Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(6): 16983–17008.
DOI: 10.3934/math.2026695
Received: 28 March 2026
Revised: 19 May 2026
Accepted: 22 May 2026
Published: 12 June 2026

Research article

Asymptotic properties of solutions to Caputo-Hadamard fractional differential equations

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Abstract: This paper investigates the stability properties of Caputo-Hadamard fractional differential equations. We first analyze the asymptotic behavior and rigorously prove a specific convergence rate for these equations. Then, a novel stability criterion called logarithmic Mittag-Leffler stability is proposed. By employing the fixed-point theorem in an innovative Banach space equipped with a designed weighted norm, we demonstrate that when the linearization spectrum of the Caputo-Hadamard fractional differential equation lies within a prescribed sector, the equilibrium point of the equation exhibits logarithmic Mittag-Leffler stability. This result leads to a version of Lyapunov's first method for Caputo-Hadamard fractional differential equations, demonstrating its stability in the presence of logarithmic memory.

Keywords: logarithmic Mittag-Leffler stability; Lyapunov's first method; Mittag-Leffler functions; Caputo-Hadamard fractional differential equation
Mathematics Subject Classification: 34A08, 34D05, 34D20

1. Introduction

In recent years, fractional differential equations have attracted increasing interest due to being able to model many mathematical problems with memory and heredity in science and engineering (see [11, 17, 27]). Fractional calculus has evolved into multiple distinct formulations, each developed to address specific application scenarios. Prominent definitions include the Riemann-Liouville [15, 18], Caputo [12, 21], Grünwald-Letnikov [30], Riesz [4, 7, 29], Erdélyi-Kober [5, 34], Hadamard [1, 16, 26], and Katugampola [26, 19, 13] operators, along with their respective generalized variants. These formulations show significant differences in both mathematical structure and practical interpretation. Fractional derivatives come in many forms. As an example, a neural-network-based method has recently been applied to obtain accurate analytical solutions of the fractional Phi-4 equation [37].

As another example of nonlinear evolution equations, the stochastic nonlinear Kodama equation has been investigated via qualitative analysis and traveling wave solutions [36]. Of particular note, the Hadamard fractional operator has emerged as an indispensable tool for modeling ultra-slow dynamical processes, with well-documented applications in material fracture analysis and igneous rock creep characterization (see [32, 26]). The Caputo-Hadamard formulation (a regularized variant of the classical Hadamard fractional derivative) shows superior behavior compared to its Riemann-Liouville counterpart, as shown in [24]. This advantage is primarily manifested in two notable distinguishing characteristics. First, unlike conventional Hadamard derivatives, where non-zero constants yield counterintuitive variable outputs, the Caputo-Hadamard operator preserves the fundamental property that its fractional derivative of a constant is still zero. Secondly, the Cauchy problem for Caputo-Hadamard fractional differential systems admits physically meaningful initial conditions analogous to integer-order calculus, whereas standard Hadamard systems require nonlocal fractional initial conditions with memory effects to ensure validity. Consequently, it is imperative to further study of the Caputo modification of Hadamard fractional calculus.

One of the most fundamental problems in the qualitative theory of fractional differential equations is the stability theory. The stability analysis of fractional differential equations has received growing scholarly interest in recent years, and some existing contributions on stability analysis for Hadamard fractional differential equations (abbreviated as FDEs) deserve to be mentioned. In [25], the author dealt with the asymptotic stability and fold bifurcation for a Caputo-Hadamard-type fractional differential system and the effection of one peculiar parameter. In [35], the author gave some new fractional differential inequalities along the given nonlinear Hadamard fractional differential system. Then, on the basis of these inequalities, some sufficient conditions for stability and Mittag-Leffler stability were given. Moreover, the boundary control matched disturbance rejection problem for Caputo-Hadamard fractional heat equations with time delay was discussed in [8].

Following Lyapunov's seminal 1892 thesis [23], two methods are expected to also work for fractional differential equations.

- Lyapunov's first method: The method of linearization of the nonlinear equation along an orbit, the study of the resulting linear variational equation by means of Lyapunov exponents (exponential growth rates of solutions), and the transfer of asymptotic stability from the linear to the nonlinear equation (the so-called theorem of linearized stability).

- Lyapunov's second method: The method of Lyapunov functions, i.e., of scalar functions on the state space that decrease along orbits.

There are many publications on Lyapunov's second method for Caputo-Hadamard FDEs, and we refer the reader to [14] or [6] for a survey.

In 2016, Cong, in his seminal work [10], extended Lyapunov's first method to the Caputo FDE

$$ {}^{C}D_{a}^{\alpha}x(t) = Ax(t) + f\left(x(t)\right), $$

where $A \in \mathbb{R}^{d\times d}$, $f : \mathbb{R}^d \to \mathbb{R}^d$, $\alpha \in (0, 1)$ and ${}^{C}D_{a}^{\alpha}$ denotes the Caputo differential operator, establishing the relationship between the stability of the equation and the spectrum of its linear part. In 2020, Cong further introduced the concept of Mittag-Leffler stability in [9] and demonstrated the Mittag-Leffler stability of the aforementioned equation using the Banach fixed-point theorem in a new normed space.

It is important to note that the work of Cong [9, 10] dealt with the Caputo fractional derivative, whose kernel is a power law t^{α}. For Caputo equations, the decay rate of solutions is of Mittag-Leffler type, i.e., $E_{\alpha}(-ct^{\alpha})$, and the stability analysis relies on standard estimates of the Mittag-Leffler function without additional singular weight. In contrast, the Caputo-Hadamard derivative considered in this paper involves the kernel $(\ln t - \ln a)^{\alpha-1}$, which exhibits a logarithmic singularity. As a result, the memory decay is much slower, and solutions cannot decay exponentially or even at a power rate; their decay rate is related to the logarithmic factor $\ln t - \ln a$ (see Proposition 3.2).

Thus, in this paper, we focus on the stability analysis of the Caputo-Hadamard FDE

$$^{CH}D_a^{\alpha}x(t) = Ax(t) + f\,(x(t))\,.$$

The asymptotic stability of (the trivial solution of) its linerization

$$^{CH}D_a^{\alpha}x(t) = Ax(t)$$

is known to be equivalent to its spectrum lying in the sector $\left\{\lambda \in \mathbb{C} : |arg\lambda| > \frac{\alpha\pi}{2}\right\}$ (see [22]). What remains to be shown is the stability behavior of the nonlinear equation. We will prove that the Caputo-Hadamard FDE

$$^{CH}D_a^{\alpha}x(t) = Ax(t) + f\,(x(t))$$

does not converge exponentially. Instead, its convergence rate is logarithmically correlated. Motivated by Cong's work, we also introduce a new type of stability analogous to Mittag-Leffler stability, referred to as logarithmic Mittag-Leffler stability. From a technical perspective, this slow decay forces us to construct a new weighted norm $\|x\|_w$ and to derive sharp estimates in preliminary work that control the logarithmic singularities. These estimates are not required in the Caputo case. Hence, while we follow the spirit of Cong's fixed-point approach, the adaptation to the Caputo-Hadamard setting is nontrivial and constitutes the main novelty of this work. Based on the above analysis, we construct a novel Banach space to analyze the logarithmic Mittag-Leffler stability of Caputo-Hadamard fractional differential equations.

In fact, the linearization method is a useful tool in the investigation of the stability of equilibria of nonlinear systems. It reduces the problem to a much simpler problem on the stability of autonomous linear systems, which can be solved explicitly. Hence, it establishes a verifiable criterion for assessing the stability of equilibria in nonlinear systems. Our theorems play analogous roles in the stability analysis of nonlinear Caputo-Hadamard FDEs as their classical counterparts do for nonlinear ordinary differential equations. However, achieving this analogy is non-trivial due to the weakly singular logarithmic kernel, requiring novel estimates and a specially designed weighted norm.

The structure of this paper is as follows: in Section 2, we present fundamental concepts in fractional calculus, with particular emphasis on Caputo-Hadamard operators and necessary lemmas. In Section 3, by means of the Mittag-Leffler function, we conjecture and verify a certain convergence rate for the Caputo-Hadamard equation. We show that the decay convergence rate of the Caputo-Hadamard equation with specific Lipschitz conditions is not exponential and not faster than the logarithmic rate $(\ln t - \ln a)^{\alpha}$ where α is the order of the equation. Subsequently, we introduce a new stability definition named logarithmic Mittag-Leffler stability, which is stronger than asymptotic stability. The practical significance and feasibility of this new definition will be demonstrated in the following sections. In Section 4, our main conclusions are presented, where we develop a Lyapunov's first method for the Caputo-Hadamard equation to characterize its stability behavior. The approach primarily involves constructing a new weighted norm and applying the Banach fixed-point theorem. In Section 5, we provide an application example of the theorem. Finally, in Section 6, we put forward a reasonable conjecture on the convergence rate of the Caputo-Hadamard equation and the future research directions.

2. Preliminaries

In this paper, all square matrices are equipped with the l_2-norm. Let $\mathbb{R}^d$ denote the d-dimensional real space equipped with the maximum norm, i.e.,

$$\|x\| = max\,(|x_1|, \ldots, |x_d|) \text{ for all } x = (x_1, \ldots, x_d)^T \in \mathbb{R}^d.$$

Let $\mathbb{R}_{(\geq a)}$ be the set of all real numbers greater than or equal to a, and $\left(\mathbb{C}(\mathbb{R}_{\geq a}, \mathbb{R}^d), \|\cdot\|_\infty\right)$ be the space of all continuous functions $\xi : \mathbb{R}_{\geq a} \to \mathbb{R}^d$ such that

$$\|\xi\|_\infty := \sup_{t\in\mathbb{R}_{\geq a}} \|\xi(t)\| < \infty.$$

It is well-known that $\left(\mathbb{C}(\mathbb{R}_{\geq a}, \mathbb{R}^d), \|\cdot\|_\infty\right)$ is a Banach space. Let $B_X(x, r)$ be the closed ball of the space X with the center at x and the radius $r > 0$.

We briefly recall a framework of Hadamard fractional calculus and Caputo-Hadamard fractional differentiation. We refer the reader to the book [1] for more details. Let $\alpha \geq 0$, and the Hadamard fractional integral of a given function $x(t)$ with order α is defined by

$$^{H}I_a^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln\frac{t}{w}\right)^{\alpha-1} x(w)\frac{1}{w}dw, \qquad t > a > 0,$$

where the Euler gamma function $\Gamma : (0, \infty) \to \mathbb{R}$ is defined as

$$\Gamma(a) = \int_0^\infty \tau^{a-1} e^{-\tau} d\tau.$$

The Caputo-Hadamard fractional derivative of a given function $x(t)$ with order α is defined by

$$^{CH}D_a^\alpha x(t) = {^H} I_a^{n-\alpha}\delta^n x(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\ln\frac{t}{w}\right)^{n-\alpha-1} \delta^n x(w)\frac{1}{w}dw,$$

where $t > a > 0, \delta = t\frac{d}{dt}$, and $n = \lceil\alpha\rceil$ being the smallest integer greater than or equal to α (see [1]). For the d-dimensional vector-valued function $x(t) = (x_1(t), \cdots, x_d(t))^T$, its Caputo-Hadamard derivative is defined as

$$^{CH}D_a^\alpha x(t) = \left({^{CH}D_a^\alpha} x_1(t), \cdots, {^{CH}D_a^\alpha} x_d(t)\right)^T.$$

The Mittag-Leffler matrix functions $E_{\alpha,\beta}(A)$ and $E_\alpha(A)$ are defined by

$$E_{\alpha,\beta}(A) = \sum_{k=0}^\infty \frac{A^k}{\Gamma(\alpha k + \beta)} \text{ and } E_\alpha(A) := E_{\alpha,1}(A).$$

Here, we also give the definition of the beta function $B(\cdot,\cdot)$ as

$$B(p,q)=\int_0^1 \tau^{p-1}(1-\tau)^{q-1}d\tau=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Let $x_a\in\mathbb{R}^d$, $K>0$, $G=\{(t,x): a\le t\le T, \|x-x_a\|\le K\}$ and $f: G\to\mathbb{R}^d$ is continuous. Consider the initial value problem of order α in the form

$$^{CH}D_a^\alpha x(t)=f\left(t,x(t)\right), \tag{2.1}$$

$$x(a)=x_a. \tag{2.2}$$

With [16, Lemma 2.5], we obtain the following result.

Lemma 2.1. *A function $y\in B_{C([a,T],\mathbb{R}^d)}(x_a,K)$ is a solution of the problem* (2.1) *and* (2.2) *if and only if it satisfies the Volterra integral equation*

$$y(t)=x_a+\frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{w}\right)^{\alpha-1}f\left(w,y(w)\right)\frac{1}{w}dw,\quad t\in[a,T].$$

Theorem 2.2. *Assume that the function $f(\cdot,\cdot)$ is uniformly Lipschitz continuous with respect to the second variable on G. Then, there exists a constant $h\in\mathbb{R}^+$ such that the problem* (2.1) *and* (2.2) *has a unique solution on the interval* $[a,a+h]\subseteq[a,T]$.

Proof. The proof is followed from [2], with $ln(\cdot)$ being the kernel function. □

In the remaining part of this section, we establish some estimates involving the Mittag-Leffler functions. These estimates will be used to prove the contraction property of the Lyapunov-Perron operator introduced in Section 4.

Lemma 2.3. *[11, Theorem 7.3] Let $\mu>0, 0<n<1$, and $u_0(x)=E_n(-\mu x^n)$. For $x\to\infty$, we have*

$$u_0(x)=\frac{x^{-n}}{\mu\Gamma(1-n)}\left(1+o(1)\right).$$

Lemma 2.4. *[10] Let $0<\alpha<1$ and λ be an arbitrary complex number with $\frac{\alpha\pi}{2}<|arg(\lambda)|\le\pi$. There exists a positive constant $M(\alpha,\lambda)$ and a positive real number t_0 such that*

$$\left|E_{\alpha,\alpha}\left(\lambda(\ln\frac{t}{a})^\alpha\right)\right|<\frac{M(\alpha,\lambda)}{(\ln t-\ln a)^{2\alpha}}\qquad \text{when } \ln t-\ln a>t_0.$$

Lemma 2.5. *Let* $0<\alpha<1$.

(i) For any λ with $\frac{\alpha\pi}{2}<|arg(\lambda)|\le\pi$, there exists a constant C_1 such that

$$\left|E_{\alpha,\alpha}\left(\lambda(\ln\frac{t}{a})^\alpha\right)\right|\le C_1E_{\alpha,\alpha}\left(-(\ln\frac{t}{a})^\alpha\right).$$

(ii)

$$\int_a^t\left(\ln\frac{t}{\tau}\right)^{\alpha+\alpha k-1}(\ln\tau-\ln a)^{-\alpha}\frac{d\tau}{\tau}=(\ln t-\ln a)^{\alpha k}\frac{\Gamma(\alpha+\alpha k)\Gamma(-\alpha+1)}{\Gamma(\alpha k+1)}.$$

Proof. (i) We first prove the non-negativity of the function $E_{\alpha,\alpha}\left(-(\ln\frac{t}{a})^{\alpha}\right)$, and then demonstrate the validity of the conclusion.

Let $u_0(t) = E_\alpha(-t^\alpha)$, and we can know from [11] that $u_0(t)$ is completely monotonic on $(0,\infty)$, i.e., $u_0'(t) = -E_\alpha'(-t^\alpha)\alpha t^{\alpha-1} \le 0$, thus $\alpha E_\alpha'(-t^\alpha) \ge 0$. Note that $\alpha E_\alpha'(-t^\alpha) = E_{\alpha,\alpha}(-t^\alpha)$. Hence

$$E_{\alpha,\alpha}(-t^\alpha) \ge 0 \qquad \text{for any } t > 0.$$

Thus, we can easily get that

$$E_{\alpha,\alpha}\left(-(\ln\frac{t}{a})^{\alpha}\right) \ge 0 \qquad \text{for any } t > a.$$

Note from Lemma 2.4 that $E_{\alpha,\alpha}\left(\lambda(\ln\frac{t}{a})^{\alpha}\right)$ and $E_{\alpha,\alpha}\left(-(\ln\frac{t}{a})^{\alpha}\right)$ are both bounded, and thus the conclusion follows immediately.

(ii)

$$\begin{aligned}
&\int_a^t \left(\ln\frac{t}{\tau}\right)^{\alpha+\alpha k-1} (\ln\tau - \ln a)^{-\alpha}\frac{d\tau}{\tau}\\
&= \int_a^t \left(\ln\frac{t}{\tau}\right)^{\alpha+\alpha k-1} (\ln\tau - \ln a)^{-\alpha} d\ln\tau \qquad (\text{Let } u = \ln\tau)\\
&= \int_{\ln a}^{\ln t} (\ln t - u)^{\alpha+\alpha k-1} (u - \ln a)^{-\alpha} du\\
&\quad (\text{Let } u - \ln a = s(\ln t - \ln a), du = (\ln t - \ln a)ds)\\
&= (\ln t - \ln a)^{\alpha k} \int_0^1 (1-s)^{\alpha+\alpha k-1} s^{-\alpha} ds\\
&= (\ln t - \ln a)^{\alpha k} \frac{\Gamma(\alpha+\alpha k)\Gamma(-\alpha+1)}{\Gamma(\alpha k+1)}.
\end{aligned}$$

□

Lemma 2.6. *Let* $0 < \alpha < 1$ *and* λ *be an arbitrary complex number with* $\frac{\alpha\pi}{2} < |arg(\lambda)| \le \pi$. *There exists a positive constant* $C(\alpha,\lambda)$ *such that*

$$\sup_{t\ge a}\int_a^t \left(\ln\frac{t}{\tau}\right)^{\alpha-1} \left|E_{\alpha,\alpha}\left(\lambda(\ln\frac{t}{\tau})^{\alpha}\right)\right| \frac{d\tau}{\tau} < C(\alpha,\lambda).$$

Proof. Choose and fix the constant t_0 from Lemma 2.4.

Case 1: $\ln t - \ln a \le t_0$, i.e., $t \le ae^{t_0}$: Note that

$$\begin{aligned}
&\int_a^t (\ln t - \ln\tau)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda(\ln\frac{t}{\tau})^{\alpha}\right)\frac{d\tau}{\tau}\\
&= -\int_a^t (\ln t - \ln\tau)^{\alpha-1} \sum_{k=0}^{\infty} \frac{\lambda^k(\ln t - \ln\tau)^{\alpha k}}{\Gamma(\alpha k+\alpha)} d(\ln t - \ln\tau)\\
&= -\sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+\alpha)} \int_a^t (\ln t - \ln\tau)^{\alpha k+\alpha-1} d(\ln t - \ln\tau)
\end{aligned}$$

$$
\begin{aligned}
&=\sum_{k=0}^{\infty}\frac{\lambda^k(\ln t-\ln a)^{\alpha k+\alpha}}{\Gamma(\alpha k+\alpha+1)}\\
&=(\ln t-\ln a)^{\alpha}E_{\alpha,\alpha+1}\left(\lambda(\ln t-\ln a)^{\alpha}\right).
\end{aligned}
$$

Therefore, we get that

$$
\begin{aligned}
&\int_a^t\left(\ln\frac{t}{\tau}\right)^{\alpha-1}\left|E_{\alpha,\alpha}\left(\lambda(\ln\frac{t}{\tau})^{\alpha}\right)\right|\frac{d\tau}{\tau}\\
\leq&\int_a^t\left(\ln\frac{t}{\tau}\right)^{\alpha-1}E_{\alpha,\alpha}\left(|\lambda|(\ln\frac{t}{\tau})^{\alpha}\right)\frac{d\tau}{\tau}\\
=&\left(\ln\frac{t}{a}\right)^{\alpha}E_{\alpha,\alpha+1}\left(|\lambda|(\ln\frac{t}{a})^{\alpha}\right)\\
\leq& t_0^{\alpha}E_{\alpha,\alpha+1}\left(|\lambda|t_0^{\alpha}\right).
\end{aligned}\tag{2.3}
$$

Case 2: $\ln t-\ln a\geq t_0$, i.e., $t\geq ae^{t_0}$: From Lemma 2.4, we see that

$$
\begin{aligned}
&\int_a^{te^{-t_0}}\left(\ln\frac{t}{\tau}\right)^{\alpha-1}\left|E_{\alpha,\alpha}\left(\lambda(\ln\frac{t}{\tau})^{\alpha}\right)\right|\frac{d\tau}{\tau}\\
\leq& M(\alpha,\lambda)\int_a^{te^{-t_0}}(\ln\frac{t}{\tau})^{-\alpha-1}\frac{d\tau}{\tau}\leq\frac{M(\alpha,\lambda)}{\alpha t_0^{\alpha}}.
\end{aligned}\tag{2.4}
$$

Note that

$$
\int_0^t\tau^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\tau^{\alpha}\right)d\tau=t^{\alpha}E_{\alpha,\alpha+1}(\lambda t^{\alpha}),
$$

which is easy to prove by definition. Thus, we get that

$$
\begin{aligned}
&\int_{te^{-t_0}}^t\left(\ln\frac{t}{\tau}\right)^{\alpha-1}\left|E_{\alpha,\alpha}\left(\lambda(\ln\frac{t}{\tau})^{\alpha}\right)\right|\frac{d\tau}{\tau}\\
\leq&\int_{te^{-t_0}}^t\left(\ln\frac{t}{\tau}\right)^{\alpha-1}E_{\alpha,\alpha}\left(|\lambda|(\ln\frac{t}{\tau})^{\alpha}\right)\frac{d\tau}{\tau}\\
=&\int_0^{t_0}s^{\alpha-1}E_{\alpha,\alpha}\left(|\lambda|s^{\alpha}\right)ds=t_0^{\alpha}E_{\alpha,\alpha+1}\left(|\lambda|t_0^{\alpha}\right).
\end{aligned}\tag{2.5}
$$

From (2.3)–(2.5), we get that

$$
\int_a^t\left(\ln\frac{t}{\tau}\right)^{\alpha-1}\left|E_{\alpha,\alpha}\left(\lambda(\ln\frac{t}{\tau})^{\alpha}\right)\right|\frac{d\tau}{\tau}<C(\alpha,\lambda),
$$

where $C(\alpha,\lambda):=t_0^{\alpha}E_{\alpha,\alpha+1}\left(|\lambda|t_0^{\alpha}\right)+\frac{M(\alpha,\lambda)}{\alpha t_0^{\alpha}}$.

□

Lemma 2.7. *Let* $0<\alpha<1$ *and* λ *be an arbitrary complex number with* $\frac{\alpha\pi}{2}<|arg(\lambda)|\leq\pi$. *There exists a positive constant* $C(\alpha,\lambda)$ *such that*

$$
\sup_{t\geq a}(\ln t-\ln a)^{\alpha}\int_a^t\left(\ln\frac{t}{\tau}\right)^{\alpha-1}\left|E_{\alpha,\alpha}\left(\lambda(\ln\frac{t}{\tau})^{\alpha}\right)\right|(\ln\tau-\ln a)^{-\alpha}\frac{d\tau}{\tau}<C(\alpha,\lambda).
$$

Proof. By Lemma 2.5 (i) and Lemma 2.5 (ii), we get that

$$
\begin{aligned}
&(\ln t - \ln a)^{\alpha} \int_a^t \left(\ln \frac{t}{\tau}\right)^{\alpha-1} \left| E_{\alpha,\alpha}\left(\lambda(\ln \frac{t}{\tau})^{\alpha}\right)\right| (\ln \tau - \ln a)^{-\alpha} \frac{d\tau}{\tau} \\
\leq & C_1(\ln t - \ln a)^{\alpha} \int_a^t \left(\ln \frac{t}{\tau}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-(\ln \frac{t}{\tau})^{\alpha}\right) (\ln \tau - \ln a)^{-\alpha} \frac{d\tau}{\tau} \\
\leq & C_1(\ln t - \ln a)^{\alpha} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\alpha k + \alpha)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{\alpha+\alpha k-1} (\ln \tau - \ln a)^{-\alpha} \frac{d\tau}{\tau} \\
= & C_1(\ln t - \ln a)^{\alpha} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\alpha k + \alpha)} (\ln t - \ln a)^{\alpha k} \frac{\Gamma(\alpha + \alpha k)\Gamma(-\alpha + 1)}{\Gamma(\alpha k + 1)} \\
= & C_1(\ln t - \ln a)^{\alpha}\Gamma(-\alpha + 1)E_{\alpha}\left(-(\ln t - \ln a)^{\alpha}\right) \\
\leq & \sup_{t\geq a} C_1(\ln t - \ln a)^{\alpha}\Gamma(-\alpha + 1)E_{\alpha}\left(-(\ln t - \ln a)^{\alpha}\right) \\
:= & C(\alpha, \lambda).
\end{aligned}
$$

The boundedness of $C(\alpha, \lambda)$ in the final step is justified by Lemma 2.3. □

3. Asymptotic behavior of solutions to Caputo-Hadamard FDEs

In this section, we study asymptotic properties of solutions to Caputo-Hadamard FDEs and show some distinct features compared to that of solutions to ordinary differential equations. We first show that a solution of Caputo-Hadamard FDEs does not converge to an equilibrium point with exponential rate. Then, we present the notion of logarithmic Mittag-Leffler stability to FDEs.

3.1. Solution of Caputo-Hadamard FDEs cannot decay faster than the logarithmic rate

Consider a nonlinear fractional system of order $\alpha \in (0, 1)$ in the form

$$ {}^{CH}D_a^{\alpha}x(t) = g\left(t, x(t)\right), \quad t > a, \tag{3.1} $$

where $g : \mathbb{R}_{\geq a} \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies the following three conditions:

(g.1) $g(\cdot, \cdot)$ is continuous;

(g.2) $g(t, 0) = 0$ for all $t \geq a$;

(g.3) $g(\cdot, \cdot)$ is global Lipschitz continuous with respect to the second variable, i.e., there exists a constant $L > 0$ such that $\|g(t, x) - g(t, y)\| \leq L\|x - y\|$ for all $t \geq a$ and $x, y \in \mathbb{R}^d$.

It is well-known that the initial value problem for the fractional differential equation (3.1) has a unique solution defined on the whole $\mathbb{R}_{\geq a}$ with any given initial value in $\mathbb{R}^d$ (see [2] with suitable kernel function). We will prove that there is no nontrivial solution of (3.1) converging to the origin with exponential rate.

Proposition 3.1. *Every nontrivial solution of (3.1) does not converge to the origin with exponential rate.*

Proof. Due to the existence and uniqueness of solution to (3.1), for any $x_a \neq 0$, the initial value problem (3.1) with the condition $x(a) = x_a$ has the unique solution $\Phi(\cdot, x_a)$ on the interval $[a, \infty)$. Assume that this solution converges to the origin with the exponential rate, then there exist positive constants λ and T_1 such that

$$\|\Phi(t, x_a)\| < e^{-\lambda(t-a)}, \quad \text{for all } t \geq T_1. \tag{3.2}$$

By Lemma 2.3,

$$E_\alpha\left(-L(lnt - lna)^\alpha\right) \sim \frac{c}{(\ln t - \ln a)^\alpha},$$

which decays slower than the exponential function. Hence, for any $K > 0$, satisfying $K\|x_a\| > 1$, there exists $T_2 > a$ such that

$$e^{-\lambda(t-a)} < \frac{1}{k} E_\alpha\left(-L(lnt - lna)^\alpha\right), \quad \text{for all } t \geq T_2. \tag{3.3}$$

Put $T_0 = \max\{T_1, T_2\}$. From Lemma 2.1, we get the equivalent integral form of (3.1),

$$\Phi(t, x_a) = x_a + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} g\left(s, \Phi(s, x_a)\right) \frac{1}{s} ds.$$

By virtue of the Lipschitz condition, we have

$$\begin{aligned}
\|x_a\| &\leq \|\Phi(t, x_a)\| + \frac{1}{\Gamma(\alpha)} \int_a^t (\ln t - \ln s)^{\alpha-1} \|g\left(s, \Phi(s, x_a)\right) - g\left(s, 0\right)\| \frac{ds}{s} \\
&\leq \|\Phi(t, x_a)\| + \frac{L}{\Gamma(\alpha)} \int_a^t (\ln t - \ln s)^{\alpha-1} \|\Phi(s, x_a)\| \frac{ds}{s}.
\end{aligned}$$

Take the limit superior on both sides of the above equation and use (3.2) and (3.3), we obtain

$$\begin{aligned}
\frac{\Gamma(\alpha)}{L}\|x_a\| \leq\ & \limsup_{t\to\infty} \int_a^{T_0} (\ln t - \ln s)^{\alpha-1} \|\Phi(s, x_a)\| \frac{ds}{s} \\
&+ \limsup_{t\to\infty} \int_{T_0}^t (\ln t - \ln s)^{\alpha-1} \|\Phi(s, x_a)\| \frac{ds}{s} \\
\leq\ & \sup_{a\leq s\leq T_0} \|\Phi(s, x_a)\| \limsup_{t\to\infty} \int_a^{T_0} (\ln t - \ln s)^{\alpha-1} \frac{ds}{s} \\
&+ \limsup_{t\to\infty} \int_{T_0}^t (\ln t - \ln s)^{\alpha-1} e^{-\lambda(s-a)} \frac{ds}{s} \\
\leq\ & \sup_{a\leq s\leq T_0} \|\Phi(s, x_a)\| \limsup_{t\to\infty} \frac{(\ln t - \ln a)^\alpha - (\ln t - \ln T_0)^\alpha}{\alpha} \\
&+ \frac{1}{K} \limsup_{t\to\infty} \int_{T_0}^t (\ln t - \ln s)^{\alpha-1} E_\alpha\left(-L(\ln s - \ln a)^\alpha\right) \frac{ds}{s} \\
\leq\ & \frac{1}{K} \limsup_{t\to\infty} \int_a^t (\ln t - \ln s)^{\alpha-1} E_\alpha\left(-L(\ln s - \ln a)^\alpha\right) \frac{ds}{s},
\end{aligned}$$

where $\limsup_{t\to\infty} \dfrac{(\ln t - \ln a)^\alpha - (\ln t - \ln T_0)^\alpha}{\alpha} = 0$.

Let

$$J := \frac{1}{K}\limsup_{t\to\infty}\int_a^t (\ln t - \ln s)^{\alpha-1}E_\alpha\left(-L(\ln s - \ln a)^\alpha\right)\frac{ds}{s}.$$

From [3, Lemma 2] with $\psi(\cdot) = \ln(\cdot)$, it is worth mentioning that $E_\alpha\left(-L(\ln t - \ln a)^\alpha\right)$ is the solution of the initial value problem

$$\begin{aligned} {}^{CH}D_a^\alpha x(t) &= -Lx(t),\\ x(a) &= 1. \end{aligned}$$

Hence,

$$E_\alpha\left(-L(\ln t - \ln a)^\alpha\right) - 1 = -\frac{L}{\Gamma(\alpha)}\int_a^t (\ln t - \ln s)^{\alpha-1}E_\alpha\left(-L(\ln s - \ln a)^\alpha\right)\frac{ds}{s}.$$

Considering the fact that $E_\alpha\left(-L(\ln t - \ln a)^\alpha\right) \to 0$, as $t \to \infty$, we get that

$$\int_a^t (\ln t - \ln s)^{\alpha-1}E_\alpha\left(-L(\ln s - \ln a)^\alpha\right)\frac{ds}{s} \to \frac{\Gamma(\alpha)}{L}, \text{as } t \to \infty.$$

So $J = \frac{1}{K}\frac{\Gamma(\alpha)}{L}$, and we get that

$$\frac{\Gamma(\alpha)}{L}\|x_a\| \le \frac{1}{K}\frac{\Gamma(\alpha)}{L},$$

which is a contradiction with $K\|x_a\| > 1$. Therefore, there does not exist any nontrivial solution of (3.1) converging to the origin with the exponential rate. □

A close look at the proof of Proposition 3.1 allows us to have an even stronger statement on the decaying rate of solutions to Caputo-Hadamard FDEs.

Proposition 3.2. *(Logarithmic decay rate of solution to Caputo-Hadamard FDEs) Any nontrivial solution of the Caputo-Hadamard FDEs* (3.1) *cannot decay to* 0 *faster than* $(\ln t - \ln a)^\alpha$. *More precisely, let* $\Phi(\cdot, x_a)$ *be an arbitrary solution of* (3.1) *with initial value* $\Phi(a, x_a) = x_a \neq 0$ *and* $\beta > 0$ *be an arbitrary positive number satisfying* $\beta > \alpha$. *Then*

$$\limsup_{t\to\infty}(\ln t - \ln a)^\beta\|\Phi(t, x_a)\| = +\infty.$$

Proof. We use a similar argument in the proof of Proposition 3.1. In contrast, assume that there exists a positive number β with $\beta > \alpha$ such that

$$\limsup_{t\to\infty}(\ln t - \ln a)^\beta\|\Phi(t, x_a)\| = M < \infty.$$

Then, there exists $T_1 > a$, such that

$$\|\Phi(t, x_a)\| < \frac{M+1}{(\ln t - \ln a)^\beta} \text{ for all } t > T_1.$$

Take and fix a positive number K satisfying $K\|x_a\| > 1$. In light of Lemma 2.3, we get that

$$E_\alpha\left(-L(\ln t - \ln a)^\alpha\right) = \frac{1}{L\Gamma(1-\alpha)}\frac{(1+o(1))}{(\ln t - \ln a)^\alpha}, \text{ as } t \to \infty.$$

In view of $\beta > \alpha$, we can easily infer the sum-up: There exists $T_2 > a$, such that

$$\frac{M+1}{(\ln t - \ln a)^{\beta}} < \frac{1}{K} E_\alpha\left(-L(\ln t - \ln a)^{\alpha}\right) \text{ for } t > T_2.$$

Put $T_0 = \max\{T_1, T_2\}$. Use the same argument in Proposition 3.1, we obtain

$$\begin{aligned}\frac{\Gamma(\alpha)}{L}\|x_a\| &\leq \limsup_{t\to\infty}\int_a^{T_0}(\ln t - \ln s)^{\alpha-1}\|\Phi(s,x_a)\|\frac{ds}{s}\\ &+ \limsup_{t\to\infty}\int_{T_0}^{t}(\ln t - \ln s)^{\alpha-1}\|\Phi(s,x_a)\|\frac{ds}{s}\\ &\leq \sup_{a\leq s\leq T_0}\|\Phi(s,x_a)\|\limsup_{t\to\infty}\int_a^{T_0}(\ln t - \ln s)^{\alpha-1}\frac{ds}{s}\\ &+ \limsup_{t\to\infty}\int_{T_0}^{t}(\ln t - \ln s)^{\alpha-1}\frac{M+1}{(\ln s - \ln a)^{\beta}}\frac{ds}{s}\\ &\leq \sup_{a\leq s\leq T_0}\|\Phi(s,x_a)\|\limsup_{t\to\infty}\frac{(\ln t - \ln a)^{\alpha} - (\ln t - \ln T_0)^{\alpha}}{\alpha}\\ &+ \frac{1}{K}\limsup_{t\to\infty}\int_{T_0}^{t}(\ln t - \ln s)^{\alpha-1}E_\alpha\left(-L(\ln s - \ln a)^{\alpha}\right)\frac{ds}{s}\\ &\leq \frac{1}{K}\limsup_{t\to\infty}\int_a^{t}(\ln t - \ln s)^{\alpha-1}E_\alpha\left(-L(\ln s - \ln a)^{\alpha}\right)\frac{ds}{s},\end{aligned}$$

and then it suffices to use the same arguments left in the proof of Proposition 3.1. □

Remark 3.3. *Proposition 3.1 remains true if we replace the strong condition of global Lipschitz property (g.3) by a weaker condition of local Lipschitz property of g at the origin:*

(g.3') There are positive constants $\omega > 0, L > 0$ such that $\|g(t,x) - g(t,y)\| \leq L\|x - y\|$ for all $t \geq a$ and $x, y \in \mathbb{R}^d, \|x\| \leq \omega, \|y\| \leq \omega$.

Similarly, nonuniform Lipschitz property (g.3') suffices for Proposition 3.2.

3.2. Notions of stability for FDEs

Consider the nonlinear fractional differential equation (3.1)

$$^{CH}D_a^{\alpha}x(t) = g\left(t, x(t)\right), \quad t > a,$$

where $g : \mathbb{R}_{\geq a} \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous and satisfies the condition (g.1) – (g.2) – (g.3′). Since g is local Lipschitz continuous, Theorem 2.2 and Remark 3.3 imply the unique existence of solution to the initial value problem (3.1), $x(a) = x_a$ withr $x_a \in \mathbb{R}^d, \|x\| \leq \omega$. Let $\Phi : I \times \mathbb{R}^d \to \mathbb{R}^d$ be the solution of (3.1), $x(a) = x_a$. We recall notions of stability and asymptotic stability of the trivial solution of (3.1), which directly extend the stability concepts from the classical theory of ordinary differential equations to the context of FDEs, cf. [11].

Definition 3.4.

(i) The trivial solution of the nonlinear fractional differential equation (3.1) is called stable if the solution exists on $[a, \infty)$, and for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$ such that for all $\|x_a\| < \delta$, we have $\|\Phi(t, x_a)\| < \varepsilon$ for all $t \geq a$.

(ii) The trivial solution is called asymptotically stable if it is stable and there exists some $\hat{\delta} > 0$ *such that* $\lim_{t\to\infty} \|\Phi(t, x_a)\| = 0$ *whenever* $\|x_a\| < \hat{\delta}$.

It is well-known that there is a notion of exponential stability of the solution of ordinary differential equations that is related to the exponential rate of convergence to solutions. However, the results of Section 3.1 show that the non-trivial solution to Caputo-Hadamard FDEs cannot decay with exponential rate but, at most, logarithmic rate. Therefore, it makes sense to investigate the logarithmic rate of decay of the solution to Caputo-Hadamard FDEs.

In the Eq (3.1), if $g(t, x) = Ax$ for all $t \geq a, x \in \mathbb{R}^d$ and $A \in \mathbb{R}^{d\times d}$, then for any $x_a \in \mathbb{R}^d$, this system with the initial condition $x(a) = x_a$ has the unique solution $x_a E_\alpha\left(A(\ln t - \ln a)^\alpha\right)$ on the interval $[a, \infty)$. This suggests us to use the Mittag-Leffler function in establishing a suitable stability definition for Caputo-Hadamard FDEs.

Motivated by Propositions 3.1 and 3.2, we now propose a new definition to characterize the convergent rate to the equilibrium points of solutions to Caputo-Hadamard FDEs. This is similar to that introduced by several authors (see [28, 20, 33]).

Definition 3.5. *The equilibrium point* $x^* = 0$ *of* (3.1) *is called logarithmic Mittag-Leffler stable if there exist positive constants* $\beta, m,$ *and* δ *such that*

$$\sup_{t\geq a}(\ln t - \ln a)^\beta \|\Phi(t, x_a)\| \leq m \tag{3.4}$$

for all $\|x_a\| \leq \delta$.

Remark 3.6. *(i) Logarithmic Mittag-Leffler stability implies asymptotic stability. This presents a novel stability definition that exceeds asymptotic stability in strength.*

(ii) Our formulation of logarithmic Mittag-Leffler stability is deliberately structured in parallel with the conventional exponential stability framework in classical ODE theory. This mathematical correspondence systematically quantifies the logarithmic decay rate inherent in Mittag-Leffler stable dynamical systems.

(iii) Proposition 3.2 imposes an essential structural constraint on β *of Definition 3.5, requiring it to maintain the inequality condition* $\beta \leq \alpha$ *throughout the stability analysis.*

4. Linearized logarithmic Mittag-Leffler stability of Caputo-Hadamard FDEs

This section presents a Lyapunov's first method framework for analyzing the asymptotic behavior of solutions to Caputo-Hadamard fractional differential equations. Through the synergistic application of a modified Laplace transform technique, Mittag-Leffler function properties, the Lyapunov-Perron approach, and a novel weighted norm introduced in this work, we establish the logarithmic Mittag-Leffler stability of fixed points associated with a newly defined operator.

4.1. Formulation of the result

Let ℓ be the Lipschitz constant

$$\ell_f(r) := \sup_{\substack{x,y\in B_{\mathbb{R}^d}(0,r)\\ x\neq y}} \frac{\|f(x) - f(y)\|}{\|x - y\|}$$

of a locally Lipschitz continuous function f on the ball $B_{\mathbb{R}^d}(0,r) := \{x \in \mathbb{R}^d : \|x\| \le r\}$. Consider a nonlinear Caputo-Hadamard fractional differential equation in the form

$$^{CH}D_a^{\alpha}x(t) = Ax(t) + f(x(t)), \tag{4.1}$$

where $A \in \mathbb{R}^{d\times d}$ and $f : \mathbb{R}^d \to \mathbb{R}^d$ is continuous on $\mathbb{R}^d$ and Lipschitz continuous in the neighborhood of the origin satisfying

$$f(0) = 0, \qquad \lim_{r\to 0} \ell_f(r) = 0. \tag{4.2}$$

Theorem 4.1. *For Eq (4.1), which satisfies (4.2), let $\hat{\lambda}_1, \cdots, \hat{\lambda}_d$ be the eigenvalues of A such that*

$$|\arg(\hat{\lambda}_i)| > \frac{\alpha\pi}{2}, \qquad i = 1, \cdots, d.$$

Then, the trivial solution of (4.1) is logarithmic Mittag-Leffler stable.

4.2. Construction of an appropriate Lyapunov-Perron operator

Before going to the proof of this theorem, we need two preparatory steps.

• Transformation of the linear part: The aim of this step is to transform the linear part of (4.1) to a quasi-diagonal matrix. This technical step reduces the difficulty in the estimation of the operators constructed in the next step.

• Construction of an appropriate Lyapunov-Perron operator: In this step, our aim is to present a family of operators with the property that any solution of the nonlinear system (4.1) can be interpreted as a fixed point of these operators. Furthermore, we show that these operators are contractive, and hence, the fixed points of these operators can be estimated and shown to be logarithmic Mittag-Leffler stable.

We now present the details of these preparatory steps.

4.2.1. Transformation of the linear part

From [31]. there exists a nonsingular matrix $T \in \mathbb{C}^{d\times d}$ transforming A into the Jordan normal form, i.e.,

$$T^{-1}AT = \operatorname{diag}(A_1, \cdots, A_n),$$

where for $i = 1, \cdots, n$ the block A_i is of the following form:

$$A_i = \lambda_i \operatorname{id}_{d_i\times d_i} + \beta_i N_{d_i\times d_i},$$

where $\beta_i \in \{0, 1\}$, $\lambda_i \in \{\hat{\lambda}_1, \cdots, \hat{\lambda}_d\}$, and the nilpotent matrix $N_{d_i\times d_i}$ is given by

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{d_i\times d_i}.$$

Let η be an arbitrary but fixed positive number. Applying the transformation

$$P = \operatorname{diag}(P_1, \cdots, P_n), P_i = \operatorname{diag}(1, \eta, \cdots, \eta^{d_i-1})$$

leads to

$$P_i^{-1} A_i P_i = \lambda_i \operatorname{id}_{d_i\times d_i} + \eta_i N_{d_i\times d_i}, \eta_i \in \{0, \eta\}.$$

Hence, we get that

$$P^{-1}T^{-1}ATP = \operatorname{diag}(P_1^{-1}A_1P_1, \cdots, P_n^{-1}A_nP_n) := \phi,$$
$$A = TP\phi(TP)^{-1}.$$

Under the transformation $y = (TP)^{-1}x$, system (4.1) becomes

$$\begin{aligned} {}^{CH}D_a^{\alpha} y(t) &= \phi y(t) + (TP)^{-1} f(TPy(t)) \\ &= \operatorname{diag}(J_1, \cdots, J_n) y(t) \\ &\quad + \operatorname{diag}(\eta_1 N_{d_1\times d_1}, \cdots, \eta_n N_{d_n\times d_n}) y(t) \\ &\quad + (TP)^{-1} f(TPy(t)). \end{aligned}$$

Then,

$$ {}^{CH}D_a^{\alpha} y(t) = \operatorname{diag}(J_1, \cdots, J_n) y(t) + h(y(t)), \tag{4.3}$$

where $J_i = \lambda_i \operatorname{id}_{d_i\times d_i}$, $i = 1, \cdots, n$, and the function h is given by

$$h(y) = \operatorname{diag}(\eta_1 N_{d_1\times d_1}, \cdots, \eta_n N_{d_n\times d_n}) y + (TP)^{-1} f(TPy). \tag{4.4}$$

Remark 4.2. *The map*

$$x \to \operatorname{diag}(\eta_1 N_{d_1\times d_1}, \cdots, \eta_n N_{d_n\times d_n}) x$$

is a Lipschitz continuous function with Lipschitz constant η or is identically zero. Thus, by (4.2), we have

$$h(0) = 0, \qquad \lim_{r\to 0} \ell_h(r) = \begin{cases} \eta & \text{if there exists } \eta_i = \eta, \\ 0 & \text{otherwise.} \end{cases}$$

In fact, the first proposition is trivially true, and we can easily see that $h(0) = 0$ from the fact $f(0) = 0$. As for the last part, we observe that

$$\begin{aligned} &\|h(y_1) - h(y_2)\| \\ \le&\| \operatorname{diag}(\eta_1 N_{d_1\times d_1}, \cdots, \eta_n N_{d_n\times d_n})\|\|y_1 - y_2\| \\ &+ \|(TP)^{-1}\|\|f(TPy_1) - f(TPy_2)\| \\ \le&\| \operatorname{diag}(\eta_1 N_{d_1\times d_1}, \cdots, \eta_n N_{d_n\times d_n})\|\|y_1 - y_2\| \\ &+ \ell_f(\max\{\|TPy_1\|, \|TPy_2\|\}) \|(TP)^{-1}\|\|TPy_1 - TPy_2\| \\ \le&\| \operatorname{diag}(\eta_1 N_{d_1\times d_1}, \cdots, \eta_n N_{d_n\times d_n})\|\|y_1 - y_2\| \\ &+ \ell_f(r)\|(TP)^{-1}\|\|TP\|\|y_1 - y_2\|, \end{aligned}$$

where $y_1, y_2 \in B_{\mathbb{R}_d}(0, r)$, hence the conclusion holds in view of (4.2).

Remark 4.3. *The type of stability of the trivial solution of Eqs (4.1) and (4.3) are the same, i.e., they are both stable (asymptotic/logarithmic Mittag-Leffler stable) or unstable (not asymptotic/logarithmic Mittag-Leffler stable).*

4.2.2. Construction of an appropriate Lyapunov-Perron operator

In this subsection, we concentrate only on Eq (4.3). We are now introducing a Lyapunov-Perron operator associated with Eq (4.3). Before doing this, we discuss some conventions that are used in the remaining part of this section. The space $\mathbb{R}^d$ can be written as $\mathbb{R}^d = \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_n}$. A vector $x \in \mathbb{R}^d$ can be written component-wise as $x = (x^1, \ldots, x^n)$.

For any $x = (x^1, \ldots, x^n) \in \mathbb{R}^d = \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_n}$, the operator $\mathscr{T}_x : C_\infty(\mathbb{R}_{\geq a}, \mathbb{R}^d) \to C_\infty(\mathbb{R}_{\geq a}, \mathbb{R}^d)$ is defined by

$$(\mathscr{T}_x\xi)(t) = \left((\mathscr{T}_x\xi)^1(t), \cdots, (\mathscr{T}_x\xi)^n(t)\right) \qquad \text{for } t \in \mathbb{R}_{\geq a},$$

where

$$\begin{aligned}&(\mathscr{T}_x\xi)^i(t)\\ =&E_\alpha\left(J_i(\ln\frac{t}{a})^\alpha\right)x^i\\ +&\int_a^t \left(\ln\frac{t}{\tau}\right)^{\alpha-1} E_{\alpha,\alpha}\left(J_i(\ln\frac{t}{\tau})^\alpha\right)h^i\left(\xi(\tau)\right)\frac{d\tau}{\tau}, i = 1, \ldots, n\end{aligned}$$

is called the Lyapunov-Perron operator associated with (4.3). The role of this operator is stated in the following theorem.

Theorem 4.4. *Let $x \in \mathbb{R}^d$ be arbitrary, and $\xi : \mathbb{R}_{\geq a} \to \mathbb{R}^d$ be a continuous function satisfying that $\xi(a) = x$. Then, the following statements are equivalent:*

(i) ξ is a solution of (4.3) satisfying the initial condition $x(a) = x$.

(ii) ξ is a fixed point of the operator $\mathscr{T}_x$.

Proof. The assertion follows from the modified Laplace transforms for Caputo-Hadamard fractional differential equations. See [25]. □

In $C\left([a, \infty), \mathbb{R}^d\right)$, we define a weighted norm as follows: For any $x \in C\left([a, \infty), \mathbb{R}^d\right)$, we endow it with the norm

$$\|x\|_w = \max\{\sup_{t\in[a,ae]} \|x(t)\|, \sup_{t\geq ae}(\ln t - \ln a)^\alpha\|x(t)\|\}.$$

$C_w := \{x \in C\left([a, \infty), \mathbb{R}^d\right), \|x\|_w < \infty\}$ is a Banach space.

Remark 4.5. *When constructing the weighted norm $\|\cdot\|_w$, we choose to partition at $t = ae$. This choice serves two primary purposes. On the one hand, it ensures that the operators in the Banach space C_w remain well-defined under the general maximum modulus norm $\|\cdot\|_\infty$. On the other hand, for $t > ae$, we have $\ln t - \ln a > 1$, hence $(\ln t - \ln a)^\alpha > 1$. Consequently,*

$$\|x\|_w = \max\{\sup_{t\in[a,ae]} \|x(t)\|, \sup_{t\geq ae}(\ln t - \ln a)^\alpha\|x(t)\|\} \geq \sup_{t\geq ae}\|x(t)\|,$$

which implies that $\|\cdot\|_\infty \leq \|\cdot\|_w$. This allows us to directly estimate the weighted norm when bounding the magnitudes in Proposition 4.7.

Next, we give some estimates concerning the operator $\mathscr{T}_x$ in the space C_w. For our estimates, we need the following technical lemma.

Lemma 4.6. *Following the definitions established above, we formally obtain that for any* $\xi : [a, \infty) \to \mathbb{R}^d$,

$$\sup_{t\in[a,ae]} \|\xi(t)\| = \max_{1\le i\le n} \sup_{t\in[a,ae]} \|\xi^i(t)\|,$$

$$\sup_{t\ge ae}(\ln \frac{t}{a})^\alpha \|\xi(t)\| = \max_{1\le i\le n} \sup_{t\ge ae}(\ln \frac{t}{a})^\alpha \|\xi^i(t)\|.$$

Proof. From the definition of the max norm, it is easy to see that

$$\|\xi(t)\| \ge \|\xi^i(t)\|, i = 1, \dots, n.$$

Hence, $\sup_{t\in[a,ae]} \|\xi(t)\| \ge \sup_{t\in[a,ae]} \|\xi^i(t)\|$, $\sup_{t\in[a,ae]} \|\xi(t)\| \ge \max_{1\le i\le n} \sup_{t\in[a,ae]} \|\xi^i(t)\|$. To the other side, as a result of $\max_{1\le i\le n} \|\xi^i(t)\| = \|\xi(t)\|$, we get that

$$\max_{1\le i\le n} \sup_{t\in[a,ae]} \|\xi^i(t)\| \ge \max_{1\le i\le n} \|\xi^i(t)\| = \|\xi(t)\|.$$

Hence, right-hand side exceeds the left-hand side. The proof of the remaining content follows the same procedure as above. □

Proposition 4.7. *For Eq* (4.3)*, suppose that*

$$|\arg(\lambda_i)| > \frac{\alpha\pi}{2}, \qquad i = 1, \dots, n.$$

Then there exists a constant $C(\alpha, \lambda)$ *depending on* α *and* $\lambda := (\lambda_1, \dots, \lambda_n)$ *such that for all* $x, \hat{x} \in \mathbb{R}^d$ *and* $\xi, \hat{\xi} \in C_w$*, the following inequality holds:*

$$\begin{aligned} &\|\mathscr{T}_x\xi - \mathscr{T}_{\hat{x}}\hat{\xi}\|_w \\ \le &\max_{1\le i\le n}\{ \sup_{t\in[a,ae]} |E_\alpha(\lambda_i(\ln \frac{t}{a})^\alpha)| + \sup_{t\ge ae}(\ln \frac{t}{a})^\alpha |E_\alpha(\lambda_i(\ln \frac{t}{a})^\alpha)|\}\|x - \hat{x}\| \\ &+ C(\alpha, \lambda)\ell_h(max(\|\xi\|_\infty, \|\hat{\xi}\|_\infty))\|\xi - \hat{\xi}\|_w. \end{aligned} \tag{4.5}$$

The validity of operator $\mathscr{T}_x \in C_w$ *can be demonstrated also by (4.5), and we can easily conjecture that*

$$\|\mathscr{T}_x\xi - \mathscr{T}_x\hat{\xi}\|_\infty \le C(\alpha, \lambda)\ell_h \left(max(\|\xi\|_\infty, \|\hat{\xi}\|_\infty)\right) \|\xi - \hat{\xi}\|_w. \tag{4.6}$$

Proof. By the definition of $(\mathscr{T}_x\xi)^i(t), i = 1, \dots, n$,

$$\begin{aligned} &\|(\mathscr{T}_x\xi - \mathscr{T}_{\hat{x}}\hat{\xi})^i(t)\| \\ \le &\left\|E_\alpha\left(J_i(\ln \frac{t}{a})^\alpha\right)\right\| \|x^i - \hat{x}^i\| \\ &+ \int_a^t \left(\ln \frac{t}{\tau}\right)^{\alpha-1} \left\|E_{\alpha,\alpha}\left(J_i(\ln \frac{t}{\tau})^\alpha\right)\right\| \\ &\times \left\|h^i\left(\xi(\tau)\right) - h^i(\hat{\xi}(\tau)\right)\| \frac{d\tau}{\tau} \\ \le &\left\|E_\alpha\left(J_i(\ln \frac{t}{a})^\alpha\right)\right\| \|x - \hat{x}\| \end{aligned}$$

$$
\begin{aligned}
&+\int_a^t \left(\ln\frac{t}{\tau}\right)^{\alpha-1} \left\|E_{\alpha,\alpha}\left(J_i(\ln\frac{t}{\tau})^{\alpha}\right)\right\| \\
&\times \left\|h\left(\xi(\tau)\right) - h(\hat{\xi}(\tau)\right)\| \frac{d\tau}{\tau} \\
\leq &\left|E_{\alpha}\left(\lambda_i(\ln\frac{t}{a})^{\alpha}\right)\right| \|x-\hat{x}\| + \ell_h\left(\max(\|\xi\|_{\infty}, \|\hat{\xi}\|_{\infty})\right)\|\xi-\hat{\xi}\|_w \\
&\times \int_a^t \left(\ln\frac{t}{\tau}\right)^{\alpha-1} |E_{\alpha,\alpha}(\lambda_i(\ln\frac{t}{\tau})^{\alpha}|\frac{d\tau}{\tau}.
\end{aligned}
$$

In the case $t \in [a, ae]$, we have

$$
\begin{aligned}
&\sup_{t\in[a,ae]} \|(\mathscr{T}_x\xi - \mathscr{T}_{\hat{x}}\hat{\xi})^i(t)\| \\
\leq &\sup_{t\in[a,ae]} \left|E_{\alpha}\left(\lambda_i(\ln\frac{t}{a})^{\alpha}\right)\right| \|x-\hat{x}\| \\
&+ \ell_h\left(\max(\|\xi\|_{\infty}, \|\hat{\xi}\|_{\infty})\right)\|\xi-\hat{\xi}\|_w \\
&\times \sup_{t\in[a,ae]} \int_a^t \left(\ln\frac{t}{\tau}\right)^{\alpha-1} |E_{\alpha,\alpha}(\lambda_i(\ln\frac{t}{\tau})^{\alpha}|\frac{d\tau}{\tau}.
\end{aligned}
$$

According to Lemma 2.6, there exists a positive constant β_i such that

$$
\begin{aligned}
&\sup_{t\in[a,ae]} \|(\mathscr{T}_x\xi - \mathscr{T}_{\hat{x}}\hat{\xi})^i(t)\| \\
\leq &\sup_{t\in[a,ae]} \left|E_{\alpha}\left(\lambda_i(\ln\frac{t}{a})^{\alpha}\right)\right| \|x-\hat{x}\| \\
&+\ell_h\left(\max(\|\xi\|_{\infty}, \|\hat{\xi}\|_{\infty})\right)\|\xi-\hat{\xi}\|_w\beta_i.
\end{aligned}
$$

Furthermore, in the case $t \geq ae$, we have

$$
\begin{aligned}
&(\ln t - \ln a)^{\alpha}\|(\mathscr{T}_x\xi - \mathscr{T}_{\hat{x}}\hat{\xi})^i\| \\
\leq &(\ln t - \ln a)^{\alpha}|E_{\alpha}\left(\lambda_i(\ln\frac{t}{a})^{\alpha}\right)|\|x-\hat{x}\| \\
&+\ell_h\left(\max(\|\xi\|_{\infty}, \|\hat{\xi}\|_{\infty})\right)\|\xi-\hat{\xi}\|_w(\ln\frac{t}{a})^{\alpha} \\
&\int_a^t \left(\ln\frac{t}{\tau}\right)^{\alpha-1} |E_{\alpha,\alpha}(\lambda_i(\ln\frac{t}{\tau})^{\alpha}|\frac{d\tau}{\tau}.
\end{aligned}
$$

According to Lemma 2.7, there exists a positive constant C_i such that

$$
\begin{aligned}
&\sup_{t\geq ae}(\ln t - \ln a)^{\alpha}\|(\mathscr{T}_x\xi - \mathscr{T}_{\hat{x}}\hat{\xi})^i\| \\
\leq &\sup_{t\geq ae}(\ln t - \ln a)^{\alpha}|E_{\alpha}\left(\lambda_i(\ln\frac{t}{a})^{\alpha}\right)|\|x-\hat{x}\| \\
&+\ell_h\left(\max(\|\xi\|_{\infty}, \|\hat{\xi}\|_{\infty})\right)\|\xi-\hat{\xi}\|_w C_i.
\end{aligned}
$$

Let $C(\alpha,\lambda) = \max\{\beta_1,\dots,\beta_n, C_1,\dots,C_n\}$.

By Lemma 4.6 and the fact that for any $\kappa \in C_w$,

$$\|\kappa\|_w \le \max_{1\le i\le n}\left\{\sup_{t\in[a,ae]}\|\kappa^i(t)\| + \sup_{t\ge ae}(\ln t-\ln a)^\alpha\|\kappa^i(t)\|\right\},$$

we obtain that

$$\begin{aligned}&\|\mathscr{T}_x\xi - \mathscr{T}_{\hat{x}}\hat{\xi}\|_w\\ &\le \max_{1\le i\le n}\{\sup_{t\in[a,ae]}|E_\alpha(\lambda_i(\ln t-\ln a)^\alpha)| + \sup_{t\ge ae}(\ln t-\ln a)^\alpha|E_\alpha(\lambda_i(\ln t-\ln a)^\alpha)|\}\|x-\hat{x}\|\\ &+\ell_h\left(\max(\|\xi\|_\infty,\|\hat{\xi}\|_\infty)\right)\|\xi-\hat{\xi}\|_w C(\alpha,\lambda),\end{aligned}$$

and

$$\|\mathscr{T}_x\xi - \mathscr{T}_x\hat{\xi}\|_w \le C(\alpha,\lambda)\ell_h\left(\max(\|\xi\|_\infty,\|\hat{\xi}\|_\infty)\right)\|\xi-\hat{\xi}\|_w.$$

From the definition of operator $\mathscr{T}_x$, we can also see that $\mathscr{T}_0(0) = 0$. □

So far, we have proved that the Lyapunov-Perron operator $\mathscr{T}_x$ is well-defined and Lipschitz continuous. Note that the Lipschitz constant is decided by the constant η, which is hidden in the coefficients of Eq (4.3). From now on, we choose and fix the constant η as $\eta = \frac{1}{2C(\alpha,\lambda)}$, where $C(\alpha,\lambda)$ is the constant defined in Proposition 4.7. The remaining difficulty is now to choose a ball with a small radius in $C_w(\mathbb{R}_{\ge a},\mathbb{R}^d)$ such that the restriction of the Lyapunov-Perron operator on this ball is strictly contractive. A positive answer to this question is given in the following technical lemma.

Lemma 4.8. *The following statements hold:*

(i) There exists $r > 0$ such that

$$q := C(\alpha,\lambda)\ell_h(r) < 1. \tag{4.7}$$

(ii) Let $r > 0$ be fixed and satisfy (4.7). Let

$$\nu = \max_{1\le i\le n}\{\sup_{t\in[a,ae]}|E_\alpha(\lambda_i(\ln\frac{t}{a})^\alpha)| + \sup_{t\ge ae}(\ln\frac{t}{a})^\alpha|E_\alpha(\lambda_i(\ln\frac{t}{a})^\alpha)|\},$$

$$r^* := \frac{r(1-q)}{\nu}, \tag{4.8}$$

and $B_{C_w}(0,r) := \{\xi \in C_w(\mathbb{R}_{\ge a},\mathbb{R}^d) : \|\xi\|_w \le r\}$. For any $x \in B_{\mathbb{R}^d}(0,r^*)$, we have $\mathscr{T}_x(B_{C_w}(0,r)) \subset B_{C_w}(0,r)$ and

$$\|\mathscr{T}_x\xi - \mathscr{T}_x\hat{\xi}\|_w \le q\|\xi-\hat{\xi}\|_w \qquad \textit{for all } \xi,\hat{\xi} \in B_{C_w}(0,r).$$

Proof. (i) By Remark 3.3,

$$\lim_{r\to 0}\ell_h(r) = \begin{cases}\eta & \text{if there exists } \eta_i = \eta,\\ 0 & \text{otherwise,}\end{cases} \text{ we get } \lim_{r\to 0}\ell_h(r) \le \eta.$$

Thus, we establish the desired conclusion from $\eta = \frac{1}{2C(\alpha,\lambda)}$.

(ii) For any $x \in B_{\mathbb{R}^d}(0, r^*)$ and $\xi \in B_{C_w}(0, r)$, it is necessary to estimate $\|\mathscr{T}_x\xi\|_w$. According to (4.5) in Proposition 4.7, we obtain that

$$\begin{aligned}\|\mathscr{T}_x\xi\|_w &\le \nu\|x\| + C(\alpha,\lambda)\ell_h\left(\|\xi\|_\infty\right)\|\xi\|_w \\ &\le \nu r^* + C(\alpha,\lambda)\ell_h\left(r\right) r \\ &= (1-q)r + qr \\ &= r,\end{aligned}$$

which prove that $\mathscr{T}_x\left(B_{C_w}(0, r)\right) \subset B_{C_w}(0, r)$. Furthermore, by (4.6) for all $x \in B_{\mathbb{R}^d}(0, r^*)$ and $\xi.\hat{\xi} \in B_{C_w}(0, r)$, we have

$$\|\mathscr{T}_x\xi - \mathscr{T}_x\hat{\xi}\|_w \le C(\alpha,\lambda)\ell_h(r)\|\xi - \hat{\xi}\|_w = q\|\xi - \hat{\xi}\|_w,$$

which concludes the proof.

□

4.3. *Proof of Theorem 4.1*

Due to Remark 4.3, it is sufficient to prove the stability for the trivial solution of Eq (4.3). For this purpose, let r^* be defined as in (4.8) and $\eta = \frac{1}{2C(\alpha,\lambda)}$. Let $x \in B_{\mathbb{R}^d}(0, r^*)$ be arbitrary. Combining Lemma 4.8 with the Banach fixed-point theorem, there exists $r > 0$ such that the Lyapunov-Perron operator $\mathscr{T}_x$ maps $B_{C_w}(0, r)$ into itself and admits a unique fixed point $\xi \in B_{C_w}(0, r)$ with $\xi(a) = x$. This point is also a solution of Eq (4.3) with the initial condition $\xi(a) = x$. In view of Remark 4.2, for any $\varepsilon \ge 0$, we can further choose r mentioned above with the additional requirement that $r \le \varepsilon$, which still satisfies (4.7). Hence, we get that

$$\|\xi(t)\| = \|\mathscr{T}_x\xi(t)\| \le r \le \varepsilon \qquad \text{for any } t \ge a,$$

which implies the stability. Furthermore, we get that

$$\sup_{t\ge a}(\ln t - \ln a)^\alpha\|\xi(t)\| \le r$$

from $\|\xi\|_w \le r$, which shows that the trivial solution of (4.3) is Logarithmic Mittag-Leffler stable. Thus, the trivial solution of (4.1) is logarithmic Mittag-Leffler stable.

5. Application

In this section, one example is provided to demonstrate the obtained theorem and illustrate its effectiveness.

Example 5.1. *The Caputo-Hadamard fractional Lotka-Volterra system is described as*

$$\begin{aligned}{}^{CH}D_a^\alpha x_1(t) &= x_1(t)\left(h + ex_1(t) + bx_2(t)\right), \\ {}^{CH}D_a^\alpha x_2(t) &= x_2(t)\left(-r + cx_1(t)\right),\end{aligned} \tag{5.1}$$

where the parameters h, r are positives, and $0 < \alpha < 1$ is the fractional order. This system can be rewritten as follows:

$${}^{CH}D_a^\alpha x(t) = Ax(t) + f(x(t)), \tag{5.2}$$

where

$$A = \begin{pmatrix} h & 0 \\ 0 & -r \end{pmatrix}, f(x) = \begin{pmatrix} ex_1^2 + bx_1x_2 \\ cx_1x_2 \end{pmatrix}.$$

In the following proposition, we first prove the instability of the trivial solution of system (5.1). Then, we show the logarithmic Mittag-Leffler stability of its controlled system with linear feedback.

Proposition 5.2. *The following statements hold.*

(i) The trivial solution of (5.1) is unstable.

(ii) For the system (5.2), its controlled system can be described as

$$^{CH}D_a^\alpha x(t) = Ax(t) + f(x(t)) + BKx(t), \tag{5.3}$$

where $B \in \mathbb{R}^{2\times 1}$ *and* $K \in \mathbb{R}^{1\times 2}$*. Letting* $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, k = \begin{pmatrix} -2h & 0 \end{pmatrix}$*, then, the trivial solution of (5.3) is logarithmic Mittag-Leffler stable.*

Proof. (i) Choose and fix an arbitrary positive number ϵ such that $\epsilon|e| < \frac{h}{2}$. Suppose in contrast that the trivial solution of (5.1) is stable. Then, there exists $\delta \in (0, \epsilon)$ such that for any solution $(x_1(t), x_2(t))^T$ of (5.1) satisfying $x_1(a) = \frac{\delta}{2}$ and $x_2(a) = 0$, we have $|x_1(t) + x_2(t)| < \epsilon$, for any $t \geq a$. Meanwhile, for all $t \geq a$, we have $x_1(t) > 0$ and $x_2(t) = 0$. Since $\epsilon|e| < \frac{h}{2}$ and $|x_1(t)| < \epsilon$, we get $x_1(t) < \frac{h}{2|e|}$. It follows that

$$hx_1 + ex_1^2 \geq \frac{h}{2}x_1 \text{ for all } t \geq a.$$

Thus we get

$$^{CH}D_a^\alpha x_1(t) \geq \frac{h}{2}x_1(t) \qquad \text{for all } t \geq a.$$

So we can find a positive function $m(t) \geq 0$, that

$$^{CH}D_a^\alpha x_1(t) - \frac{h}{2}x_1(t) = m(t). \tag{5.4}$$

Apply the modified Laplace transform in [22] to (5.4), we get that

$$\begin{aligned} x_1(t) =& E_\alpha\left(\frac{h}{2}(\ln t - \ln a)^\alpha\right)x_1(a) \\ &+ \int_a^t (\ln t - \ln\tau)^{\alpha-1} E_{\alpha,\alpha}\left(\frac{h}{2}(\ln t - \ln\tau)^\alpha\right)m(\tau)\frac{d\tau}{\tau}. \end{aligned}$$

In view of the non-negativity of $m(\cdot)$, it follows that

$$x_1(t) \geq x_1(a)E_\alpha\left(\frac{h}{2}(\ln t - \ln a)^\alpha\right) \qquad \text{for all } t \geq a.$$

This contradicts the fact that $\lim_{t\to\infty} E_\alpha\left(\frac{h}{2}(\ln t - \ln a)^\alpha\right) = \infty$. The proof of this part is complete.

(ii) The linear part of the system (5.3) is

$$A + BK = \begin{pmatrix} -h & 0 \\ -2h & -r \end{pmatrix},$$

which makes the spectrum of $A + BK$ contained in the sector

$$\{\lambda \in C : |\arg \lambda| > \frac{\alpha\pi}{2}\}$$

by the eigenvalues of $A + BK$ being $-h$ and $-r$. According to Theorem 4.1, the trivial solution of the system (5.3) is logarithmic Mittag-Leffler stable.

□

Next, we present the simulation results of a specific numerical example. The equation from Example 5.1 is adopted. For $e = 0, b = 0, c = 1, h = 10, r = 2$, *and* $\alpha = 0.5$, *the outcomes are illustrated in the following figures (see Figures 1–4).*

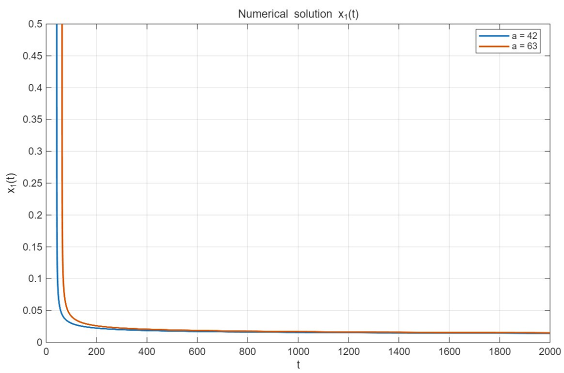

Figure 1. Time evolution of $x_1(t)$ for $a = 42$ and $a = 63$ with initial values $x_1(a) = 0.5, x_2(a) = 0.3$, and fractional order $\alpha = 0.5$.

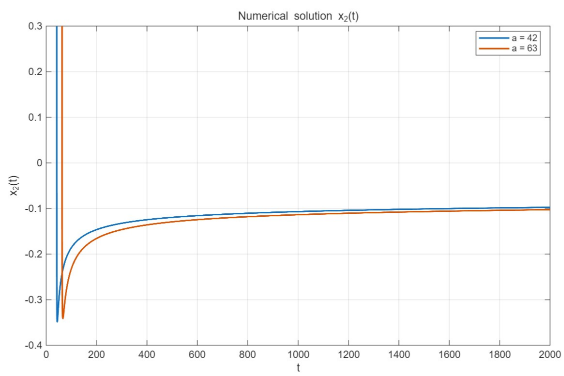

Figure 2. Time evolution of $x_2(t)$ under the same parameter setting as in Figure 1. Both components exhibit a slow, non-exponential decay characteristic of Caputo–Hadamard fractional systems.

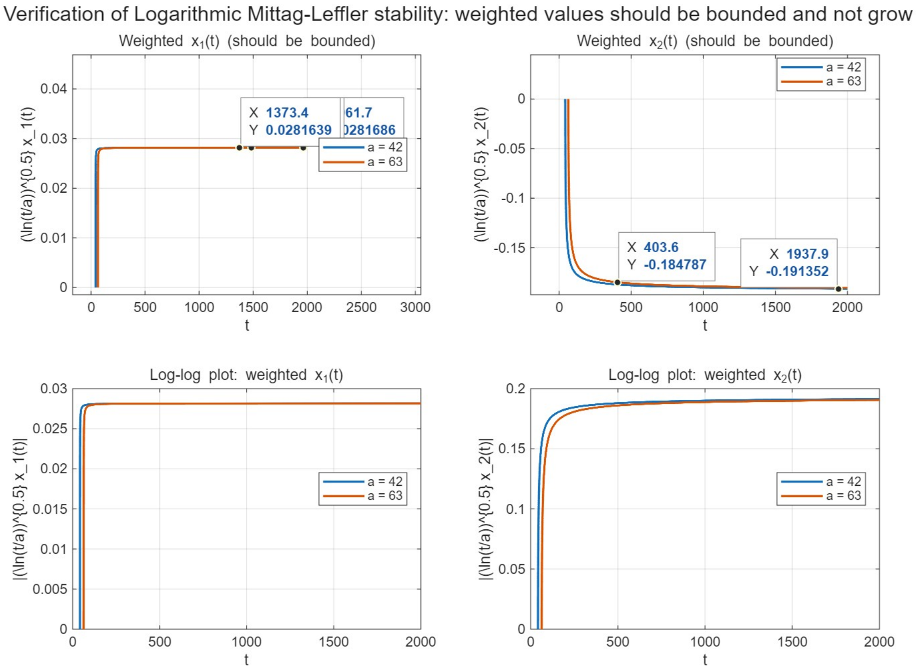

Figure 3. Top left: Weighted function $(\ln t - \ln a)^{\frac{1}{2}} x_1(t)$; top right: $(\ln t - \ln a)^{\frac{1}{2}} x_2(t)$; bottom left: log-log plot of $|(\ln t - \ln a)^{\frac{1}{2}} x_1(t)|$; bottom right: log-log plot of $|(\ln t - \ln a)^{\frac{1}{2}} x_2(t)|$. The weighted quantities remain bounded (approaching non-zero constants).

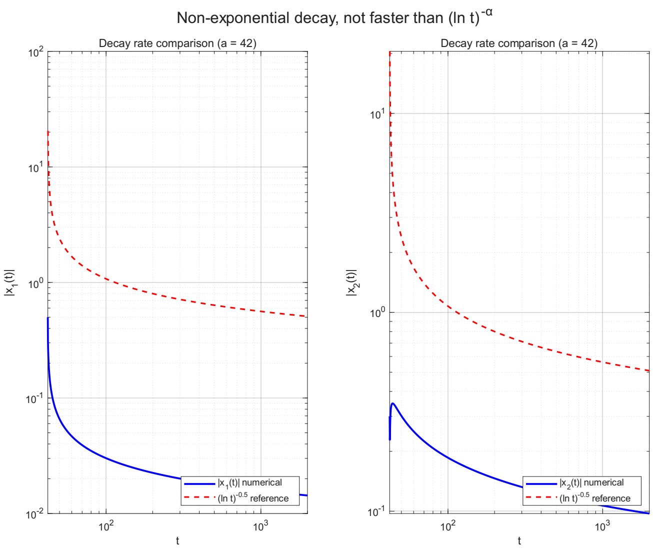

Figure 4. Decay rate comparison. **Left**: $|x_1(t)|$ (blue solid) against the reference line $(\ln t)^{-0.5}$ (red dashed) in log-log scale; **right**: $|x2(t)|$ against the same reference.

From Figure 1, we can see that the solution decays to zero as t increases, but the decay is not exponential. In Figure 2, the result also shows the rate of the solution $x_2(t)$.

The numerical results in Figure 3 show that $(\ln t - \ln a)^{\frac{1}{2}} x_1(t)$ *and* $(\ln t - \ln a)^{\frac{1}{2}} x_2(t)$ *remain bounded and approach non-zero constants as* $t \to \infty$, *which confirms the logarithmic Mittag-Leffler stability from Definition 3.5. See from Figure 4, the numerical solution does not decay faster than the logarithmic rate, which supports Proposition 3.2.*

6. Conjectures and future work

Conjecture 6.1. *(Power decay rate of solution to Caputo-Hadamard FDEs) Any nontrivial solution of the Caputo-Hadamard FDEs* (3.1) *cannot decay to* 0 *faster than* $(t-a)^{\alpha}$. *More precisely, let* $\Phi(\cdot, x_a)$ *be an arbitrary solution of* (3.1) *with initial value* $\Phi(a, x_a) = x_a \neq 0$, *and* $\beta > 0$ *be an arbitrary positive number satisfying* $\beta > \alpha$. *Then*

$$\limsup_{t\to\infty} (t-a)^{\beta} \|\Phi(t, x_a)\| = +\infty.$$

Due to the estimate in Conjecture 6.1 on the algebraic decay of the solution Φ, we can reasonably conjecture that it decays not merely logarithmically, but may in fact exhibit algebraic rate.

7. Conclusions

This paper systematically investigates the stability and convergence properties of the Caputo-Hadamard FDEs. By leveraging the properties of the Mittag-Leffler function, we have provided a numerical estimation for the equivalent integral term of the Caputo-Hadamard FDEs. By employing the Mittag-Leffler function to conjecture and verify the convergence rate of the solution, we conducted a detailed study on the convergence rate of the solutions, proving that under specific Lipschitz conditions, the decay rate is non-exponential and no faster than the logarithmic rate $(\ln t - \ln a)^{\alpha}$ (where α denotes the equation's order). Thus, we proposes a new "logarithmic Mittag-Leffler stability" definition, which is stronger than asymptotic stability, and highlights its practical significance. A variant of Lyapunov's first method, tailored to the logarithmic decay nature of Caputo-Hadamard systems, is developed, involving the construction of a new weighted norm and the application of the Banach fixed-point theorem to the stability. Thereby, we have established criteria for judging the logarithmic Mittag-Leffler stability of the solutions to Caputo-Hadamard FDEs. Finally, we puts forward reasonable conjectures on the equation's convergence rate and outlines future research directions, indicating potential extensions of the study.

Author contributions

Haonan Zhang: Conceptualization, methodology, formal analysis, validation, writing – original draft; Qixiang Dong: Method-validation, Supervision, writing – review and editing; Zidi Zhao: writing – review. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The Artificial Intelligence tools were used for language polishing and grammar improvement. No AI was used in research design, data analysis, interpretation, or conclusion. All scientific content is entirely the authors' own.

Conflict of interest

All authors state no conflicts of interest in this article.

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