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Theory article

An improved q-rung orthopair fuzzy REGIME method and its application for the selection of early esophageal cancer screening schemes

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Abstract: Esophageal cancer is a common malignant tumor of the digestive tract with a high incidence rate in China. Early detection, early diagnosis, and early treatment can greatly improve the survival rate of patients.To address the uncertainty and fuzziness existing in the selection of early esophageal cancer screening schemes, this paper establishes an extended REGIME decision-making framework under the q-rung orthopair fuzzy environment. First, a novel order relation of q-rung orthopair fuzzy numbers is proposed, which is strictly proved to be a q-rung orthopair fuzzy admissible order. Second, a new q-rung orthopair fuzzy distance operator is constructed to measure the differences between q-rung orthopair fuzzy sets, and the proposed order relation and distance operator are integrated into the construction process of the preference matrix of the REGIME method. Simulation results show that compared with existing methods, the proposed q-rung orthopair fuzzy order relation can obtain more reliable and convincing ranking results of fuzzy numbers, and the newly constructed distance operator can accurately identify subtle differences among q-rung orthopair fuzzy sets with better stability than state-of-the-art distance operators. Furthermore, the extended q-rung orthopair fuzzy REGIME method is applied to select optimal early esophageal cancer screening schemes. Practical application results reveal that when the parameter q ranges from 3 to 20, the proposed method achieves the optimal stability, and the relative ranking order of all alternatives remains unchanged.

Keywords: q-rung orthopair fuzzy sets; the REGIME method; early esophageal cancer screening
Mathematics Subject Classification: 03E72, 90B50

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For the convenience of readers, all core variables involved in this paper are defined in Table 1.

Table 1. Definitions of variables.

No.	Variable	Meaning
1	$X=\{x_1,x_2,...,x_n\}$	The universe of discourse
2	M , N	The q-rung orthopair fuzzy set
3	A , B , α , β , α_i	The q-rung orthopair fuzzy number
4	q	The order parameter of a q-rung orthopair fuzzy set
5	u , u_M , u_N , u_i , u_{ij}	The membership degree of a q-rung orthopair fuzzy set
6	v , v_M , v_N , v_i , v_{ij}	The non-membership degree of a q-rung orthopair fuzzy set
7	π_M , π_α , π_{ij}	The hesitation degree of a q-rung orthopair fuzzy set
8	$S(\cdot)$, $S_i(\cdot)$, S_i	The score function of a q-rung orthopair fuzzy number
9	$H(\cdot)$, $H_i(\cdot)$	The accuracy function of a q-rung orthopair fuzzy number
10	$A_i\ (i=1,2,...,m)$	The alternatives
11	$C_i\ (i=1,2,...,n)$	The criteria
12	$E_k(k=1,2,\ldots,K)$	The experts
13	$w=(w_1,w_2,\ldots,w_n)$	The criteria weight vector
14	$d(M,N)$	The distance between two q-rung orthopair fuzzy sets
15	$d(\alpha_1,\alpha_2)$	The distance between two q-rung orthopair fuzzy numbers
16	$d_i\ (i=1,2,...,32)$	The distance operator of q-rung orthopair fuzzy set
17	$D^{(k)}$	The decision matrices from expert k
18	D	The aggregated decision matrix
19	Φ_s	The net preference flow of the alternative A_s
20	η	Spearman's rank correlation coefficient
21	Z	Z-test statistic for Spearman's rank correlation
22	$\{\chi^i\}$, $\{\gamma^i\}$	The rank sequence

1. Introduction

1.1. Practical and methodological aims of the study

Rising public health awareness and advancing medical diagnostic techniques make early malignant tumor screening essential for reducing mortality and improving clinical diagnosis and treatment. As a prevalent digestive tract malignancy with insidious early symptoms, esophageal cancer seriously threatens public health security [1]. Scientific, efficient screening strategies are critical for its early detection and treatment and constitute a complex multi-criteria decision-making problem in medicine. In practical screening evaluations, indicators are highly fuzzy, uncertain, and incomplete, which traditional crisp numbers and single-type fuzzy sets cannot accurately represent and quantify.

As a generalized form of fuzzy set [2], intuitionistic fuzzy set [3], and Pythagorean fuzzy set [4], the q-rung orthopair fuzzy set (q-ROFS) [5] was proposed by Yager, breaking through the constraint limitations of membership degrees. By adjusting the parameter q, they can flexibly characterize uncertain information within a larger hesitation space, demonstrating stronger information expression capability and broader applicability when dealing with complex fuzzy multi-criteria decision-making problems. In recent years, q-rung orthopair fuzzy (q-ROF) sets have rapidly become a research frontier in the field of fuzzy decision-making due to their significant advantages in modeling complex uncertain information, with a growing number of theoretical achievements and application explorations. Among these, scholars have conducted extensive research and achieved fruitful results in areas such as the order relations of q-rung orthopair fuzzy numbers (q-ROFNs), q-ROF distance operators, and multi-criteria decision-making methods under q-ROF environments.

Recently, fuzzy logic and intelligent decision-making methods have been widely applied to the prognostic evaluation and survival prediction of esophageal cancer [6,7], and fuzzy data mining technology has also been applied to the analysis of esophageal cancer-related gene promoters [8], which provides a methodological reference for the application of fuzzy decision-making theory in the field of esophageal cancer. Meanwhile, q-ROFS has been successfully applied to the quality evaluation of manufacturing workers [9], offshore wind farm site selection [10], and sustainable urban innovation decision-making [11], which verifies the effectiveness of q-ROFS in dealing with complex multi-criteria decision making (MCDM) problems with uncertainty.

The practical goal of this study is to provide a scientific, objective, and robust decision-making framework for the optimal selection of early esophageal cancer screening schemes, so as to assist medical institutions at all levels to formulate personalized and efficient screening strategies. The methodological goal is to construct a reasonable order relation and high-stability distance operator for q-ROFNs and extend the classical REGIME method to the q-ROF environment, so as to make up for the application limitations of traditional methods in fuzzy decision-making.

1.2. Motivation for conducting research

Although the existing research has made considerable progress in q-ROFS theory and has extended the REGIME method to the fuzzy decision-making environment, the research on q-ROFS theory and REGIME method still faces the following bottlenecks:

(1) The current order relations for q-ROFNs have certain limitations in distinguishing similar fuzzy numbers, and some ranking results are contrary to intuitive judgment, which damages the reliability of decision-making results.

(2) Most of the existing q-ROF distance operators cannot accurately capture the subtle differences between q-ROFSs, and show unsatisfactory stability when processing highly hesitant and highly uncertain evaluation information.

(3) The research on REGIME method is mainly carried out in intuitionistic fuzzy, Pythagorean fuzzy, and spherical fuzzy environments, which makes the method not able to be directly applied to the q-ROF environment.

In terms of practical application, the existing research on early esophageal cancer screening mainly focuses on the clinical effect evaluation of a single screening technology and lacks systematic MCDM research on the comprehensive optimization of screening strategies. Although some fuzzy decision-making methods have been applied to medical prognosis prediction [6,7] and biological information analysis [8], there is still a lack of targeted decision-making methods for the selection of early esophageal cancer screening schemes. Meanwhile, the existing q-ROF-MCDM methods rely too much on subjective parameters, and cannot meet the objective and fair decision-making requirements in medical scenarios.

In terms of methodology, the existing q-ROF score functions and distance operators have defects such as weak discrimination, easy to produce counterintuitive results, and mathematical abnormalities. The REGIME method, as a MCDM technique based on pairwise comparison, has unique advantages in dealing with qualitative information and complex decision-making scenarios, but it has not been extended to the q-ROF environment. Therefore, it is urgent to construct a fully objective, high-discrimination, and high-stability q-ROF-REGIME decision-making method to solve the above theoretical and practical problems.

1.3. Contributions of the study

To address the challenges of fuzzy multi-criteria decision-making in early esophageal cancer screening, this paper establishes a more reasonable order relation for q-ROFNs, designs a highly stable distance operator, and extends the REGIME method to the q-ROF environment. The main contributions of this paper are as follows:

(1) A novel order relation for q-ROFNs is proposed and rigorously proven to conform to the definition of admissible orders for q-ROFNs, thereby improving the rationality and credibility of ranking q-ROFNs.

(2) A novel q-ROF distance operator is constructed, which can effectively capture subtle variations between fuzzy sets. Compared with existing mainstream distance operators, it exhibits superior stability and discriminability, thus enriching the theoretical system of q-ROFS.

(3) The classical REGIME decision-making method is extended to the q-ROF environment by integrating the proposed order relation and distance operator into the preference matrix construction process, thereby overcoming the application limitations of the traditional REGIME method in fuzzy decision-making.

(4) The extended q-ROF REGIME method is applied to the selection of optimal esophageal cancer screening schemes, which validates the robustness and practicality of the proposed method and provides a new decision-making framework for similar multi-criteria decision-making problems.

1.4. Research gaps and necessity of the proposed approach

Through the comprehensive analysis of the existing literature, the following prominent research gaps are clarified:

(1) The existing q-ROF order relations and distance operators have poor discrimination and stability, and cannot effectively deal with highly hesitant fuzzy information; the REGIME method has not been extended to the q-ROF environment, and there is a lack of a fully objective q-ROF-MCDM method that integrates superior order relations and distance operators.

(2) The existing research rarely involves the MCDM optimization of early esophageal cancer screening schemes; although fuzzy methods have been applied to esophageal cancer prognosis prediction [6,7] and gene analysis [8], there is no specialized fuzzy MCDM framework for screening strategy selection, and the existing methods rely on subjective parameters and are not suitable for medical decision-making scenarios.

(3) The existing q-ROF-MCDM methods mostly adopt the weighted sum model, which requires manual setting of parameters, lacking the pairwise comparison and net flow ranking mechanism with stronger robustness and higher objectivity.

The proposed q-ROF-REGIME method is necessary because it achieves fully objective decision-making free of subjective parameters, delivers improved discrimination and stability via custom order relation and distance operator, and first extends REGIME to the q-ROF environment to fill theoretical and application gaps.

1.5. Organization of the paper

The remainder of this paper is structured as follows: Section 2 provides a comprehensive literature review on existing q-rung orthopair fuzzy score functions, distance operators, and REGIME-based decision-making methods. Section 3 introduces the fundamental definitions and operational rules of q-ROFSs as well as the classical REGIME method. Section 4 proposes a novel score function and establishes an admissible order relation for q-ROFNs. Section 5 develops a new q-ROF distance operator and verifies its advantages through comparative simulation experiments. Section 6 extends the REGIME approach to the q-ROF context and elaborates the detailed decision-making procedure of the presented q-ROF-REGIME method. Section 7 applies the proposed model to the selection problem of early esophageal cancer screening strategies, followed by sensitivity analysis and comparative validation to demonstrate its effectiveness and robustness. Section 8 systematically discusses the limitations of the proposed model. Finally, Section 9 concludes the paper, highlights the unique contributions, and suggests future research directions.

2. Literature review

2.1. Existing score function

In the theoretical framework of q-ROFSs, the order relation of q-ROFNs is a fundamental issue for fuzzy information comparison and decision-making, which is typically implemented by defining score functions and accuracy functions. Yager introduced the basic score function $S(x)$ and accuracy function $H(x)$, establishing a foundational ranking framework for q-ROFNs [5]. Nevertheless, this approach cannot effectively distinguish distinct q-ROFNs that share identical score and accuracy values, leading to ranking failure and revealing obvious limitations. To address this shortcoming, successive researchers have proposed improved strategies from diverse perspectives. The discriminative capability has been strengthened by utilizing a weighted combination of membership and non-membership degrees [12]. A parametric score function based on weighted averaging has been developed [13]. Adjustable parameters have been introduced to characterize the risk attitudes of decision-makers [14,15]. Innovative score functions have been presented to tackle the incomparability issue among q-ROFNs [16,17]. With the advancement of research, scholars have further identified that several existing score functions may produce counterintuitive ranking results [18,19] and even exhibit mathematical defects such as negative inputs in logarithmic operations or zero division anomalies [20,21]. In addition, some studies have combined score functions with traditional decision-making approaches, designing customized score functions for specific application scenarios [22–25]. Matching score functions have also been constructed in the development of aggregation operators, establishing a complete decision-making pipeline spanning information representation, aggregation, and alternative ranking [19,26].

For a q-ROFN $\alpha = (u, v)$ with a hesitation degree of π_α, various score functions are presented in Table 2.

Table 2. Previous q-ROF ranking methods.

References	Score function	Accuracy function	Range
Liu [12]	$S_1(\alpha)=u^q-v^q$	$H_1(\alpha)=u^q+v^q$	[-1, 1]
Peng [16]	$S_2(\alpha)=u^q-v^q+\left(\frac{e^{u^q-v^q}}{e^{u^q-v^q}+1}-\frac{1}{2}\right)\pi_\alpha^q$	$H_2(\alpha)=\pi_\alpha$	[-1, 1]
Li [14]	$S_3(\alpha)=\frac{\left((1-v)^q+1-v^q\right)^{1/q}}{\left((1-v)^q+1-v^q\right)^{\frac{1}{q}}+\left((1-u)^q+1-u^q\right)^{1/q}}$	$H_3(\alpha)=u^q+v^q$	[0, 1]
Mi [22]	$S_4(\alpha)=\frac{2+u^q-v^q}{\left(2-u^q+v^q\right)\left(2-u^q-v^q\right)}$		[1/3, 3]
Peng [18]	$S_5(\alpha)=\frac{u^q-2v^q-1}{3}+\frac{\lambda}{3}\left(u^q+v^q+2\right), \lambda\in[0,1]$		[-0.5, 0.5]
Peng [17]	$S_6(\alpha)=\frac{e^{u^q-v^q}}{\pi_\alpha^q+1}$		[e^{-1}, e]
Xing [27]	$S_7(\alpha)=\left\|u^q-v^q\right\|^{1/q}$	$H_7(\alpha)=\left(u^q+v^q\right)^{1/q}$	[0, 1]
Yager [5] and Wei [28]	$S_8(\alpha)=\frac{u^q-v^q+1}{2}$		[0, 1]
Du [29]	$S_9(\alpha)=\frac{1}{1+\left(\frac{(1-u)^q+v^q}{u^q+(1-v)^q}\right)^{1/q}}$		[0, 1]
Banerjee [30]	$S_{10}(\alpha)=\frac{1-v^q}{2-u^q-v^q}$		[0, 1]
Garg [31]	$S_{11}(\alpha)=\left(u^q-v^q\right)\left(1+\pi_\alpha^q\right)$		[-1, 1]
Peng [23] and Wang [26]	$S_{12}(\alpha)=u^q-v^q-\ln\left(1+\pi_\alpha^q\right)$		[-1, 1]
Rani [24]	$S_{13}(\alpha)=u^q\left(1+\pi_\alpha\right)$		[0, 1]
Aydemir [19] and Ali [25]	$S_{14}(\alpha)=u^q-v^q-\pi_\alpha^q\times\frac{\log_2\left(1+\pi_\alpha^q\right)}{100}$		[-1, 1]
Farhadinia [13]	$S_{15}(\alpha)=u^q+\lambda\pi_\alpha^q, 0<\lambda<1$		[0, 1]
Peng [15]	$S_{16}(\alpha)=\frac{1}{2}\left(u^2+\left(\sqrt[q]{1-v^q}\right)^2\right)$		[0, 1]
Li [32]	$S_{17}(\alpha)=\frac{1}{2}\left(1+u^q-v^q-\frac{1}{2}\sin\frac{\pi_\alpha^q\pi}{2}\right)$		[0, 1]
Garg [33]	$S_{18}(\alpha)=\frac{e^{u^q-v^q}}{1+\pi_\alpha^q}$		[e^{-1}, e]
Xiao [20]	$S_{19}(\alpha)=\frac{1+u^q-v^q+\log_2\left(\sqrt[10]{\frac{1+u^q+v^q}{2}}\right)}{2}$		[0, 1]

Continued on next page

References	Score function	Accuracy function	Range
Liu [21]	$S_{20}(\alpha) = u^q - v^q - \pi_\alpha^q \times \dfrac{e^{1-v^q}}{e^{1-u^q} + e^{1-v^q}}$		[-1, 1]
Rawat [34]	$S_{21}(\alpha) = \dfrac{1+u^q-v^q}{2} - \dfrac{\cos\left(\dfrac{\left(1-\pi_\alpha^q\right)\pi}{2}\right)}{2\pi_\alpha^3}$		[0, 1]

2.2. q-ROF distance operators

Within the theoretical framework of q-ROFSs, distance operators serve as critical instruments for measuring discrepancies in fuzzy information. Early investigations primarily extended conventional distance metrics to the q-ROF context. Minkowski-type distance operators, encompassing Hamming, Euclidean, and Chebyshev distances, have been systematically established, providing a fundamental basis for quantifying disparities in q-ROF information [35]. Starting from the cosine similarity measure, corresponding cosine similarity and Euclidean distance operators were defined, and the conversion relationship between similarity and distance metrics was further investigated [36]. The intrinsic connections among distance, similarity, entropy, and inclusion measures have been further explored, and a systematic transformation framework for information measures under q-ROFSs has been established [37]. As research advanced, scholars began to address the limitations of existing distance operators. It has been identified that conventional distance operators may yield counterintuitive results, and a novel distance operator has been proposed and applied to q-ROF-based TOPSIS and ELECTRE methods [38]. A distance operator grounded in score functions has been introduced and integrated into the TAOV decision-making method, which effectively avoids counterintuitive phenomena as well as mathematical defects, including negative logarithmic arguments and division by zero [15]. Regarding innovative methodological designs, researchers have incorporated diverse mathematical tools to refine relevant metrics. Distance and similarity measures based on the Hausdorff metric have been proposed, and a q-ROF TODIM decision-making framework has been constructed accordingly [39]. The Soergel distance has been extended to the q-ROF context, with weighted Soergel distance and similarity coefficients developed, and their superiority in preventing counterintuitive outcomes has been fully validated [40]. Scholars have further expanded the application scenarios and mathematical expressions of distance operators. A novel distance operator that comprehensively accounts for interaction effects and information volatility has been developed and combined with DEMATEL and taxonomy methods to realize the evaluation of pharmaceutical cold chain logistics [41]. Grounded in Jensen-Shannon divergence, the D-JS_2D and D-JS_3D distance operators have been put forward [42]. Logarithmic-based distance operators have been introduced and integrated with the TOPSIS method to solve practical vehicle selection problems [43]. Furthermore, several studies have combined distance operators with score functions to boost decision-making performance. A distance-based similarity measure coupled with an improved score function has been proposed and successfully applied to medical waste management decision-making [44]. Distance operators have been integrated with SPC, PIPRECIA, WASPAS, and other approaches, constructing a comprehensive decision-making framework for the sustainability assessment of e-scooter micromobility systems [45].

These operators aim to quantify the dissimilarity between two q-ROFSs and are fundamental in various decision-making, pattern recognition, and clustering tasks. Let M and N be two q-ROFSs defined on a finite universe $X = \{x_1, x_2, \ldots, x_n\}$, where $M = \{\langle x_i, (u_M(x_i), v_M(x_i))\rangle | x_i \in X\}$,

$N=\{\langle x_i,\ (u_N(x_i),\ v_N(x_i))\rangle | x_i \in X\}$. We present a comprehensive summary of the representative q-ROFDOs proposed in the existing literature in Table 3.

Table 3. Previous q-ROF distance operators.

References	Distance operators
Du [35]	$d_1(M,N)=\left(\frac{1}{3n}\sum_{i=1}^{n}\left(\left\|u_M(x_i)-u_N(x_i)\right\|^p+\left\|v_M(x_i)-v_N(x_i)\right\|^p+\left\|\pi_M(x_i)-\pi_N(x_i)\right\|^p\right)\right)^{1/p}$
Peng [37]	$d_2(M,N)=\frac{1}{2n}\sum_{i=1}^{n}\left(\left\|u_M^q(x_i)-u_N^q(x_i)\right\|+\left\|v_M^q(x_i)-v_N^q(x_i)\right\|+\left\|\pi_M^q(x_i)-\pi_N^q(x_i)\right\|\right)$
	$d_3(M,N)=\frac{1}{2n}\sum_{i=1}^{n}\left\|u_M^q(x_i)-u_N^q(x_i)-\left(v_M^q(x_i)-v_N^q(x_i)\right)\right\|$
	$d_4(M,N)=\frac{1}{4n}\sum_{i=1}^{n}\left(\left\|u_M^q(x_i)-u_N^q(x_i)\right\|+\left\|v_M^q(x_i)-v_N^q(x_i)\right\|+\left\|\pi_M^q(x_i)-\pi_N^q(x_i)\right\|\right.$ $\left.+\left\|\left(u_M^q(x_i)-v_M^q(x_i)\right)-\left(u_N^q(x_i)-v_N^q(x_i)\right)\right\|\right)$
	$d_5(M,N)=\frac{1}{n}\sum_{i=1}^{n}\left\|u_M^q(x_i)-u_N^q(x_i)\right\|\vee\left\|v_M^q(x_i)-v_N^q(x_i)\right\|$
	$d_6(M,N)=\frac{2}{n}\sum_{i=1}^{n}\frac{\left\|u_M^q(x_i)-u_N^q(x_i)\right\|\vee\left\|v_M^q(x_i)-v_N^q(x_i)\right\|}{1+\left\|u_M^q(x_i)-u_N^q(x_i)\right\|\vee\left\|v_M^q(x_i)-v_N^q(x_i)\right\|}$
	$d_7(M,N)=\frac{2\sum_{i=1}^{n}\left(\left\|u_M^q(x_i)-u_N^q(x_i)\right\|\vee\left\|v_M^q(x_i)-v_N^q(x_i)\right\|\right)}{\sum_{i=1}^{n}\left(1+\left\|u_M^q(x_i)-u_N^q(x_i)\right\|\vee\left\|v_M^q(x_i)-v_N^q(x_i)\right\|\right)}$
	$d_8(M,N)=1-\alpha\frac{\sum_{i=1}^{n}\left(u_M^q(x_i)\wedge u_N^q(x_i)\right)}{\sum_{i=1}^{n}\left(u_M^q(x_i)\vee u_N^q(x_i)\right)}-\beta\frac{\sum_{i=1}^{n}\left(v_M^q(x_i)\wedge v_N^q(x_i)\right)}{\sum_{i=1}^{n}\left(v_M^q(x_i)\vee v_N^q(x_i)\right)}$
	$d_9(M,N)=1-\frac{\alpha}{n}\sum_{i=1}^{n}\frac{\left(u_M^q(x_i)\wedge u_N^q(x_i)\right)}{\left(u_M^q(x_i)\vee u_N^q(x_i)\right)}-\frac{\beta}{n}\sum_{i=1}^{n}\frac{\left(v_M^q(x_i)\wedge v_N^q(x_i)\right)}{\left(v_M^q(x_i)\vee v_N^q(x_i)\right)}$
	$d_{10}(M,N)=1-\frac{1}{n}\sum_{i=1}^{n}\frac{\left(u_M^q(x_i)\wedge u_N^q(x_i)\right)+\left(v_M^q(x_i)\wedge v_N^q(x_i)\right)}{\left(u_M^q(x_i)\vee u_N^q(x_i)\right)+\left(v_M^q(x_i)\vee v_N^q(x_i)\right)}$
	$d_{11}(M,N)=1-\frac{\sum_{i=1}^{n}\left(\left(u_M^q(x_i)\wedge u_N^q(x_i)\right)+\left(v_M^q(x_i)\wedge v_N^q(x_i)\right)\right)}{\sum_{i=1}^{n}\left(\left(u_M^q(x_i)\vee u_N^q(x_i)\right)+\left(v_M^q(x_i)\vee v_N^q(x_i)\right)\right)}$
	$d_{12}(M,N)=1-\frac{1}{n}\sum_{i=1}^{n}\frac{\left(u_M^q(x_i)\wedge u_N^q(x_i)\right)+\left(\left(1-v_M^q(x_i)\right)\wedge\left(1-v_N^q(x_i)\right)\right)}{\left(u_M^q(x_i)\vee u_N^q(x_i)\right)+\left(\left(1-v_M^q(x_i)\right)\vee\left(1-v_N^q(x_i)\right)\right)}$
	$d_{13}(M,N)=1-\frac{\sum_{i=1}^{n}\left(\left(u_M^q(x_i)\wedge u_N^q(x_i)\right)+\left(1-v_M^q(x_i)\right)\wedge\left(1-v_N^q(x_i)\right)\right)}{\sum_{i=1}^{n}\left(\left(u_M^q(x_i)\vee u_N^q(x_i)\right)+\left(1-v_M^q(x_i)\right)\vee\left(1-v_N^q(x_i)\right)\right)}$

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References	Distance operators
Peng [37]	$d_{14}(M,N)=\left(\frac{1}{2n(t_1+1)^p}\sum_{i=1}^{n}\left(\left\lvert t_1\left(u_M^q(x_i)-u_N^q(x_i)\right)-\left(v_M^q(x_i)-v_N^q(x_i)\right)\right\rvert^p\right)+\frac{1}{2n(t_2+1)^p}\sum_{i=1}^{n}\left(\left\lvert t_2\left(v_M^q(x_i)-v_N^q(x_i)\right)-\left(u_M^q(x_i)-u_N^q(x_i)\right)\right\rvert^p\right)\right)^{\frac{1}{p}}$
Liu [36]	$d_{15}(M,N)=\left(\frac{1}{2n}\sum_{i=1}^{n}\left(\left\lvert u_M^q(x_i)-u_N^q(x_i)\right\rvert^2+\left\lvert v_M^q(x_i)-v_N^q(x_i)\right\rvert^2\right)\right)^{\frac{1}{2}}$
	$d_{16}(M,N)=\left(\frac{1}{2}\sum_{i=1}^{n}\omega_i\left(\left\lvert u_M^q(x_i)-u_N^q(x_i)\right\rvert^2+\left\lvert v_M^q(x_i)-v_N^q(x_i)\right\rvert^2\right)\right)^{\frac{1}{2}}$
	$d_{17}(M,N)=\frac{1}{2}\left(1-\sum_{i=1}^{n}\omega_i\frac{u_M^q(x_i)u_N^q(x_i)+v_M^q(x_i)v_N^q(x_i)}{\sqrt{u_M^{2q}(x_i)+v_M^{2q}(x_i)}\sqrt{u_N^{2q}(x_i)+v_N^{2q}(x_i)}}+\left(\frac{1}{2}\sum_{i=1}^{n}\omega_i\left(\left\lvert u_M^q(x_i)-u_N^q(x_i)\right\rvert^2+\left\lvert v_M^q(x_i)-v_N^q(x_i)\right\rvert^2\right)\right)^{\frac{1}{2}}\right)$
	$d_{18}(M,N)=\frac{1}{2}\left(1-\frac{1}{n}\sum_{i=1}^{n}\frac{u_M^q(x_i)u_N^q(x_i)+v_M^q(x_i)v_N^q(x_i)}{\sqrt{u_M^{2q}(x_i)+v_M^{2q}(x_i)}\sqrt{u_N^{2q}(x_i)+v_N^{2q}(x_i)}}+\left(\frac{1}{2n}\sum_{i=1}^{n}\left(\left\lvert u_M^q(x_i)-u_N^q(x_i)\right\rvert^2+\left\lvert v_M^q(x_i)-v_N^q(x_i)\right\rvert^2\right)\right)^{\frac{1}{2}}\right)$
Pinar [38]	$d_{19}(M,N)=\left(\frac{1}{2n}\sum_{i=1}^{n}\left(\left\lvert(1-k)\left(u_M(x_i)-u_N(x_i)\right)+k\left(\left(1-v_M^q(x_i)\right)^{\frac{1}{q}}-\left(1-v_N^q(x_i)\right)^{\frac{1}{q}}\right)\right\rvert^p+\left\lvert(1-k)\left(v_M(x_i)-v_N(x_i)\right)+k\left(\left(1-u_M^q(x_i)\right)^{\frac{1}{q}}-\left(1-u_N^q(x_i)\right)^{\frac{1}{q}}\right)\right\rvert^p\right)\right)^{\frac{1}{p}}$, where $k=\frac{\frac{1}{2}q^2+\frac{3}{2}q-\frac{1}{3}}{q^2+3q+1}, k\in\left[\frac{1}{3},\frac{1}{2}\right]$
Peng [15]	$d_{20}(M,N)=\frac{1}{2n}\sum_{i=1}^{n}\left(\left(u_M(x_i)-u_N(x_i)\right)^2+\left(\sqrt[q]{1-v_M^q(x_i)}-\sqrt[q]{1-v_N^q(x_i)}\right)^2\right)$
Hussain [39]	$d_{21}(M,N)=\frac{1}{n}\sum_{i=1}^{n}\max\left\{\left\lvert u_M^q(x_i)-u_N^q(x_i)\right\rvert,\left\lvert 1-v_M^q(x_i)-\left(1-v_N^q(x_i)\right)\right\rvert\right\}$
Kamaci [40]	$d_{22}(M,N)=\frac{1}{n}\sum_{i=1}^{n}\frac{\left\lvert u_M^q(x_i)-u_N^q(x_i)\right\rvert+\left\lvert v_M^q(x_i)-v_N^q(x_i)\right\rvert}{\max\left\{u_M^q(x_i),u_N^q(x_i)\right\}+\max\left\{v_M^q(x_i),v_N^q(x_i)\right\}}$
	$d_{23}(M,N)=\frac{1}{n}\sum_{i=1}^{n}\frac{\left\lvert u_M^q(x_i)-u_N^q(x_i)\right\rvert+\left\lvert v_M^q(x_i)-v_N^q(x_i)\right\rvert+\left\lvert \pi_M^q(x_i)-\pi_N^q(x_i)\right\rvert}{\max\left\{u_M^q(x_i),u_N^q(x_i)\right\}+\max\left\{v_M^q(x_i),v_N^q(x_i)\right\}+\max\left\{\pi_M^q(x_i),\pi_N^q(x_i)\right\}}$
	$d_{24}(M,N)=\frac{\sum_{i=1}^{n}\left(\left\lvert u_M^q(x_i)-u_N^q(x_i)\right\rvert+\left\lvert v_M^q(x_i)-v_N^q(x_i)\right\rvert+\left\lvert \pi_M^q(x_i)-\pi_N^q(x_i)\right\rvert\right)}{\sum_{i=1}^{n}\left(\max\left\{u_M^q(x_i),u_N^q(x_i)\right\}+\max\left\{v_M^q(x_i),v_N^q(x_i)\right\}+\max\left\{\pi_M^q(x_i),\pi_N^q(x_i)\right\}\right)}$

Continued on next page

References	Distance operators
Rong [41]	$d_{25}(M,N)=\frac{1}{3n}\sum_{i=1}^{n}\left(\frac{1}{2}\left(\left\|u_M^q(x_i)-u_N^q(x_i)\right\|+\left\|v_M^q(x_i)-v_N^q(x_i)\right\| +\left\|u_M^q(x_i)-v_M^q(x_i)-\left(u_N^q(x_i)-v_N^q(x_i)\right)\right\|+\left(\pi_M^q(x_i)+\pi_N^q(x_i)\right)\right) +\left\|\max_i\left\{u_M^q(x_i),v_N^q(x_i)\right\}-\max_i\left\{v_M^q(x_i),u_N^q(x_i)\right\}\right\|\right)$
Wang [42]	$d_{26}(M,N)=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{2}\left(u_M^q(x_i)\log\frac{2u_M^q(x_i)}{u_M^q(x_i)+u_N^q(x_i)}+u_N^q(x_i)\log\frac{2u_N^q(x_i)}{u_M^q(x_i)+u_N^q(x_i)} +v_M^q(x_i)\log\frac{2v_M^q(x_i)}{v_M^q(x_i)+v_N^q(x_i)}+v_N^q(x_i)\log\frac{2v_N^q(x_i)}{v_M^q(x_i)+v_N^q(x_i)}\right)\right)^{\frac{1}{2}}$ $d_{27}(M,N)=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{2}\left(u_M^q(x_i)\log\frac{2u_M^q(x_i)}{u_M^q(x_i)+u_N^q(x_i)}+u_N^q(x_i)\log\frac{2u_N^q(x_i)}{u_M^q(x_i)+u_N^q(x_i)} +v_M^q(x_i)\log\frac{2v_M^q(x_i)}{v_M^q(x_i)+v_N^q(x_i)}+v_N^q(x_i)\log\frac{2v_N^q(x_i)}{v_M^q(x_i)+v_N^q(x_i)} +\pi_M^q(x_i)\log\frac{2\pi_M^q(x_i)}{\pi_M^q(x_i)+\pi_N^q(x_i)}+\pi_N^q(x_i)\log\frac{2\pi_N^q(x_i)}{\pi_M^q(x_i)+\pi_N^q(x_i)}\right)\right)^{\frac{1}{2}}$
Mishra [45]	$d_{28}(M,N)=\frac{1}{5n}\sum_{i=1}^{n}\left(\left\|u_M^q(x_i)-u_N^q(x_i)\right\|+\left\|v_M^q(x_i)-v_N^q(x_i)\right\| +\left\|u_M^q(x_i)v_N^q(x_i)-u_N^q(x_i)v_M^q(x_i)\right\| +\left\|\min\left\{u_M^q(x_i),v_N^q(x_i)\right\}-\min\left\{u_N^q(x_i),v_M^q(x_i)\right\}\right\| +\left\|\max\left\{u_M^q(x_i),v_N^q(x_i)\right\}-\max\left\{u_N^q(x_i),v_M^q(x_i)\right\}\right\|\right)$
Basu [44]	$d_{29}(M,N)=\frac{1}{2n}\sum_{i=1}^{n}\left(\left\|\cos\left(\left(\left\|u_M^q(x_i)-u_N^q(x_i)\right\|+3\right)\frac{\pi}{2}\right)\right\|+\left\|\cos\left(\left(\left\|v_M^q(x_i)-v_N^q(x_i)\right\|+3\right)\frac{\pi}{2}\right)\right\|\right)$
Anum [43]	$d_{30}(M,N)=\left(\frac{1}{3n}\sum_{i=1}^{n}\left(u_M^q(x_i)\log\frac{2u_M^q(x_i)}{u_M^q(x_i)+u_N^q(x_i)}+u_N^q(x_i)\log\frac{2u_N^q(x_i)}{u_M^q(x_i)+u_N^q(x_i)}\right) +\frac{1}{3n}\sum_{i=1}^{n}\left(v_M^q(x_i)\log\frac{2v_M^q(x_i)}{v_M^q(x_i)+v_N^q(x_i)}+v_N^q(x_i)\log\frac{2v_N^q(x_i)}{v_M^q(x_i)+v_N^q(x_i)}\right) +\frac{1}{3n}\sum_{i=1}^{n}\left(\pi_M^q(x_i)\log\frac{2\pi_M^q(x_i)}{\pi_M^q(x_i)+\pi_N^q(x_i)}+\pi_N^q(x_i)\log\frac{2\pi_N^q(x_i)}{\pi_M^q(x_i)+\pi_N^q(x_i)}\right)\right)^{\frac{1}{2}}$ $d_{31}(M,N)=\frac{1}{3n}\left(\sum_{i=1}^{n}\left(u_M^q(x_i)\log\frac{2u_M^q(x_i)}{u_M^q(x_i)+u_N^q(x_i)}+u_N^q(x_i)\log\frac{2u_N^q(x_i)}{u_M^q(x_i)+u_N^q(x_i)}\right) +\sum_{i=1}^{n}\left(v_M^q(x_i)\log\frac{2v_M^q(x_i)}{v_M^q(x_i)+v_N^q(x_i)}+v_N^q(x_i)\log\frac{2v_N^q(x_i)}{v_M^q(x_i)+v_N^q(x_i)}\right) +\sum_{i=1}^{n}\left(\pi_M^q(x_i)\log\frac{2\pi_M^q(x_i)}{\pi_M^q(x_i)+\pi_N^q(x_i)}+\pi_N^q(x_i)\log\frac{2\pi_N^q(x_i)}{\pi_M^q(x_i)+\pi_N^q(x_i)}\right)\right)$

2.3. The REGIME decision-making method

As a multi-criteria decision-making technique based on pairwise comparisons, the REGIME decision-making method [46] has attracted extensive attention owing to its ability to handle qualitative information and its suitability for complex decision-making scenarios. This method realizes the ranking and selection of alternatives by constructing mechanisms such as superiority identifiers, guide indices, and REGIME matrices. In recent years, scholars have integrated the REGIME method with various fuzzy set theories to strengthen its ability to process uncertain information. The REGIME method has been extended to the Pythagorean fuzzy environment, yielding the PF-REGIME method that has been successfully applied to the problem of waste disposal site selection [47]. The REGIME method was further extended to the T-spherical fuzzy environment to develop the T-SF REGIME method, which enhanced its applicability and discrimination power by introducing Minkowski-type distance operators and Gaussian preference functions [48,49]. Within the picture fuzzy set framework, a CRITIC-REGIME method has been constructed and implemented for wearable health technology selection [50]. The CRITIC-REGIME method has been further extended to the spherical fuzzy environment, and the improved approach has been utilized for breast cancer treatment plan selection [51]. Moreover, the PFR-CRITIC-REGIME method has been proposed by combining Pythagorean fuzzy rough numbers and applied to sustainable supply chain decision-making for electric ferries [52].

3. Preliminary

This section establishes the foundational framework for the subsequent research by systematically reviewing and elaborating on the core concepts, mathematical definitions, and essential operational rules associated with q-ROFNs.

3.1. The q-rung orthopair fuzzy sets

To facilitate the subsequent research on the score function, distance operator, and the improved REGIME method under the q-ROF environment, we first give the basic definition of q-ROFSs and q-ROFNs, and then further introduce their operational laws and weighted averaging operator.

Definition 3.1. [5] *Assume $X = \{x_1, x_2, \ldots, x_n\}$ is a nonempty set, and a q-ROFS M on X is defined as:*

$$M = \langle x, u_M(x), v_M(x) | x \in X \rangle \tag{1}$$

where $u_M(x)$ and $v_M(x)$ are the membership degree and the non-membership degree of x to the set M, respectively. Simultaneously, $u_M(x)$ and $v_M(x)$ satisfy the constraints $0 \le u_M(x) \le 1$, $0 \le v_M(x) \le 1$, and $0 \le \left(u_M(x)\right)^q + \left(v_M(x)\right)^q \le 1$, and q is a positive integer. In addition, the hesitation degree $\pi_M(x)$ can be calculated by $\pi_M(x) = \left(1 - (u_M(x)^q + v_M(x)^q)\right)^{\frac{1}{q}}$.

For a q-ROFS M, an element A in q-ROFS is called q-ROFN, i.e., the pair $\langle u_A(x), v_A(x) \rangle$ is called q-ROFN. For any two q-ROFN A and B, $A \succ B$ is used to denote that A is superior to B, $A \sim B$ denotes that A is indifferent to B, and $A \succcurlyeq B$ represents that A is not inferior to B.

Definition 3.2. [5,12,53] *Let $\alpha = (u, v)$, $\alpha_1 = (u_1, v_1)$, and $\alpha_2 = (u_2, v_2)$ be three q-ROFNs, and λ is a positive real number. Then the following operations can be defined:*

$$\alpha_1 \subseteq \alpha_2 \ iff \ u_1 \le u_2 \ and \ v_1 \ge v_2,$$

$$\alpha_1 = \alpha_2 \ iff \ u_1 = u_2 \ and \ v_1 = v_2,$$

$$\alpha^c = (v, u),$$

$$\alpha_1 \vee \alpha_2 = (max\{u_1, u_2\}, min\{v_1, v_2\}),$$

$$\alpha_1 \wedge \alpha_2 = (min\{u_1, u_2\}, max\{v_1, v_2\}),$$

$$\alpha_1 \oplus \alpha_2 = \left(\left(u_1^q + u_2^q - u_1^q u_2^q\right)^{\frac{1}{q}}, v_1 v_2\right)$$

$$\alpha_1 \otimes \alpha_2 = \left(u_1 u_2, \left(v_1^q + v_2^q - v_1^q v_2^q\right)^{\frac{1}{q}}\right),$$

$$\lambda\alpha = \left(\left(1 - (1 - u^q)^\lambda\right)^{\frac{1}{q}}, v^\lambda\right),$$

$$\alpha^\lambda = \left(u^\lambda, \left(1 - (1 - v^q)^\lambda\right)^{\frac{1}{q}}\right).$$

Definition 3.3. [12] *Suppose* $\alpha_i = (u_i, v_i)$ $(i = 1,2,\ldots,n)$ *is a collection of q-ROFNs, and q-ROFWA:* $\Omega^n \rightarrow \Omega$, *if*

$$\text{q-ROFWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = w_1\alpha_1 \oplus w_2\alpha_2 \oplus \ldots \oplus w_n\alpha_n$$
$$= \left((1 - \prod_{i=1}^n (1 - u_i{}^q)^{w_i})^{\frac{1}{q}}, \prod_{i=1}^n v_i{}^{w_i}\right), \tag{2}$$

where Ω *is the set of all q-ROFNs, and* $w = (w_1, w_2, \ldots, w_n)^T$ *is the weight vector of* $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, *such that* $0 \leq w_i \leq 1$ *and* $\sum_{i=1}^n w_i = 1$. *Then, the q-ROFWA is called the q-rung orthopair fuzzy weighted averaging operator.*

3.2. The classical REGIME method

The REGIME method, first proposed by Hinloopen, is a classic multi-attribute decision-making approach known for its simple calculation logic and intuitive preference comparisons in practical applications. This subsection outlines the basic steps of the classical REGIME method.

Given m alternatives A_i $(i = 1,2,\ldots,m)$, n criteria C_i $(i = 1,2,\ldots,n)$, and a criterion weight vector $w = (w_1, w_2, \ldots, w_n)$ satisfying $\sum_{j=1}^n w_j = 1$ and $w_j \geq 0$. The decision matrix is $X = [x_{ij}]_{m\times n}$, where x_{ij} denotes the evaluation value of alternative A_i under criterion C_j. Then the decision-making steps of the REGIME method are presented as follows.

Step 1. Let I_b be the set of benefit criteria, and I_c be the set of cost criteria. Standardize the decision matrix X to obtain the matrix $R = [r_{ij}]_{m\times n}$, where:

$$r_{ij} = \begin{cases} \dfrac{x_{ij}}{\max\limits_i x_{ij}}, \text{if } j \in I_b \\ \dfrac{\min\limits_i x_{ij}}{x_{ij}}, \text{if j} \in I_c \end{cases}.$$

Step 2. Calculate the weighted standardized values:

$$y_{ij} = w_j \times r_{ij}, i = 1, \ldots, m,\ j = 1, \ldots, n.$$

Step 3. For each pair of alternatives (A_s, A_t), compare their performance under each criterion and define the preference indicator function as follows:

$$\delta_{stj} = \begin{cases} 1, & \text{if } y_{sj} > y_{tj} \\ 0, & \text{if } y_{sj} = y_{tj} \\ -1, & \text{if } y_{sj} < y_{tj} \end{cases}.$$

Step 4. Construct the preference matrix $\boldsymbol{G} = [G_{st}]_{m \times m}$, where

$$G_{st} = \begin{cases} \sum_{j=1}^{n} w_j \cdot \delta_{stj}, & s \neq t \\ 0, & s = t \end{cases},$$

represents the preference intensity of alternative A_s over A_t.

Step 5. For each alternative A_s, calculate its preference flow Φ_s:

$$\Phi_s = \sum_{t=1}^{m} G_{st} = \sum_{t=1, t \neq s}^{m} \sum_{j=1}^{n} w_j \cdot \delta_{stj}.$$

Step 6. Rank the alternatives in descending order of the preference flow Φ_s.

4. A novel q-ROF score function

As a core tool for ranking q-ROFNs, the score function directly determines the rationality and effectiveness of multi-criteria decision-making results under the q-ROF environment. Although numerous score functions for q-ROFNs have been proposed by scholars, most of them have limitations such as unreasonable ranking results, narrow application scope, or failure to fully consider the influence of hesitation degree. To address these deficiencies, this section proposes a novel score function for q-ROFNs by comprehensively integrating the membership degree, non-membership degree, and hesitation degree with a more reasonable weight allocation. Based on the proposed score function, an order relation for q-ROFNs is constructed, and it is proven to be an admissible order.

4.1. The novel score function for q-ROFNs

This subsection introduces a novel score function for q-ROFNs by explicitly considering the interplay among membership, non-membership, and hesitation degrees. We rigorously prove its fundamental properties and subsequently define a new order relation that is proven to be an admissible order.

Definition 4.1. *Let* $\alpha = (u, v)$ *be a q-ROFN, then the score function* $S(\alpha)$ *can be defined as follows:*

$$S(\alpha) = \left[\frac{u^q}{1+\pi_\alpha^q} - \frac{v^q}{1+\pi_\alpha}\right] \times \left(1 + \frac{1-\pi_\alpha^q}{2}\right), \tag{3}$$

where $u, v \in [0,1]$, $0 \leq u^q + v^q \leq 1$, *and* $q \geq 1$, *the hesitancy degree is* $\pi_\alpha = \left(1 - (u^q + v^q)\right)^{\frac{1}{q}}$.

Theorem 4.1. *For a q-ROFN* $\alpha = (u, v)$, $S(\alpha)$ *monotonically increases with respect to* u *and monotonically decreases with respect to* v.

Proof. The complete proof is provided in the appendix.

Theorem 4.2. *For two q-ROFNs* $\alpha_1 = (u_1, v_1)$ *and* $\alpha_2 = (u_2, v_2)$, *if* $u_1 \geq u_2$ *and* $v_1 \leq v_2$, *then* $S(\alpha_1) \geq S(\alpha_2)$, *the equality holds if and only if* $u_1 = u_2$ *and* $v_1 = v_2$.

Proof. The complete proof is provided in the appendix.

Theorem 4.3. *For a q-ROFN* $\alpha = (u, v)$, *the proposed score function* $S(\alpha)$ *satisfies:*

(1) $-\frac{3}{2} \leq S(\alpha) \leq \frac{3}{2}$;

(2) $S(\alpha) = \frac{3}{2}\ iff\ \alpha = (1,0)$;

(3) $S(\alpha) = -\frac{3}{2}\ iff\ \alpha = (0,1)$;

(4) $S(\alpha) = 0\ iff\ \alpha = (0,0)$.

Proof. The complete proof is provided in the appendix.

Definition 4.2. *Let* $\alpha_i = (u_i, v_i)\ (i = 1,2)$ *be two q-ROFNs, the new order relation of q-ROFNs is specified as follows:*
(1) If $S(\alpha_1) > S(\alpha_2)$, *then* α_1 *is bigger than* α_2, *denoted as* $\alpha_1 \succ \alpha_2$*;*

(2) If $S(\alpha_1) = S(\alpha_2)$, *then* α_1 *is indifferent to* α_2, *denoted as* $\alpha_1 \sim \alpha_2$*;*

(3) If $S(\alpha_1) \geq S(\alpha_2)$, *then* α_1 *is bigger than or indifferent to* α_2, *denoted as* $\alpha_1 \succcurlyeq \alpha_2$.

Proposition 4.1. *Let* $\alpha_i = (u_i, v_i)\ (i = 1,2,3)$ *be q-ROFNs. The relation "* $\succcurlyeq$ *" defined in Definition 4.2 satisfies the following properties:*
(1) Reflexivity: $\alpha_i \succcurlyeq \alpha_i\ (i = 1,2,3)$*;*

(2) Antisymmetry: $\alpha_1 \succcurlyeq \alpha_2, \alpha_2 \succcurlyeq \alpha_1 \Rightarrow \alpha_1 \sim \alpha_2$*;*

(3) Transitivity: $\alpha_1 \succcurlyeq \alpha_2, \alpha_2 \succcurlyeq \alpha_3 \Rightarrow \alpha_1 \succcurlyeq \alpha_3$.

Proof. The complete proof is provided in the appendix.

Inspired by Bustince et al.'s concept of an admissible order for intervals [54], we extend this concept to q-ROFSs and introduce the following definition:

Definition 4.3. *Let* $\alpha_i = (u_i, v_i)\ (i = 1,2)$ *be two q-ROFNs. A nature quasi-ordering on q-ROFNs is defined as* $\alpha_i \succcurlyeq_N \alpha_j$ *if and only if* $u_1 \geq u_2$ *and* $v_1 \leq v_2$, *where* $\succcurlyeq_N$ *means "bigger than or indifferent to".*

Definition 4.4. *Let* Ω *be a set of q-ROFNs with order* $\succcurlyeq_A$. *The order* $\succcurlyeq_A$ *is a linear order on* Ω *if* $\succcurlyeq_A$ *satisfies three fundamental properties for all* $\alpha_1, \alpha_2, \alpha_3 \in \Omega$:
(1) Reflexivity: $\alpha_i \succcurlyeq_A \alpha_i\ (i = 1,2,3)$*;*

(2) Antisymmetry: $\alpha_1 \succcurlyeq_A \alpha_2, \alpha_2 \succcurlyeq_A \alpha_1 \Rightarrow \alpha_1 \sim_A \alpha_2$*;*

(3) Totality: for all $\alpha_i, \alpha_j \in \Omega$, *either* $\alpha_i \succcurlyeq_A \alpha_j$ *or* $\alpha_j \succcurlyeq_A \alpha_i$ *must hold.*

Definition 4.5. *Let* Ω *be a set of q-ROFNs with an order* $\succcurlyeq_A$. *The order* $\succcurlyeq_A$ *is called an admissible order if the following two conditions hold:*
(1) $\succcurlyeq_A$ *is a linear order on* Ω*;*
(2) For all $\alpha_i, \alpha_j \in \Omega$, $\alpha_i \succcurlyeq_A \alpha_j$ *whenever* $\alpha_i \succcurlyeq_N \alpha_j$.

Proposition 4.2. *The order relation of q-ROFNs defined in Definition 4.2 is an admissible order.*

Proof. The complete proof is provided in the appendix.

4.2. A brief comparison between the proposed score function and previous score functions

To validate the effectiveness and superiority of the proposed score function S_{22}, and to systematically reveal its differences from existing score functions in terms of performance, two sets of simulation experiments are designed in this section. The comparative methods involved correspond to the numbering in Table 2, and the experimental results are shown in Figures 1 and 2.

The first experiment is designed to observe the distribution trends of different score function values and to intuitively illustrate the discriminative ability of each score function for q-ROFNs. This experiment only considers the score function, without involving the accuracy function. The parameter λ for S_5 and S_{15} is set to 0.5, and q is set to 3. A total of 20,000 pairs of values $A = (a, b)$ are randomly generated, with $a, b \in [0,1]$. Pairs that satisfy the q-ROFN condition are retained and their scores are calculated, while those that do not meet the condition are discarded. In Figure 1, the brightness of each point represents the magnitude of the score value; brighter points indicate higher scores, and darker points indicate lower scores.

As shown in Figure 1, as the membership degree u increases, the images of almost all score functions gradually transition from dark to bright; as the non-membership degree v increases, the images gradually transition from bright to dark. This trend aligns with intuition, as the higher the membership degree and the lower the non-membership degree, the higher the score should be.

However, the performance of score function S_7 significantly deviates from this pattern. Its image appears notably darker near the diagonal region and unusually brighter near the extreme point (0,1). This indicates that S_7 fails to effectively distinguish q-ROFNs with similar membership and non-membership degrees, and may even assign excessively high scores to q-ROFNs with ambiguous or conflicting information, revealing inherent deficiencies in its discriminative ability.

In contrast, the proposed score function S_{22} exhibits a smoother and more stable distribution trend, with brightness changes strictly following the monotonic relationship between u and v. This demonstrates favorable discriminative consistency and monotonicity, laying a solid foundation for subsequent ranking and comparison tasks.

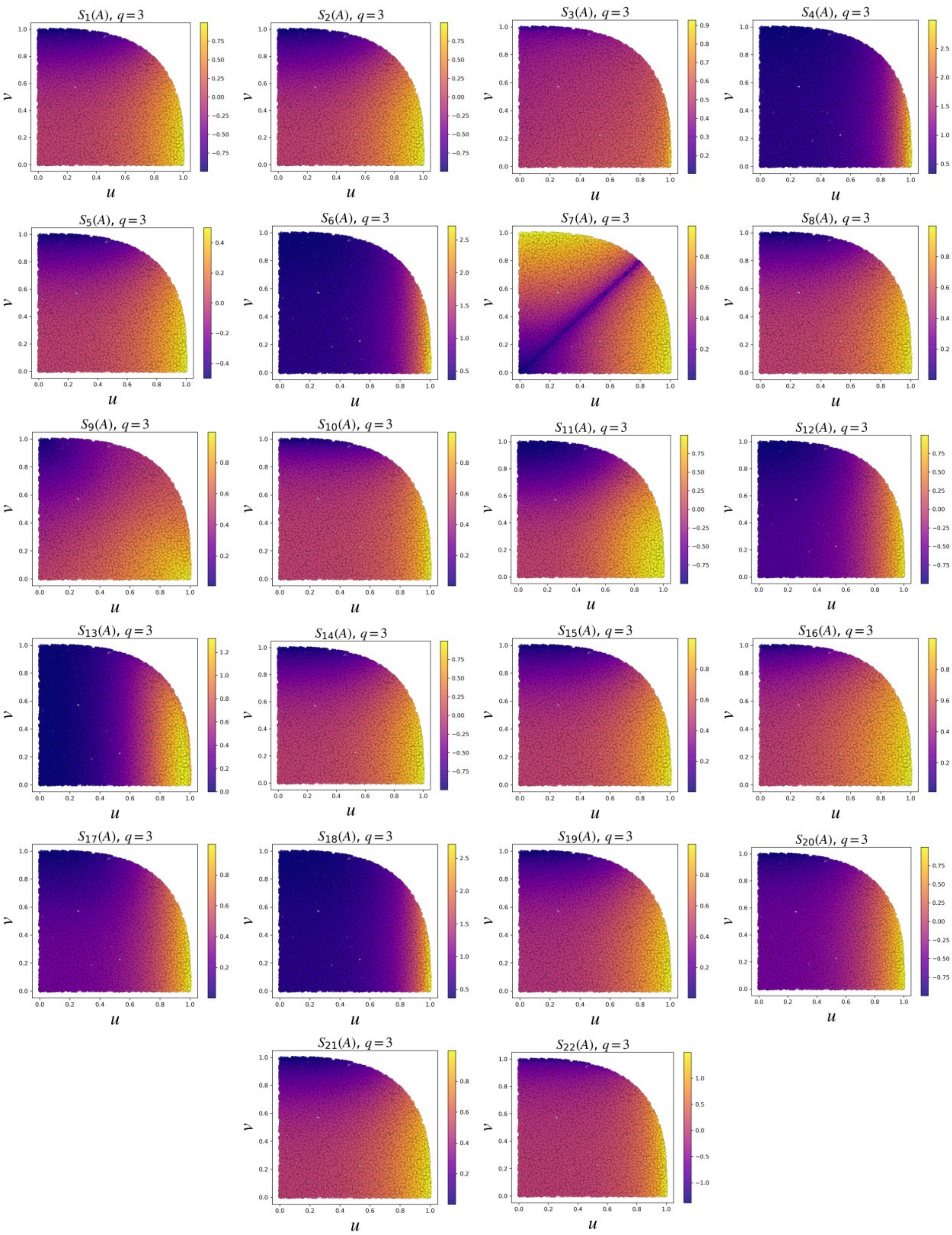


Figure 1. Distribution trend of scores for q-ROFNs.

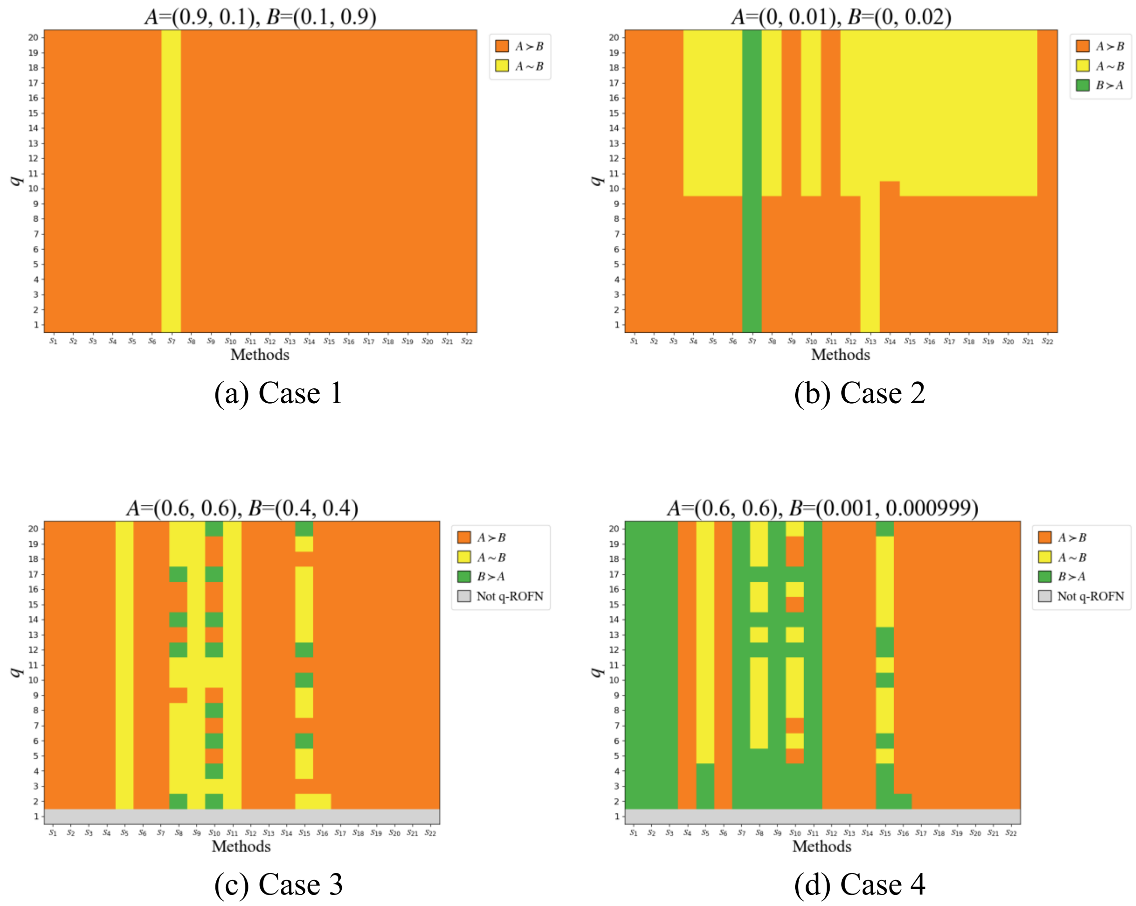


Figure 2. Comparison results between the proposed method and previous ranking methods.

The second experiment further incorporates the accuracy function to compare the rankings of q-ROFNs $A = (u_A, v_A)$ and $B = (u_B, v_B)$ under different q values. The parameter λ for S_5 and S_{15} is set to 0.5, and q is varied from 1 to 20 to examine the discriminative ability of different score functions as q changes. In Figure 2, orange represents $A \succ B$, yellow represents $A \sim B$, green represents $B \succ A$, and gray indicates that at least one of A or B does not satisfy the definition of a q-ROFN. From the cases in Figure 2, the following conclusion can be drawn:

(1) For case 1, A = (0.9, 0.1), B = (0.1, 0.9). A is clearly superior to B in both membership and non-membership degrees, representing a typical dominance relationship. According to the natural order of q-ROFNs, it should hold that $A \succ B$. However, S_7 yields $A \sim B$, indicating its insufficient ability to distinguish extremely contrasting information and thus leading to misjudgment.

(2) For case 2, A = (0, 0.01), B = (0, 0.02). With identical membership degrees, A has a lower non-membership degree and should be regarded as superior. Yet S_7 produces $B \succ A$, a directional error; S_{13} results in $A \sim B$, failing to capture the difference in non-membership degrees. Moreover, methods such as S_4, S_5, S_6, S_8, S_{10}, S_{12}, S_{14}, S_{15}, S_{16}, S_{17}, S_{18}, S_{19}, S_{20}, and S_{21} exhibit unstable ranking outcomes as q varies, alternating between $A \succ B$ and $A \sim B$. This indicates excessive sensitivity to the parameter q and a lack of robustness.

(3) For case 3, A = (0.6, 0.6), B = (0.4, 0.4). Both pairs lie on the diagonal, but A has higher membership and non-membership degrees, and its hesitancy $\pi_A = \sqrt[q]{1 - u_A^q - v_A^q}$ is significantly smaller than that of B, $\pi_B = \sqrt[q]{1 - u_B^q - v_B^q}$, implying that A carries richer information. Intuitively, $A \succ B$ should hold. However, methods such as S_5, S_9, and S_{11} consider them indifferent; for S_8, S_{10}, and S_{15}, as q varies, the ranking results even fluctuate among $A \succ B$, $A \sim B$, and $B \succ A$. Such inconsistency undermines the reliability of the ranking outcomes.

(4) For case 4, A = (0.6, 0.6), B = (0.001, 0.000999). A provides substantially more information than B, and it is generally expected that $A \succ B$. Nevertheless, methods such as S_1, S_2, S_3, S_7, S_9, and S_{11} yield $B \succ A$, a result that severely violates intuition. Furthermore, for S_5, S_8, S_{10}, and S_{15}, the ranking outcomes also oscillate among $A \succ B$, $A \sim B$, and $B \succ A$ as q varies, further confirming the lack of consistency and stability of these methods when dealing with extreme information.

According to Figures 1 and 2, combined with the score functions and accuracy functions listed in Table 2, it can be seen that the existing q-ROFN score functions S_1 to S_{21} generally suffer from various deficiencies. Specifically, methods S_1, S_7, S_8, and S_{10} are constructed only around $u^q - v^q$, completely ignoring the hesitation information, and cannot distinguish objects with the same score difference but different information certainty; moreover, S_7 exhibits obvious non-monotonic mutations and is overall sensitive to the parameter q, making it difficult to maintain a stable order relationship. S_2, S_6, S_{17}, S_{18}, and S_{21}, which introduce nonlinear transformations such as exponential and trigonometric functions, lack fuzzy semantic support and merely serve as pure mathematical fitting, which tend to cause excessive amplification of local differences, ranking oscillations, and uncontrollable value ranges. S_{13} and S_{14}, which take the hesitation degree as the dominant term, excessively weaken the core roles of membership and non-membership degrees, violate the basic logic of q-ROFN ranking, fail to identify subtle differences in non-membership degrees, and are prone to equivalent misjudgment. S_{12}, S_{19}, and S_{20}, which adopt multi-level nesting of logarithms, exponents, and fractions, are overly complex in structure, hard to theoretically prove their monotonicity with respect to membership and non-membership degrees, and prone to gradient explosion when variables approach 0, leading to reversal of extreme value ranking; meanwhile, their scores lack interpretability. S_5 and S_{15} with manually adjustable parameter λ rely entirely on artificial parameter settings, resulting in non-unique and non-objective ranking results that cannot maintain stable and consistent order relationships under different q values and different datasets. S_5, S_9, S_{11}, and S_{16}, which adopt symmetric or square structures, completely lose their discrimination ability in the diagonal region where $u \approx v$, cannot identify differences in information content, and even produce counter-intuitive ranking results, failing to satisfy the basic principle that higher information certainty corresponds to higher scores. In addition, functions such as S_4, S_6, and S_{18} have value ranges beyond the conventional interval, which may affect the stability of subsequent aggregation operations and decision-making results.

5. A novel q-ROF distance operator

The distance operator is a fundamental topic within the q-ROF framework, widely applied in pattern recognition, multi-attribute decision-making, and cluster analysis to quantify the discrepancy between two q-ROFSs. This section begins by systematically reviewing existing q-ROF distance operators (q-ROFDOs) reported in the literature. Subsequently, we propose a novel distance operator that fully accounts for the membership, non-membership, and hesitation degrees of q-ROFNs, as well as their inherent correlations. The proposed distance operator is rigorously proven to satisfy five fundamental axiomatic properties. Finally, a comparative analysis is conducted between our proposed distance operator and existing distance operators.

5.1. The novel q-ROF distance operator

This subsection proposes a novel distance operator for q-ROFSs, which combines the mathematical characteristics of harmonic analysis with the intuitive geometric interpretation of q-ROF information. This operator makes up for the deficiencies of traditional distance measures in depicting the similarity and difference relationships between q-ROFSs, and more accurately characterizes the distance attributes of fuzzy information in the q-ROF environment.

Definition 5.1. *Let* $M = \{\alpha_M(x_i) = (u_M(x_i), v_M(x_i)) \mid i = 1,2,\ldots,n\}$ *and* $N = \{\alpha_N(x_i) = (u_N(x_i), v_N(x_i)) \mid i = 1,2,\ldots,n\}$ *be two q-ROFSs defined on the universe of discourse* $X = \{x_1, x_2, \ldots, x_n\}$. *For each element* $x_i \in X$, *let* $\pi_M(x_i) = \left(1 - (u_M(x_i))^q - (v_M(x_i))^q\right)^{1/q}$ *and* $\pi_M(x_i) = \left(1 - (u_M(x_i))^q - (v_M(x_i))^q\right)^{1/q}$ *denote the hesitancy degrees of* $\alpha_M(x_i)$ *and* $\alpha_N(x_i)$, *respectively. The distance operator between q-ROFSs* M *and* N *is defined as:*

$$d(M,N) = 1 - \frac{4}{n}\sum_{i=1}^{n} \frac{\sqrt{u_M^q(x_i)u_N^q(x_i)} + \sqrt{v_M^q(x_i)v_N^q(x_i)} + \sqrt{\pi_M^q(x_i)\pi_N^q(x_i)}}{\left(\sqrt{u_M^q(x_i)} + \sqrt{u_N^q(x_i)}\right)^2 + \left(\sqrt{v_M^q(x_i)} + \sqrt{v_N^q(x_i)}\right)^2 + \left(\sqrt{\pi_M^q(x_i)} + \sqrt{\pi_N^q(x_i)}\right)^2}. \tag{4}$$

Definition 5.2. *Let* $\alpha_1 = (u_1, v_1)$ *and* $\alpha_2 = (u_2, v_2)$ *be two q-ROFNs. Let* $\pi_1 = \left(1 - u_1^q - v_1^q\right)^{1/q}$ *and* $\pi_2 = \left(1 - u_2^q - v_2^q\right)^{1/q}$ *denote the hesitancy degrees of* α_1 *and* α_2, *respectively. The distance between* α_1 *and* α_2 *is defined as:*

$$d(\alpha_1, \alpha_2) = 1 - \frac{4\left(\sqrt{u_1^q u_2^q} + \sqrt{v_1^q v_2^q} + \sqrt{\pi_1^q \pi_2^q}\right)}{\left(\sqrt{u_1^q} + \sqrt{u_2^q}\right)^2 + \left(\sqrt{v_1^q} + \sqrt{v_2^q}\right)^2 + \left(\sqrt{\pi_1^q} + \sqrt{\pi_2^q}\right)^2}. \tag{5}$$

Theorem 5.1. *Let* $X = \{x_1,\ x_2,\ \ldots,\ x_n\}$ *be a finite universe of discourse. Let* $M,\ N,$ *and* P *be arbitrary q-ROFSs on* X, *then the distance operator* $d(M,N)$ *satisfies the following five properties:*
(1) Boundedness: $d(M,N) \in [0,1]$.
(2) Symmetry: $d(M,N) = d(N,M)$.
(3) Triangle inequality: $d(M,N) \leq d(M,P) + d(P,N)$.
(4) Monotonicity: if $M \subseteq N \subseteq P$, *then* $d(M,P) \geq d(M,N)$ *and* $d(M,P) \geq d(N,P)$.
(5) Identity: $d(M,N) = 0$ *if and only if* $M = N$.

Proof. The complete proof is provided in the appendix.

5.2. A brief comparison between the proposed distance operator and previous distance operators

To fully verify the effectiveness and superiority of the q-rung orthopair fuzzy distance operator proposed in this paper, a systematic comparative experiment is carried out with existing mainstream q-ROF distance operators under two representative application scenarios of q-ROFSs, namely "the membership degree micro-disturbance scenario" (named "Scenario 1") and "the set cardinality variation scenario" (named "Scenario 2"), with the results illustrated in Figures 3 and 4, respectively. For convenience, the proposed distance operator is denoted as d_{32}, and all comparative experiments adopt standardized equal weights to ensure fairness and objectivity.

In Scenario 1, two q-ROFSs $M_1 = \{(0.2,0.1),(0.3,0.2),(0.4,0.3)\}$ and $N_1 = \{(0.15,0.1),(0.25,0.2),(0.35,0.3)\}$ are constructed. N_1 has a strict micro-disturbance correspondence with M_1, both contain exactly the same number of elements, the non-membership degrees of all elements remain unchanged, and only a small uniform increment of 0.05 is applied to the membership degrees. This design simulates weak fluctuations of characteristic parameters in practical decision-making and is used to test the sensitivity of each operator in identifying minor differences. The equal weight vector of elements in this scenario is $\omega_{M_1} = \omega_{N_1} = \{\frac{1}{3},\frac{1}{3},\frac{1}{3}\}$.

In Scenario 2, the sets are extended on the basis of Scenario 1 to construct large-cardinality q-ROFSs M_2 and N_2, where

$$M_2 = \{(0.2,0.1),(0.3,0.2),(0.4,0.3),(0.5,0.4),(0.6,0.5),(0.7,0.6)\},$$

$$N_2 = \{(0.15,0.1),(0.25,0.2),(0.35,0.3),(0.45,0.4),(0.55,0.5),(0.65,0.6)\}.$$

M_2 is a direct extended set of M_1, which completely retains all three elements of M_1 and adds three new elements following the same numerical rule. Similarly, N_2 maintains a micro-disturbance correspondence with M_2, their non-membership degrees are identical, and the membership degrees differ only by 0.05. Meanwhile, N_2 is also a direct extended set of N_1, following exactly the same micro-disturbance rule as in Scenario 1. The equal weight vector in this scenario is $\omega_{M_2} = \omega_{N_2} = \{\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6}\}$.

To ensure the standardization and reliability of the comparative experiment, parameters of all existing contrast operators are set to commonly used values. The parameter p was set to 3 for d_1 and d_{19}; both parameters $\bar{\alpha}$ and $\bar{\beta}$ were set to 0.5 for d_8 and d_9; for d_{14}, p was set to 3 and t_1, t_2 were both set to 2; and the parameter q of q-ROFSs takes typical values of 2, 3, 5, 8, and 13.

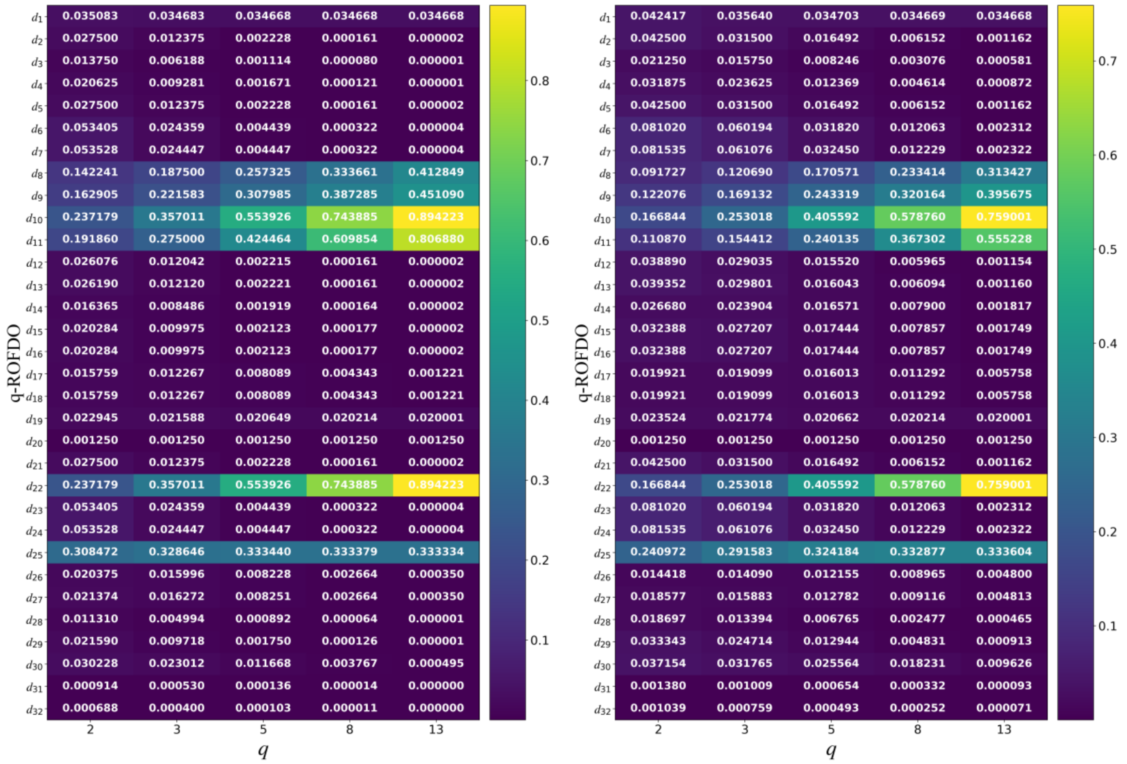
The experimental results in Figures 3 and 4 clearly show that the proposed q-ROF distance operator outperforms all the compared existing q-ROFDOs in both scenarios.

(1) The distance values calculated by the proposed q-ROF distance operator are the minimum among all q-ROFDOs under both scenarios, which indicates that the proposed operator can accurately capture the slight characteristic differences between q-ROFSs and has a more sensitive response to membership degree micro-disturbances.

(2) The proposed operator remains stable with the change of the core parameter q of q-ROFSs: when q takes typical values of 2, 3, 5, 8, and 13, the proposed operator still maintains the minimum distance value in all comparative operators, which verifies the strong robustness of the proposed operator to the variation of q value.

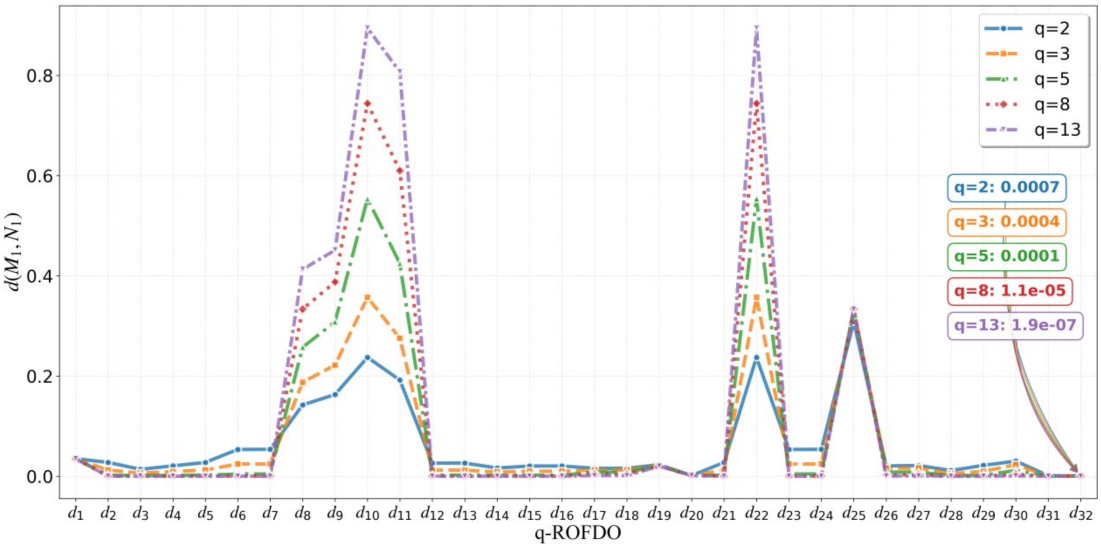
(3) The performance of the proposed operator is not affected by the expansion of the number of q-ROFS elements. It still keeps the optimal calculation effect when the number of elements is extended from 3 to 6 under the same micro-disturbance rule, reflecting the good scale adaptability of the proposed operator for q-ROFSs with different element scales.

Under the standardized equal weight and membership degree micro-disturbance experimental conditions, the proposed q-ROF distance operator shows comprehensive advantages in capturing micro differences, parameter robustness, and scale adaptability compared with the existing operators.

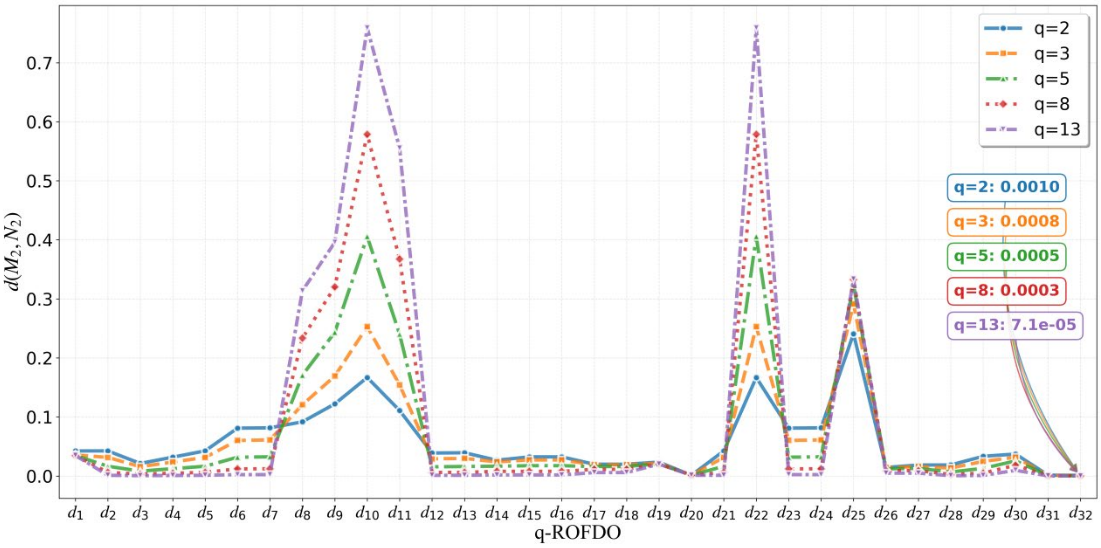


(*a*) (*b*)

Figure 3. Distance values of q-ROFDOs under micro-disturbance scenarios.



(*a*)



(*b*)

Figure 4. Distance value trends of q-ROFDOs with different q values.

6. The extended q-ROF REGIME method

Next, we elaborate on the extension of the classical REGIME method to the q-ROF environment. Given m alternatives A_i $(i = 1,2,\dots,m)$, n criteria C_i $(i = 1,2,\dots,n)$, and K experts $E_k(k = 1,2,\dots,K)$. Each expert E_k provides a decision matrix $D^{(k)} = [a_{ij}^{(k)}]_{m\times n}$ represented by q-ROFNs, where $a_{ij}^{(k)} = (u_{ij}^{(k)}, v_{ij}^{(k)})$ is a q-ROFN satisfying $u_{ij}^{(k)}, v_{ij}^{(k)} \in [0,1]$, $0 \le \left(u_{ij}^{(k)}\right)^q + \left(v_{ij}^{(k)}\right)^q \le 1$, and the hesitancy degree is $\pi_{ij}^{(k)} = \left(1 - \left(u_{ij}^{(k)}\right)^q - \left(v_{ij}^{(k)}\right)^q\right)^{\frac{1}{q}}$. To make full use of the above q-rung orthopair fuzzy evaluation information and ensure reasonable and objective decision results, we propose the detailed calculation steps of the extended q-rung orthopair fuzzy REGIME (q-ROF-REGIME) method as follows:

Step 1. Determine expert weights based on the evaluation data of alternatives provided by experts. Specifically, the lower the hesitancy degree of the evaluation data given by an expert, the more certain their evaluation of the alternatives, and the higher their weight. Calculate the average information content of each expert using the following formula:

$$IC_k = \frac{1}{m \cdot n} \sum_{i=1}^{m} \sum_{j=1}^{n} \left[1 - \left(\pi_{ij}^{(k)}\right)^q\right]. \tag{6}$$

Then, normalize to obtain the expert weights:

$$w_k^E = \frac{IC_k}{\sum_{k=1}^{K} IC_k}. \tag{7}$$

Step 2. Use the q-ROF weighted average operator [12] to aggregate the decision matrices of multiple experts to obtain the aggregated decision matrix $D = [a_{ij}]_{m \times n}$, where:

$$a_{ij} = (u_{ij}, v_{ij}) = \left(\left(1 - \prod_{k=1}^{K} \left(1 - \left(u_{ij}^{(k)}\right)^q\right)^{w_k^E}\right)^{\frac{1}{q}}, \prod_{k=1}^{K} \left(v_{ij}^{(k)}\right)^{w_k^E}\right). \tag{8}$$

Step 3. Calculate criterion weights based on the entropy weight method. Criteria with greater differences in evaluation values have stronger discrimination ability and should be assigned higher weights. First, calculate the q-ROF entropy Ent_j of each criterion:

$$Ent_j = \frac{1}{m} \sum_{i=1}^{m} \left[1 - \left|\left(u_{ij}\right)^q - \left(v_{ij}\right)^q\right| \cdot \left(1 - \left(\pi_{ij}\right)^q\right)\right], \ (\pi_{ij})^q = 1 - (u_{ij})^q - (v_{ij})^q. \tag{9}$$

Then calculate the degree of difference:

$$diff_j = 1 - Ent_j. \tag{10}$$

Finally, normalize the degree of difference of each criterion to obtain the criterion weights:

$$w_j = \frac{diff_j}{\sum_{j=1}^{n} diff_j}, \tag{11}$$

Step 4. Let I_b be the set of benefit criteria and I_c be the set of cost criteria. Standardize the decision matrix $D = [a_{ij}]_{m \times n}$ to obtain the matrix $R = [r_{ij}]_{m \times n}$, where:

$$r_{ij} = (\tilde{u}_{ij}, \tilde{v}_{ij}) = \begin{cases} (u_{ij}, v_{ij}), & j \in I_b \\ (v_{ij}, u_{ij}), & j \in I_c \end{cases}. \tag{12}$$

Step 5. Calculate the weighted scores of q-ROFNs. To compare the magnitudes of q-ROFNs, use the score function proposed in this paper:

$$S(r_{ij}) = \left[\frac{(\tilde{u}_{ij})^q}{1+(\tilde{\pi}_{ij})^q} - \frac{(\tilde{v}_{ij})^q}{1+\tilde{\pi}_{ij}}\right] \times \left(1 + \frac{1-(\tilde{\pi}_{ij})^q}{2}\right), \tag{13}$$

where $\tilde{\pi}_{ij} = \left(1 - (\tilde{u}_{ij})^q - (\tilde{v}_{ij})^q\right)^{\frac{1}{q}}$.

Calculate the weighted score matrix $Y = [y_{ij}]_{m\times n}$ of the evaluation values, where:

$$y_{ij} = w_j \cdot S(r_{ij}). \tag{14}$$

Step 6. For each pair of alternatives (A_s, A_t), compare their performance under each criterion. Define the preference indicator function:

$$\delta_{stj} = \begin{cases} 1, & y_{sj} > y_{tj} \\ 0, & y_{sj} = y_{tj}. \\ -1, & y_{sj} < y_{tj} \end{cases} \tag{15}$$

On this basis, construct the preference matrix $\boldsymbol{G} = [G_{st}]_{m\times m}$, where

$$G_{st} = \sum_{j=1}^{n} w_j \cdot \delta_{stj} \cdot d(r_{sj}, r_{tj}) \tag{16}$$

represents the preference intensity of alternative A_s over A_t. The distance formula proposed in this paper to calculate the distance between r_{sj} and r_{tj} is:

$$d(r_{sj}, r_{tj}) = 1 - \frac{4\left(\sqrt{\tilde{u}_{sj}^q \tilde{u}_{tj}^q} + \sqrt{\tilde{v}_{sj}^q \tilde{v}_{tj}^q} + \sqrt{\tilde{\pi}_{sj}^q \tilde{\pi}_{tj}^q}\right)}{\left(\sqrt{\tilde{u}_{sj}^q} + \sqrt{\tilde{u}_{tj}^q}\right)^2 + \left(\sqrt{\tilde{v}_{sj}^q} + \sqrt{\tilde{v}_{tj}^q}\right)^2 + \left(\sqrt{\tilde{\pi}_{sj}^q} + \sqrt{\tilde{\pi}_{tj}^q}\right)^2}. \tag{17}$$

It is worth noting that $G_{ss} = 0$ holds for all $s = 1, \ldots, m$.

Step 7. For each alternative A_s, calculate its net preference flow Φ_s:

$$\Phi_s = \frac{1}{m-1}\left(\sum_{t=1}^{m} G_{st} - \sum_{t=1}^{m} G_{ts}\right). \tag{18}$$

Step 8. Sort all alternatives in descending order of Φ_s, $s = 1, \ldots, m$ to obtain the ranking list of the alternatives.

The overall flowchart of the proposed q-ROF-REGIME model consists of four main phases to fulfill systematic decision analysis:

Phase 1: Define alternatives and criteria. Identify decision alternatives, evaluation criteria, and classify criteria into benefit and cost types.

Phase 2: Determine weighting coefficients. Compute expert weights using information content (Eqs (6) and (7)). Aggregate expert decision matrices (Eq (8)). Calculate criterion weights using the q-ROF entropy weight method (Eqs (9)–(11)).

Phase 3: Apply the q-ROF-REGIME method. Standardize the decision matrix (Eq (12)). Compute score values and weighted scores (Eqs (13) and (14)). Construct a preference matrix and obtain net preference flow (Eqs (15)–(18)).

Phase 4: Validate the model. Rank alternatives, conduct sensitivity analysis, and compare with existing methods to verify robustness and effectiveness.

The complete framework is shown in Figure 5.

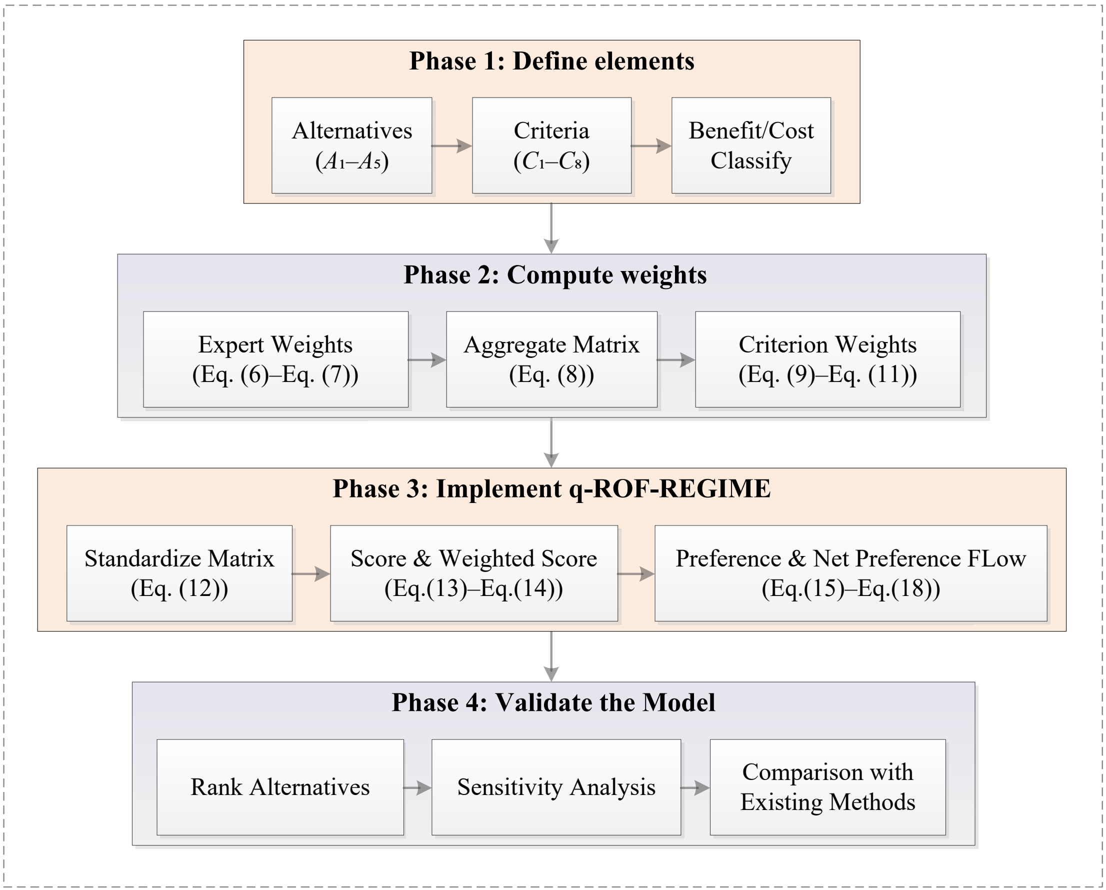


Figure 5. Framework of the q-ROF-REGIME method.

To facilitate the implementation and computational application of the extended q-rung orthopair fuzzy REGIME method, the complete decision-making procedure described in Steps 1–8 can be structured into a concise algorithmic form. The pseudo-code of the proposed method, denoted as Algorithm 1, is presented as follows:

Algorithm 1. q-ROF-REGIME method.

1: Input the q-ROF decision matrices $D^{(k)} = [a_{ij}^{(k)}]_{m\times n}$ provided by K experts.

2: Calculate expert weights w_k^E based on information content using Eq (6) and normalize by Eq (7).
3: Aggregate the individual decision matrices into a collective decision matrix $D = [a_{ij}]_{m\times n}$ using the q-ROF weighted average operator by Eq (8).
4: Compute criterion weights w_j using the entropy weight method, calculate q-ROF entropy Ent_j by Eq (9), degree of difference $diff_j$ by Eq (10), and normalize by Eq (11).
5: Transform the matrix $D = [a_{ij}]_{m\times n}$ into a normalized matrix $R = [r_{ij}]_{m\times n}$ by Eq (12), where benefit criteria remain unchanged and cost criteria have membership and non-membership swapped.
6: Calculate the score matrix $S = [S(r_{ij})]_{m\times n}$ of $R = [r_{ij}]_{m\times n}$ by the score function in Eq (13).
7: Compute the weighted score matrix $Y = [y_{ij}]_{m\times n}$ by multiplying criterion weights with scores using Eq (14).
8: Construct the preference matrix $\boldsymbol{G} = [G_{st}]_{m\times m}$ by Eqs (15)–(17).
9: Calculate the net preference flow of each alternative Φ_s by Eq (18).
10: Rank the alternatives according to the decreasing values of net preference flow Φ_s.

7. Case study

Esophageal cancer is a malignant tumor of the digestive tract with a high incidence in China. Early detection, early diagnosis, and early treatment are critical to improving patient survival rates. In this section, the proposed q-ROF-REGIME model is used to conduct a case study on the selection of early esophageal cancer screening strategies. Subsequently, a sensitivity analysis is performed to verify its robustness. Finally, by comparing it with other existing multi-criteria decision-making methods, the feasibility and effectiveness of the proposed model are demonstrated.

7.1. Background description

Early detection and early treatment are key to improving the survival rate of patients with esophageal cancer. However, the current diagnostic rate of early esophageal cancer in China remains relatively low. Based on this, this study constructed a multi-criteria decision-making model that includes five screening strategies and eight evaluation criteria, aiming to provide a scientific basis for medical institutions at all levels to select appropriate screening strategies for early upper gastrointestinal cancer. To scientifically screen for the optimal early esophageal cancer screening strategy, this study established a decision-making team consisting of three experts (E_1, E_2, E_3) to select from five candidate strategies, named Endoscopic examination (A_1), Endoscopy combined with narrow-band imaging (A_2), Artificial intelligence-assisted endoscopic screening system (A_3), Serum tumor marker testing combined with endoscopic screening (A_4), and Endoscopy combined with chromoendoscopy (A_5). To ensure the comprehensiveness and objectivity of the evaluation, the decision-making team constructed an indicator system consisting of eight evaluation criteria, named Sensitivity (C_1), Specificity (C_2), Examination cost (C_3), Operational complexity (C_4), Patient tolerance (C_5), False positive rate (C_6), Examination time (C_7), and Accessibility in primary hospitals (C_8).

7.2. Initial assessment information

Based on the five screening strategies and eight evaluation criteria established in Section 7.1, three digestive endoscopy experts (E_1, E_2, E_3) independently evaluated the performance of each strategy against each criterion using q-ROFNs, thereby constructing three initial decision matrices $D^{(k)} = [a_{ij}^{(k)}]_{m\times n}$ ($k = 1,2,3$; $m = 5$; $n = 8$). Each expert assigned values for satisfaction degree (membership u) and dissatisfaction degree (non-membership v) based on their clinical experience and professional knowledge, ensuring that $u^q + v^q \leq 1$ with $q = 3$, adopted in this paper. On this basis, the extended q-ROF REGIME method elaborated in Section 6 is utilized to process the following evaluation information and derive the optimal screening strategy.

$$D^{(2)} = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{array} \begin{array}{c} \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \end{array} \\ \begin{pmatrix} (0.73, 0.34) & (0.84, 0.18) & (0.43, 0.64) & (0.58, 0.50) & (0.70, 0.26) \\ (0.76, 0.32) & (0.83, 0.26) & (0.46, 0.62) & (0.60, 0.48) & (0.68, 0.28) \\ (0.63, 0.45) & (0.34, 0.72) & (0.33, 0.74) & (0.50, 0.58) & (0.38, 0.70) \\ (0.60, 0.48) & (0.56, 0.58) & (0.36, 0.72) & (0.53, 0.55) & (0.34, 0.63) \\ (0.66, 0.42) & (0.49, 0.64) & (0.38, 0.70) & (0.56, 0.52) & (0.30, 0.66) \\ (0.68, 0.40) & (0.37, 0.66) & (0.40, 0.68) & (0.54, 0.54) & (0.32, 0.64) \\ (0.70, 0.38) & (0.65, 0.60) & (0.42, 0.66) & (0.52, 0.56) & (0.35, 0.72) \\ (0.71, 0.36) & (0.88, 0.15) & (0.44, 0.64) & (0.57, 0.51) & (0.72, 0.24) \end{pmatrix}^T \end{array}.$$

$$D^{(3)} = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{array} \begin{array}{c} \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \end{array} \\ \begin{pmatrix} (0.74, 0.33) & (0.71, 0.32) & (0.44, 0.63) & (0.59, 0.49) & (0.71, 0.45) \\ (0.77, 0.31) & (0.82, 0.28) & (0.47, 0.61) & (0.61, 0.47) & (0.69, 0.27) \\ (0.64, 0.44) & (0.63, 0.73) & (0.34, 0.73) & (0.51, 0.57) & (0.46, 0.61) \\ (0.61, 0.47) & (0.55, 0.60) & (0.37, 0.71) & (0.54, 0.54) & (0.33, 0.64) \\ (0.67, 0.41) & (0.68, 0.45) & (0.39, 0.69) & (0.57, 0.51) & (0.39, 0.67) \\ (0.69, 0.39) & (0.46, 0.58) & (0.41, 0.67) & (0.55, 0.53) & (0.31, 0.65) \\ (0.71, 0.36) & (0.84, 0.22) & (0.43, 0.65) & (0.53, 0.55) & (0.34, 0.63) \\ (0.72, 0.35) & (0.85, 0.26) & (0.45, 0.62) & (0.58, 0.50) & (0.73, 0.43) \end{pmatrix}^T \end{array}.$$

The average information content IC_k of each expert can be calculated by using Eq (6), and the expert weights w_k^E are obtained by normalizing IC_k using Eq (7).

Taking expert E_1 as an example, the average information content is computed as $IC_1 = 0.3524$ by Eq (6); similarly, $IC_2 = 0.3293$ and $IC_3 = 0.3317$ are obtained for E_2 and E_3, respectively. Then the expert weights are calculated as $w_1^E = \frac{IC_1}{(IC_1+IC_2+IC_3)} = 0.3476$, $w_2^E = 0.3251$, $w_3^E = 0.3273$ by Eq (7).

Based on the obtained expert weights, the individual decision matrices are aggregated into a collective decision matrix $D = [a_{ij}]_{m\times n}$ by using Eq (8).

Taking the element corresponding to A_1 and C_1 as an example, the aggregated q-ROFN is calculated as (0.7404, 0.3297) via Eq (8). The complete aggregated decision matrix D is shown below.

$$D = \begin{array}{cc} & \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \end{array} \\ \begin{array}{c} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{array} & \begin{pmatrix} (0.7404, 0.3297) & (0.8036, 0.1887) & (0.4404, 0.6297) & (0.5904, 0.4897) & (0.6794, 0.3026) \\ (0.7704, 0.3097) & (0.8200, 0.2344) & (0.4704, 0.6097) & (0.6104, 0.4697) & (0.6904, 0.2697) \\ (0.6404, 0.4397) & (0.5005, 0.7302) & (0.3404, 0.7297) & (0.5104, 0.5662) & (0.4019, 0.6415) \\ (0.6104, 0.4661) & (0.5533, 0.6261) & (0.3704, 0.7097) & (0.5404, 0.5362) & (0.3300, 0.6402) \\ (0.6704, 0.4097) & (0.5342, 0.6371) & (0.3904, 0.6897) & (0.5704, 0.5097) & (0.3301, 0.6702) \\ (0.6904, 0.3897) & (0.3711, 0.6936) & (0.4104, 0.6662) & (0.5504, 0.5297) & (0.3100, 0.6502) \\ (0.7104, 0.3628) & (0.7982\ 0.4603) & (0.4304, 0.6462) & (0.5304, 0.5497) & (0.3400, 0.6958) \\ (0.7204, 0.3497) & (0.8606, 0.2174) & (0.4504, 0.6229) & (0.5804, 0.4997) & (0.7304, 0.2818) \end{pmatrix}^T \end{array}.$$

Then the criterion weights are determined using the entropy weight method. Specifically, the q-ROF entropy Ent_j of each criterion is calculated by Eq (9), the degree of divergence $diff_j$ is obtained by Eq (10), and the final weights w_j are normalized by Eq (11).

Taking criterion C_1 as an example, the q-ROF entropy is $Ent_1 = 0.8312$ by Eq (9), the divergence degree is $diff_1 = 0.1688$ by Eq (10), and the criterion weight is $w_1 = 0.1514$ by Eq (11). The complete criterion weight vector is obtained as follows.

$$w = (w_1, w_2,\dots,w_8)^T = (0.1514, 0.1730, 0.1055, 0.0658, 0.0839, 0.0929, 0.1379, 0.1895)^T.$$

Afterward, the matrix $D = [a_{ij}]_{m\times n}$ is transformed into a normalized matrix $R = [r_{ij}]_{m\times n}$ by using Eq (12), where benefit criteria C_1, C_2, and C_8 (i.e., $I_b = \{1,2,8\}$) remain unchanged, and cost criteria C_3 – C_7 (i.e., $I_c = \{3,4,5,6,7\}$) have membership and non-membership swapped.

Taking criterion C_3 and alternative A_1 as an example, the normalized value is changed from (0.6404, 0.4397) to (0.4397, 0.6404) by Eq (12). The complete normalized matrix R is shown below.

$$R = \begin{array}{cc} & \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \end{array} \\ \begin{array}{c} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{array} & \begin{pmatrix} (0.7404, 0.3297) & (0.8036, 0.1887) & (0.4404, 0.6297) & (0.5904, 0.4897) & (0.6794, 0.3026) \\ (0.7704, 0.3097) & (0.8200, 0.2344) & (0.4704, 0.6097) & (0.6104, 0.4697) & (0.6904, 0.2697) \\ (0.4397, 0.6404) & (0.7302, 0.5005) & (0.7297, 0.3404) & (0.5662, 0.5104) & (0.6415, 0.4019) \\ (0.4661, 0.6104) & (0.6261, 0.5533) & (0.7097, 0.3704) & (0.5362, 0.5404) & (0.6402, 0.3300) \\ (0.4097, 0.6704) & (0.6371, 0.5342) & (0.6897, 0.3904) & (0.5097, 0.5704) & (0.6702, 0.3301) \\ (0.3897, 0.6904) & (0.6936, 0.3711) & (0.6662, 0.4104) & (0.5297, 0.5504) & (0.6502, 0.3100) \\ (0.3628, 0.7104) & (0.4603, 0.7982) & (0.6462, 0.4304) & (0.5497, 0.5304) & (0.6958, 0.3400) \\ (0.7204, 0.3497) & (0.8606, 0.2174) & (0.4504, 0.6229) & (0.5804, 0.4997) & (0.7304, 0.2818) \end{pmatrix}^T \end{array}.$$

The score matrix $S = [S(r_{ij})]_{m\times n}$ of $R = [r_{ij}]_{m\times n}$ is then obtained by applying Eq (13).

Taking the element of A_1 under C_1 as an example, the score value is calculated as $S(r_{11}) = 0.29401609$ by Eq (13). The complete score matrix S is shown below.

$$
S = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{array}
\begin{array}{c}
\begin{array}{ccccc} \quad A_1 \quad & \quad A_2 \quad & \quad A_3 \quad & \quad A_4 \quad & \quad A_5 \quad \end{array} \\
\begin{pmatrix}
0.29401609 & 0.43976924 & -0.09578054 & 0.06991462 & 0.20392627 \\
0.35531386 & 0.48300338 & -0.06823973 & 0.09438053 & 0.22168513 \\
-0.10469713 & 0.24127468 & 0.27392994 & 0.04282918 & 0.14369323 \\
-0.07061389 & 0.07555589 & 0.23719187 & 0.00865861 & 0.15530097 \\
-0.14224351 & 0.09617552 & 0.20452469 & -0.02313711 & 0.18903726 \\
-0.1686884 & 0.21345596 & 0.16982436 & -0.00045022 & 0.16860495 \\
-0.1983084 & -0.2912594 & 0.14186157 & 0.02234252 & 0.22129787 \\
0.25716597 & 0.61609179 & -0.0865343 & 0.05799241 & 0.28125455
\end{pmatrix}^T
\end{array} .
$$

The weighted score matrix $Y = [y_{ij}]_{m\times n}$ is then computed by multiplying the criterion weights with the scores via Eq (14).

Taking $y_{11} = w_1 \cdot S(r_{11})$ as an example, the weighted score is $y_{11} = 0.1514 \times 0.29401609 = 0.04452350$ by Eq (14). The complete weighted score matrix Y is shown below.

$$
Y = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{array}
\begin{array}{c}
\begin{array}{ccccc} \quad A_1 \quad & \quad A_2 \quad & \quad A_3 \quad & \quad A_4 \quad & \quad A_5 \quad \end{array} \\
\begin{pmatrix}
0.04452350 & 0.06659522 & -0.01450426 & 0.01058732 & 0.03088100 \\
0.06148397 & 0.08357953 & -0.01180829 & 0.01633173 & 0.03836068 \\
-0.01104755 & 0.02545910 & 0.02890485 & 0.00451930 & 0.01516239 \\
-0.00464896 & 0.00497432 & 0.01561583 & 0.00057005 & 0.01022444 \\
-0.01192884 & 0.00806548 & 0.01715187 & -0.00194033 & 0.01585306 \\
-0.01566785 & 0.01982588 & 0.01577336 & -0.00004182 & 0.01566010 \\
-0.02735182 & -0.04017216 & 0.01956636 & 0.00308161 & 0.03052266 \\
0.04873370 & 0.11675118 & -0.01639850 & 0.01098973 & 0.05329855
\end{pmatrix}^T
\end{array} .
$$

The preference matrix $\boldsymbol{G} = [G_{st}]_{m\times m}$ is obtained by applying Eqs (15)–(17).

Taking the pair (A_1, A_2) as an example, the preference indicator δ_{12j} is determined by Eq (15), the distance $d(r_{1j}, r_{2j})$ is calculated by Eq (16), and then $G_{12} = \sum_{j=1}^{8} w_j \cdot \delta_{12j} \cdot d(r_{1j}, r_{2j}) = -0.0157$ is obtained by Eq (17). The complete preference matrix $\boldsymbol{G}$ is shown below.

$$
\boldsymbol{G} = \begin{pmatrix}
0 & -0.0157 & 0.0041 & 0.0023 & -0.0256 \\
0.0157 & 0 & 0.0436 & 0.0231 & -0.0074 \\
-0.0041 & -0.0436 & 0 & 0.0010 & -0.0257 \\
-0.0023 & -0.0231 & -0.0010 & 0 & -0.0066 \\
0.0256 & 0.0074 & 0.0257 & 0.0148 & 0
\end{pmatrix}
$$

For each alternative A_s, its net preference flow Φ_s is calculated via Eq (18).

Taking alternative A_2 as an example, the net preference flow is $\Phi_2 = [(G_{21} + G_{23} + G_{24} + G_{25}) - (G_{12} + G_{32} + G_{42} + G_{52})]/4 = 0.0376$ by Eq (18). The net preference flows of all alternatives are obtained as follows.

$$
\Phi_1 = \text{-}0.0175,\ \Phi_2 = 0.0376,\ \Phi_3 = \text{-}0.0363,\ \Phi_4 = \text{-}0.0206,\ \Phi_5 = \text{-}0.0368.
$$

Finally, the ranking of alternatives in descending order of net preference flow Φ_s yields the sequence $A_2 \succ A_5 \succ A_1 \succ A_4 \succ A_3$, indicating that the screening strategy "Endoscopy combined with chromoendoscopy" is the optimal alternative.

7.3. Sensitivity analysis

In the proposed q-ROF-REGIME model, the parameter q enables experts to flexibly express their preferences in different decision-making scenarios. In this paper, the value of parameter q is set to 3, and variations in q may exert a significant impact on the entire evaluation process. Accordingly, this section investigates the effects of different q values on the evaluation results. Expert E_1 provided a preference evaluation of (0.84 0.72) for alternative A_2 under criterion C_7, which requires the value of q to be no less than 3. This is because the preference value becomes invalid when $q = 1$ or $q = 2$. Specifically, for $q = 1$, the sum 0.84+0.72=1.56>1; for $q = 2$, the sum of squares $0.84^2 + 0.72^2 = 1.224 > 1$.

In the proposed q-ROF-REGIME model, the Φ value serves as the basis for ranking the alternatives. In this paper, 18 scenarios with q ranging from 3 to 20 are simulated to analyze the variation trend of the Φ values of different alternatives with the change in q, and the simulation results are presented in Figure 6.

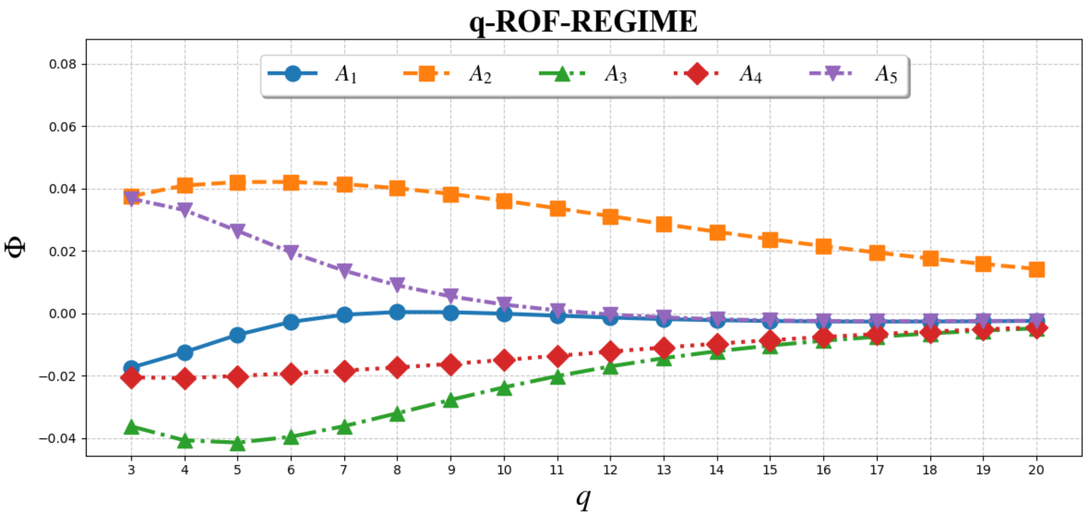


Figure 6. Variation trend of Φ values with q ranging from 3 to 20.

As can be seen from Figure 6, variations in parameter q affect the values Φ of the five esophageal cancer screening alternatives, yet such variations do not alter the ranking of the alternatives. The final ranking of the alternatives remains $A_2 \succ A_5 \succ A_1 \succ A_4 \succ A_3$ across all 18 scenarios, which verifies the robustness of the evaluation results obtained by the proposed model. Furthermore, 30 additional scenarios with q ranging from 3 to 50 are simulated to analyze the variation trend of the Φ values of different alternatives with the change in q, and the simulation results are shown in Figure 7.

It can be observed from Figure 7 that the Φ values of the alternatives change with the variation in q, while the final ranking of the alternatives remains $A_2 \succ A_5 \succ A_1 \succ A_4 \succ A_3$. Both the proposed score function and the distance operator in the q-ROF-REGIME model incorporate the power exponent q. As the value of q increases, the Φ values gradually decrease and approach *0*. However, slight differences among the Φ values of different alternatives still persist. This allows for the effective ranking of the alternatives based on these values. The final ranking results are presented in Figure 8, which further confirms that the ranking order remains $A_2 \succ A_5 \succ A_1 \succ A_4 \succ A_3$. Thus, the robustness of the proposed q-ROF-REGIME method is further verified.

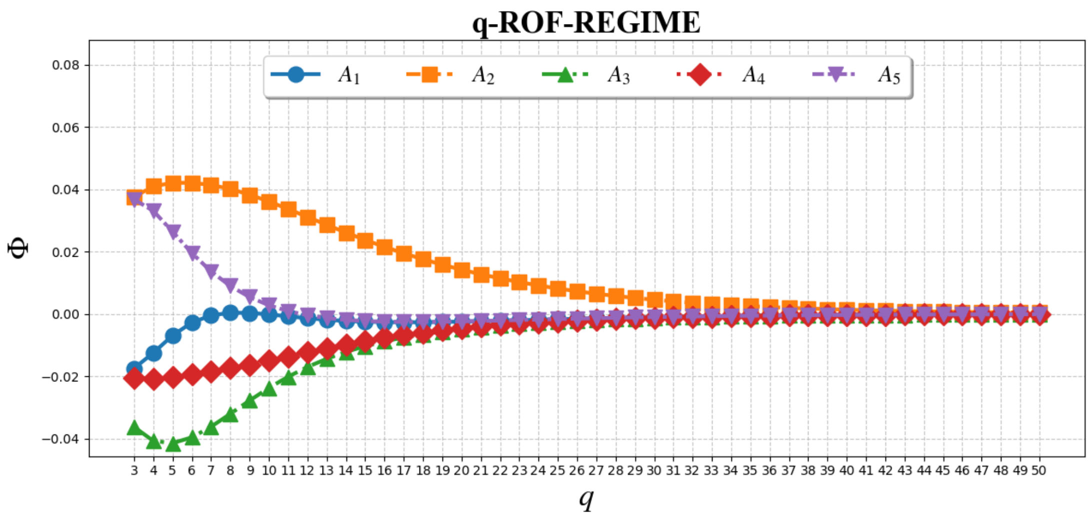


Figure 7. Variation trend of Φ values with q ranging from 3 to 50.

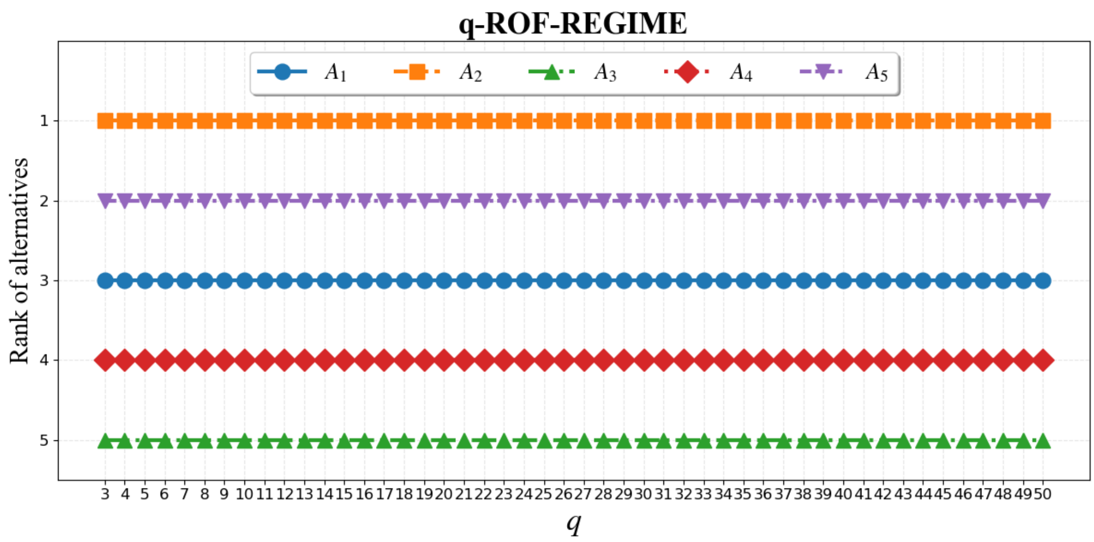


Figure 8. Ranking results of the proposed q-ROF-REGIME method with q ranging from 3 to 50.

7.4. Comparisons and discussions

To validate the effectiveness and superiority of the proposed q-ROF-REGIME method, this section presents a comparative analysis with several classical q-rung orthopair fuzzy multi-criteria group decision-making methods. Specifically, three methods, namely q-ROF-MABAC [55], q-ROF-WSM [24], and q-ROF-VIKOR [56], are selected for comparison. All these methods are tailored to solve decision-making problems under the q-ROF environment.

In the comparative experiment, the parameters of each method were set in accordance with the values specified in their original papers. Specifically, for q-ROF-MABAC, $q = 3$; for q-ROF-WSM, $q = 3$, $\lambda = 0.5$, $\gamma_1 = 0.7$, and $\gamma_2 = 0.3$; and for q-ROF-VIKOR, $q = 3$ and $\phi = 0.5$. Based on the initial evaluation data provided by three experts in early esophageal cancer diagnosis and treatment in Section 7.2, we applied the three comparative methods to rank the five screening strategies. The ranking results obtained by the q-ROF-MABAC method are $A_5 \succ A_2 \succ A_3 \succ A_4 \succ A_1$, by the q-ROF-WSM method are $A_2 \succ A_5 \succ A_1 \succ A_4 \succ A_3$, and by the q-ROF-VIKOR method are $A_5 \succ A_2 \succ A_3 \succ A_4 \succ A_1$. The ranking results obtained by the proposed q-ROF-REGIME method are $A_2 \succ A_5 \succ A_1 \succ A_4 \succ A_3$.

To compare these ranking orders by the q-ROF-MABAC method, q-ROF-WSM method, q-ROF-VIKOR method, and the proposed q-ROF-REGIME method, Spearman's rank-correlation test [57] is considered to estimate whether there is statistical significance of the ranking differences among them. During the process of Spearman's test, a rank-correlation coefficient η and a test statistic Z are defined to determine the similarity of the rankings between two sets of ranking $\{\chi^i\}$ and $\{\gamma^i\}$ for m alternatives, where

$$Z = \eta\sqrt{m-1},\ \eta = 1 - 6\sum_{i=1}^{m} \frac{(\rho^i)^2}{m(m^2-1)},\ \rho^i = \chi^i - \gamma^i,\ (i = 1,2,\ldots,m).$$

The closer η is to ± 1, the stronger the relationship between $\{\chi^i\}$ and $\{\gamma^i\}$. Especially, when the rank-correlation coefficient η varies to $+1$, it denotes a perfect positive relationship between $\{\chi^i\}$ and $\{\gamma^i\}$. If the relative measure η varies to -1, it implies a perfect negative relationship between $\{\chi^i\}$ and $\{\gamma^i\}$. In addition, the test statistic Z is utilized to compare with a pre-determined level of significance value τ. Usually, set $\tau = 0.05$, the critical Z value is 1.645, i.e., $Z_{0.05} = 1.96$. When Z exceeds 1.645, it can be derived that $\{\chi^i\}$ and $\{\gamma^i\}$ are similar. Otherwise, it can be considered as there is no evidence of a positive relationship between $\{\chi^i\}$ and $\{\gamma^i\}$.

When solving the above investment example, there are three sets of preference rankings obtained by the proposed method q-ROF-REGIME and methods q-ROF-MABAC, q-ROF-WSM, q-ROF-VIKOR, denoted by A, B, C, and D, respectively.To compare these ranking orders, the rank-correlation coefficients and the test statistics are shown in Table 4.

Table 4. Comparison of ranking results from four methods.

Alternatives	Ranking				Ranking difference		
	q-ROF-REGIME (A)	q-ROF-MABAC (B)	q-ROF-WSM (C)	q-ROF-VIKOR (D)	A-B	A-C	A-D
A_1	2	5	2	5	-3	0	-3
A_2	5	2	5	2	3	0	3
A_3	1	3	1	3	-2	0	-2
A_4	4	4	4	4	0	0	0
A_5	3	1	3	1	2	0	2
				Spearman's rank-correlation coefficient η	-0.3	1	-0.3
				Test value Z	-0.6	2	-0.6

As presented in Table 4, the ranking (A) of this study is identical to (C) with a perfect positive rank correlation. Meanwhile, weak negative correlations exist between (A) and (B), (A) and (D), and none of these correlations are statistically significant at the significance level of $\tau = 0.05$.

It is observed that Alternative A_4 is always ranked fourth across the four ranking results. The two pairs of alternatives with prominent ranking discrepancies are (A_2, A_5) and (A_1, A_3). In ranking (A) and ranking (C), A_2 ranks first and A_3 ranks last. By contrast, in ranking (B) and ranking (D), A_5 takes the first place, while A_1 is placed at the bottom. A detailed and in-depth analysis is presented as follows.

This evaluation involves five alternatives and eight evaluation criteria. Among them, C_1, C_2, and C_8 are benefit-type criteria, while C_3, C_4, C_5, C_6, and C_7 are cost-type criteria. Based on the evaluation data of q-ROFNs provided by three experts, a comparative analysis is conducted according to the two categories of criteria.

In terms of the three benefit-type criteria C_1, C_2, and C_8, the membership degrees of alternative A_2 range from 0.81 to 0.88, and the non-membership degrees are between 0.12 and 0.28. The high membership degrees combined with low non-membership degrees indicate an outstanding benefit performance. By contrast, the membership degrees of alternative A_5 vary from 0.62 to 0.71, and the non-membership degrees range from 0.24 to 0.45, revealing a markedly weaker benefit level than A_2. For the five cost-type criteria C_3, C_4, C_5, C_6, and C_7 , a lower membership degree and a higher non-membership degree represent a better performance in cost assessment. Alternative A_2 generally has lower membership degrees and higher non-membership degrees, showing prominent advantages in cost control. In comparison, A_5 presents relatively higher membership degrees and lower non-membership degrees, leading to an inferior cost control effect. Combined with the consistent evaluation tendencies of the three experts, it can be concluded that A_2 outperforms A_5 comprehensively in both benefit output and cost management, so A_2 achieves a better overall performance.

A comparative analysis between Alternative A_3 and A_1 is further carried out. In the aspect of benefit-type criteria, the membership degrees of A_1 are within 0.70–0.77, and the non-membership degrees are 0.31–0.38, which means its overall benefit is at a moderate level. The membership degrees of A_3 are only 0.43–0.47, while the non-membership degrees reach 0.61–0.64. The sharp drop in membership degrees and the remarkable rise in non-membership degrees demonstrate that A_3 is far less competitive in benefit output. In terms of cost-type criteria, the membership degrees of A_1 are 0.60–0.68, and the non-membership degrees are 0.40–0.48, suggesting its cost is moderately controllable. A_3 has low membership degrees and high non-membership degrees, which implies higher costs and potential risks as well as poor cost control capacity. The evaluation results from the three experts are highly consistent. Since A_3 lags behind A_1 across all benefit-type and cost-type criteria, its overall performance is significantly inferior to that of A_1.

To further investigate the ranking stability of each method under different q values, we conducted a sensitivity analysis with q ranging from 3 to 20. Figures 9–12 illustrate the variation trends of ranking results obtained by the proposed q-ROF-REGIME method and the three comparative methods, namely q-ROF-VIKOR, q-ROF-WSM, and q-ROF-MABAC, under different q values.

It can be observed from the figures that within the range of q from 3 to 20, the proposed q-ROF-REGIME method exhibits optimal stability, with the relative ranking positions of all alternatives remaining unchanged throughout. The q-ROF-VIKOR method demonstrates relatively stable rankings overall, with the relative positions of alternatives remaining largely unchanged. However, when $q > 12$, this method ranks alternative A_4 in fifth place, indicating a certain degree of sensitivity. The q-ROF-MABAC method exhibits more pronounced sensitivity during q value variations, showing a trend where the ranking position of alternative A_2 gradually declines as q increases.

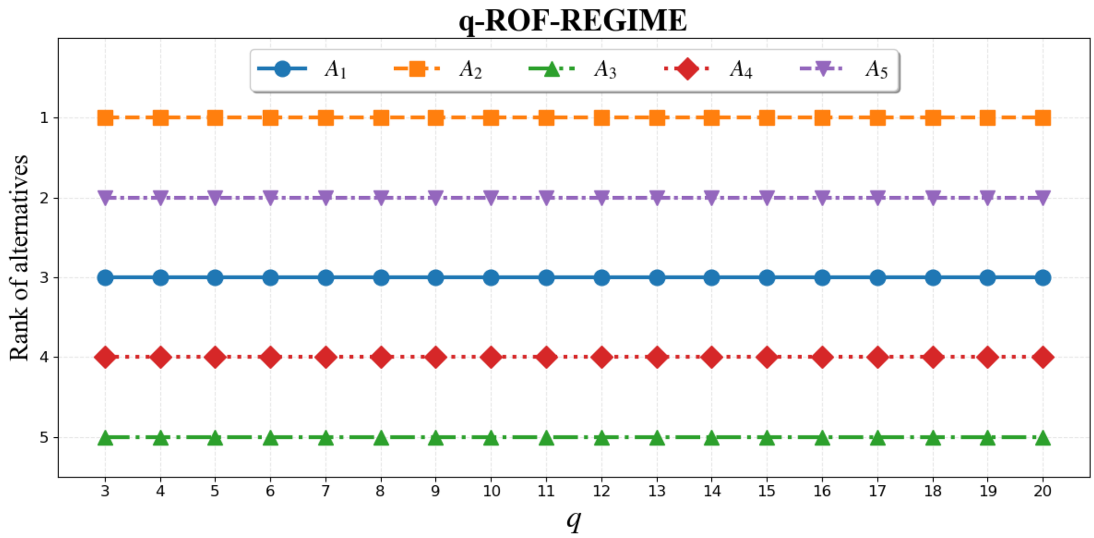


Figure 9. Ranking results of the proposed q-ROF-REGIME method with q ranging from 3 to 20.

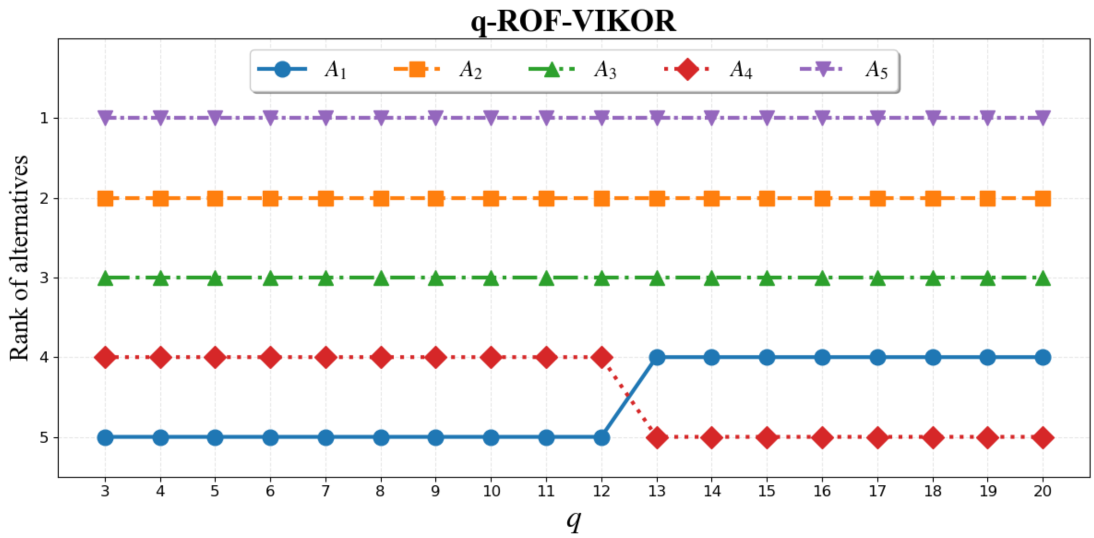


Figure 10. Ranking results of the q-ROF-VIKOR method with q ranging from 3 to 20.

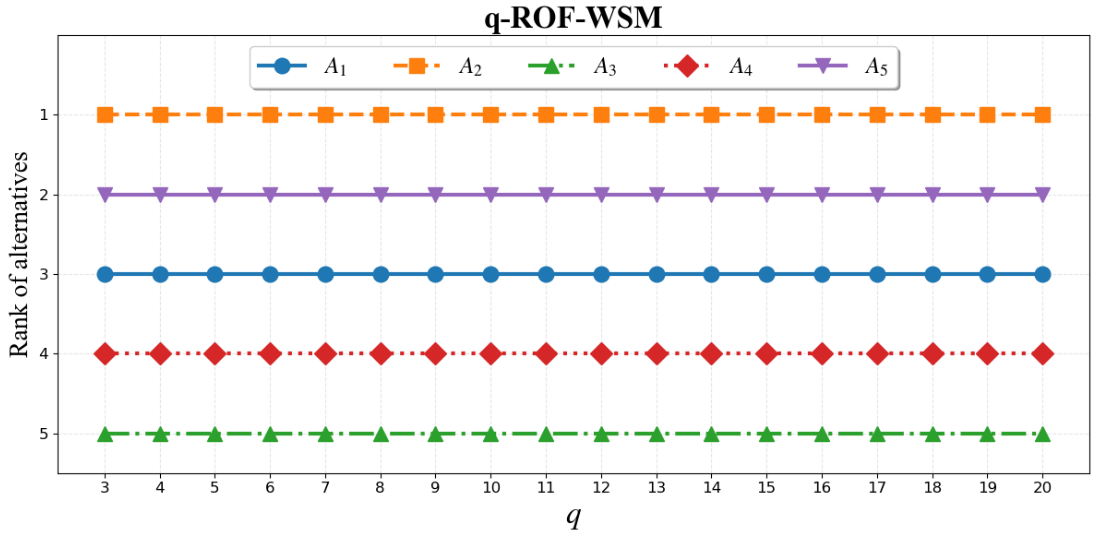


Figure 11. Ranking results of the q-ROF-WSM method with q ranging from 3 to 20.

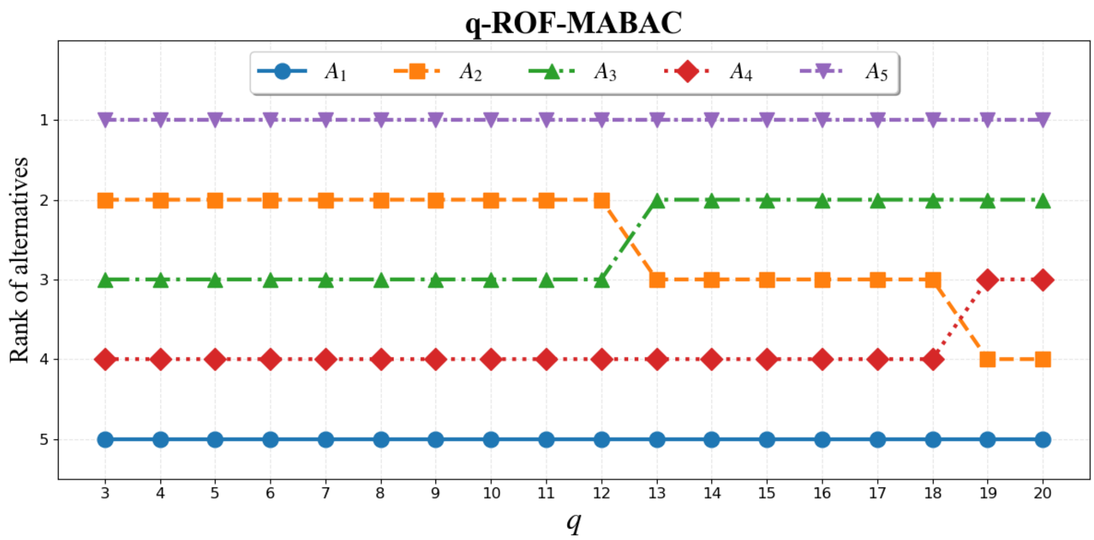


Figure 12. Ranking results of the q-ROF-MABAC method with q ranging from 3 to 20.

Although the q-ROF-REGIME method and the q-ROF-WSM method obtain the same ranking and show the same robustness to the parameter q, the q-ROF-WSM method requires four artificial parameters q, λ, γ_1, and γ_2, which introduces strong subjectivity and leads to unstable and unreproducible results. In contrast, the proposed q-ROF-REGIME method is completely data-driven and parameter-free, avoiding subjective intervention. It determines expert weights and criterion weights objectively, and ranks alternatives by net preference flow with better stability, rationality, and wider applicability.

The comprehensive comparative analysis demonstrates that the proposed q-ROF-REGIME method not only effectively identifies the optimal screening strategy but also exhibits excellent stability under parameter variations. Compared with existing methods, the q-ROF-REGIME method, by introducing the REGIME preference ranking mechanism, can more comprehensively capture the superiority relationships among alternatives. Meanwhile, its preference matrix construction approach based on distance operators enhances the discriminative power of the decision results. Therefore, this method provides an effective and reliable decision-making tool for the selection of early esophageal cancer screening strategies.

This case study is grounded in the real-world scenario of early esophageal cancer screening in China and carries clear and practical medical implications. The indicator system established in this paper covers sensitivity, specificity, examination cost, accessibility in primary hospitals, patient tolerance, and other dimensions, which fully reflects the core concerns in clinical practice and public health. The optimal strategy identified by the proposed model helps improve the early detection rate of esophageal cancer, reduce misdiagnosis risks, cut medical costs, and enhance patient compliance, being better suited for primary medical institutions, thereby effectively promoting the clinical translation and practical implementation of screening techniques. This decision-support framework can also be extended to other medical decision-making scenarios such as cancer diagnosis, treatment selection, and health resource allocation.

8. Limitations of the proposed model

Although the proposed q-ROF-REGIME decision-making framework achieves satisfactory performance in the application of early esophageal cancer screening strategy selection and demonstrates theoretical rationality and practical effectiveness, it still has certain limitations that need to be addressed in future research.

(1) The proposed score function and distance operator are both constructed based on static q-rung orthopair fuzzy information, which can only process cross-sectional evaluation data at a single time point. The model does not integrate dynamic time-series fuzzy information and thus fails to capture the evolutionary characteristics of alternative performance and expert evaluations over time.

(2) The current model assumes the independence of evaluation criteria without considering the interactive effects and hierarchical correlations among criteria, which may lead to a certain deviation between weight distribution and actual decision-making logic. The lack of a criterion interaction mechanism reduces the adaptability of the model to complex multi-criteria systems with interdependent relationships.

(3) The expert weight determination adopts an objective weighting approach based on information content, which cannot reflect the differences in professional authority and practical experience among experts in the field of digestive system tumors. Similarly, criterion weights are obtained by the entropy weight method, which relies entirely on data distribution characteristics without integrating domain knowledge guidance, resulting in insufficient integration of objective data and subjective experience in the weight calculation process.

(4) The integration with machine learning and deep learning methods to realize adaptive parameter adjustment and intelligent weight learning has not been explored, which limits the efficiency and intelligent level of the model when processing massive fuzzy decision-making information.

9. Conclusions

This paper systematically addresses the key theoretical and application bottlenecks in integrating q-rung orthopair fuzzy sets with the REGIME multi-criteria decision-making method, and constructs a complete q-ROF-REGIME decision-making framework for uncertain multi-criteria group decision-making problems. The research outcomes not only improve the theoretical system of q-ROFS information measurement and sorting but also provide a scientific and reliable decision-making tool for practical applications such as early esophageal cancer screening strategy selection, which has important theoretical value and practical significance.

The most prominent unique contribution of this paper is the proposal of a novel admissible order relation based on an improved score function, which overcomes the defects of existing score functions such as counterintuitive sorting results, weak discrimination ability, and sensitivity to parameter q. By comprehensively weighting membership degree, non-membership degree, and hesitation degree, the proposed score function maintains strict monotonicity and boundedness, and the constructed order relation is proven to be an admissible order, filling the theoretical gap of robust and reasonable sorting of q-ROFNs. The second unique contribution is the development of a new q-ROF distance operator that integrates membership, non-membership, and hesitation degrees harmoniously. This operator exhibits high sensitivity to micro-disturbances of fuzzy information and strong robustness to parameter q and set scale changes, which solves the problem of traditional distance operators that cannot accurately capture subtle differences between q-ROFSs. The third unique contribution is the innovative extension of the classical REGIME method to the q-ROF environment, integrating the proposed score function and distance operator into the preference matrix construction process, and combining expert weight determination based on information content and criterion weight calculation based on entropy weight method to form a fully fused multi-criteria group decision-making pipeline, which realizes the effective application of the REGIME method in uncertain fuzzy decision-making scenarios for the first time.

In the case study of early esophageal cancer screening strategy selection, the proposed q-ROF-REGIME framework stably identifies the optimal alternative and maintains consistent sorting results under a wide range of parameter q variations, showing better stability and discrimination than traditional q-ROF-MABAC, q-ROF-WSM, and q-ROF-VIKOR methods. This application verifies that the framework can effectively assist medical decision-makers in conducting scientific and objective screening program selection, and provides a referable decision-making paradigm for clinical medical technology evaluation and selection under uncertain information.

Future research will expand and optimize the proposed model in multiple dimensions based on the limitations of this paper. First, a dynamic q-ROF-REGIME model will be constructed to integrate time-series fuzzy information and realize long-term tracking and evolutionary decision-making of alternatives. Second, an interactive criterion weight calculation mechanism will be introduced to capture the internal correlation between indicators and improve the fitting degree of the model to complex decision systems. Third, a hybrid weighting method combining subjective expert experience and objective data characteristics will be designed to enhance the rationality and credibility of weight distribution. Fourth, the application scope of the model will be expanded to more medical and non-medical multi-criteria decision-making scenarios to verify its universality. Finally, the model will be integrated with deep learning and intelligent optimization algorithms to reduce computational complexity and realize adaptive and intelligent decision-making, so as to better meet the needs of practical large-scale complex decision-making problems.

Author contributions

Conceptualization: Xiaofang Deng, Zhen Jin; Formal Analysis: Zhen Jin, Xiaofang Deng; Methodology: Xiaofang Deng, Zhen Jin, Dan Xiang; Resources: Xiaofang Deng, Zhen Jin, Dan Xiang, Yukai Shen; Software: Xiaofang Deng, Hui Cao, Yukai Shen; Writing - Original Draft Preparation: Xiaofang Deng, Zhen Jin, Dan Xiang, Hui Cao, Yukai Shen; Writing - Review & Editing: Xiaofang Deng, Zhen Jin, Dan Xiang, Hui Cao, Yukai Shen. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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Appendix

Proof of Theorem 4.4. From Definition 3.1, we have $\pi_\alpha^q = 1 - (u^q + v^q)$. It follows that:

$$1 + \pi_\alpha^q = 1 + 1 - (u^q + v^q) = 2 - u^q - v^q,\ 1 - \pi_\alpha^q = u^q + v^q.$$

Substitute the above identities into the score function $S(\alpha)$, then the simplified form is:

$$S(\alpha) = \left[\frac{u^q}{2-u^q-v^q} - \frac{v^q}{1+\pi_\alpha}\right] \times \left(1 + \frac{u^q+v^q}{2}\right),$$

where $\pi_\alpha = (1 - u^q - v^q)^{\frac{1}{q}}$, denoted as $\pi_\alpha = h(u, v)$. Obviously, $h(u, v)$ is monotonically decreasing with respect to both u and v, because $1 - u^q - v^q$ is a decreasing function of u^q and v^q, and the q-th root operation preserves the monotonicity of the function for non-negative real numbers.

(1) Proof of: $S(\alpha)$ is monotonically increasing with respect to u.

Fix v and regard $S(u)$ as a unary function of u, denoted as:

$$S(u) = \left[\frac{u^q}{2 - u^q - v^q} - \frac{v^q}{1 + h(u)}\right] \times \left(1 + \frac{u^q + v^q}{2}\right).$$

Our goal is to prove $S'(u) > 0$ for all $u \in 0,1$ and $S'(0) = 0$

Let $A(u) = \frac{u^q}{2-u^q-v^q}$, $B(u) = \frac{v^q}{1+h(u)}$, $C(u) = 1 + \frac{u^q+v^q}{2}$.

Then $S(u) = \big(A(u) - B(u)\big) \cdot C(u)$, and the first derivative of $S(u)$ with respect to u is given by the product rule:

$$S'(u) = \left(A'(u) - B'(u)\right) C(u) + \big(A(u) - B(u)\big) C'(u),$$

$$A'(u) = \frac{qu^{q-1}(2-u^q-v^q)+qu^{2q-1}}{(2-u^q-v^q)^2} = \frac{2qu^{q-1}-qu^{q-1}v^q}{(2-u^q-v^q)^2}.$$

Since $u, v \in [0,1]$ and $q \geq 1$, then numerator $2qu^{q-1} - qu^{q-1}v^q = qu^{q-1}(2-v^q) \geq 0$, and the denominator $(2-u^q-v^q)^2 \geq 0$. For $u \in [0,1]$, $A'(u) > 0$; for $u = 0$, $A'(u) = 0$.

$$h'(u) = \frac{1}{q}(1-u^q-v^q)^{\frac{1}{q}-1} \cdot (-qu^{q-1}) = -u^{q-1} \cdot (1-u^q-v^q)^{\frac{1-q}{q}} < 0.$$

Then the derivative of $B(u)$ is:

$$B'(u) = -v^q \cdot \frac{h'(u)}{\left(1+h(u)\right)^2}.$$

Substitute $h'(u) < 0$, into the above formula, we get $B'(u) > 0$ for $u \in [0,1]$.

$$C'(u) = \frac{qu^{q-1}}{2}.$$

Clearly, $C'(u) \geq 0$ for all $u \in [0,1]$, and $C'(u) > 0$ for $u \in (0,1]$. In addition, $C(u) = 1 + \frac{u^q+v^q}{2} > 0$ holds for all $u, v \in [0,1]$.

From $0 \leq u^q + v^q \leq 1$, we have $2 - u^q - v^q \geq 1$, so $\frac{u^q}{2-u^q-v^q} \geq u^q$. Meanwhile, $\pi_\alpha \in [0,1]$ implies $1 + \pi_\alpha \geq 1$, so $\frac{v^q}{1+\pi_\alpha} \leq v^q$. Thus:

$$A(u) - B(u) = \frac{u^q}{2-u^q-v^q} - \frac{v^q}{1+\pi_\alpha} \geq u^q - v^q.$$

Combined with the core property of the q-ROFN score function, $A(u) - B(u) > 0$ for all valid u, v.

For $A'(u) - B'(u)$, $A'(u)$ is a positive function dominated by u^{q-1} with a coefficient $q(2-v^q)/(2-u^q-v^q)^2 \geq q/2$, while $B'(u)$ is a positive function with an upper bound of $v^q u^{q-1}/(1+0)^2 = v^q u^{q-1} \leq u^{q-1}$. Since $q \geq 1$, we have $A'(u) > B'(u)$, i.e., $A'(u) - B'(u) > 0$.

Since $A'(u) - B'(u) > 0, C(u) > 0$, we have $\left(A'(u) - B'(u)\right)C(u) > 0$;

Since $A(u) - B(u) > 0, C'(u) \geq 0$, we have $\left(A(u) - B(u)\right)C'(u) \geq 0$.

The sum of two non-negative terms is non-negative, and $S'(u) > 0$ for $u \in 0,1$, $S'(0) = 0$. Therefore, $S(\alpha)$ is monotonically increasing with respect to u.

(2) Proof of: $S(\alpha)$ is monotonically decreasing with respect to v.

Fix u and regard $S(\alpha)$ as a unary function of v, denoted as:

$$S(v) = \left[\frac{u^q}{2-u^q-v^q} - \frac{v^q}{1+h(v)}\right] \times \left(1 + \frac{u^q+v^q}{2}\right),$$

Our goal is to prove $S'(v) < 0$ for all $v \in 0,1$ and $S'(0) = 0$.

Let $A(v) = \frac{u^q}{2-u^q-v^q}$, $B(v) = \frac{v^q}{1+h(v)}$, $C(v) = 1 + \frac{u^q+v^q}{2}$.

Then $S(v) = \left(A(v) - B(v)\right) \cdot C(v)$, and the first derivative of $S(v)$ with respect to v is:

$$S'(v) = \left(A'(v) - B'(v)\right)C(v) + \left(A(v) - B(v)\right)C'(v).$$

$$A'(v) = \frac{qu^q v^{q-1}}{(2-u^q-v^q)^2}.$$

Since $u, v \in [0,1]$, $q \geq 1$, we have $A'(v) \geq 0$ for all $v \in [0,1]$, and $A'(v) > 0$ for $v \in 0,1$.

The derivative of $h(v)$ with respect to v is:

$$h^{'}(v) = -v^{q-1} \cdot (1 - u^q - v^q)^{\frac{1-q}{q}} < 0, v \in 0{,}1\cdot$$

Using the quotient rule for differentiation, the derivative of $B(v)$ is:

$$B^{'}(v) = \frac{qv^{q-1}(1+h(v)) - v^q \cdot h^{'}(v)}{(1+h(v))^2}.$$

Since $h'(v) < 0$, let $|h'(v)| = -h'(v)$, then:

$$B^{'}(v) = \frac{qv^{q-1}(1+h(v)) + v^q \cdot |h'(v)|}{(1+h(v))^2}.$$

All terms in the numerator are positive, so $B'(v) > 0$ for $v \in 0{,}1$. Moreover, $B(v)$ is a strongly increasing function of v, while $A(v)$ is only an indirectly increasing function of v. For $q \geq 1$ and $v \in 0{,}1$, we have $B'(v) \gg A'(v)$, which implies $A'(v) - B'(v) < 0$.

$$C'(v) = \frac{qv^{q-1}}{2} > 0, v \in 0{,}1.$$

And $C(v) = 1 + \frac{u^q + v^q}{2} > 0$ for all $v \in [0,1]$.

Since $A'(v) - B'(v) < 0$ and $C(v) > 0$, the term $\big(A'(v) - B'(v)\big)C(v)$ is negative with a large magnitude, which is the dominant term of $S'(v)$.

Although $\big(A(v) - B(v)\big)C'(v) > 0$, the increment amplitude of this term is much smaller than the decrement amplitude of $\big(A'(v) - B'(v)\big)C(v)$. This is because $\frac{v^q}{1+h(v)}$ directly dominates the decrease of the score function as a strongly increasing function of v.

Thus, the sum of the two terms satisfies $S'(v) < 0$ for $v \in 0{,}1$, and $S'(0) = 0$. Therefore, $S(\alpha)$ is monotonically decreasing with respect to v. □

Proof of Theorem 4.5. From Theorem 4.1, we have the core monotonicity properties of the score function $S(\alpha)$ for a q-ROFN $\alpha = (u, v)$:

Fixing v, $S(\alpha)$ is monotonically increasing with respect to the membership degree u. Fixing u, $S(\alpha)$ is monotonically decreasing with respect to the non-membership degree v.

Given $u_1 \geq u_2$ and $v_1 \leq v_2$, we use the monotonicity for deduction:

Fix the non-membership degree as v_1, by the monotonic increase of $S(\alpha)$ with u:

$$u_1 \geq u_2 \;\Rightarrow\; S(u_1, v_1) \geq S(u_2, v_1).$$

Fix the membership degree as u_2, by the monotonic decrease of $S(\alpha)$ with v:

$$v_1 \leq v_2 \;\Rightarrow\; S(u_2, v_1) \geq S(u_2, v_2) = S(\alpha_2).$$

By the transitivity of inequalities, we obtain:

$$S(\alpha_1) = S(u_1, v_1) \geq S(u_2, v_1) \geq S(\alpha_2).$$

Now we prove that the equality holds if and only if $u_1 = u_2$ and $v_1 = v_2$.

Necessity (⇐): If $u_1 = u_2$ and $v_1 = v_2$, it is straightforward that $S(\alpha_1) = S(\alpha_2)$.

Sufficiency (⇒): Assume that $S(\alpha_1) = S(\alpha_2)$ but the condition $u_1 = u_2$ and $v_1 = v_2$ does not hold. Then there are three possible cases:

(1) $u_1 > u_2$ and $v_1 = v_2$.

By the monotonic increase of $S(\alpha)$ with u,

$$u_1 > u_2 \Rightarrow S(u_1, v_1) > S(u_2, v_1), \text{ i.e., } S(\alpha_1) > S(\alpha_2).$$

This contradicts the assumption $S(\alpha_1) = S(\alpha_2)$.

(2) $u_1 = u_2$ and $v_1 < v_2$.

By the monotonic decrease of $S(\alpha)$ with v,

$$v_1 < v_2 \Rightarrow S(u_1, v_1) > S(u_1, v_2), \text{ i.e., } S(\alpha_1) > S(\alpha_2).$$

This contradicts the assumption $S(\alpha_1) = S(\alpha_2)$.

(3) $u_1 > u_2$ and $v_1 < v_2$.

Combining the two monotonicity properties, we have $S(\alpha_1) > S(\alpha_2)$. This also contradicts the assumption $S(\alpha_1) = S(\alpha_2)$.

All three cases lead to a contradiction, which means the initial assumption is invalid. Thus, the only valid condition for $S(\alpha_1) = S(\alpha_2)$ is $u_1 = u_2$ and $v_1 = v_2$. □

Proof of Theorem 4.6. Substitute the identity $\pi_\alpha^q = 1 - u^q - v^q$ into the score function, we have $1 + \pi_\alpha^q = 2 - u^q - v^q$ and $1 - \pi_\alpha^q = u^q + v^q$. Then the score function can be simplified as:

$$S(\alpha) = \left[\frac{u^q}{2 - u^q - v^q} - \frac{v^q}{1 + \pi_\alpha}\right] \times \frac{2 + u^q + v^q}{2}.$$

According to Definition 3.1, we obtain the ranges of variables: $u^q \in [0,1], v^q \in [0,1]$, $\pi_\alpha \in [0,1]$, $1 + \pi_\alpha \in [1,2]$, and $2 - (u^q + v^q) \in [1,2]$.

(1) Proof of $-\frac{3}{2} \leq S(\alpha) \leq \frac{3}{2}$.

It is obvious that:

$$-1 \leq \frac{u^q}{2-u^q-v^q} - \frac{v^q}{1+\pi_\alpha} \leq 1 \text{ and } 1 \leq \frac{2+u^q+v^q}{2} \leq \frac{3}{2}.$$

By multiplying the two inequalities above, we get: $-\frac{3}{2} \leq S(\alpha) \leq \frac{3}{2}$.

(2) Proof of $S(\alpha) = \frac{3}{2}$ iff $\alpha = (1,0)$.

Necessity (⇐): If $\alpha = (1,0)$, then $u = 1$, $v = 0$, $u^q = 1$, and $v^q = 0$. The hesitancy degree is $\pi_\alpha = (1 - 1 - 0)^{1/q} = 0$. Substitute into $S(\alpha)$:

$$S(\alpha) = \left(\frac{1}{1+0} - \frac{0}{1+0}\right) \times \left(1 + \frac{1-0}{2}\right) = 1 \times \frac{3}{2} = \frac{3}{2}.$$

Sufficiency (⇒): If $S(\alpha) = \frac{3}{2}$, the product attains its maximum value if and only if both factors reach their maximum values, respectively:

$$\frac{u^q}{2-u^q-v^q} - \frac{v^q}{1+\pi_\alpha} = 1 \text{ and } \frac{2+u^q+v^q}{2} = \frac{3}{2}.$$

The first equality implies $\frac{u^q}{2-u^q-v^q} = 1$ and $\frac{v^q}{1+\pi_\alpha} = 0$, which gives $v = 0$ and $u = 1$. The second equality is satisfied with $u = 1$, $v = 0$. Thus, $\alpha = (1,0)$.

(3) Proof of $S(\alpha) = -\frac{3}{2} \Leftrightarrow \alpha = (0,1)$.

Necessity (⇐): If $\alpha = (0,1)$, then $u = 0$, $v = 1$, $u^q = 0$, and $v^q = 1$. The hesitancy degree is $\pi_\alpha = (1 - 0 - 1)^{1/q} = 0$. Substitute into $S(\alpha)$:

$$S(\alpha)=\left(\frac{0}{1+0}-\frac{1}{1+0}\right)\times\left(1+\frac{1-0}{2}\right)=-1\times\frac{3}{2}=-\frac{3}{2}.$$

Sufficiency (⇒): If $S(\alpha)=-\frac{3}{2}$, the product attains its maximum value if and only if both factors reach their maximum values, respectively:

$$\frac{u^q}{2-u^q-v^q}-\frac{v^q}{1+\pi_\alpha}=-1 \text{ and } \frac{2+u^q+v^q}{2}=\frac{3}{2}.$$

The first equality implies $\frac{u^q}{2-u^q-v^q}=0$ and $\frac{v^q}{1+\pi_\alpha}=1$, which gives $v=1$ and $u=0$. The second equality is satisfied with $u=0,\ v=1$. Thus, $\alpha=(0,1)$.

(4) Proof of $S(\alpha)=0\Leftrightarrow\alpha=(0,0)$.

Necessity (⇐): If $\alpha=(0,0)$, then $u=0,\ v=0,\ u^q=0,\ v^q=0$. The hesitancy degree is $\pi_\alpha=(1-0-0)^{1/q}=1$. Substitute into $S(\alpha)$:

$$S(\alpha)=\left(\frac{0}{1+1}-\frac{0}{1+1}\right)\times\left(1+\frac{1-1}{2}\right)=0\times 1=0.$$

Sufficiency (⇒): If $S(\alpha)=0$, let $A=\frac{u^q}{1+\pi_\alpha^q}-\frac{v^q}{1+\pi_\alpha}$ and $B=1+\frac{1-\pi_\alpha^q}{2}$. We have $S(\alpha)=A\times B$. Note that $B\in\left[1,\frac{3}{2}\right]>0$ for all q-ROFN α, so $A=0$ is the necessary and sufficient condition for $S(\alpha)=0$. $A=0$ means $\frac{u^q}{1+\pi_\alpha^q}=\frac{v^q}{1+\pi_\alpha}$.

Combined with $0\le u^q+v^q\le 1$, the only solution is $u^q=0$ and $v^q=0$, i.e., $u=0,\ v=0$, so $\alpha=(0,0)$. □

Proof of Proposition 4.3. (1) For any q-ROFN α_i, the score function $S(\alpha_i)$ is a unique real number. By the reflexivity of real number equality:

$$S(\alpha_i)=S(\alpha_i)\ \Rightarrow\ S(\alpha_i)\ge S(\alpha_i).$$

From the definition of "$\succcurlyeq$" in Definition 4.2, this directly implies $\alpha_i\succcurlyeq\alpha_i$.

(2) Assume $\alpha_1\succcurlyeq\alpha_2$ and $\alpha_2\succcurlyeq\alpha_1$. We have $S(\alpha_1)>S(\alpha_2)$ and $S(\alpha_2)\ge S(\alpha_1)$, then $S(\alpha_1)=S(\alpha_2)$. From the definition of "$\sim$" in Definition 4.2, this means $\alpha_1\sim\alpha_2$.

(3) Assume $\alpha_1\succcurlyeq\alpha_2$ and $\alpha_2\succcurlyeq\alpha_3$, we have $S(\alpha_1)\ge S(\alpha_2)$ and $S(\alpha_2)\ge S(\alpha_3)$, which imply $S(\alpha_1)\ge S(\alpha_3)$, this directly means $\alpha_1\succcurlyeq\alpha_3$. □

Proof of Proposition 4.4. (1) According to Proposition 4.1, the order relation of q-ROFNs defined in Definition 4.2 satisfies reflexivity, antisymmetry, and transitivity. Now we prove that it also satisfies totality.

For any two q-ROFNs $\alpha_1,\alpha_2\in\Omega$, $S(\alpha_1)$ and $S(\alpha_2)$ are real numbers. By the trichotomy property of real numbers, either $S(\alpha_1)>S(\alpha_2)$, $S(\alpha_1)=S(\alpha_2)$, or $S(\alpha_2)>S(\alpha_1)$.

By Definition 4.2, this is equivalent to: $\alpha_1\succ\alpha_2$, $\alpha_1\sim\alpha_2$ or $\alpha_1\succ\alpha_2$.

In all cases, either $\alpha_1\succcurlyeq\alpha_2$ or $\alpha_2\succcurlyeq\alpha_1$ holds for any pair $\alpha_1,\alpha_2\in\Omega$.

Since $\succcurlyeq$ satisfies reflexivity, antisymmetry, and totality, it is a linear order on Ω.

(2) It is a direct corollary of Theorem 4.2. □

Proof of Theorem 5.2. For each $x_i\in X$ define

$$a_i=\sqrt{u_M^q(x_i)},\ b_i=\sqrt{u_N^q(x_i)},$$

$$c_i = \sqrt{v_M^q(x_i)},\ d_i = \sqrt{v_N^q(x_i)},$$

$$e_i = \sqrt{\pi_M^q(x_i)},\ f_i = \sqrt{\pi_N^q(x_i)}.$$

These are non-negative numbers with

$$a_i^2 + c_i^2 + e_i^2 = 1,\ b_i^2 + d_i^2 + f_i^2 = 1.$$

Define vectors

$$\boldsymbol{p}_i = (a_i, c_i, e_i),\ \boldsymbol{q}_i = (b_i, d_i, f_i),$$

so $\|\boldsymbol{p}_i\| = \|\boldsymbol{q}_i\| = 1$.

Let

$$t_i = \boldsymbol{p}_i \cdot \boldsymbol{q}_i = a_i b_i + c_i d_i + e_i f_i \in [0,1].$$

The denominator is

$$D_i = (a_i + b_i)^2 + (c_i + d_i)^2 + (e_i + f_i)^2 = \|\boldsymbol{p}_i\|^2 + \|\boldsymbol{q}_i\|^2 + 2\boldsymbol{p}_i \cdot \boldsymbol{q}_i = 2 + 2t_i.$$

The numerator is $N_i = t_i$.

Thus

$$\frac{N_i}{D_i} = \frac{t_i}{2(1 + t_i)}.$$

Define $f(t) = \frac{t}{1+t}$ for $t \geq 0$.

Then

$$\frac{t_i}{2(1 + t_i)} = \frac{1}{2} f(t_i),$$

$$d(M, N) = 1 - \frac{4}{n} \sum_{i=1}^{n} \frac{t_i}{2(1 + t_i)} = 1 - \frac{2}{n} \sum_{i=1}^{n} f\,(t_i).$$

(1) Since $t_i \in [0,1]$, we have

$$0 \leq f(t_i) \leq \frac{1}{2}.$$

Hence

$$0 \leq \frac{2}{n} \sum_{i=1}^{n} f\,(t_i) \leq 1,$$

and therefore,

$$d(M, N) = 1 - \frac{2}{n} \sum_{i=1}^{n} f\,(t_i) \in [0,1].$$

(2) From $t_i = \boldsymbol{p}_i \cdot \boldsymbol{q}_i = \boldsymbol{q}_i \cdot \boldsymbol{p}_i$, it is obvious that $d(M, N) = d(N, M)$.

(3) Let $\boldsymbol{r}_i$ be the vector for P, with $y_i = \boldsymbol{p}_i \cdot \boldsymbol{r}_i, \quad z_i = \boldsymbol{r}_i \cdot \boldsymbol{q}_i$.
Then

$$d(M,P) = 1 - \frac{2}{n}\sum_{i=1}^{n} f(y_i), \quad = 1 - \frac{2}{n}\sum_{i=1}^{n} f(z_i).$$

The triangle inequality $d(M,N) \le d(M,P) + d(P,N)$ is equivalent to

$$\frac{2}{n}\sum_{i=1}^{n}[f(y_i) + f(z_i) - f(t_i)] \le 1.$$

Since $f(t) = 1 - \frac{1}{1+t}$,

$$f(y_i) + f(z_i) - f(t_i) = 1 - \frac{1}{1+y_i} - \frac{1}{1+z_i} + \frac{1}{1+t_i}.$$

From the Euclidean triangle inequality

$$\|\boldsymbol{p}_i - \boldsymbol{q}_i\| \le \|\boldsymbol{p}_i - \boldsymbol{r}_i\| + \|\boldsymbol{r}_i - \boldsymbol{q}_i\|,$$

one can deduce $t_i \ge y_i + z_i - 1$. Consequently,

$$\frac{1}{1+t_i} \le \frac{1}{y_i + z_i}.$$

Thus

$$f(y_i) + f(z_i) - f(t_i) \le 1 - \frac{1}{1+y_i} - \frac{1}{1+z_i} + \frac{1}{y_i + z_i}.$$

The right-hand side, as a function of (y_i, z_i) with $y_i + z_i = s$, attains its maximum at $y_i = z_i = \frac{s}{2}$, giving

$$g(s) = 1 - \frac{2}{1+\frac{s}{2}} + \frac{1}{s}.$$

One can verify that $g(s) \le \frac{1}{2}$ for $s \in (0,2]$. Hence,

$$f(y_i) + f(z_i) - f(t_i) \le \frac{1}{2}.$$

Therefore,

$$\frac{2}{n}\sum_{i=1}^{n}[f(y_i) + f(z_i) - f(t_i)] \le \frac{2}{n} \cdot n \cdot \frac{1}{2} \le 1,$$

which completes the proof of the triangle inequality.

(4) If $M \subseteq N \subseteq P$ in the q-ROFS sense (i.e., $u_M \le u_N \le u_P$ and $v_M \ge v_N \ge v_P$), then it can be shown that

$$t_{MP} \le t_{MN},\ t_{MP} \le t_{NP},$$

where

$$t_{MN} = \boldsymbol{p}_i \cdot \boldsymbol{q}_i,\ t_{MP} = \boldsymbol{p}_i \cdot \boldsymbol{r}_i,\ t_{NP} = \boldsymbol{q}_i \cdot \boldsymbol{r}_i.$$

Since function f is increasing,

$$f(t_{MP}) \leq f(t_{MN}),\ f(t_{MP}) \leq f(t_{NP}).$$

Averaging over i yields

$$\frac{2}{n}\sum_{i=1}^{n} f\,(t_{MP}) \leq \frac{2}{n}\sum_{i=1}^{n} f\,(t_{MN}),\ \frac{2}{n}\sum_{i=1}^{n} f\,(t_{MP}) \leq \frac{2}{n}\sum_{i=1}^{n} f\,(t_{NP}),$$

which is equivalent to

$$d(M,P) \geq d(M,N),\ d(M,P) \geq d(N,P).$$

(5) From (1), we already have $d(M,N) \geq 0$.

If $M = N$, then $\boldsymbol{p}_i = \boldsymbol{q}_i$, so $t_i = 1$ and $f(t_i) = \frac{1}{2}$. Hence,

$$\frac{2}{n}\sum_{i=1}^{n} f(t_i) = 1,$$

then we have $d(M,N) = 0$.

Conversely, if $d(M,N) = 0$, then

$$\frac{2}{n}\sum_{i=1}^{n} f\,(t_i) = 1.$$

Since each $f(t_i) \leq \frac{1}{2}$, this forces $f(t_i) = \frac{1}{2}$ for all i, and thus $t_i = 1$.

Then, $\|\boldsymbol{p}_i - \boldsymbol{q}_i\|^2 = 2 - 2t_i = 0$, so $\boldsymbol{p}_i = \boldsymbol{q}_i$. This implies

$$a_i = b_i,\ c_i = d_i,\ e_i = f_i,$$

that is

$$u_M^q(x_i) = u_N^q(x_i),\ v_M^q(x_i) = v_N^q(x_i),\ \pi_M^q(x_i) = \pi_N^q(x_i).$$

Therefore, $M = N$. □