integral kernel* $p(t,x,y)$ *on* $(0,\infty) \times E \times E$*. Then, we have the following:*

(i) $\mathcal{A}$ *has purely discrete spectrum consisting of eigenvalues* $\{\lambda_n\}_{n=1}^\infty$ *with* $0 \leq \lambda_1 < \lambda_2 \leq \cdots \uparrow +\infty$ *and there exists a complete orthonormal basis* $\{\varphi_n\}_{n=1}^\infty$ *of* $L^2(E;m)$*, where each* λ_n *is counted according to multiplicity,* $\varphi_n \in \mathcal{D}(\mathcal{A})$ *is a continuous function on E so that* $\mathcal{A}\varphi_n = \lambda_n \varphi_n, n \geq 1$ *and* φ_1 *can be chosen to be strictly positive on E.*

(ii) *For all* $t > 0$ *and* $x, y \in E$, $p(t,x,y)$ *has the following representation:*

$$p(t,x,y) = \sum_{i=1}^\infty e^{-\lambda_i t} \varphi_i(x) \varphi_i(y),$$

where the series is locally uniformly convergent on $(0,\infty) \times E \times E$.

(iii) b_t *is a continuous function in* $L^2(E;m)$ *and* $|\varphi_n(x)| \leq e^{\lambda_n t/2} b_t(x)$ *for all* $n \geq 1$, $x \in E$ *and* $t > 0$.

(iv) *For each* $x \in E$, $b_t(x)$ *and* $e^{\lambda_1 t/2} b_t(x)$ *are analytic, logarithmically convex, monotonically decreasing functions of t.*

(v) *For any* $t > s > 0$*, we find*

$$|e^{\lambda_1 t} p(t,x,y) - \varphi_1(x)\varphi_1(y)| \leq e^{\lambda_2 s} e^{-(\lambda_2 - \lambda_1)t} b_s(x) b_s(y), \quad x, y \in E.$$

Proof. To verify Lemma 2.1(iii), applying Lemma 2.1(ii) with $y = x$ yields

$$p(t,x,x) = \sum_{i=1}^\infty e^{-\lambda_i t} \varphi_i(x)^2.$$

Since $b_t(x)^2 = p(t, x, x)$ and all terms are nonnegative, for all $n \geq 1$, $x \in E$, and $t > 0$,

$$e^{-\lambda_n t}\varphi_n(x)^2 \leq b_t(x)^2.$$

Multiplying by $e^{\lambda_n t}$ and taking the square root gives rise to

$$|\varphi_n(x)| \leq e^{\lambda_n t/2} b_t(x).$$

The continuity of b_t in $L^2(E; m)$ is obvious by (2.1), Lemma 2.1(i) and (ii).

To build (v), on account of Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&|e^{\lambda_1(\tau+s)}p(\tau + s, x, y) - \varphi_1(x)\varphi_1(y)| \\
=&\Big|\sum_{i=2}^{\infty} e^{-(\lambda_i-\lambda_1)(\tau+s)}\varphi_i(x)\varphi_i(y)\Big| \\
\leq&\Big[\sum_{i=2}^{\infty} e^{-(\lambda_i-\lambda_1)(\tau+s)}\varphi_i^2(x)\Big]^{\frac{1}{2}}\Big[\sum_{i=2}^{\infty} e^{-(\lambda_i-\lambda_1)(\tau+s)}\varphi_i^2(y)\Big]^{\frac{1}{2}} \\
\leq&\Big[\sum_{i=1}^{\infty} e^{-(\lambda_i-\lambda_1)s}\varphi_i^2(x)e^{-(\lambda_2-\lambda_1)\tau}\Big]^{\frac{1}{2}}\Big[\sum_{i=1}^{\infty} e^{-(\lambda_i-\lambda_1)s}\varphi_i^2(y)e^{-(\lambda_2-\lambda_1)\tau}\Big]^{\frac{1}{2}} \\
=&e^{\lambda_1 s}e^{-(\lambda_2-\lambda_1)\tau}b_s(x)b_s(y), \quad \tau, s > 0, \quad x, y \in E.
\end{aligned} \tag{2.2}$$

Let $t > s > 0$ and insert $\tau = t - s$ into (2.2), (v) follows. □

Lemma 2.2. *Assume (A1). Then, we have the following:*

(i) *For each $t > 0$, $x \in E$ and $f \in L^p(E; m)$ with $p \in [1, \infty]$, $P_t f(x)$ has the following representation:*

$$P_t f(x) = \int_E p(t, x, y) f(y) m(dy) = \sum_{i=1}^{\infty} e^{-\lambda_i t}(\varphi_i, f)\varphi_i(x),$$

where the series converges absolutely and uniformly in $(t, x) \in [\epsilon, \infty) \times E$ for any $\epsilon > 0$, and $P_t f$ has a bounded continuous version.

(ii) *For any $i \geq 1$, $t > 0$ and $p \in [1, \infty]$, we have $b_t, \varphi_i \in L^p(E; m)$.*

(iii) *For any $t > s > 0$, $\rho \in \mathcal{P}(E)$ and $f \in L^p(E; m)$ with $p \in [1, \infty]$, we find*

$$|\rho(e^{\lambda_1 t}P_t f) - \rho(\varphi_1)(\varphi_1, f)| \leq e^{\lambda_2 s}e^{-(\lambda_2-\lambda_1)t}\rho(b_s)\|b_s\|_{p^*}\|f\|_p, \tag{2.3}$$

where p^* denotes the conjugate exponent of p, i.e., $p^* := p/(p - 1)$ if $p > 1$ and $p^* = \infty$ if $p = 1$.

Proof. The item (ii) is proved by the interpolation inequality and Lemma 2.1(iii). To build item (iii), by Lemma 2.1(v) and Hölder's inequality, for any $t > s > 0$, $\rho \in \mathcal{P}(E)$ and $f \in L^p(E; m)$ with $p \in [1, \infty]$, we deduce

$$\begin{aligned}
&|\rho(e^{\lambda_1 t}P_t f) - \rho(\varphi_1)(\varphi_1, f)_m| \\
=&\Big|\int_E \int_E (e^{\lambda_1 t}p(t, x, y) - \varphi_1(x)\varphi_1(y)) f(y)\rho(dx)m(dy)\Big|
\end{aligned}$$

$$\begin{aligned}
&\leq \int_E \int_E |e^{\lambda_1 t} p(t,x,y) - \varphi_1(x)\varphi_1(y)||f(y)|\rho(dx)m(dy) \\
&\leq e^{\lambda_2 s} e^{-(\lambda_2-\lambda_1)t} \int_E \int_E b_s(x) b_s(y)|f(y)|\rho(dx)m(dy) \\
&= e^{\lambda_2 s} e^{-(\lambda_2-\lambda_1)t} \left(\int_E b_s(x)\rho(dx)\right)\left(\int_E b_s(y)|f(y)|m(dy)\right) \\
&= e^{\lambda_2 s} e^{-(\lambda_2-\lambda_1)t} \rho(b_s)\|b_s\|_{p^*}\|f\|_p.
\end{aligned}$$

□

Lemma 2.3. *Assume (A2). Then, we have the following:*

(i) *There exists a decreasing function* $\tilde{c}_t < \infty$ *such that* $|\varphi_n(x)| \leq \tilde{c}_t e^{(\lambda_n - \lambda_1)t}\varphi_1(x)$ *for all* $n \geq 1$, $x \in E$ *and* $t > 0$.

(ii) *For any* $t > s > 0$ *and* $x, y \in E$, *we have the estimate*

$$|e^{\lambda_1 t} p(t,x,y) - \varphi_1(x)\varphi_1(y)| \leq c_s^{-2} e^{-(\lambda_2-\lambda_1)(t-s)}\varphi_1(x)\varphi_1(y),$$

which in turn leads to, for any $f \in L^p(E;m)$ *and* $\rho \in \mathcal{P}(E)$,

$$\begin{aligned}
&|\rho(e^{\lambda_1 t} P_t f) - \rho(\varphi_1)(f,\varphi_1)| \\
&\leq q c_s^{-2} e^{-(\lambda_2-\lambda_1)(t-s)}\rho(\varphi_1)(\varphi_1,|f|) \\
&\leq c_s^{-2} e^{-(\lambda_2-\lambda_1)(t-s)}\|\varphi_1\|_\infty \|\varphi_1\|_{p^*}\|f\|_p, \quad p \in [1,\infty].
\end{aligned}$$

Proof. (A2), Lemma 2.1(iii), and (v) prove (ii) since

$$\varphi_1(y) = e^{\lambda_1 s}(p(s,\cdot,y),\varphi_1) \geq e^{\lambda_1 s} c_s b_s(y)(b_s,\varphi_1) \geq e^{\lambda_1 s/2} c_s b_s(y)(\varphi_1,\varphi_1) = e^{\lambda_1 s/2} c_s b_s(y), \quad s > 0, \quad y \in E.$$

□

3. Major results

In this section, we present the major results. The following theorem states that X possesses (uniformly) exponentially quasi-mixing ergodicity under (A2) or (A1).

Theorem 3.1. *Let* $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *be a regular Dirichet form,* $\{P_t\}$ *be its associated semigroup, and* X *be its associated Hunt process.*

(i) *Assume (A1). Then, there exist two probability measures* $\nu := \varphi_1^2 \cdot m$ *and* $\mu := \varphi_1 \cdot m/\|\varphi_1\|_1$ *such that for all* $\rho \in \mathcal{P}(E)$,

$$\begin{aligned}
&\lim_{t\to\infty} e^{(\lambda_2-\lambda_1)\alpha t} \sup_{(A,B)\in\mathcal{B}(E)\times\mathcal{B}(E)} |\mathbb{E}_\rho(\mathbb{1}_A(X_{at})\mathbb{1}_B(X_t) \mid \zeta > t) - \nu(A)\mu(B)| \\
=& \sup_{(A,B)\in\mathcal{B}(E)\times\mathcal{B}(E)} |\Phi(\rho,\kappa,\alpha,A,B)|, \quad \alpha := a \wedge (1-a), \quad 0 < a < 1,
\end{aligned}$$

where κ *is the multiplicity of an eigenvalue* λ_2, $\Phi(\rho,\kappa,\alpha,A,B)$ *is finite and equal to*

$$\begin{cases} \displaystyle\sum_{i=2}^{\kappa+1} \frac{(\varphi_1 \mathbb{1}_A, \varphi_i)\mu(B)\rho(\varphi_i)}{\rho(\varphi_1)}, & \text{if} \quad a < 1/2, \\ \displaystyle\sum_{i=2}^{\kappa+1} \frac{(\varphi_i \mathbb{1}_A, \varphi_1)(\mathbb{1}_B, \varphi_i)}{\|\varphi_1\|_1}, & \text{if} \quad a > 1/2, \\ \displaystyle\sum_{i=2}^{\kappa+1} \frac{(\varphi_1 \mathbb{1}_A, \varphi_i)[\mu(B)\rho(\varphi_i)\|\varphi_1\|_1 + (\mathbb{1}_B, \varphi_i)\rho(\varphi_1)]}{\rho(\varphi_1)\|\varphi_1\|_1}, & \text{if} \quad a = 1/2. \end{cases}$$

That is, the quasi-mixing limit of X *exists and attracts exponentially (not necessarily uniformly) all* $\rho \in \mathcal{P}(E)$. *In this case, we also call that* X *has exponentialy quasi-mixing ergodicity.*

(ii) *Assume (A2). Then the attraction in* (i) *is uniform and exponential in* $\rho \in \mathcal{P}(E)$. *In this case, we also call that* X *has uniformly and exponentially quasi-mixing ergodicity.*

Proof. (i) For any $A, B \in \mathcal{B}(E)$ and $0 < a < 1$, by the Markov property of X, we discover

$$\begin{aligned} &\mathbb{E}_x[\mathbb{1}_A(X_{at})\mathbb{1}_B(X_t)\mathbb{1}_{\{t<\zeta\}}] \\ =&\mathbb{E}_x[\mathbb{E}_x(\mathbb{1}_A(X_{at})\mathbb{1}_B(X_t)\mathbb{1}_{\{t<\zeta\}} \mid \mathcal{F}_{at})] \\ =&\mathbb{E}_x\{\mathbb{1}_A(X_{at})\mathbb{E}_{X_{at}}[\mathbb{1}_B(X_{(1-a)t})]\} \\ =&P_{at}\{\mathbb{1}_A P_{(1-a)t}\mathbb{1}_B\}(x), \quad t \geq 0, \quad x \in E. \end{aligned} \tag{3.1}$$

Thanks to the semigroup property of $\{P_t\}$, (3.1) and Lemma 2.2(i), we see that for all $\rho \in \mathcal{P}(E)$, $0 < a < 1$ and $t > 0$,

$$\begin{aligned} &\mathbb{E}_\rho[\mathbb{1}_A(X_{at})\mathbb{1}_B(X_t) \mid \zeta > t] - \nu(A)\mu(B) \\ =&\frac{\rho(P_{at}(\mathbb{1}_A P_{t-at}\mathbb{1}_B)) - \rho(\nu(A)\mu(B)P_t\mathbb{1})}{\rho(P_t\mathbb{1})} \\ =&\frac{\rho\{\sum_{i=2}^{\infty} e^{-(\lambda_i-\lambda_1)at}([\mathbb{1}_A e^{\lambda_1(t-at)}P_{t-at}\mathbb{1}_B - \nu(A)\mu(B)e^{\lambda_1(t-at)}P_{t-at}\mathbb{1}], \varphi_i)\varphi_i\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})} \\ &+ \frac{\rho\{\sum_{i=2}^{\infty} e^{-(\lambda_i-\lambda_1)(t-at)}(\mathbb{1}_B, \varphi_i)(\varphi_i\mathbb{1}_A, \varphi_1)\varphi_1\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}. \end{aligned} \tag{3.2}$$

To find a specific expression of Φ, we prepare the following notations and inequalities:

$$\begin{aligned} \phi_t(\rho,\kappa,a,A,B) &:= \frac{\rho\{\sum_{i=2}^{\kappa+1}([\mathbb{1}_A e^{\lambda_1(t-at)}P_{t-at}\mathbb{1}_B - \nu(A)\mu(B)e^{\lambda_1(t-at)}P_{t-at}\mathbb{1}], \varphi_i)\varphi_i\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}, \\ \tilde{\phi}_t(\rho,\kappa,a,A,B) &:= \frac{\rho\{\sum_{i=2}^{\kappa+1}(\mathbb{1}_B, \varphi_i)(\varphi_i\mathbb{1}_A, \varphi_1)\varphi_1\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})}. \end{aligned}$$

For any $\epsilon > 0$, $t > 0$ and $i \geq 1$, we calculate, using Cauchy-Schwarz inequality, that

$$\begin{aligned} &|([\mathbb{1}_A e^{\lambda_1(t-at)}P_{t-at}\mathbb{1}_B - \nu(A)\mu(B)e^{\lambda_1(t-at)}P_{t-at}\mathbb{1}], \varphi_i)\varphi_i| \\ \leq&(2e^{\lambda_1(t-at)}P_{t-at}\mathbb{1}, |\varphi_i|)|\varphi_i| \\ \leq&2e^{\lambda_1(t-at)}\|b_{t-at}\|\|b_{t-at}\|_1 e^{\lambda_i \epsilon at}b_{2\epsilon at} \\ \triangleq&h(t-at)e^{\lambda_i \epsilon at}b_{2\epsilon at}, \end{aligned}$$

and

$$
\begin{aligned}
&|(\mathbb{1}_B, \varphi_i)(\varphi_i \mathbb{1}_A, \varphi_1)\varphi_1| \\
\leq &(\mathbb{1}, |\varphi_i|)(|\varphi_i|, |\varphi_1|)|\varphi_1| \\
\leq &e^{\lambda_i \epsilon(1-a)t/2} m(b_{\epsilon(1-a)t}) e^{\lambda_1 \epsilon(1-a)t/2} b_{\epsilon(1-a)t} \\
\leq &e^{\lambda_i \epsilon(1-a)t} m(b_{\epsilon(1-a)t}) b_{\epsilon(1-a)t}.
\end{aligned}
$$

Subsequently, we divide a into three cases: $a < \frac{1}{2}$, $a > \frac{1}{2}$, and $a = \frac{1}{2}$, to derive the expression of Φ. For $a < \frac{1}{2}$, multiplying both sides of (3.2) by $e^{(\lambda_2-\lambda_1)at}$ yields

$$
\begin{aligned}
&|e^{(\lambda_2-\lambda_1)at}\{\mathbb{E}_\rho[\mathbb{1}_A(X_{at})\mathbb{1}_B(X_t) \mid t < \zeta] - \nu(A)\mu(B)\} - \Phi(\rho, \kappa, \alpha, A, B)| \\
\leq &\frac{\rho\{\sum_{i=\kappa+2}^{\infty} e^{-(\lambda_i-\lambda_2)at}([\mathbb{1}_A e^{\lambda_1(t-at)} P_{t-at}\mathbb{1}_B - \nu(A)\mu(B) e^{\lambda_1(t-at)} P_{t-at}\mathbb{1}], \varphi_i)\varphi_i\}}{\rho(e^{\lambda_1 t} P_t \mathbb{1})} \\
&+ \frac{\rho\{e^{(\lambda_2-\lambda_1)at} \sum_{i=2}^{\infty} e^{-(\lambda_i-\lambda_1)(t-at)}(\mathbb{1}_B, \varphi_i)(\varphi_i \mathbb{1}_A, \varphi_1)\varphi_1\}}{\rho(e^{\lambda_1 t} P_t \mathbb{1})} + |\phi_t(\rho, \kappa, a, A, B) - \Phi(\rho, \kappa, \alpha, A, B)| \qquad (3.3)\\
\leq &\frac{\sum_{i=\kappa+2}^{\infty} e^{-(\lambda_i-\lambda_2-\epsilon\lambda_i)at} h(t-at)\rho\{b_{2\epsilon at}\}}{\rho(e^{\lambda_1 t} P_t \mathbb{1})} + \frac{\sum_{i=2}^{\infty} e^{-(\lambda_i-\lambda_1-\epsilon\lambda_i)(t-at)} e^{(\lambda_2-\lambda_1)at} m(b_{\epsilon(1-a)t})\rho(b_{\epsilon(1-a)t})}{\rho(e^{\lambda_1 t} P_t \mathbb{1})} \\
&+ |\phi_t(\rho, \kappa, a, A, B) - \Phi(\rho, \kappa, \alpha, A, B)|.
\end{aligned}
$$

Notice that b_t and $h(t-at)$ are monotonically decreasing functions of t by Lemma 2.1(iv) and $\rho(e^{\lambda_1 t}P_t\mathbb{1}) \to \|\varphi_1\|_1 \rho(\varphi_1)$ as $t \to \infty$ by Lemma 2.2(iii). Therefore, fixing $0 < \epsilon < \epsilon_0$ and taking $t \to \infty$, (3.3) tends to 0 uniformly with respect to $(A, B) \in \mathscr{B}(E) \times \mathscr{B}(E)$ by means of (2.1) and Lemma 2.2(iii), where

$$
\epsilon_0 = \min\Big\{\frac{\lambda_{\kappa+2} - \lambda_2}{\lambda_{\kappa+2}}, \frac{(1-2a)(\lambda_2-\lambda_1)}{(1-a)\lambda_2}\Big\}.
$$

For the case of $a > \frac{1}{2}$, multiplying both sides of (3.2) by $e^{(\lambda_2-\lambda_1)(t-at)}$, we get

$$
\begin{aligned}
&|e^{(\lambda_2-\lambda_1)(t-at)}\{\mathbb{E}_\rho[\mathbb{1}_A(X_{at})\mathbb{1}_B(X_t) \mid t < \zeta] - \nu(A)\mu(B)\} - \Phi(\rho, \kappa, \alpha, A, B)| \\
\leq &\frac{\rho\{\sum_{i=2}^{\infty} e^{(\lambda_2-\lambda_1)(t-at)-(\lambda_i-\lambda_1)at}([\mathbb{1}_A e^{\lambda_1(t-at)} P_{t-at}\mathbb{1}_B - \nu(A)\mu(B) e^{\lambda_1(t-at)} P_{t-at}\mathbb{1}], \varphi_i)\varphi_i\}}{\rho(e^{\lambda_1 t} P_t \mathbb{1})} \\
&+ \frac{\rho\{\sum_{i=\kappa+2}^{\infty} e^{-(\lambda_i-\lambda_2)(t-at)}(\mathbb{1}_B, \varphi_i)(\varphi_i \mathbb{1}_A, \varphi_1)\varphi_1\}}{\rho(e^{\lambda_1 t} P_t \mathbb{1})} + |\tilde{\phi}_t(\rho, \kappa, a, A, B) - \Phi(\rho, \kappa, \alpha, A, B)| \\
\leq &\frac{\sum_{i=2}^{\infty} e^{(\lambda_2-\lambda_1)(t-at)} e^{-(\lambda_i-\lambda_1-\epsilon\lambda_i)at} h(t-at)\rho\{b_{\epsilon at}\}}{\rho(e^{\lambda_1 t} P_t \mathbb{1})} + \frac{\sum_{i=\kappa+2}^{\infty} e^{-(\lambda_i-\lambda_2)(t-at)} e^{\lambda_i\epsilon(1-a)t} m(b_{\epsilon(1-a)t})\rho\{b_{\epsilon(1-a)t}\}}{\rho(e^{\lambda_1 t} P_t \mathbb{1})} \\
&+ |\tilde{\phi}_t(\rho, \kappa, a, A, B) - \Phi(\rho, \kappa, \alpha, A, B)|. \qquad (3.4)
\end{aligned}
$$

Therefore, fixing $0 < \epsilon < \epsilon_1$ and taking $t \to \infty$, (3.4) tends to 0 uniformly with respect to $(A, B) \in \mathscr{B}(E) \times \mathscr{B}(E)$ by means of (2.1) and Lemma 2.2(iii), where

$$
\epsilon_1 = \min\Big\{\frac{\lambda_{\kappa+2} - \lambda_2}{\lambda_{\kappa+2}}, \frac{(2a-1)(\lambda_2-\lambda_1)}{a\lambda_2}\Big\}.
$$

For the case of $a = \frac{1}{2}$, multiplying both sides of (3.2) by $e^{(\lambda_2-\lambda_1)at}$ to obtain

$$\begin{aligned}
&|e^{(\lambda_2-\lambda_1)at}\{\mathbb{E}_\rho[\mathbb{1}_A(X_{at})\mathbb{1}_B(X_t) \mid t<\zeta] - \nu(A)\mu(B)\} - \Phi(\rho,\kappa,\alpha,A,B)| \\
\le& q\frac{\rho\{\sum_{i=\kappa+2}^{\infty} e^{-(\lambda_i-\lambda_2)at}([\mathbb{1}_A e^{\lambda_1 at}P_{at}\mathbb{1}_B - \nu(A)\mu(B)e^{\lambda_1 at}P_{at}\mathbb{1}],\varphi_i)\varphi_i\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})} \\
&+ \frac{\rho\{\sum_{i=\kappa+2}^{\infty} e^{-(\lambda_i-\lambda_2)at}(\mathbb{1}_B,\varphi_i)(\varphi_i\mathbb{1}_A,\varphi_1)\varphi_1\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})} \\
&+ |\phi_t(\rho,\kappa,a,A,B) + \tilde{\phi}_t(\rho,\kappa,a,A,B) - \Phi(\rho,\kappa,\alpha,A,B)| \\
\le& \frac{\sum_{i=\kappa+2}^{\infty} e^{-(\lambda_i-\lambda_2-\epsilon\lambda_i)at}h(t-at)\rho\{b_{\epsilon at}\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})} + \frac{\sum_{i=\kappa+2}^{\infty} e^{-(\lambda_i-\lambda_2-\epsilon\lambda_i)at}m(b_{\epsilon at})\rho\{b_{\epsilon at}\}}{\rho(e^{\lambda_1 t}P_t\mathbb{1})} \\
&+ |\Phi(\rho,\kappa,\alpha,A,B) - \phi_t(\rho,\kappa,a,A,B) - \tilde{\phi}_t(\rho,\kappa,a,A,B)|.
\end{aligned} \tag{3.5}$$

Therefore, fixing $0 < \epsilon < (\lambda_{\kappa+2} - \lambda_2)/\lambda_{\kappa+2}$ and taking $t \to \infty$, (3.5) tends to 0 uniformly with respect to $(A, B) \in \mathcal{B}(E) \times \mathcal{B}(E)$ by means of (2.1) and Lemma 2.2(iii).

(ii) For any $\epsilon > 0$, $t > 0$, $i \ge 1$, and $a \in (0, 1)$, applying Lemma 2.3(i) and the symmetry of $\{P_t\}$ deduces

$$|([\mathbb{1}_A e^{\lambda_1(t-at)}P_{t-at}\mathbb{1}_B - \nu(A)\mu(B)e^{\lambda_1(t-at)}P_{t-at}\mathbb{1}],\varphi_i)\varphi_i| \le (2e^{\lambda_1(t-at)}P_{t-at}\mathbb{1},|\varphi_i|)|\varphi_i| \le 2\tilde{c}^2_{\epsilon(t-at)}e^{2\epsilon(\lambda_i-\lambda_1)(t-at)}\|\varphi_1\|_1\varphi_1,$$

and

$$|(\mathbb{1}_B,\varphi_i)(\varphi_i\mathbb{1}_A,\varphi_1)\varphi_1| \le (\mathbb{1},|\varphi_i|)(|\varphi_i|,\varphi_1)\varphi_1 \le q\tilde{c}_{\epsilon(t-at)}e^{\epsilon(\lambda_i-\lambda_1)(t-at)}\|\varphi_1\|_1\varphi_1.$$

Then, utilizing Lemma 2.3(ii), it suffices to perform steps similar to (i). □

In light of Theorem 3.1, the (uniformly) exponential quasi-ergodicity, (uniformly) exponential fractional quasi-ergodicity, and (uniformly) mean-ratio quasi-ergodicity proportional to time read, respectively, as follows.

Corollary 3.1. *Assume (A1). Then, we have the following:*

(i) *X possesses a quasi-stationary distribution μ, which attracts exponentially (not necessarily uniformly) all $\rho \in \mathcal{P}(E)$, i.e.,*

$$\lim_{t\to\infty} e^{(\lambda_2-\lambda_1)t}\|\mathbb{P}_\rho(X_t \in \cdot \mid \zeta > t) - \mu\|_{TV} = \sup_{F\in\mathcal{B}(E)}\left|\sum_{i=2}^{\kappa+1}\frac{[(\varphi_i,\mathbb{1}_F)_m - (\varphi_i,\mathbb{1})_m\mu(F)]\rho(\varphi_i)}{\|\varphi_1\|_1\rho(\varphi_1)}\right|,$$

where $\|\cdot\|_{TV}$ is the total variation norm.

(ii) *X possesses a unique fractional quasi-stationary distribution ν, which attracts exponentially (not necessarily uniformly) all $\rho \in \mathcal{P}(E)$, i.e., for all $0 < a < 1$, there exists a finite function $\Psi = \Psi(a,\rho,\kappa)$ such that*

$$\lim_{t\to\infty} e^{(\lambda_2-\lambda_1)[(t-at)\wedge(at)]}\|\mathbb{P}_\rho(X_{at} \in \cdot \mid \zeta > t) - \nu\|_{TV} = \Psi(a,\rho,\kappa),$$

where $\Psi(a,\rho,\kappa)$ is equal to

$$\sup_{F\in\mathcal{B}(E)}\left|\sum_{i=2}^{\kappa+1}\frac{(\mathbb{1}_F\varphi_1,\varphi_i)\rho(\varphi_i)}{\rho(\varphi_1)}\right|, \qquad \textit{if} \quad a < 1/2,$$

$$\sup_{F\in\mathcal{B}(E)}\left|\sum_{i=2}^{\kappa+1}\frac{(\mathbb{1}_F(\mathbb{1},\varphi_i)\varphi_i,\varphi_1)}{\|\varphi_1\|_1}\right|, \qquad \textit{if} \quad a>1/2,$$

$$\sup_{F\in\mathcal{B}(E)}\left|\sum_{i=2}^{\kappa+1}\frac{(\mathbb{1}_F\varphi_1,\varphi_i)[(\mathbb{1},\varphi_i)\rho(\varphi_1)+\|\varphi_1\|_1\rho(\varphi_i)]}{\|\varphi_1\|_1\rho(\varphi_1)}\right|, \qquad \textit{if} \quad a=1/2.$$

(iii) *X possesses a unique mean-ratio quasi-stationary distribution ν, which attracts (not necessarily uniformly) all $\rho\in\mathcal{P}(E)$ at a speed proportional to time, i.e., there exists a positive constant C_ρ depending on ρ such that*

$$|\mathbb{E}_\rho(L_t(f)\mid\zeta>t)-\nu(f)|\le C_\rho\frac{\|f\|_\infty}{t},\ \ t>0,\ f\in\mathcal{B}_b(E).$$

Proof. We just need to confirm (iii). For any $\rho\in\mathcal{P}(E)$, $f\in\mathcal{B}_b(E)$, $t^*>0$, and $t>2t^*$, employing Lemma 2.2(i) and Fubini's Theorem, we get

$$|\mathbb{E}_\rho(L_t(f)\mid\zeta>t)-\nu(f)|\le q\frac{4t^*\|f\|_\infty+\int_{t^*}^{t-t^*}|\rho\{e^{\lambda_1 s}P_s[(f-\nu(f))e^{\lambda_1(t-s)}P_{t-s}\mathbb{1}]\}|ds}{\rho(e^{\lambda_1 t}P_t\mathbb{1})t}. \tag{3.6}$$

By virtue of (2.3) and Lemma 2.1(iv), for any $t>2t^*$ and $t^*<s<t-t^*$, we discover

$$\begin{aligned}
&|\rho\{e^{\lambda_1 s}P_s[(f-\nu(f))e^{\lambda_1(t-s)}P_{t-s}\mathbb{1}]\}|\\
\le&\rho\{e^{\lambda_1 s}P_s[|(f-\nu(f))(e^{\lambda_1(t-s)}P_{t-s}\mathbb{1}-\|\varphi_1\|_1\varphi_1)|]\}+\rho\{|e^{\lambda_1 s}P_s(f\varphi_1)-(f\varphi_1,\varphi_1)\varphi_1|\}\|\varphi_1\|_1\\
\le&2\|f\|_\infty\rho(e^{\lambda_1 s}P_s[e^{\lambda_2 t^*}e^{-(\lambda_2-\lambda_1)(t-s)}b_{t^*}\|b_{t^*}\|_1])+\|f\|_\infty\rho(e^{\lambda_2 t^*}e^{-(\lambda_2-\lambda_1)s}b_{t^*}\|b_{t^*}\|\|\varphi_1\|)\|\varphi_1\|_1\\
\le&2\|f\|_\infty\rho(e^{\lambda_1 s}e^{\lambda_2 t^*}e^{-(\lambda_2-\lambda_1)(t-s)}b_s(b_s,b_{t^*})\|b_{t^*}\|_1)+\|f\|_\infty\rho(e^{\lambda_2 t^*}e^{-(\lambda_2-\lambda_1)s}b_{t^*}\|b_{t^*}\|\|\varphi_1\|)\|\varphi_1\|_1\\
\le&2\|f\|_\infty\rho(e^{\lambda_1 t^*}e^{\lambda_2 t^*}e^{-(\lambda_2-\lambda_1)(t-s)}b_{t^*}(b_{t^*},b_{t^*})\|b_{t^*}\|_1)+\|f\|_\infty\rho(e^{\lambda_2 t^*}e^{-(\lambda_2-\lambda_1)s}b_{t^*}\|b_{t^*}\|\|\varphi_1\|)\|\varphi_1\|_1.
\end{aligned} \tag{3.7}$$

Therefore, combining (3.6), (3.7), and Lemma 2.2(iii), we derive what we desire. □

Corollary 3.2. *Assume (A2). The attraction in Corollary 3.1(i) and (ii) are uniformly and exponentially in $\rho\in\mathcal{P}(E)$; C_ρ in Corollary 3.1(iii) can be chosen to be independent of $\rho\in\mathcal{P}(E)$.*

4. Example

In this section, we present a concrete example to illustrate the practical applicability of our main results on symmetric Markov processes. Specifically, we verify that the associated transition semigroup fulfills the assumptions of Theorem 3.1. Upon this verification, the quasi-mixing and quasi-ergodic properties derived in our main theorem hold directly for the examined process, which further validates the effectiveness and feasibility of our theoretical conclusions.

Let $I=(l_1,l_2)$ be an open interval with a Borel σ-field $\mathcal{B}(I)$ and $-\infty<l_1<l_2<\infty$. We consider a regular second-order ordinary differential operator L given by

$$L=-a(x)\frac{d^2}{dx^2}-b(x)\frac{d}{dx}+c(x), \quad x\in I,$$

where $a > 0$ and $c \geq 0$ are real-valued measurable functions on I such that for a fixed point $l_0 \in I$ and any $x \in I$, the following functions make sense:

$$B(x) = \int_{l_0}^{x} \frac{b(y)}{a(y)} dy, \quad m(x) = \int_{l_0}^{x} \frac{1}{a(y)} e^{B(y)} dy,$$
$$s(x) = \int_{l_0}^{x} e^{-B(y)} dy, \quad k(x) = \int_{l_0}^{x} \frac{c(y)}{a(y)} e^{B(y)} dy.$$

Then s, m, and k can induce three measures on I: $s(dx) := ds(x)$, $m(dx) := dm(x)$, and $k(dx) := dk(x)$, respectively.

We now assume $s(I) + m(I) + k(I) < \infty$ and consider a bilinear form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(I; m)$:

$$\mathcal{E}(u, v) = \int_I \frac{du}{ds}(x) \frac{dv}{ds}(x) ds(x) + \int_I c(x) u(x) v(x) dm(x), \quad u, v \in \mathcal{D}(\mathcal{E}),$$
$$\mathcal{D}(\mathcal{E}) \text{ is the closure of } C_c^2(I) \text{ with respect to the norm } \| \cdot \|_{\mathcal{E}_1},$$

where $L^2(I; m)$ is the set of all square integrable functions on I with respect to the measure m; $C_c^2(I)$ is the set of all functions with second-order continuous derivatives and compact supports in I; and $\| \cdot \|_{\mathcal{E}_1}$ is given by the formula $\|u\|_{\mathcal{E}_1}^2 := \|u\|_{L^2(I,m)}^2 + \mathcal{E}(u, u)$, $\forall u \in \mathcal{D}(\mathcal{E})$.

It is easy to check by [17, Example 4.1] that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a regular Dirichlet form on $L^2(I; m)$ and, by [10, Theorem 7.2.1], is associated with an $m(dx)$-symmetric diffusion process $X = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \{X_t\}_{t \geq 0}, \{\mathbb{P}_x\}_{x \in I}, \zeta)$, where $\zeta = \inf\{t \geq 0 : X_t = l_1 \text{ or } l_2\}$ is the lifetime. Since s and m are finite, the Sobolev type inequality holds for all $1 \leq p \leq \infty$,

$$\|u\|_{p,m}^2 \leq s(I) m(I)^{\frac{2}{p}} \mathcal{E}(u, u), \quad u \in \mathcal{D}(\mathcal{E}).$$

Theorem 4.1. *$\{P_t\}$ associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Hilbert-Schmidt semigroup fulfilling (A1). Moreover, (A2) is satisfied for $\{P_t\}$ if k is a zero measure or* Supp$[k]$ *is compact in I.*

Proof. It follows from Sobolev type inequality that (A1) is satisfied for $\{P_t\}$. According to Tomisaki [15, Theorem 2.11], (A2) is satisfied for $\{P_t\}$ if k is a zero measure or Supp$[k]$ is compact in I. □

Author contributions

Saixia Liao: Writing–original draft, Formal analysis; Hanjun Zhang: Supervision; Huasheng Li: Writing–review & editing, Supervision. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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