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Research article

A note on nonlinear mixed bi-skew Jordan and bi-skew Lie n-derivations on $*$-algebras

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Abstract: Let $\mathfrak{A}$ be a unital $*$-algebra with identity $\mathscr{I}$. For $\mathscr{A}, \mathscr{B} \in \mathfrak{A}$, define the bi-skew Jordan product

$$\mathscr{A} \bullet \mathscr{B} = \mathscr{A}\mathscr{B}^* + \mathscr{B}\mathscr{A}^*$$

and the bi-skew Lie product

$$[\mathscr{A}, \mathscr{B}]_\diamond = \mathscr{A}\mathscr{B}^* - \mathscr{B}\mathscr{A}^*.$$

Suppose that a nonlinear mapping $\Phi : \mathfrak{A} \to \mathfrak{A}$ satisfies

$$\Phi\Big([\mathscr{A}_1 \bullet \mathscr{A}_2 \bullet \cdots \bullet \mathscr{A}_{n-1}, \mathscr{A}_n]_\diamond\Big) = \sum_{j=1}^{n} [\mathscr{A}_1 \bullet \cdots \bullet \mathscr{A}_{j-1} \bullet \Phi(\mathscr{A}_j) \bullet \mathscr{A}_{j+1} \bullet \cdots \bullet \mathscr{A}_{n-1}, \mathscr{A}_n]_\diamond$$

for all suitable elements $\mathscr{A}_1, \ldots, \mathscr{A}_n \in \mathfrak{A}$, where $\mathscr{A}_1, \mathscr{A}_2 \in \{\mathscr{I}, i\mathscr{I}\}$ and $\mathscr{A}_j = \mathscr{I}$ for every $j = 3, 4, \ldots, n-2$. We prove that Φ is an additive $*$-derivation on $\mathfrak{A}$. As applications, several consequences are obtained for prime $*$-algebras, factor von Neumann algebras, von Neumann algebras without central summands of type $\mathscr{I}_1$, and standard operator algebras. Moreover, a conjecture is proposed to motivate further research in this direction. The obtained results extend and generalize a recent result of Abbasi et al. [Non-additive mixed bi-skew Jordan and bi-skew Lie triple derivations on $*$-algebras, *Ricerche Mat.*, 2026.] concerning non-additive mixed bi-skew Jordan and bi-skew Lie triple derivations on $*$-algebras.

Keywords: mixed bi-skew Jordan product; bi-skew Lie product; nonlinear derivation; $*$-derivation; prime $*$-algebra; von Neumann algebra
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1. Introduction

Consider $\mathfrak{A}$ be a $*$-algebra over $\mathbb{C}$, a complex field. Among its elements, various operations such as the bi-skew Jordan products, as well as the bi-skew Lie products, play a pivotal role. These products are defined as follows: the bi-skew Jordan product is $\mathscr{A} \bullet \mathscr{B} = \mathscr{A}\mathscr{B}^* + \mathscr{B}\mathscr{A}^*$ and the bi-skew Lie product is $[\mathscr{A}, \mathscr{B}]_\diamond = \mathscr{A}\mathscr{B}^* - \mathscr{B}\mathscr{A}^*$.

A linear map $\Phi : \mathfrak{A} \to \mathfrak{A}$ is called a $*$-derivation if $\Phi(\mathscr{A}\mathscr{B}) = \Phi(\mathscr{A})\mathscr{B} + \mathscr{A}\Phi(\mathscr{B})$ and $\Phi(\mathscr{A}^*) = \Phi(\mathscr{A})^*$ for all $\mathscr{A}, \mathscr{B} \in \mathfrak{A}$. A map $\Phi : \mathfrak{A} \to \mathfrak{A}$, that is not necessarily linear, is referred to as a nonlinear bi-skew Jordan derivation (respectively, a nonlinear bi-skew Jordan triple derivation) if $\Phi(\mathscr{A} \bullet \mathscr{B}) = \Phi(\mathscr{A}) \bullet \mathscr{B} + \mathscr{A} \bullet \Phi(\mathscr{B})$ (respectively, $\Phi(\mathscr{A} \bullet \mathscr{B} \bullet \mathscr{G}) = \Phi(\mathscr{A}) \bullet \mathscr{B} \bullet \mathscr{G} + \mathscr{A} \bullet \Phi(\mathscr{B}) \bullet \mathscr{G} + \mathscr{A} \bullet \mathscr{B} \bullet \Phi(\mathscr{G})$) for all $\mathscr{A}, \mathscr{B}, \mathscr{G} \in \mathfrak{A}$. Further, Φ is known as a nonlinear bi-skew Lie derivation (respectively, nonlinear bi-skew Lie triple derivation) if $\Phi([\mathscr{A}, \mathscr{B}]_\diamond) = [\Phi(\mathscr{A}), \mathscr{B}]_\diamond + [\mathscr{A}, \Phi(\mathscr{B})]_\diamond$ (respectively, $\Phi([[\mathscr{A}, \mathscr{B}]_\diamond, \mathscr{G}]_\diamond) = [[\Phi(\mathscr{A}), \mathscr{B}]_\diamond, \mathscr{G}]_\diamond + [[\mathscr{A}, \Phi(\mathscr{B})]_\diamond, \mathscr{G}]_\diamond + [[\mathscr{A}, \mathscr{B}]_\diamond, \Phi(\mathscr{G})]_\diamond$).

In recent years, many researchers have investigated various forms of nonlinear bi-skew Jordan and nonlinear bi-skew Lie derivations (see [2–11] and the references therein). In [12], Darvish et al. explored nonlinear bi-skew Jordan derivations on prime $*$-algebras. In [13], Khan and Alhazmi proved that nonlinear multiplicative bi-skew Jordan triple derivations on prime $*$-algebras are additive $*$-derivations. According to [14], nonlinear bi-skew Lie derivations under suitable self-adjointness assumptions reduce to additive $*$-derivations.

The products $\mathscr{A}_1 \bullet \cdots \bullet \mathscr{A}_n$ and $[[\ldots[\mathscr{A}_1, \mathscr{A}_2]_\diamond, \ldots, \mathscr{A}_{n-1}]_\diamond, \mathscr{A}_n]_\diamond$ are called the bi-skew Jordan n-product and the bi-skew Lie n-product, respectively. Nonlinear maps satisfying the corresponding identities were investigated in [15–17]. For additional information, the reader i s referred to the references cited therein.

Further developments concerning mixed Jordan and Lie structures appear in [18–20]. Recently, Zhao et al. [21] showed that nonlinear bi-skew Jordan-type derivations on unital $*$-algebras are additive $*$-derivations. Quite recently, Alam et al. [22] showed that nonlinear mixed skew Lie-type derivations on unital $*$-algebras are additive $*$-derivations. In [6], Ferreira and Wei studied mixed $*$-Jordan-type derivations on $*$-algebras. Motivated by these developments, in this paper we mixed the bi-skew Jordan product $\mathscr{A} \bullet \mathscr{B} = \mathscr{A}\mathscr{B}^* + \mathscr{B}\mathscr{A}^*$ and the bi-skew Lie product $[\mathscr{A}, \mathscr{B}]_\diamond = \mathscr{A}\mathscr{B}^* - \mathscr{B}\mathscr{A}^*$, and define a nonlinear mixed bi-skew Jordan and bi-skew Lie n-derivation as a map $\Phi : \mathfrak{A} \to \mathfrak{A}$ satisfying

$$\Phi([\mathscr{A}_1 \bullet \cdots \bullet \mathscr{A}_{n-1}, \mathscr{A}_n]_\diamond) = \sum_{j=1}^{n} [\mathscr{A}_1 \bullet \cdots \bullet \Phi(\mathscr{A}_j) \bullet \cdots \bullet \mathscr{A}_{n-1}, \mathscr{A}_n]_\diamond,$$

for all $\mathscr{A}_1, \ldots, \mathscr{A}_n \in \mathfrak{A}$, where $\mathscr{A}_1, \mathscr{A}_2 \in \{\mathscr{I}, i\mathscr{I}\}$ and $\mathscr{A}_j = \mathscr{I}$ for $3 \le j \le n-2$, with $n \ge 3$. Our main objective is to characterize the structure of such derivations on a $*$-algebra. Under suitable assumptions, we prove that Φ is a nonlinear mixed bi-skew Jordan and bi-skew Lie n-derivation if and only if it is an additive $*$-derivation.

2. Main result

To begin with, we present a fundamental lemma that plays a crucial role in the proof of our main theorem.

Lemma 2.1. *(Peirce multiplication table) Let $\mathfrak{A}$ be a unital $*$-algebra, and let $\mathscr{P}_1, \mathscr{P}_2 \in \mathfrak{A}$ be two orthogonal projections such that*

$$\mathscr{P}_1 + \mathscr{P}_2 = \mathscr{I}.$$

For $i, j \in \{1, 2\}$, define

$$\mathfrak{A}_{ij} = \mathscr{P}_i \mathfrak{A} \mathscr{P}_j.$$

Then, the following Peirce multiplication relations hold:

$$\mathfrak{A}_{ij}\mathfrak{A}_{kl} \subseteq \begin{cases} \mathfrak{A}_{il}, & \text{if } j = k, \\ \{0\}, & \text{if } j \neq k. \end{cases}$$

In particular,

$$\begin{aligned} &\mathfrak{A}_{11}\mathfrak{A}_{11} \subseteq \mathfrak{A}_{11}, \qquad \mathfrak{A}_{22}\mathfrak{A}_{22} \subseteq \mathfrak{A}_{22}, \\ &\mathfrak{A}_{11}\mathfrak{A}_{12} \subseteq \mathfrak{A}_{12}, \qquad \mathfrak{A}_{12}\mathfrak{A}_{22} \subseteq \mathfrak{A}_{12}, \\ &\mathfrak{A}_{22}\mathfrak{A}_{21} \subseteq \mathfrak{A}_{21}, \qquad \mathfrak{A}_{21}\mathfrak{A}_{11} \subseteq \mathfrak{A}_{21}, \\ &\mathfrak{A}_{12}\mathfrak{A}_{21} \subseteq \mathfrak{A}_{11}, \qquad \mathfrak{A}_{21}\mathfrak{A}_{12} \subseteq \mathfrak{A}_{22}, \\ &\mathfrak{A}_{12}\mathfrak{A}_{11} = \mathfrak{A}_{22}\mathfrak{A}_{12} = \mathfrak{A}_{21}\mathfrak{A}_{22} = \mathfrak{A}_{11}\mathfrak{A}_{21} = \{0\}. \end{aligned}$$

Moreover, the algebra $\mathcal{A}$ admits the Peirce decomposition

$$\mathfrak{A} = \mathfrak{A}_{11} \oplus \mathfrak{A}_{12} \oplus \mathfrak{A}_{21} \oplus \mathfrak{A}_{22}.$$

Our main theorem is stated as follows.

Theorem 2.2. *Let $\mathfrak{A}$ be a unital $*$-algebra with unity $\mathscr{I}$. Assume that a nontrivial projection $\mathscr{P}$ in $\mathfrak{A}$ satisfies*

$$\mathscr{A}\mathfrak{A}\,\mathscr{P} = 0 \implies \mathscr{A} = 0 \tag{2.1}$$

and

$$\mathscr{A}\mathfrak{A}\,(\mathscr{I} - \mathscr{P}) = 0 \implies \mathscr{A} = 0. \tag{2.2}$$

Then a map $\Phi : \mathfrak{A} \to \mathfrak{A}$ satisfies

$$\begin{aligned} &\Phi\left([\mathscr{A}_1 \bullet \mathscr{A}_2 \bullet \cdots \bullet \mathscr{A}_{n-1},\ \mathscr{A}_n]_\diamond\right) \\ =\ &\sum_{j=1}^{n} [\mathscr{A}_1 \bullet \mathscr{A}_2 \bullet \cdots \bullet \mathscr{A}_{j-1} \bullet \Phi(\mathscr{A}_j) \bullet \mathscr{A}_{j+1} \bullet \cdots \bullet \mathscr{A}_{n-1},\ \mathscr{A}_n]_\diamond \end{aligned}$$

for all $\mathscr{A}_{n-1}, \mathscr{A}_n \in \mathfrak{A}$, where $\mathscr{A}_1, \mathscr{A}_2 \in \{\mathscr{I}, i\mathscr{I}\}$ and $\mathscr{A}_j = \mathscr{I}$ for every $j \in \{3, 4, \ldots, n-2\}$. Then, Φ is an additive $*$-derivation on $\mathfrak{A}$.

Define $\mathscr{P}_1 = \mathscr{P}$ and $\mathscr{P}_2 = \mathscr{I} - \mathscr{P}$ as two distinct nontrivial projections in $\mathfrak{A}$. Let $\mathscr{H}$ be a complex Hilbert space, and let $\mathfrak{B}(\mathscr{H})$ denote the algebra of all bounded linear operators acting on $\mathscr{H}$. For any operator $\mathscr{A} \in \mathfrak{B}(\mathscr{H})$ can be written in the form $\mathscr{A} = \mathscr{A}_1 + i\mathscr{A}_2$, where $i \in \mathbb{C}$ (set of complex numbers) such that $i^2 = -1$ and $\mathscr{A}_1, \mathscr{A}_2$ are self-adjoint operators. Define $\mathfrak{X} = \{\mathscr{X} \in \mathfrak{A} | \ \mathscr{X}^* = -\mathscr{X}\}$, $\mathfrak{X}_{12} = \{\mathscr{P}_1\mathscr{X}\mathscr{P}_2 + \mathscr{P}_2\mathscr{X}\mathscr{P}_1 | \ \mathscr{X} \in \mathfrak{X}\}$, and $\mathfrak{X}_{ii} = \ \mathscr{P}_i\mathfrak{X}\mathscr{P}_i$ $(i = 1, 2)$. Thus, for every $\mathscr{X} \in \mathfrak{X}$, $\mathscr{X} = \mathscr{X}_{11} + \mathscr{X}_{12} + \mathscr{X}_{22}$, for every $\mathscr{X}_{12} \in \mathfrak{X}_{12}$, and $\mathscr{X}_{ii} \in \mathfrak{X}_{ii}$ $(i = 1, 2)$.

The proof of the main result is developed through a series of auxiliary lemmas, which we present below.

Lemma 2.3. $\Phi(0) = 0$.

Proof. By the hypothesis, we have

$$\begin{aligned}\Phi(0) =&\Phi\left([\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet 0,\ 0]_\diamond\right)\\ =&[\Phi(\mathscr{I}) \bullet \cdots \bullet \mathscr{I} \bullet 0,\ 0]_\diamond + \cdots + [\mathscr{I} \bullet \cdots \bullet \Phi(\mathscr{I}) \bullet 0,\ 0]_\diamond\\ &+[\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \Phi(0),\ 0] + [\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet 0,\ \Phi(0)]_\diamond\\ =&0.\end{aligned}$$

□

Lemma 2.4. *For any $\mathscr{X} \in \mathfrak{X}$, we have $\Phi(\mathscr{X}) \in \mathfrak{X}$.*

Proof. For any $\mathscr{X} \in \mathfrak{X}$, we can write

$$\mathscr{X} = \left[\mathscr{I} \bullet \mathscr{I} \bullet \cdots \bullet \mathscr{I}, \frac{-\mathscr{X}}{2^{n-1}}\right]_\diamond.$$

Therefore,

$$\begin{aligned}\Phi(\mathscr{X}) &= \Phi\left(\left[\mathscr{I} \bullet \mathscr{I} \bullet \cdots \bullet \mathscr{I}, \frac{-\mathscr{X}}{2^{n-1}}\right]_\diamond\right)\\ &= \left[\Phi(\mathscr{I}) \bullet \mathscr{I} \bullet \cdots \bullet \mathscr{I}, \frac{-\mathscr{X}}{2^{n-1}}\right]_\diamond + \left[\mathscr{I} \bullet \Phi(\mathscr{I}) \bullet \cdots \bullet \mathscr{I}, \frac{-\mathscr{X}}{2^{n-1}}\right]_\diamond\\ &\quad + \cdots + \left[\mathscr{I} \bullet \mathscr{I} \bullet \cdots \bullet \Phi(\mathscr{I}), \frac{-\mathscr{X}}{2^{n-1}}\right]_\diamond + \left[\mathscr{I} \bullet \mathscr{I} \bullet \cdots \bullet \mathscr{I}, \Phi\left(\frac{-\mathscr{X}}{2^{n-1}}\right)\right]_\diamond.\end{aligned}$$

This yields

$$\Phi(\mathscr{X}) = \frac{1}{2}(n-1)\left(\Phi(\mathscr{I})^*\frac{\mathscr{X}}{2^{n-1}} + \frac{\mathscr{X}}{2^{n-1}}\Phi(\mathscr{I})\right) + 2^{n-2}\left(\Phi\left(\frac{-\mathscr{X}}{2^{n-1}}\right) - \Phi\left(\frac{-\mathscr{X}}{2^{n-1}}\right)^*\right).$$

Then, we obtain

$$\Phi(\mathscr{X})^* = -\frac{1}{2}(n-1)\left(\Phi(\mathscr{I})^*\frac{\mathscr{X}}{2^{n-1}} + \frac{\mathscr{X}}{2^{n-1}}\Phi(\mathscr{I})\right) - 2^{n-2}\left(\Phi\left(\frac{-\mathscr{X}}{2^{n-1}}\right) - \Phi\left(\frac{-\mathscr{X}}{2^{n-1}}\right)^*\right).$$

From the above two, we conclude that

$$\Phi(\mathscr{X})^* = -\Phi(\mathscr{X}).$$

Hence, $\Phi(\mathscr{X}) \in \mathfrak{X}$. □

Lemma 2.5. *For any* $\mathscr{X}_{11} \in \mathfrak{X}_{11}$, $\mathscr{Y}_{12} \in \mathfrak{X}_{12}$, *and* $\mathscr{Z}_{22} \in \mathfrak{X}_{22}$, *we have*

(i) $\Phi(\mathscr{X}_{11} + \mathscr{Y}_{12}) = \Phi(\mathscr{X}_{11}) + \Phi(\mathscr{Y}_{12})$;

(ii) $\Phi(\mathscr{Y}_{12} + \mathscr{Z}_{22}) = \Phi(\mathscr{Y}_{12}) + \Phi(\mathscr{Z}_{22})$.

Proof. (*i*) Let $\mathscr{M} = \Phi(\mathscr{X}_{11} + \mathscr{Y}_{12}) - \Phi(\mathscr{X}_{11}) - \Phi(\mathscr{Y}_{12})$. By Lemma 2.4, it follows that $\mathscr{M}^* = -\mathscr{M}$. Moreover, since $[\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \mathscr{P}_2, \mathscr{X}_{11}]_\diamond = 0$, we obtain

$$\begin{aligned}
&[\Phi(\mathscr{I}) \bullet \cdots \bullet \mathscr{I} \bullet \mathscr{P}_2,\ \mathscr{X}_{11} + \mathscr{Y}_{12}]_\diamond + \cdots + [\mathscr{I} \bullet \cdots \bullet \Phi(\mathscr{I}) \bullet \mathscr{P}_2,\ \mathscr{X}_{11} + \mathscr{Y}_{12}]_\diamond \\
&+[\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \Phi(\mathscr{P}_2),\ \mathscr{X}_{11} + \mathscr{Y}_{12}]_\diamond + [\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \mathscr{P}_2,\ \Phi(\mathscr{X}_{11} + \mathscr{Y}_{12})]_\diamond \\
&=\Phi([\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \mathscr{P}_2,\ \mathscr{X}_{11} + \mathscr{Y}_{12}]_\diamond) \\
&=\Phi([\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \mathscr{P}_2,\ \mathscr{X}_{11}]_\diamond) + \Phi([\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \mathscr{P}_2,\ \mathscr{Y}_{12}]_\diamond) \\
&=[\Phi(\mathscr{I}) \bullet \cdots \bullet \mathscr{I} \bullet \mathscr{P}_2,\ \mathscr{X}_{11} + \mathscr{Y}_{12}]_\diamond + \cdots + [\mathscr{I} \bullet \cdots \bullet \Phi(\mathscr{I}) \bullet \mathscr{P}_2,\ \mathscr{X}_{11} + \mathscr{Y}_{12}]_\diamond \\
&+[\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \Phi(\mathscr{P}_2),\ \mathscr{X}_{11} + \mathscr{Y}_{12}]_\diamond + [\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \mathscr{P}_2,\ \Phi(\mathscr{X}_{11}) + \Phi(\mathscr{Y}_{12})]_\diamond.
\end{aligned}$$

This yields $[\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \mathscr{P}_2, \mathscr{M}]_\diamond = 0$, and hence $\mathscr{P}_1\mathscr{M}\mathscr{P}_2 = \mathscr{P}_2\mathscr{M}\mathscr{P}_1 = \mathscr{P}_2\mathscr{M}\mathscr{P}_2 = 0$.

Furthermore, using the relation $[\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet (\mathscr{P}_2 - \mathscr{P}_1),\ \mathscr{Y}_{12}]_\diamond = 0$, we have

$$\begin{aligned}
&[\Phi(\mathscr{I}) \bullet \cdots \bullet \mathscr{I} \bullet (\mathscr{P}_2 - \mathscr{P}_1),\ \mathscr{X}_{12} + \mathscr{Y}_{12}]_\diamond \\
&+ \cdots + [\mathscr{I} \bullet \cdots \bullet \Phi(\mathscr{I}) \bullet (\mathscr{P}_2 - \mathscr{P}_1),\ \mathscr{X}_{12} + \mathscr{Y}_{12}]_\diamond \\
&+ [\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \Phi(\mathscr{P}_2 - \mathscr{P}_1),\ \mathscr{X}_{12} + \mathscr{Y}_{12}]_\diamond + [\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet (\mathscr{P}_2 - \mathscr{P}_1),\ \Phi(\mathscr{X}_{12} + \mathscr{Y}_{12})]_\diamond \\
&=\Phi\left([\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet (\mathscr{P}_2 - \mathscr{P}_1),\ \mathscr{X}_{12} + \mathscr{Y}_{12}]_\diamond\right) \\
&=\Phi\left([\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet (\mathscr{P}_2 - \mathscr{P}_1),\ \mathscr{X}_{12}]_\diamond\right) + \Phi\left([\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet (\mathscr{P}_2 - \mathscr{P}_1),\ \mathscr{Y}_{12}]_\diamond\right) \\
&= [\Phi(\mathscr{I}) \bullet \cdots \bullet \mathscr{I} \bullet (\mathscr{P}_2 - \mathscr{P}_1),\ \mathscr{X}_{12} + \mathscr{Y}_{12}]_\diamond \\
&+ \cdots + [\mathscr{I} \bullet \cdots \bullet \Phi(\mathscr{I}) \bullet (\mathscr{P}_2 - \mathscr{P}_1),\ \mathscr{X}_{12} + \mathscr{Y}_{12}]_\diamond \\
&+ [\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \Phi(\mathscr{P}_2 - \mathscr{P}_1),\ \mathscr{X}_{12} + \mathscr{Y}_{12}]_\diamond \\
&+ [\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet (\mathscr{P}_2 - \mathscr{P}_1),\ \Phi(\mathscr{X}_{12}) + \Phi(\mathscr{Y}_{12})]_\diamond .
\end{aligned}$$

This further implies that $[\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet (\mathscr{P}_2 - \mathscr{P}_1),\ \mathscr{M}]_\diamond = 0$, which yields $\mathscr{P}_1\mathscr{M}\mathscr{P}_1 = 0$. Therefore, $\mathscr{M} = 0$.

By a similar argument, we also obtain $\Phi(\mathscr{Y}_{12} + \mathscr{Z}_{22}) = \Phi(\mathscr{Y}_{12}) + \Phi(\mathscr{Z}_{22})$. □

Lemma 2.6. *For any* $\mathscr{X}_{11} \in \mathfrak{X}_{11}, \mathscr{Y}_{12} \in \mathfrak{X}_{12}$ *and* $\mathscr{Z}_{22} \in \mathfrak{X}_{22}$, *we have*

$$\Phi(\mathscr{X}_{11} + \mathscr{Y}_{12} + \mathscr{Z}_{22}) = \Phi(\mathscr{X}_{11}) + \Phi(\mathscr{Y}_{12}) + \Phi(\mathscr{Z}_{22}).$$

Proof. Define $\mathscr{M} = \Phi(\mathscr{X}_{11} + \mathscr{Y}_{12} + \mathscr{Z}_{22}) - \Phi(\mathscr{X}_{11}) - \Phi(\mathscr{Y}_{12}) - \Phi(\mathscr{Z}_{22})$. Then by Lemma 2.4, we get $\mathscr{M}^* = -\mathscr{M}$.

Further, observe that $[\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \mathscr{P}_1,\ \mathscr{Z}_{22}]_\diamond = 0$, and by applying Lemma 2.5, we obtain

$$\begin{aligned}
&[\Phi(\mathscr{I}) \bullet \cdots \bullet \mathscr{I} \bullet \mathscr{P}_1,\ \mathscr{X}_{11} + \mathscr{Y}_{12} + \mathscr{Z}_{22}]_\diamond \\
&+ \cdots + [\mathscr{I} \bullet \cdots \bullet \Phi(\mathscr{I}) \bullet \mathscr{P}_1,\ \mathscr{X}_{11} + \mathscr{Y}_{12} + \mathscr{Z}_{22}]_\diamond \\
&+ [\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \Phi(\mathscr{P}_1),\ \mathscr{X}_{11} + \mathscr{Y}_{12} + \mathscr{Z}_{22}]_\diamond + [\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \mathscr{P}_1,\ \Phi(\mathscr{X}_{11} + \mathscr{Y}_{12} + \mathscr{Z}_{22})]_\diamond
\end{aligned}$$

$$\begin{aligned}
&=\Phi\left([\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \mathscr{P}_1,\ \mathscr{X}_{11}+\mathscr{Y}_{12}+\mathscr{Z}_{22}]_\diamond\right)\\
&=\Phi\left([\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \mathscr{P}_1,\ \mathscr{X}_{11}+\mathscr{Y}_{12}]_\diamond\right)+\Phi\left([\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \mathscr{P}_1,\ \mathscr{Z}_{22}]_\diamond\right)\\
&=[\Phi(\mathscr{I}) \bullet \cdots \bullet \mathscr{I} \bullet \mathscr{P}_1,\ \mathscr{X}_{11}+\mathscr{Y}_{12}+\mathscr{Z}_{22}]_\diamond\\
&+\cdots+[\mathscr{I} \bullet \cdots \bullet \Phi(\mathscr{I}) \bullet \mathscr{P}_1,\ \mathscr{X}_{11}+\mathscr{Y}_{12}+\mathscr{Z}_{22}]_\diamond\\
&+[\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \Phi(\mathscr{P}_1),\ \mathscr{X}_{11}+\mathscr{Y}_{12}+\mathscr{Z}_{22}]_\diamond\\
&+[\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \mathscr{P}_1,\ \Phi(\mathscr{X}_{11})+\Phi(\mathscr{Y}_{12})+\Phi(\mathscr{Z}_{22})]_\diamond .
\end{aligned}$$

This implies that $[\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \mathscr{P}_1, \mathscr{M}]_\diamond = 0$. Consequently, $\mathscr{P}_1\mathscr{M}\mathscr{P}_1 = \mathscr{P}_1\mathscr{M}\mathscr{P}_2 = \mathscr{P}_2\mathscr{M}\mathscr{P}_1 = 0$.

Next, using Lemma 2.5 and the fact that $[\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \mathscr{P}_2,\ \mathscr{X}_{11}]_\diamond = 0$, we have

$$\begin{aligned}
&[\Phi(\mathscr{I}) \bullet \cdots \bullet \mathscr{I} \bullet \mathscr{P}_2,\ \mathscr{X}_{11}+\mathscr{Y}_{12}+\mathscr{Z}_{22}]_\diamond\\
&+\cdots+[\mathscr{I} \bullet \cdots \bullet \Phi(\mathscr{I}) \bullet \mathscr{P}_2,\ \mathscr{X}_{11}+\mathscr{Y}_{12}+\mathscr{Z}_{22}]_\diamond\\
&+[\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \Phi(\mathscr{P}_2),\ \mathscr{X}_{11}+\mathscr{Y}_{12}+\mathscr{Z}_{22}]_\diamond+[\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \mathscr{P}_2,\ \Phi(\mathscr{X}_{11}+\mathscr{Y}_{12}+\mathscr{Z}_{22})]_\diamond\\
&=\Phi\left([\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \mathscr{P}_2,\ \mathscr{X}_{11}+\mathscr{Y}_{12}+\mathscr{Z}_{22}]_\diamond\right)\\
&=\Phi\left([\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \mathscr{P}_2,\ \mathscr{X}_{11}]_\diamond\right)+\Phi\left([\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \mathscr{P}_2,\ \mathscr{Y}_{12}+\mathscr{Z}_{22}]_\diamond\right)\\
&=[\Phi(\mathscr{I}) \bullet \cdots \bullet \mathscr{I} \bullet \mathscr{P}_2,\ \mathscr{X}_{11}+\mathscr{Y}_{12}+\mathscr{Z}_{22}]_\diamond\\
&+\cdots+[\mathscr{I} \bullet \cdots \bullet \Phi(\mathscr{I}) \bullet \mathscr{P}_2,\ \mathscr{X}_{11}+\mathscr{Y}_{12}+\mathscr{Z}_{22}]_\diamond\\
&+[\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \Phi(\mathscr{P}_2),\ \mathscr{X}_{11}+\mathscr{Y}_{12}+\mathscr{Z}_{22}]_\diamond\\
&+[\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \mathscr{P}_2,\ \Phi(\mathscr{X}_{11})+\Phi(\mathscr{Y}_{12})+\Phi(\mathscr{Z}_{22})]_\diamond .
\end{aligned}$$

This yields $[\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \mathscr{P}_2, \mathscr{M}]_\diamond = 0$. Hence $\mathscr{P}_2\mathscr{M}\mathscr{P}_2 = 0$. Therefore, we conclude that $\mathscr{M} = 0$. □

Lemma 2.7. *For any* $\mathscr{X}_{12}, \mathscr{Y}_{12} \in \mathfrak{X}_{12}$, *we have*

$$\Phi(\mathscr{X}_{12}+\mathscr{Y}_{12}) = \Phi(\mathscr{X}_{12}) + \Phi(\mathscr{Y}_{12}).$$

Proof. Let $\mathscr{X}_{12}, \mathscr{Y}_{12} \in \mathfrak{X}_{12}$. Then,

$$\left[\mathscr{I} \bullet \cdots \mathscr{I} \bullet \frac{(\mathscr{P}_1 + i\mathscr{X}_{12})}{2^{n-2}}, (i\mathscr{P}_2 - \mathscr{Y}_{12})\right]_\diamond = \mathscr{X}_{12} + \mathscr{Y}_{12} + i\mathscr{X}_{12}\mathscr{Y}_{12} + i\mathscr{Y}_{12}\mathscr{X}_{12}.$$

Clearly,

$$\mathscr{X}_{12} + \mathscr{Y}_{12} \in \mathfrak{X}_{12}.$$

Moreover,

$$\begin{aligned}
&i\mathscr{X}_{12}\mathscr{Y}_{12} + i\mathscr{Y}_{12}\mathscr{X}_{12}\\
=&\mathscr{P}_1(i(\mathscr{X}\mathscr{P}_2\mathscr{Y} + \mathscr{Y}\mathscr{P}_2\mathscr{X}))\mathscr{P}_1 + \mathscr{P}_2(i(\mathscr{X}\mathscr{P}_1\mathscr{Y} + \mathscr{Y}\mathscr{P}_1\mathscr{X}))\mathscr{P}_2 \in \mathfrak{X}_{11} + \mathfrak{X}_{22}.
\end{aligned}$$

Applying Lemmas 2.5 and 2.6, we obtain

$$\Phi(\mathscr{X}_{12} + \mathscr{Y}_{12}) + \Phi(i\mathscr{X}_{12}\mathscr{Y}_{12} + i\mathscr{Y}_{12}\mathscr{X}_{12})$$

$$\begin{aligned}
&=\Phi(\mathscr{X}_{12}+\mathscr{Y}_{12}+i\mathscr{X}_{12}\mathscr{Y}_{12}+i\mathscr{Y}_{12}\mathscr{X}_{12})\\
&=\Phi\left(\left[\mathscr{I}\bullet\cdots\mathscr{I}\bullet\frac{(\mathscr{P}_1+i\mathscr{X}_{12})}{2^{n-2}},(i\mathscr{P}_2-\mathscr{Y}_{12})\right]_\diamond\right)\\
&=\left[\Phi(\mathscr{I})\bullet\cdots\mathscr{I}\bullet\frac{(\mathscr{P}_1+i\mathscr{X}_{12})}{2^{n-2}},(i\mathscr{P}_2-\mathscr{Y}_{12})\right]_\diamond\\
&+\cdots+\left[\mathscr{I}\bullet\cdots\Phi(\mathscr{I})\bullet\frac{(\mathscr{P}_1+i\mathscr{X}_{12})}{2^{n-2}},(i\mathscr{P}_2-\mathscr{Y}_{12})\right]_\diamond\\
&+\left[\mathscr{I}\bullet\cdots\mathscr{I}\bullet\left(\Phi\left(\frac{\mathscr{P}_1}{2^{n-2}}\right)+\Phi\left(\frac{i\mathscr{X}_{12}}{2^{n-2}}\right)\right),(i\mathscr{P}_2-\mathscr{Y}_{12})\right]_\diamond\\
&+\left[\mathscr{I}\bullet\cdots\mathscr{I}\bullet\frac{(\mathscr{P}_1+i\mathscr{X}_{12})}{2^{n-2}},(\Phi(i\mathscr{P}_2)-\Phi(\mathscr{Y}_{12}))\right]_\diamond\\
&=\Phi\left(\left[\mathscr{I}\bullet\cdots\mathscr{I}\bullet\frac{\mathscr{P}_1}{2^{n-2}},i\mathscr{P}_2\right]_\diamond\right)\\
&+\Phi\left(\left[\mathscr{I}\bullet\cdots\mathscr{I}\bullet\frac{\mathscr{P}_1}{2^{n-2}},-\mathscr{Y}_{12}\right]_\diamond\right)\\
&+\Phi\left(\left[\mathscr{I}\bullet\cdots\mathscr{I}\bullet\frac{i\mathscr{X}_{12}}{2^{n-2}},i\mathscr{P}_2\right]_\diamond\right)\\
&+\Phi\left(\left[\mathscr{I}\bullet\cdots\mathscr{I}\bullet\frac{i\mathscr{X}_{12}}{2^{n-2}},-\mathscr{Y}_{12}\right]_\diamond\right)\\
&=\Phi(\mathscr{Y}_{12})+\Phi(\mathscr{X}_{12})+\Phi(i\mathscr{X}_{12}\mathscr{Y}_{12}+i\mathscr{Y}_{12}\mathscr{X}_{12}).
\end{aligned}$$

This implies that

$$\Phi(\mathscr{X}_{12}+\mathscr{Y}_{12})=\Phi(\mathscr{X}_{12})+\Phi(\mathscr{Y}_{12}).$$

□

Lemma 2.8. *For every* $\mathscr{X}_{ii},\mathscr{Y}_{ii}\in\mathfrak{X}_{ii}$ $(i=1,2)$, *we have*

$$\Phi(\mathscr{X}_{ii}+\mathscr{Y}_{ii})=\Phi(\mathscr{X}_{ii})+\Phi(\mathscr{Y}_{ii}).$$

Proof. Let $\mathscr{M}=\Phi(\mathscr{X}_{11}+\mathscr{Y}_{11})-\Phi(\mathscr{X}_{11})-\Phi(\mathscr{Y}_{11})$. Then $\mathscr{M}^*=-\mathscr{M}$. Observe that

$$[\mathscr{I}\bullet\cdots\bullet\mathscr{I}\bullet\mathscr{P}_2,\ \mathscr{X}_{11}]_\diamond=[\mathscr{I}\bullet\cdots\bullet\mathscr{I}\bullet\mathscr{P}_2,\ \mathscr{Y}_{11}]_\diamond=0.$$

Hence,

$$\begin{aligned}
&[\Phi(\mathscr{I})\bullet\cdots\bullet\mathscr{I}\bullet\mathscr{P}_2,\ \mathscr{X}_{11}+\mathscr{Y}_{11}]_\diamond+\cdots+[\mathscr{I}\bullet\cdots\bullet\Phi(\mathscr{I})\bullet\mathscr{P}_2,\ \mathscr{X}_{11}+\mathscr{Y}_{11}]_\diamond\\
&+[\mathscr{I}\bullet\cdots\bullet\mathscr{I}\bullet\Phi(\mathscr{P}_2),\ \mathscr{X}_{11}+\mathscr{Y}_{11}]_\diamond+[\mathscr{I}\bullet\cdots\bullet\mathscr{I}\bullet\mathscr{P}_2,\ \Phi(\mathscr{X}_{11}+\mathscr{Y}_{11})]_\diamond\\
&=\Phi\left([\mathscr{I}\bullet\cdots\bullet\mathscr{I}\bullet\mathscr{P}_2,\ \mathscr{X}_{11}+\mathscr{Y}_{11}]_\diamond\right)\\
&=\Phi\left([\mathscr{I}\bullet\cdots\bullet\mathscr{I}\bullet\mathscr{P}_2,\ \mathscr{X}_{11}]_\diamond\right)+\Phi\left([\mathscr{I}\bullet\cdots\bullet\mathscr{I}\bullet\mathscr{P}_2,\ \mathscr{Y}_{11}]_\diamond\right)\\
&=[\Phi(\mathscr{I})\bullet\cdots\bullet\mathscr{I}\bullet\mathscr{P}_2,\ \mathscr{X}_{11}+\mathscr{Y}_{11}]_\diamond+\cdots+[\mathscr{I}\bullet\cdots\bullet\Phi(\mathscr{I})\bullet\mathscr{P}_2,\ \mathscr{X}_{11}+\mathscr{Y}_{11}]_\diamond\\
&+[\mathscr{I}\bullet\cdots\bullet\mathscr{I}\bullet\Phi(\mathscr{P}_2),\ \mathscr{X}_{11}+\mathscr{Y}_{11}]_\diamond+[\mathscr{I}\bullet\cdots\bullet\mathscr{I}\bullet\mathscr{P}_2,\ \Phi(\mathscr{X}_{11})+\Phi(\mathscr{Y}_{11})]_\diamond.
\end{aligned}$$

Hence, we deduce that $[\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \mathscr{P}_2, \mathscr{M}]_\diamond = 0$, which implies that $\mathscr{P}_1\mathscr{M}\mathscr{P}_2 = \mathscr{P}_2\mathscr{M}\mathscr{P}_1 = \mathscr{P}_2\mathscr{M}\mathscr{P}_2 = 0$.

Let $\mathscr{A}_{12} \in \mathfrak{A}_{12}$ and $\mathscr{Z}_{12} = \mathscr{A}_{12} - \mathscr{A}_{12}^* \in \mathfrak{X}_{12}$. Then,

$$\left[\mathscr{I} \bullet \cdots \mathscr{I} \bullet \frac{i\mathscr{Z}_{12}}{2^{n-2}}, \mathscr{X}_{11}\right]_\diamond, \left[\mathscr{I} \bullet \cdots \mathscr{I} \bullet \frac{i\mathscr{Z}_{12}}{2^{n-2}}, \mathscr{Y}_{11}\right]_\diamond \in \mathfrak{X}_{12}.$$

In view of Lemma 2.7, we deduce that

$$\begin{aligned}
&\left[\Phi(\mathscr{I}) \bullet \cdots \bullet \mathscr{I} \bullet \frac{i\mathscr{Z}_{12}}{2^{n-2}},\ \mathscr{X}_{11} + \mathscr{Y}_{11}\right]_\diamond + \cdots + \left[\mathscr{I} \bullet \cdots \bullet \Phi(\mathscr{I}) \bullet \frac{i\mathscr{Z}_{12}}{2^{n-2}},\ \mathscr{X}_{11} + \mathscr{Y}_{11}\right]_\diamond \\
&+\left[\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \Phi(\frac{i\mathscr{Z}_{12}}{2^{n-2}}),\ \mathscr{X}_{11} + \mathscr{Y}_{11}\right]_\diamond + \left[\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \frac{i\mathscr{Z}_{12}}{2^{n-2}},\ \Phi(\mathscr{X}_{11} + \mathscr{Y}_{11})\right]_\diamond \\
=&\Phi\left(\left[\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \frac{i\mathscr{Z}_{12}}{2^{n-2}},\ \mathscr{X}_{11} + \mathscr{Y}_{11}\right]_\diamond\right) \\
=&\Phi\left(\left[\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \frac{i\mathscr{Z}_{12}}{2^{n-2}},\ \mathscr{X}_{11}\right]_\diamond\right) + \Phi\left(\left[\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \frac{i\mathscr{Z}_{12}}{2^{n-2}},\ \mathscr{Y}_{11}\right]_\diamond\right) \\
=&\left[\Phi(\mathscr{I}) \bullet \cdots \bullet \mathscr{I} \bullet \frac{i\mathscr{Z}_{12}}{2^{n-2}},\ \mathscr{X}_{11} + \mathscr{Y}_{11}\right]_\diamond + \cdots + \left[\mathscr{I} \bullet \cdots \bullet \Phi(\mathscr{I}) \bullet \frac{i\mathscr{Z}_{12}}{2^{n-2}},\ \mathscr{X}_{11} + \mathscr{Y}_{11}\right]_\diamond \\
&+\left[\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \Phi\left(\frac{i\mathscr{Z}_{12}}{2^{n-2}}\right),\ \mathscr{X}_{11} + \mathscr{Y}_{11}\right]_\diamond + \left[\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \frac{i\mathscr{Z}_{12}}{2^{n-2}},\ \Phi(\mathscr{X}_{11}) + \Phi(\mathscr{Y}_{11})\right]_\diamond .
\end{aligned}$$

This yields $\left[\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \frac{i\mathscr{Z}_{12}}{2^{n-2}}, \mathscr{M}\right]_\diamond = 0$, and consequently,

$$\mathscr{A}_{12}\mathscr{M}^* - \mathscr{A}_{12}^*\mathscr{M}^* - \mathscr{M}\mathscr{A}_{12} + \mathscr{M}\mathscr{A}_{12}^* = 0.$$

Multiplying the above equation by $\mathscr{P}_1$ from left and by $\mathscr{P}_2$ from right, we obtain $\mathscr{P}_1\mathscr{M}\mathscr{A}_{12} = 0$. By (2.2), it follows that $\mathscr{P}_1\mathscr{M}\mathscr{P}_1 = 0$. Hence $\mathscr{M} = 0$.

Similarly, one can show that $\Phi(\mathscr{X}_{22} + \mathscr{Y}_{22}) = \Phi(\mathscr{X}_{22}) + \Phi(\mathscr{Y}_{22})$. □

Combining Lemmas 2.6–2.8, we can readily derive the following lemma:

Lemma 2.9. Φ *is additive on* $\mathfrak{X}$.

Lemma 2.10. $\Phi(\mathscr{I}) = \Phi(i\mathscr{I}) = 0$.

Proof. Since $0 = [\mathscr{I} \bullet \cdots \bullet \mathscr{I}, \mathscr{I}]_\diamond$. Then,

$$\begin{aligned}
0 =&\Phi\left([\mathscr{I} \bullet \mathscr{I} \bullet \cdots \bullet \mathscr{I},\ \mathscr{I}]_\diamond\right) \\
=&[\Phi(\mathscr{I}) \bullet \mathscr{I} \bullet \cdots \bullet \mathscr{I},\ \mathscr{I}]_\diamond + [\mathscr{I} \bullet \Phi(\mathscr{I}) \bullet \cdots \bullet \mathscr{I},\ \mathscr{I}]_\diamond \\
&+ \cdots + [\mathscr{I} \bullet \mathscr{I} \bullet \cdots \bullet \Phi(\mathscr{I}),\ \mathscr{I}]_\diamond + [\mathscr{I} \bullet \mathscr{I} \bullet \cdots \bullet \mathscr{I},\ \Phi(\mathscr{I})]_\diamond \\
=&[\mathscr{I} \bullet \mathscr{I} \bullet \cdots \bullet \mathscr{I},\ \Phi(\mathscr{I})]_\diamond \\
=&2^{n-2}\left(\Phi(\mathscr{I})^* - \Phi(\mathscr{I})\right).
\end{aligned}$$

It gives

$$\Phi(\mathscr{I})^* = \Phi(\mathscr{I}). \tag{2.3}$$

Moreover, noting that $-2^{n-1}i\mathscr{I} = [\mathscr{I} \bullet \cdots \bullet \mathscr{I},\ i\mathscr{I}]_\diamond$, it follows that

$$\begin{aligned}-2^{n-1}\Phi(i\mathscr{I}) =&\Phi\left([\mathscr{I} \bullet \mathscr{I} \bullet \cdots \bullet \mathscr{I},\ i\mathscr{I}]_\diamond\right)\\ =&[\Phi(\mathscr{I}) \bullet \mathscr{I} \bullet \cdots \bullet \mathscr{I},\ i\mathscr{I}]_\diamond + [\mathscr{I} \bullet \Phi(\mathscr{I}) \bullet \cdots \bullet \mathscr{I},\ i\mathscr{I}]_\diamond\\ &+ \cdots + [\mathscr{I} \bullet \mathscr{I} \bullet \cdots \bullet \Phi(\mathscr{I}),\ i\mathscr{I}]_\diamond + [\mathscr{I} \bullet \mathscr{I} \bullet \cdots \bullet \mathscr{I},\ \Phi(i\mathscr{I})]_\diamond\\ =&2^{n-2}i(n-1)\left(\Phi(\mathscr{I})^* + \Phi(\mathscr{I})\right) - 2^{n-1}\Phi(i\mathscr{I}).\end{aligned}$$

This yields that

$$\Phi(\mathscr{I})^* = -\Phi(\mathscr{I}). \tag{2.4}$$

From (2.3) and (2.4), we obtain

$$\Phi(\mathscr{I}) = 0.$$

In view of Lemmas 2.4 and 2.9 and using $\Phi(\mathscr{I}) = 0$, it follows that

$$\begin{aligned}-2^{n-1}\Phi(i\mathscr{I}) =&\Phi(-2^{n-1}i\mathscr{I}) = \Phi\left([i\mathscr{I} \bullet i\mathscr{I} \bullet \mathscr{I} \bullet \cdots \bullet \mathscr{I},\ i\mathscr{I}]_\diamond\right)\\ =&[\Phi(i\mathscr{I}) \bullet i\mathscr{I} \bullet \mathscr{I} \bullet \cdots \bullet \mathscr{I},\ i\mathscr{I}]_\diamond + [i\mathscr{I} \bullet \Phi(i\mathscr{I}) \bullet \mathscr{I} \bullet \cdots \bullet \mathscr{I},\ i\mathscr{I}]_\diamond\\ &+ [i\mathscr{I} \bullet i\mathscr{I} \bullet \mathscr{I} \bullet \cdots \bullet \mathscr{I},\ \Phi(i\mathscr{I})]_\diamond\\ =&-2^{n-1}\Phi(i\mathscr{I}) - 2^{n}\Phi(i\mathscr{I}).\end{aligned}$$

From this, it follows that $\Phi(i\mathscr{I}) = 0$. ☐

Lemma 2.11. *$\Phi(\mathscr{A})^* = \Phi(\mathscr{A})$ and $\Phi(i\mathscr{A}) = i\Phi(\mathscr{A})$ for all $\mathscr{A} = \mathscr{A}^* \in \mathfrak{A}$.*

Proof. Let $\mathscr{A} \in \mathfrak{A}$ such that $\mathscr{A}^* = \mathscr{A}$. By using Lemma 2.10

$$\begin{aligned}0 =&\Phi\left([\mathscr{I} \bullet \cdots \bullet \mathscr{I},\ \mathscr{A}]_\diamond\right)\\ =&[\mathscr{I} \bullet \cdots \bullet \mathscr{I},\ \Phi(\mathscr{A})]_\diamond\\ =&2^{n-2}(\Phi(\mathscr{A})^* - \Phi(\mathscr{A})).\end{aligned}$$

Hence $\Phi(\mathscr{A})^* = \Phi(\mathscr{A})$ for all $\mathscr{A}^* = \mathscr{A}$.

Moreover, for any $\mathscr{A} \in \mathfrak{A}$ with $\mathscr{A}^* = \mathscr{A}$, we have

$$\begin{aligned}-2^{n-1}\Phi(i\mathscr{A}) =&\Phi(-2^{n-1}i\mathscr{A}) = \Phi([\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \mathscr{A},\ i\mathscr{I}]_\diamond)\\ =&[\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \Phi(\mathscr{A}),\ i\mathscr{I}]_\diamond\\ =&-2^{n-1}i\,\Phi(\mathscr{A}).\end{aligned}$$

Thus, $\Phi(i\mathscr{A}) = i\Phi(\mathscr{A})$ for all $\mathscr{A}^* = \mathscr{A}$. ☐

Lemma 2.12. *For any $\mathscr{A}_1, \mathscr{A}_2 \in \mathfrak{A}$ such that $\mathscr{A}_1^* = \mathscr{A}_1, \mathscr{A}_2^* = \mathscr{A}_2$, we have*

$$\Phi(\mathscr{A}_1 + \mathscr{A}_2) = \Phi(\mathscr{A}_1) + \Phi(\mathscr{A}_2)$$

and

$$\Phi(\mathscr{A}_1 + i\mathscr{A}_2) = \Phi(\mathscr{A}_1) + i\Phi(\mathscr{A}_2).$$

Proof. Let $\mathscr{A}_1, \mathscr{A}_2 \in \mathfrak{A}$ be self-adjoint, that is, $\mathscr{A}_1^* = \mathscr{A}_1, \mathscr{A}_2^* = \mathscr{A}_2$. By Lemmas 2.9 and 2.11, we have $i\,\Phi(\mathscr{A}_1 + \mathscr{A}_2) = \Phi(i(\mathscr{A}_1 + \mathscr{A}_2)) = \Phi(i\mathscr{A}_1) + \Phi(i\mathscr{A}_2) = i(\Phi(\mathscr{A}_1) + \Phi(\mathscr{A}_2))$.
Which gives

$$\Phi(\mathscr{A}_1 + \mathscr{A}_2) = \Phi(\mathscr{A}_1) + \Phi(\mathscr{A}_2).$$

To begin with, we note that

$$\begin{aligned} -2^{n-1} i\Phi(\mathscr{A}_1) =& \Phi(-2^{n-1} i\mathscr{A}_1) \\ =& \Phi\left([\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet (\mathscr{A}_1 + i\mathscr{A}_2),\ i\mathscr{I}]_\diamond\right) \\ =& [\mathscr{I} \bullet \cdots \bullet \mathscr{I} \bullet \Phi(\mathscr{A}_1 + i\mathscr{A}_2),\ i\mathscr{I}]_\diamond \\ =& -2^{n-2} i(\Phi(\mathscr{A}_1 + i\mathscr{A}_2) + \Phi(\mathscr{A}_1 + i\mathscr{A}_2)^*). \end{aligned} \tag{2.5}$$

On the other hand, we have

$$\begin{aligned} -2^{n-1} i\Phi(\mathscr{A}_2) =& \Phi(-2^{n-1} i\mathscr{A}_2) \\ =& \Phi([\mathscr{I} \bullet \cdots \bullet \mathscr{I},\ (\mathscr{A}_1 + i\mathscr{A}_2)]_\diamond) \\ =& ([\mathscr{I} \bullet \cdots \bullet \mathscr{I},\ \Phi(\mathscr{A}_1 + i\mathscr{A}_2)]_\diamond \\ =& 2^{n-2}(\Phi(\mathscr{A}_1 + i\mathscr{A}_2)^* - \Phi(\mathscr{A}_1 + i\mathscr{A}_2)). \end{aligned} \tag{2.6}$$

From (2.5) and (2.6), we get $\Phi(\mathscr{A}_1 + i\mathscr{A}_2) = \Phi(\mathscr{A}_1) + i\Phi(\mathscr{A}_2)$. □

Lemma 2.13. *For any* $\mathscr{A} \in \mathfrak{A}$*, we have*

(i) $\Phi(i\mathscr{A}) = i\,\Phi(\mathscr{A})$ *and* $\Phi(\mathscr{A}^*) = \Phi(\mathscr{A})^*$*;*

(ii) Φ *is additive on* $\mathfrak{A}$*.*

Proof. Let $\mathscr{A} \in \mathfrak{A}$. Then it can be written as

$$\mathscr{A} = \mathscr{A}_1 + i\mathscr{A}_2,$$

where $\mathscr{A}_1^* = \mathscr{A}_1, \mathscr{A}_2^* = \mathscr{A}_2$. By Lemmas 2.11 and 2.12, we compute

$$\begin{aligned} \Phi(i\mathscr{A}) =& \Phi(i\mathscr{A}_1 - \mathscr{A}_2) = i\Phi(\mathscr{A}_1) - \Phi(\mathscr{A}_2) \\ =& i\,(\Phi(\mathscr{A}_1) + i\,\Phi(\mathscr{A}_2)) = i\Phi(\mathscr{A}_1 + i\mathscr{A}_2) \\ =& i\Phi(\mathscr{A}). \end{aligned}$$

Similarly, applying Lemmas 2.11 and 2.12, we obtain

$$\begin{aligned} \Phi(\mathscr{A}^*) =& \Phi(\mathscr{A}_1 - i\mathscr{A}_2) = \Phi(\mathscr{A}_1) - i\Phi(\mathscr{A}_2) \\ =& (\Phi(\mathscr{A}_1) + i\Phi(\mathscr{A}_2))^* = (\Phi(\mathscr{A}_1 + i\mathscr{A}_2))^* \\ =& \Phi(\mathscr{A})^*. \end{aligned}$$

Let $\mathscr{A}, \mathscr{B} \in \mathfrak{A}$, such that $\mathscr{A} = \mathscr{A}_1 + i\mathscr{A}_2$ and $\mathscr{B} = \mathscr{B}_1 + i\mathscr{B}_2$, where $\mathscr{A}_1^* = \mathscr{A}_1, \mathscr{A}_2^* = \mathscr{A}_2, \mathscr{B}_1^* = \mathscr{B}_1, \mathscr{B}_2^* = \mathscr{B}_2$. Using Lemma 2.12, we derive

$$\Phi(\mathscr{A} + \mathscr{B}) = \Phi((\mathscr{A}_1 + \mathscr{B}_1) + i(\mathscr{A}_2 + \mathscr{B}_2))$$

$$
\begin{aligned}
&=\Phi(\mathscr{A}_1+\mathscr{B}_1)+i\Phi(\mathscr{A}_2+\mathscr{B}_2)\\
&=\Phi(\mathscr{A}_1)+i\Phi(\mathscr{A}_2)+\Phi(\mathscr{B}_1)+i\Phi(\mathscr{B}_2)\\
&=\Phi(\mathscr{A})+\Phi(\mathscr{B}).
\end{aligned}
$$

□

Lemma 2.14. Φ *is an additive* $*$*-derivation on* $\mathfrak{A}$*.*

Proof. Let $\mathscr{A},\mathscr{B}\in\mathfrak{A}$, on the one hand, using Lemmas 2.10 and 2.13, we get

$$
\begin{aligned}
&2^{n-3}\,\Phi(\mathscr{A}\mathscr{B}^*+\mathscr{A}^*\mathscr{B}^*-\mathscr{B}\mathscr{A}-\mathscr{B}\mathscr{A}^*)\\
=&\Phi(2^{n-3}(\mathscr{A}\mathscr{B}^*+\mathscr{A}^*\mathscr{B}^*-\mathscr{B}\mathscr{A}-\mathscr{B}\mathscr{A}^*))\\
=&\Phi([\mathscr{I}\bullet\cdots\bullet\mathscr{I}\bullet\mathscr{A},\ \mathscr{B}]_\diamond)\\
=&[\mathscr{I}\bullet\cdots\bullet\mathscr{I}\bullet\Phi(\mathscr{A}),\ \mathscr{B}]_\diamond+[\mathscr{I}\bullet\cdots\bullet\mathscr{I}\bullet\mathscr{A},\ \Phi(\mathscr{B})]_\diamond\\
=&2^{n-3}(\Phi(\mathscr{A})\mathscr{B}^*+\Phi(\mathscr{A})^*\mathscr{B}^*-\mathscr{B}\Phi(\mathscr{A})-\mathscr{B}\Phi(\mathscr{A})^*\\
&+\mathscr{A}\Phi(\mathscr{B})^*+\mathscr{A}^*\Phi(\mathscr{B})^*-\Phi(\mathscr{B})\mathscr{A}-\Phi(\mathscr{B})\mathscr{A}^*).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\Phi(\mathscr{A}\mathscr{B}^*+\mathscr{A}^*\mathscr{B}^*-\mathscr{B}\mathscr{A}-\mathscr{B}\mathscr{A}^*)\\
=&\Phi(\mathscr{A})\mathscr{B}^*+\Phi(\mathscr{A})^*\mathscr{B}^*-\mathscr{B}\Phi(\mathscr{A})-\mathscr{B}\Phi(\mathscr{A})^* \qquad (2.7)\\
&+\mathscr{A}\Phi(\mathscr{B})^*+\mathscr{A}^*\Phi(\mathscr{B})^*-\Phi(\mathscr{B})\mathscr{A}-\Phi(\mathscr{B})\mathscr{A}^*. \qquad (2.8)
\end{aligned}
$$

On the other hand, we get

$$
\begin{aligned}
&-2^{n-3}i\,\Phi(\mathscr{A}\mathscr{B}^*+\mathscr{A}^*\mathscr{B}^*+\mathscr{B}\mathscr{A}+\mathscr{B}\mathscr{A}^*)\\
=&\Phi(-2^{n-3}i(\mathscr{A}\mathscr{B}^*+\mathscr{A}^*\mathscr{B}^*+\mathscr{B}\mathscr{A}+\mathscr{B}\mathscr{A}^*))\\
=&\Phi([\mathscr{I}\bullet\cdots\bullet\mathscr{I}\bullet\mathscr{A},\ i\mathscr{B}]_\diamond)\\
=&[\mathscr{I}\bullet\cdots\bullet\mathscr{I}\bullet\Phi(\mathscr{A}),\ i\mathscr{B}]_\diamond+[\mathscr{I}\bullet\cdots\bullet\mathscr{I}\bullet\mathscr{A},\ \Phi(i\mathscr{B})]_\diamond\\
=&-2^{n-3}i(\Phi(\mathscr{A})\mathscr{B}^*+\Phi(\mathscr{A})^*\mathscr{B}^*+\mathscr{B}\Phi(\mathscr{A})+\mathscr{B}\Phi(\mathscr{A})^*\\
&+\mathscr{A}\Phi(\mathscr{B})^*+\mathscr{A}^*\Phi(\mathscr{B})^*+\Phi(\mathscr{B})\mathscr{A}+\Phi(\mathscr{B})\mathscr{A}^*).
\end{aligned}
$$

Which gives

$$
\begin{aligned}
&\Phi(\mathscr{A}\mathscr{B}^*+\mathscr{A}^*\mathscr{B}^*+\mathscr{B}\mathscr{A}+\mathscr{B}\mathscr{A}^*)\\
=&\Phi(\mathscr{A})\mathscr{B}^*+\Phi(\mathscr{A})^*\mathscr{B}^*+\mathscr{B}\Phi(\mathscr{A})+\mathscr{B}\Phi(\mathscr{A})^* \qquad (2.9)\\
&+\mathscr{A}\Phi(\mathscr{B})^*+\mathscr{A}^*\Phi(\mathscr{B})^*+\Phi(\mathscr{B})\mathscr{A}+\Phi(\mathscr{B})\mathscr{A}^*. \qquad (2.10)
\end{aligned}
$$

From (2.7) and (2.9), we get

$$\Phi(\mathscr{A}\mathscr{B}^*+\mathscr{A}^*\mathscr{B}^*)=\Phi(\mathscr{A})\mathscr{B}^*+\Phi(\mathscr{A})^*\mathscr{B}^*+\mathscr{A}\Phi(\mathscr{B})^*+\mathscr{A}^*\Phi(\mathscr{B})^*. \qquad (2.11)$$

Replacing $\mathscr{A}$ by $i\mathscr{A}$ in (2.11), we get

$$\Phi(\mathscr{A}\mathscr{B}^*-\mathscr{A}^*\mathscr{B}^*)=\Phi(\mathscr{A})\mathscr{B}^*-\Phi(\mathscr{A})^*\mathscr{B}^*+\mathscr{A}\Phi(\mathscr{B})^*-\mathscr{A}^*\Phi(\mathscr{B})^*. \qquad (2.12)$$

From (2.11) and (2.12), we obtain

$$\Phi(\mathscr{A}\mathscr{B}^*) = \Phi(\mathscr{A})\mathscr{B}^* + \mathscr{A}\Phi(\mathscr{B})^*.$$

Substituting $\mathscr{B}^*$ in place of $\mathscr{B}$ in the preceding identity and applying Lemma 2.13(i), we obtain

$$\Phi(\mathscr{A}\mathscr{B}) = \Phi(\mathscr{A})\mathscr{B} + \mathscr{A}\Phi(\mathscr{B}).$$

Hence Φ is an additive $*$-derivation on $\mathfrak{A}$. This completes the proof of the main theorem. □

Example 2.15. Let $\mathfrak{A} = \mathbb{C} \oplus \mathbb{C}$ be equipped with componentwise addition and multiplication, unit $\mathscr{I} = (1, 1)$, and involution $(z_1, z_2)^* = (\overline{z_1}, \overline{z_2})$. Then $\mathfrak{A}$ is a unital commutative $*$-algebra over $\mathbb{C}$. Define the nontrivial projection $\mathscr{P} = (1, 0)$. Then $\mathscr{I} - \mathscr{P} = (0, 1)$. Let $\mathscr{A} = (0, 1) \neq 0$. For any $\mathscr{B} = (u, v) \in \mathfrak{A}$, $\mathscr{A}\mathscr{B}\mathscr{P} = (0, 0)$. Similarly, for $\mathscr{A} = (1, 0) \neq 0$, $\mathscr{A}\mathscr{B}(\mathscr{I} - \mathscr{P}) = (0, 0)$, for all $\mathscr{B} \in \mathfrak{A}$. Define $\Phi : \mathfrak{A} \to \mathfrak{A}$ by $\Phi((z_1, z_2)) = \left(0, \frac{z_2 - \overline{z_2}}{2}\right)$. Then for every $n \geq 3$, $\Phi([\mathscr{A}_1 \bullet \cdots \bullet \mathscr{A}_{n-1}, \mathscr{A}_n]_\diamond) = \sum_{j=1}^{n} [\mathscr{A}_1 \bullet \cdots \bullet \Phi(\mathscr{A}_j) \bullet \cdots \bullet \mathscr{A}_{n-1}, \mathscr{A}_n]_\diamond$, for all $\mathscr{A}_{n-1}, \mathscr{A}_n \in \mathfrak{A}$, where $\mathscr{A}_1, \mathscr{A}_2 \in \{\mathscr{I}, i\mathscr{I}\}$ and $\mathscr{A}_j = \mathscr{I}$ for every $j \in \{3, 4, \ldots, n-2\}$. Hence Φ is not an additive $*$-derivation.

3. Corollaries

Corollary 3.1. *Let $\mathfrak{A}$ be a unital $*$-algebra with unity $\mathscr{I}$. Suppose that a non-trivial projection $\mathscr{P}$ in $\mathfrak{A}$, which satisfies (2.1) and (2.2). Then a map $\Phi : \mathfrak{A} \to \mathfrak{A}$ satisfies*

$$\begin{aligned}&\Phi\left([\mathscr{A}_1 \bullet \mathscr{A}_2 \bullet \cdots \bullet \mathscr{A}_{n-1},\ \mathscr{A}_n]_\diamond\right)\\ =\ &\sum_{j=1}^{n} [\mathscr{A}_1 \bullet \mathscr{A}_2 \bullet \cdots \bullet \mathscr{A}_{j-1} \bullet \Phi(\mathscr{A}_j) \bullet \mathscr{A}_{j+1} \bullet \cdots \bullet \mathscr{A}_{n-1},\ \mathscr{A}_n]_\diamond\end{aligned}$$

for all $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n \in \mathfrak{A}$ if and only if Φ is an additive $*$-derivation.

Remember that if $\mathscr{A}\mathfrak{A}\mathscr{B} = 0$ for any $\mathscr{A}, \mathscr{B} \in \mathfrak{A}$ indicates that either $\mathscr{A} = 0$ or $\mathscr{B} = 0$, then the algebra $\mathfrak{A}$ is prime. We obtain the following result by noting that prime $*$-algebras satisfy (2.1) and (2.2).

Corollary 3.2. *Consider a unital prime $*$-algebra $\mathfrak{A}$ with a non-trivial projection $\mathscr{P}$. Consequently, a map $\Phi : \mathfrak{A} \to \mathfrak{A}$ satisfies*

$$\begin{aligned}&\Phi\left([\mathscr{A}_1 \bullet \mathscr{A}_2 \bullet \cdots \bullet \mathscr{A}_{n-1},\ \mathscr{A}_n]_\diamond\right)\\ =\ &\sum_{j=1}^{n} [\mathscr{A}_1 \bullet \mathscr{A}_2 \bullet \cdots \bullet \mathscr{A}_{j-1} \bullet \Phi(\mathscr{A}_j) \bullet \mathscr{A}_{j+1} \bullet \cdots \bullet \mathscr{A}_{n-1},\ \mathscr{A}_n]_\diamond\end{aligned}$$

for all $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n \in \mathfrak{A}$ if and only if Φ is an additive $*$-derivation.

A von Neumann algebra $\mathfrak{A}$ is defined as a weakly closed, self-adjoint operator algebra on a complex Hilbert space $\mathscr{H}$ that includes the identity operator $\mathscr{I}$. When the center of $\mathfrak{A}$ is trivial, it is classified as a factor von Neumann algebra. Noting that factor von Neumann algebras are prime $*$-algebras, the following result is obtained.

Corollary 3.3. *Assume that* $\mathfrak{A}$ *is a factor von Neumann algebra of dim*$(\mathfrak{A}) \geq 2$. *A map* $\Phi : \mathfrak{A} \to \mathfrak{A}$ *satisfies*

$$\begin{aligned}&\Phi\left([\mathscr{A}_1 \bullet \mathscr{A}_2 \bullet \cdots \bullet \mathscr{A}_{n-1},\ \mathscr{A}_n]_\diamond\right)\\ =\ & \sum_{j=1}^{n}[\mathscr{A}_1 \bullet \mathscr{A}_2 \bullet \cdots \bullet \mathscr{A}_{j-1} \bullet \Phi(\mathscr{A}_j) \bullet \mathscr{A}_{j+1} \bullet \cdots \bullet \mathscr{A}_{n-1},\ \mathscr{A}_n]_\diamond\end{aligned}$$

for all $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n \in \mathfrak{A}$ *if and only if* Φ *is an additive* $*$*-derivation.*

Corollary 3.4. *Consider a von Neumann algebra* $\mathfrak{A}$ *with no central summands of type* $\mathscr{I}_1$. *Then a map* $\Phi : \mathfrak{A} \to \mathfrak{A}$ *satisfies*

$$\begin{aligned}&\Phi\left([\mathscr{A}_1 \bullet \mathscr{A}_2 \bullet \cdots \bullet \mathscr{A}_{n-1},\ \mathscr{A}_n]_\diamond\right)\\ =\ & \sum_{j=1}^{n}[\mathscr{A}_1 \bullet \mathscr{A}_2 \bullet \cdots \bullet \mathscr{A}_{j-1} \bullet \Phi(\mathscr{A}_j) \bullet \mathscr{A}_{j+1} \bullet \cdots \bullet \mathscr{A}_{n-1},\ \mathscr{A}_n]_\diamond\end{aligned}$$

for all $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n \in \mathfrak{A}$ *if and only if* Φ *is an additive* $*$*-derivation.*

Let $\mathfrak{B}(\mathscr{H})$ represent the algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$. If $\mathfrak{F}(\mathscr{H})$, the subalgebra of all finite-rank operators on $\mathscr{H}$, is contained in $\mathfrak{A}$, then $\mathfrak{A}$, being a subalgebra of $\mathfrak{B}(\mathscr{H})$, is classified as a standard operator algebra. Since we know that standard operator algebras are prime $*$-algberas, then the following result can be established.

Corollary 3.5. *Consider an infinite dimensional complex Hilbert space* $\mathscr{H}$ *with the identity operator* $\mathscr{I}$ *contained in a standard operator algebra* $\mathfrak{A}$. *Let* $\mathfrak{A}$ *be closed under the adjoint operation. Then a map* $\Phi : \mathfrak{A} \to \mathfrak{A}$ *satisfies*

$$\begin{aligned}&\Phi\left([\mathscr{A}_1 \bullet \mathscr{A}_2 \bullet \cdots \bullet \mathscr{A}_{n-1},\ \mathscr{A}_n]_\diamond\right)\\ =\ & \sum_{j=1}^{n}[\mathscr{A}_1 \bullet \mathscr{A}_2 \bullet \cdots \bullet \mathscr{A}_{j-1} \bullet \Phi(\mathscr{A}_j) \bullet \mathscr{A}_{j+1} \bullet \cdots \bullet \mathscr{A}_{n-1},\ \mathscr{A}_n]_\diamond\end{aligned}$$

for all $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n \in \mathfrak{A}$ *if and only if* Φ *is an additive* $*$*-derivation. Additionally, for every* $\mathscr{A} \in \mathfrak{A}$, *there exists an operator* $\mathscr{L} \in \mathscr{B}(\mathscr{H})$ *that satisfies* $\mathscr{L} + \mathscr{L}^* = 0$ *such that* $\Phi(\mathscr{A}) = \mathscr{A}\mathscr{L} - \mathscr{L}\mathscr{A}$, *i.e.,* Φ *is inner.*

Proof. If Φ is an additive $*$-derivation and $\mathfrak{A}$ a standard operator algebra, then, as shown in [13], Φ is a linear inner derivation. This means there exists an operator $\mathscr{K} \in \mathscr{B}(\mathscr{H})$ such that $\Phi(\mathscr{A}) = \mathscr{A}\mathscr{K} - \mathscr{K}\mathscr{A}$. Furthermore, since $\Phi(\mathscr{A}^*) = \Phi(\mathscr{A})^*$, it follows that

$$\mathscr{A}^*\mathscr{K} - \mathscr{K}\mathscr{A}^* = \Phi(\mathscr{A}^*) = \Phi(\mathscr{A})^* = -\mathscr{A}^*\mathscr{K}^* + \mathscr{K}^*\mathscr{A}^*$$

for all $\mathscr{A} \in \mathfrak{A}$. Hence $\mathscr{A}^*(\mathscr{K} + \mathscr{K}^*) = (\mathscr{K} + \mathscr{K}^*)\mathscr{A}^*$ and then $\mathscr{K} + \mathscr{K}^* = \beta\mathscr{I}$ for some $\beta \in \mathbb{R}$. Let $\mathscr{L} = \mathscr{K} - \frac{1}{2}\beta\mathscr{I}$. It is easy to see that $\mathscr{L} + \mathscr{L}^* = 0$ such that $\Phi(\mathscr{A}) = \mathscr{A}\mathscr{L} - \mathscr{L}\mathscr{A}$. □

Motivated by the structural framework developed in this article for associative $*$-algebras and in view of the growing interest in non-associative operator-theoretic structures, we propose the following conjecture.

Conjecture 3.6. *(Alternative $*$-algebras) Let $\Re$ be a unital 2-torsion-free alternative $*$-algebra with identity $\mathscr{I}$, containing a non-trivial symmetric idempotent $\mathscr{P}_1$ and let $\mathscr{P}_2 = \mathscr{I} - \mathscr{P}_1$. Assume that $\Re$ satisfies the condition*

$$\mathscr{S}\Re\mathscr{P}_t = 0 \implies \mathscr{S} = 0, \qquad t \in \{1, 2\}, \tag{3.1}$$

for every $\mathscr{S} \in \Re$. Suppose that a (not necessarily additive) mapping $\Phi : \Re \to \Re$ satisfies

$$\Phi\Big([\mathscr{A}_1 \bullet \mathscr{A}_2 \bullet \cdots \bullet \mathscr{A}_{n-1}, \mathscr{A}_n]_\diamond\Big) = \sum_{j=1}^{n} [\mathscr{A}_1 \bullet \cdots \bullet \mathscr{A}_{j-1} \bullet \Phi(\mathscr{A}_j) \bullet \mathscr{A}_{j+1} \bullet \cdots \bullet \mathscr{A}_{n-1}, \mathscr{A}_n]_\diamond \tag{3.2}$$

for all $\mathscr{A}_1, \ldots, \mathscr{A}_n \in \Re$, where $\mathscr{A}_1, \mathscr{A}_2 \in \{\mathscr{I}, i\mathscr{I}\}$, $\mathscr{A}_j = \mathscr{I}$ for $j \in \{3, 4, \ldots, n-2\}$.

Then Φ is an additive $*$-derivation on $\Re$.

4. Conclusions

In this paper, we studied nonlinear mixed bi-skew Jordan and bi-skew Lie n-derivations on unital $*$-algebras. We proved that every such mapping is an additive $*$-derivation under suitable projection and annihilator conditions.

The proof is based on the Peirce decomposition associated with a nontrivial projection. First, we showed that the mapping preserves skew-adjoint elements. Then we established additivity on the Peirce components. This led to additivity on the whole algebra. Finally, we proved that the mapping preserves the involution and satisfies the usual derivation rule.

The main result extends the known triple-product case to the general n-fold mixed product case. It also gives several direct consequences for prime $*$-algebras, factor von Neumann algebras, von Neumann algebras without central summands of type I_1, and standard operator algebras.

These results show that mixed bi-skew Jordan and bi-skew Lie identities have a strong rigid structure. They force a nonlinear mapping to behave as a classical additive $*$-derivation. This work may also be useful for studying similar problems on nonunital $*$-algebras, rings with involution, and other operator algebraic structures.

Author contributions

Abu Zaid Ansari: Conceptualization, methodology, validation, supervision, writing–original draft preparation, writing–review and editing, funding acquisition, and project administration. Mohammad Shane Alam: Conceptualization, methodology, formal analysis, investigation, writing–original draft preparation, and writing–review and editing. Nof T. Alharbi: Formal analysis, validation, resources, and writing–review and editing. Ishraga A. Mohamed: Formal analysis, validation, resources, and writing–review and editing. All authors have read and approved the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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