Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 11(6): 16763–16787.
DOI: 10.3934/math.2026688
Received: 16 March 2026
Revised: 03 June 2026
Accepted: 08 June 2026
Published: 11 June 2026

Research article

Exact solutions of fractional differential Stein matrix equations via diagonalization and Mittag–Leffler functions

Lakhlifa Sadek[1,2] **and Ali Algefary**[3,*]

[1] Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai 602105, Tamilnadu, India
[2] Department of Mathematics, Faculty of Sciences and Technology, BP 34. Ajdir 32003 Al-Hoceima, Abdelmalek Essaadi University, Tetouan, Morocco
[3] Department of Mathematics, College of Science, Qassim University, P.O. Box 6644, Buraydah 51452, Saudi Arabia

* **Correspondence:** Email: a.algefary@qu.edu.sa.

Abstract: In this work, an explicit formula in terms of the Hadamard product was derived for the solutions to fractional differential Stein matrix equations (FDSMEs), assuming that the coefficient matrices are separately diagonalizable. To begin with, we gave a general formula using the Kronecker product notation, which does not need any diagonalizability assumption to show the existence and uniqueness of the solution. In the case of diagonalizable coefficient matrices, the matrix equation became decoupled scalar Caputo fractional differential equations. Solutions to these differential equations were given explicitly in terms of two-parameter Mittag–Leffler functions and put together using the Hadamard product. The results for integer-order differential equations are obtained as special cases. Finally, we give three examples.

Keywords: fractional derivative; fractional Stein matrix equations; Mittag–Leffler function; Diagonalization; Hadamard product
Mathematics Subject Classification: 34A08, 15A24, 26A33, 15A18

1. Introduction

Fractional calculus [1, 2] has been considered a generalization of conventional calculus. It has gained much popularity in the past few decades owing to its applications in modeling physical processes that possess memory and inheritance effects [3, 4]. Applications span psychology [3], biology [4], engineering [5, 6], chemistry [7], electromagnetics [8], epidemiology [9], medicine [10], and many other domains.

The computation of Mittag–Leffler functions with matrix arguments has attracted attention. Garrappa and Popolizio considered the algorithms for matrix Mittag–Leffler functions and their applications to fractional calculus [11]. Most recently, Cardoso has proposed a derivative-free method combining Taylor series evaluation and a Schur–Parlett/Cauchy integral approach using ordinary double precision arithmetic [12]. This inspires the following crucial observation that underpins the current manuscript: A vectorized matrix-argument formula can be computed without diagonalizability, whereas the separate diagonalizations of the coefficient matrices yield the simpler scalar Hadamard-product formula.

Contribution and Motivation: General solutions of a fractional matrix differential equation can be expressed through a matrix Mittag–Leffler function in vectorized form. The aim of this paper is to obtain a reduced scalar form Mittag–Leffler representation under the condition that both coefficient matrices are individually diagonalizable. For such a case, the decoupling through diagonalization of the vectorized form will lead to scalar Caputo-type fractional differential equations (FDEs). This Hadamard product representation circumvents the computation of a matrix Mittag–Leffler function.

This paper focuses on the fractional differential Stein matrix equation (FDSME) defined on the interval $I = [0, T]$ as

$$\begin{cases} P^{(r)}(\vartheta) = HP(\vartheta)L - P(\vartheta) + M, & \vartheta \in I, \\ P(0) = P_0, \end{cases} \tag{1.1}$$

where $H \in \mathbb{R}^{m_1 \times m_1}$, $L \in \mathbb{R}^{m_2 \times m_2}$, $M \in \mathbb{R}^{m_1 \times m_2}$, $P(\vartheta)$ is an $m_1 \times m_2$ unknown matrix function, and $r \in (0, 1]$ denotes the order of the fractional derivative in the Caputo sense. In [13], Lakhlifa Sadek presented the fractional BDF method for solving the FDSME with variable coefficients. When $r = 1$, Eq (3.19) reduces to the classical Stein matrix differential equation

$$\begin{cases} \dot{P}(\vartheta) = HP(\vartheta)L - P(\vartheta) + M, & \vartheta \in I, \\ P(0) = P_0, \end{cases} \tag{1.2}$$

which is of crucial importance in filter design theory, control systems, and model reduction [14–16]. In [17], Sadek et al. introduced Krylov subspace methods for solving Eq (1.2). The main difficulty in solving the FDSME is related to the derivation of explicit analytical solutions for this equation. Although numerical methods have been proposed in the literature [13], few analytical solutions are available in the existing literature. The contributions of this manuscript are

1. the vectorization formulation leading to the existence-uniqueness theorem without diagonalizability conditions;
2. the explicit solution in terms of the Hadamard product based on the diagonalizability condition of H and L independently;
3. an easier formula for the special case $L = H$ along with the symmetry preservation feature if further symmetry conditions hold;
4. reduction to the conventional ordinary Stein differential equation for $r = 1$;
5. illustrative examples, which include the perturbed Jordan block example.

Notation

Throughout this paper, we employ the following notation:

- $P^{(r)}$: Caputo fractional derivative of order r of the matrix function P.

- ${}^{RL}D^{r}_{\vartheta}$: Riemann-Liouville fractional derivative of order r.
- H^T: Transpose of matrix H.
- I_{m_1}: Identity matrix of size $m_1 \times m_1$.
- $0_{m_1 \times m_2}$: Zero matrix of size $m_1 \times m_2$.
- $\Gamma(\cdot)$: Euler's gamma function.
- $E_{r,\alpha}(\cdot)$: Two-parameter Mittag–Leffler function.
- $\odot$: Hadamard (element-wise) product.
- $e_i^{(m)}$: Standard basis vector in $\mathbb{R}^m$ with 1 in the i-th position.
- $\|\cdot\|_F$: Frobenius norm of a matrix.
- $C([0,T];\mathbb{R}^{m_1\times m_2})$: Space of continuous matrix-valued functions on $[0,T]$.
- $\mathrm{vec}(P)$: Column-wise vectorization of the matrix P.
- $\otimes$: Kronecker product.
- $\exp^{\odot}(A)$: Element-wise exponential of a matrix A.

2. Preliminary concepts

This section introduces essential definitions, properties, and theoretical results that form the foundation of our work. Readers familiar with fractional calculus and matrix theory may skip to Section 3 for the main results.

2.1. Special functions

Definition 1 ([18]). *Euler's gamma function is defined for Re(r) > 0 as*

$$\Gamma(r) = \int_0^\infty e^{-\lambda}\lambda^{r-1}d\lambda.$$

Definition 2 ([11, 18]). *The Mittag–Leffler function is defined as*

$$E_{r,\alpha}(\vartheta) = \sum_{j=0}^{\infty} \frac{\vartheta^j}{\Gamma(rj+\alpha)}, \quad \vartheta \in \mathbb{C}, \quad r > 0, \quad \alpha \in \mathbb{R}.$$

Important special cases include

$$E_{1,1}(\vartheta) = e^{\vartheta}, \tag{2.1}$$

$$E_{1,2}(\vartheta) = \frac{e^{\vartheta}-1}{\vartheta}, \tag{2.2}$$

$$E_{1,k}(\vartheta) = \frac{1}{\vartheta^{k-1}}\left(e^{\vartheta} - \sum_{j=0}^{k-2}\frac{\vartheta^j}{j!}\right), \quad k = 2, 3, \ldots. \tag{2.3}$$

2.2. Fractional calculus fundamentals

Definition 3 ([5]). *For $r > 0$, the left-sided Riemann–Liouville fractional integral of a matrix-valued function P is defined entrywise by*

$${}^{RL}I^{r}_{0+}P(\vartheta) = \frac{1}{\Gamma(r)}\int_0^{\vartheta}(\vartheta - s)^{r-1}P(s)\,ds. \tag{2.4}$$

For $r = 0$*, we set* ${}^{RL}I_{0+}^{0}P = P$*.*

Definition 4 ([5])**.** *For* $0 < r < 1$*, the left-sided Riemann–Liouville fractional derivative of* P *is defined entrywise by*

$$ {}^{RL}D_{0+}^{r}P(\vartheta) = \frac{1}{\Gamma(1-r)}\frac{d}{d\vartheta}\int_0^{\vartheta}(\vartheta-\tau)^{-r}P(\tau)\,d\tau. \tag{2.5}$$

Definition 5 ([5])**.** *Let* $P \in AC([0,T];\mathbb{R}^{m_1\times m_2})$*. For* $0 < r < 1$*, the left-sided Caputo fractional derivative of* P *is defined entrywise by*

$$ {}^{C}D_{0+}^{r}P(\vartheta) = \frac{1}{\Gamma(1-r)}\int_0^{\vartheta}(\vartheta-s)^{-r}\dot{P}(s)\,ds. \tag{2.6}$$

For $r = 1$*, we set* ${}^{C}D_{0+}^{1}P(\vartheta) = \dot{P}(\vartheta)$*. In the sequel,* $P^{(r)}$ *denotes* ${}^{C}D_{0+}^{r}P$*.*

Lemma 1. *Let* $0 < r \le 1$ *and* $P \in AC([0,T];\mathbb{R}^{m_1\times m_2})$*. Then,*

$$ {}^{RL}I_{0+}^{r}P^{(r)}(\vartheta) = P(\vartheta) - P(0). \tag{2.7}$$

2.3. Matrix theory essentials

Definition 6 ([19])**.** *For matrices* $M = [a_{ij}], N = [b_{ij}] \in \mathbb{R}^{m_1\times m_2}$*, the Hadamard product is defined element-wise as*

$$ M \odot N := \begin{pmatrix} a_{11}b_{11} & \cdots & a_{1m_2}b_{1m_2} \\ \vdots & \ddots & \vdots \\ a_{m_11}b_{m_11} & \cdots & a_{m_1m_2}b_{m_1m_2} \end{pmatrix}. \tag{2.8}$$

Definition 7. *A matrix* $H \in \mathbb{R}^{m_1\times m_1}$ *is diagonalizable if there exist an invertible matrix* Q *and a diagonal matrix* Λ *such that*

$$ H = Q\Lambda Q^{-1}, \tag{2.9}$$

where $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_{m_1})$ *contains the eigenvalues of* H*, and the columns of* Q *are the corresponding eigenvectors.*

2.4. Sufficient conditions for diagonalizability

Throughout this paper, the coefficient matrices are real. Useful sufficient conditions are:

- a real symmetric matrix is diagonalizable by an orthogonal matrix;
- a real matrix with distinct real eigenvalues is diagonalizable over $\mathbb{R}$.

More generally, a real matrix may be diagonalizable over $\mathbb{C}$. If complex normal or Hermitian matrices are considered, the transpose must be replaced by the conjugate transpose where appropriate.

2.5. Fundamental theoretical results

Theorem 1 ([18])**.** *Let* $0 < r \le 1$*. The unique solution of*

$$ \begin{cases} z^{(r)}(\vartheta) = az(\vartheta) + c, & \vartheta \ge 0, \\ z(0) = z_0, \end{cases} \tag{2.10}$$

is

$$z(\vartheta) = E_{r,1}(a\vartheta^r)z_0 + \int_0^{\vartheta} (\vartheta - \tau)^{r-1} E_{r,r}(a(\vartheta - \tau)^r)c \, d\tau. \tag{2.11}$$

Equivalently,

$$z(\vartheta) = E_{r,1}(a\vartheta^r)z_0 + \vartheta^r E_{r,r+1}(a\vartheta^r)c. \tag{2.12}$$

Theorem 2. *Let* $0 < r \leq 1$. *Consider the FDSME*

$$\begin{cases} P^{(r)}(\vartheta) = HP(\vartheta)L - P(\vartheta) + M, & \vartheta \geq 0, \\ P(0) = P_0. \end{cases} \tag{2.13}$$

Let $H \in \mathbb{R}^{m_1 \times m_1}$, $L \in \mathbb{R}^{m_2 \times m_2}$, *and* $M, P_0 \in \mathbb{R}^{m_1 \times m_2}$. *Then, for every* $T > 0$, *system (2.13) admits a unique solution on* $[0, T]$.

Proof. Set

$$p(\vartheta) = \operatorname{vec}(P(\vartheta)), \qquad p_0 = \operatorname{vec}(P_0), \qquad b = \operatorname{vec}(M).$$

Using the identity

$$\operatorname{vec}(HPL) = (L^T \otimes H) \operatorname{vec}(P),$$

we transform (2.13) into the linear Caputo system

$$p^{(r)}(\vartheta) = \mathcal{K} p(\vartheta) + b, \qquad p(0) = p_0, \tag{2.14}$$

where

$$\mathcal{K} = L^T \otimes H - I_{m_1 m_2}.$$

The unique solution of (2.14) is

$$p(\vartheta) = E_{r,1}(\mathcal{K}\vartheta^r)p_0 + \vartheta^r E_{r,r+1}(\mathcal{K}\vartheta^r)b. \tag{2.15}$$

Here, the Mittag–Leffler functions with matrix argument are defined by their convergent power series. Consequently,

$$P(\vartheta) = \operatorname{vec}^{-1}\left(E_{r,1}(\mathcal{K}\vartheta^r) \operatorname{vec}(P_0) + \vartheta^r E_{r,r+1}(\mathcal{K}\vartheta^r) \operatorname{vec}(M)\right). \tag{2.16}$$

□

Remark 1. *Formula (2.16) does not require diagonalizability. The separate diagonalizability assumptions introduced below are used to reduce the matrix Mittag–Leffler evaluation to scalar Mittag–Leffler evaluations and to obtain a compact Hadamard-product representation.*

3. Exact solutions

In this section, we derive an explicit Hadamard-product representation under separate diagonalizability assumptions. The general matrix-argument solution has already been given in Remark 1.

3.1. General nonsymmetric case

Consider the FDSME (3.19) with diagonalizable matrices:

$$H = Q\Lambda Q^{-1}, \quad \Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_{m_1}), \tag{3.1}$$
$$L = \Psi\Omega\Psi^{-1}, \quad \Omega = \mathrm{diag}(\omega_1, \dots, \omega_{m_2}). \tag{3.2}$$

We propose a solution of the form

$$P(\vartheta) = QZ(\vartheta)\Psi^{-1}. \tag{3.3}$$

Theorem 3. *Under the transformations (3.1) and (3.2), the matrix function $Z(\vartheta)$ in (3.3) satisfies the transformed system*

$$\begin{cases} Z^{(r)}(\vartheta) = \Lambda Z(\vartheta)\Omega - Z(\vartheta) + S, & \vartheta \geq 0, \\ Z(0) = Z_0, \end{cases} \tag{3.4}$$

where $S = Q^{-1}M\Psi$ and $Z_0 = Q^{-1}P_0\Psi$.

Proof. Assume that H and L are diagonalizable matrices with the following diagonalizations and we have

$$H = Q\Lambda Q^{-1}, \quad \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_{m_1}), \tag{3.5}$$
$$L = \Psi\Omega\Psi^{-1}, \quad \Omega = \mathrm{diag}(\omega_1, \omega_2, \dots, \omega_{m_2}). \tag{3.6}$$

Left-multiply both sides by Q^{-1} and right-multiply by Ψ of Eq (3.19) and we have

$$Q^{-1}P^{(r)}(\vartheta)\Psi = Q^{-1}HP(\vartheta)L\Psi - Q^{-1}P(\vartheta)\Psi + Q^{-1}M\Psi.$$

Next, consider each term separately. Note that the Caputo fractional derivative is linear, and since Q^{-1} and Ψ are constant matrices, we obtain

$$Q^{-1}P^{(r)}(\vartheta)\Psi = (Q^{-1}P(\vartheta)\Psi)^{(r)} = Z^{(r)}(\vartheta).$$

First term on the righthand side:

$$\begin{aligned} Q^{-1}HP(\vartheta)L\Psi &= (Q^{-1}HQ)(Q^{-1}P(\vartheta)\Psi)(\Psi^{-1}L\Psi) \\ &= \Lambda Z(\vartheta)\Omega. \end{aligned}$$

Second term on the righthand side:

$$Q^{-1}P(\vartheta)\Psi = Z(\vartheta).$$

Third term on the righthand side:

$$S = Q^{-1}M\Psi \in \mathbb{R}^{m_1 \times m_2}.$$

Putting it all together, we obtain

$$Z^{(r)}(\vartheta) = \Lambda Z(\vartheta)\Omega - Z(\vartheta) + S.$$

From the initial condition, we have

$$Z(0) = Q^{-1}P(0)\Psi = Q^{-1}P_0\Psi.$$

□

Remark 2. *If H and L are symmetric matrices, then Q and Ψ can be chosen to be orthogonal matrices ($Q^{-1} = Q^T$, $\Psi^{-1} = \Psi^T$), which simplifies numerical computations and improves stability.*

We now introduce the key construct for our solution representation:

Definition 8. *For diagonal matrices $\Lambda = diag(\lambda_1, \ldots, \lambda_{m_1})$ and $\Omega = diag(\omega_1, \ldots, \omega_{m_2})$, and parameters $\gamma, r > 0$, define*

$$\mathbb{E}^{\Omega,\Lambda}_{\gamma,r}(\mu) = \begin{pmatrix} E_{\gamma,r}((\lambda_1\omega_1 - 1)\mu) & \cdots & E_{\gamma,r}((\lambda_1\omega_{m_2} - 1)\mu) \\ \vdots & \ddots & \vdots \\ E_{\gamma,r}((\lambda_{m_1}\omega_1 - 1)\mu) & \cdots & E_{\gamma,r}((\lambda_{m_1}\omega_{m_2} - 1)\mu) \end{pmatrix}_{m_1\times m_2}. \tag{3.7}$$

Lemma 2. *Let $0 < r \le 1$, and let*

$$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_{m_1}), \qquad \Omega = \mathrm{diag}(\omega_1, \ldots, \omega_{m_2}).$$

Define the matrix

$$\Delta^{\Omega,\Lambda} = \left[\delta_{ij}\right]_{m_1\times m_2}, \qquad \delta_{ij} = \lambda_i\omega_j - 1,$$

and, for $\alpha,\beta > 0$, define

$$\mathbb{E}^{\Omega,\Lambda}_{\alpha,\beta}(\mu^r) = \left[E_{\alpha,\beta}\left(\delta_{ij}\mu^r\right)\right]_{m_1\times m_2}.$$

Then, for $\mu \ge 0$, the Caputo fractional derivative is computed entrywise and satisfies

$${}^{C}D^{r}_{0+}\mathbb{E}^{\Omega,\Lambda}_{\alpha,\beta}(\mu^r) = \sum_{k=1}^{\infty} \frac{\Gamma(rk+1)}{\Gamma(rk+1-r)\Gamma(\alpha k+\beta)}\mu^{r(k-1)}\left(\Delta^{\Omega,\Lambda}\right)^{\odot k}, \tag{3.8}$$

where $\odot$ denotes the Hadamard product and

$$\left(\Delta^{\Omega,\Lambda}\right)^{\odot k} = \left[\delta^k_{ij}\right]_{m_1\times m_2}.$$

In particular, the following identities hold:

$${}^{C}D^{r}_{0+}\mathbb{E}^{\Omega,\Lambda}_{r,1}(\mu^r) = \Delta^{\Omega,\Lambda} \odot \mathbb{E}^{\Omega,\Lambda}_{r,1}(\mu^r), \tag{3.9}$$

and

$${}^{C}D^{r}_{0+}\left[\mu^r\mathbb{E}^{\Omega,\Lambda}_{r,r+1}(\mu^r)\right] = \mathbb{E}^{\Omega,\Lambda}_{r,1}(\mu^r). \tag{3.10}$$

Proof. The Caputo fractional derivative of a matrix-valued function is understood entrywise. Let

$$\delta_{ij} = \lambda_i\omega_j - 1.$$

Using the series representation of the two-parameter Mittag–Leffler function, we have

$$E_{\alpha,\beta}\left(\delta_{ij}\mu^r\right) = \sum_{k=0}^{\infty}\frac{\delta^k_{ij}\mu^{rk}}{\Gamma(\alpha k+\beta)}.$$

Since the Mittag–Leffler series converges locally uniformly, the Caputo fractional derivative may be applied term by term. Moreover, the Caputo derivative of a constant is zero and, for $k \geq 1$,

$$ {}^{C}D_{0+}^{r}\mu^{rk} = \frac{\Gamma(rk+1)}{\Gamma(rk+1-r)}\mu^{r(k-1)}. $$

Therefore,

$$ \begin{aligned} {}^{C}D_{0+}^{r}E_{\alpha,\beta}\left(\delta_{ij}\mu^{r}\right) &= \sum_{k=1}^{\infty}\frac{\delta_{ij}^{k}}{\Gamma(\alpha k+\beta)}{}^{C}D_{0+}^{r}\mu^{rk} \\ &= \sum_{k=1}^{\infty}\frac{\delta_{ij}^{k}\Gamma(rk+1)}{\Gamma(rk+1-r)\Gamma(\alpha k+\beta)}\mu^{r(k-1)}. \end{aligned} $$

Assembling these relations entrywise gives Eq (3.8). We now prove the first particular identity. Setting $\alpha = r$ and $\beta = 1$, we obtain

$$ \begin{aligned} {}^{C}D_{0+}^{r}E_{r,1}\left(\delta_{ij}\mu^{r}\right) &= \sum_{k=1}^{\infty}\frac{\delta_{ij}^{k}\Gamma(rk+1)}{\Gamma(rk+1-r)\Gamma(rk+1)}\mu^{r(k-1)} \\ &= \sum_{k=1}^{\infty}\frac{\delta_{ij}^{k}\mu^{r(k-1)}}{\Gamma(r(k-1)+1)}. \end{aligned} $$

After introducing the new index $\ell = k-1$, it follows that

$$ \begin{aligned} {}^{C}D_{0+}^{r}E_{r,1}\left(\delta_{ij}\mu^{r}\right) &= \delta_{ij}\sum_{\ell=0}^{\infty}\frac{\delta_{ij}^{\ell}\mu^{r\ell}}{\Gamma(r\ell+1)} \\ &= \delta_{ij}E_{r,1}\left(\delta_{ij}\mu^{r}\right). \end{aligned} $$

Hence,

$$ {}^{C}D_{0+}^{r}\mathbb{E}_{r,1}^{\Omega,\Lambda}(\mu^{r}) = \Delta^{\Omega,\Lambda}\odot\mathbb{E}_{r,1}^{\Omega,\Lambda}(\mu^{r}). $$

To prove the second identity, observe that

$$ \mu^{r}E_{r,r+1}\left(\delta_{ij}\mu^{r}\right) = \sum_{k=0}^{\infty}\frac{\delta_{ij}^{k}\mu^{r(k+1)}}{\Gamma(r(k+1)+1)}. $$

Applying the Caputo derivative term by term yields

$$ \begin{aligned} {}^{C}D_{0+}^{r}\left[\mu^{r}E_{r,r+1}\left(\delta_{ij}\mu^{r}\right)\right] &= \sum_{k=0}^{\infty}\frac{\delta_{ij}^{k}}{\Gamma(r(k+1)+1)}{}^{C}D_{0+}^{r}\mu^{r(k+1)} \\ &= \sum_{k=0}^{\infty}\frac{\delta_{ij}^{k}\mu^{rk}}{\Gamma(rk+1)} \\ &= E_{r,1}\left(\delta_{ij}\mu^{r}\right). \end{aligned} $$

By assembling the resulting scalar identities entrywise, we obtain Eq (3.10). □

Theorem 4. *The exact solution of FDSME (3.4) is*

$$Z(\vartheta) = \Big[\mathbb{E}^{\Omega,\Lambda}_{r,1}(\vartheta^r) \odot Z_0 + \vartheta^r \mathbb{E}^{\Omega,\Lambda}_{r,r+1}(\vartheta^r) \odot S\Big]. \tag{3.11}$$

Proof. Let $Z(\vartheta) = [z_{ij}(\vartheta)]_{m_1\times m_2}$, $Z_0 = [z_{0,ij}]_{m_1\times m_2}$, and $S = [s_{ij}]_{m_1\times m_2}$. From the structure of Λ and Ω, the (i, j)-th component of (3.4) is

$$z^{(r)}_{ij}(\vartheta) = (\lambda_i\omega_j - 1)z_{ij}(\vartheta) + s_{ij}, \tag{3.12}$$

with initial condition

$$z_{ij}(0) = z_{0,ij}. \tag{3.13}$$

According to Theorem 1, the unique solution is

$$z_{ij}(\vartheta) = E_{r,1}(\alpha\vartheta^r)z_{0,ij} + \int_0^\vartheta (\vartheta - \tau)^{r-1} E_{r,r}(\alpha(\vartheta - \tau)^r)s_{ij}\, d\tau, \tag{3.14}$$

where $\alpha = \lambda_i\omega_j - 1$. We need to compute the integral

$$I_{ij}(\vartheta) = \int_0^\vartheta (\vartheta - \tau)^{r-1} E_{r,r}(\alpha(\vartheta - \tau)^r)s_{ij}\, d\tau.$$

Using the substitution $u = \vartheta - \tau$, $du = -d\tau$, we have

$$I_{ij}(\vartheta) = s_{ij} \int_0^\vartheta u^{r-1} E_{r,r}(\alpha u^r)\, du.$$

From the properties of Mittag–Leffler functions [11, 18], we have

$$\int_0^\vartheta u^{r-1} E_{r,r}(\alpha u^r)\, du = \vartheta^r E_{r,r+1}(\alpha\vartheta^r).$$

Therefore,

$$I_{ij}(\vartheta) = \vartheta^r E_{r,r+1}(\alpha\vartheta^r)s_{ij}. \tag{3.15}$$

Substituting (3.15) into (3.14) gives

$$z_{ij}(\vartheta) = E_{r,1}((\lambda_i\omega_j - 1)\vartheta^r)z_{0,ij} + \vartheta^r E_{r,r+1}((\lambda_i\omega_j - 1)\vartheta^r)s_{ij}. \tag{3.16}$$

Then, the solution for $Z(\vartheta)$ can now be written compactly using the Hadamard (element-wise) product $\odot$ as

$$Z(\vartheta) = \Big[\mathbb{E}^{\Omega,\Lambda}_{r,1}(\vartheta^r) \odot Z_0 + \vartheta^r \mathbb{E}^{\Omega,\Lambda}_{r,r+1}(\vartheta^r) \odot S\Big]. \tag{3.17}$$

□

Theorem 5. *Given diagonalizable matrices H and L as in (3.1) and (3.2), the exact solution of FDSME (3.19) is*

$$P(\vartheta) = Q\Big[\mathbb{E}^{\Omega,\Lambda}_{r,1}(\vartheta^r) \odot (Q^{-1}P_0\Psi) + \vartheta^r \mathbb{E}^{\Omega,\Lambda}_{r,r+1}(\vartheta^r) \odot (Q^{-1}M\Psi)\Big]\Psi^{-1}. \tag{3.18}$$

Proof. From Eqs (3.3) and (3.17), we have

$$\begin{aligned} P(\vartheta) &= QZ(\vartheta)\Psi^{-1} \\ &= Q\Big[\mathbb{E}_{r,1}^{\Omega,\Lambda}(\vartheta^r) \odot (Q^{-1}P_0\Psi) + \vartheta^r \mathbb{E}_{r,r+1}^{\Omega,\Lambda}(\vartheta^r) \odot (Q^{-1}M\Psi)\Big]\Psi^{-1}. \end{aligned}$$

□

Remark 3. *The solution obtained using formula (3.20) requires a few steps to be carried out*

(1) diagonalization of H and L: $O(m_1^3 + m_2^3)$ *operations;*
(2) evaluation of Mittag–Leffler functions at $m_1 m_2$ *points: the computational cost depends on the method used (series expansion, integral representation, etc.);*
(3) matrix multiplications: $O(m_1^2 m_2 + m_1 m_2^2)$ *operations.*

For large matrices, the diagonalization step dominates the computational cost.

Theorem 6. *Let* $0 < r \le 1$. *Consider the fractional differential Stein matrix equation*

$$\begin{cases} P^{(r)}(\vartheta) = HP(\vartheta)L - P(\vartheta) + M, & \vartheta \ge 0, \\ P(0) = P_0, \end{cases} \tag{3.19}$$

where

$$H \in \mathbb{R}^{m_1 \times m_1}, \qquad L \in \mathbb{R}^{m_2 \times m_2}, \qquad P_0, M \in \mathbb{R}^{m_1 \times m_2}.$$

Assume that H and L are diagonalizable, namely,

$$H = Q\Lambda Q^{-1}, \qquad L = \Psi\Omega\Psi^{-1},$$

where

$$\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_{m_1}), \qquad \Omega = \operatorname{diag}(\omega_1, \ldots, \omega_{m_2}).$$

Define

$$\Delta^{\Omega,\Lambda} = \left[\lambda_i \omega_j - 1\right]_{m_1 \times m_2}.$$

Then, the unique exact solution of Eq (3.19) *is*

$$P(\vartheta) = Q\Big[\mathbb{E}_{r,1}^{\Omega,\Lambda}(\vartheta^r) \odot \left(Q^{-1}P_0\Psi\right) + \vartheta^r \mathbb{E}_{r,r+1}^{\Omega,\Lambda}(\vartheta^r) \odot \left(Q^{-1}M\Psi\right)\Big]\Psi^{-1}. \tag{3.20}$$

Proof. Set

$$Z_0 = Q^{-1}P_0\Psi, \qquad S = Q^{-1}M\Psi,$$

and define

$$Z(\vartheta) = \mathbb{E}_{r,1}^{\Omega,\Lambda}(\vartheta^r) \odot Z_0 + \vartheta^r \mathbb{E}_{r,r+1}^{\Omega,\Lambda}(\vartheta^r) \odot S. \tag{3.21}$$

With this notation, Eq (3.20) can be written as

$$P(\vartheta) = QZ(\vartheta)\Psi^{-1}.$$

We first verify the initial condition. Since

$$E_{r,1}(0) = 1,$$

we have

$$\mathbb{E}_{r,1}^{\Omega,\Lambda}(0) = \mathbf{1}_{m_1\times m_2},$$

where $\mathbf{1}_{m_1\times m_2}$ denotes the matrix whose entries are all equal to one. Moreover,

$$0^r\mathbb{E}_{r,r+1}^{\Omega,\Lambda}(0) = 0_{m_1\times m_2}.$$

Consequently,

$$Z(0) = \mathbf{1}_{m_1\times m_2} \odot Z_0 = Z_0.$$

It follows that

$$P(0) = QZ(0)\Psi^{-1} = Q\left(Q^{-1}P_0\Psi\right)\Psi^{-1} = P_0.$$

We now verify the fractional differential equation. Since Q and Ψ^{-1} are constant matrices and the Caputo fractional derivative is linear, we obtain

$$P^{(r)}(\vartheta) = QZ^{(r)}(\vartheta)\Psi^{-1}.$$

By applying Lemma 2, we have

$${}^{C}D_{0+}^{r}\mathbb{E}_{r,1}^{\Omega,\Lambda}(\vartheta^r) = \Delta^{\Omega,\Lambda} \odot \mathbb{E}_{r,1}^{\Omega,\Lambda}(\vartheta^r),$$

and

$${}^{C}D_{0+}^{r}\left[\vartheta^r\mathbb{E}_{r,r+1}^{\Omega,\Lambda}(\vartheta^r)\right] = \mathbb{E}_{r,1}^{\Omega,\Lambda}(\vartheta^r).$$

Therefore,

$$Z^{(r)}(\vartheta) = \Delta^{\Omega,\Lambda} \odot \mathbb{E}_{r,1}^{\Omega,\Lambda}(\vartheta^r) \odot Z_0 + \mathbb{E}_{r,1}^{\Omega,\Lambda}(\vartheta^r) \odot S. \tag{3.22}$$

For every scalar a, the following identity holds:

$$E_{r,1}(a\vartheta^r) = 1 + a\vartheta^r E_{r,r+1}(a\vartheta^r).$$

Hence, entrywise,

$$\mathbb{E}_{r,1}^{\Omega,\Lambda}(\vartheta^r) = \mathbf{1}_{m_1\times m_2} + \vartheta^r\Delta^{\Omega,\Lambda} \odot \mathbb{E}_{r,r+1}^{\Omega,\Lambda}(\vartheta^r). \tag{3.23}$$

Using Eq (3.23), we obtain

$$\mathbb{E}_{r,1}^{\Omega,\Lambda}(\vartheta^r) \odot S = S + \vartheta^r\Delta^{\Omega,\Lambda} \odot \mathbb{E}_{r,r+1}^{\Omega,\Lambda}(\vartheta^r) \odot S.$$

Substituting this expression into Eq (3.22) gives

$$Z^{(r)}(\vartheta) = \Delta^{\Omega,\Lambda} \odot \mathbb{E}_{r,1}^{\Omega,\Lambda}(\vartheta^r) \odot Z_0 + \vartheta^r\Delta^{\Omega,\Lambda} \odot \mathbb{E}_{r,r+1}^{\Omega,\Lambda}(\vartheta^r) \odot S + S.$$

By using Eq (3.21), we conclude that

$$Z^{(r)}(\vartheta) = \Delta^{\Omega,\Lambda} \odot Z(\vartheta) + S. \tag{3.24}$$

Let

$$Z(\vartheta) = \left[z_{ij}(\vartheta)\right]_{m_1\times m_2}.$$

The (i, j)-th entry of $\Lambda Z(\vartheta)\Omega - Z(\vartheta)$ is

$$\lambda_i z_{ij}(\vartheta)\omega_j - z_{ij}(\vartheta) = (\lambda_i\omega_j - 1)z_{ij}(\vartheta).$$

Therefore,

$$\Lambda Z(\vartheta)\Omega - Z(\vartheta) = \Delta^{\Omega,\Lambda} \odot Z(\vartheta).$$

Equation (3.24) can thus be rewritten as

$$Z^{(r)}(\vartheta) = \Lambda Z(\vartheta)\Omega - Z(\vartheta) + S. \tag{3.25}$$

Finally, by using

$$P(\vartheta) = QZ(\vartheta)\Psi^{-1}, \qquad H = Q\Lambda Q^{-1}, \qquad L = \Psi\Omega\Psi^{-1},$$

we obtain

$$\begin{aligned}P^{(r)}(\vartheta) &= QZ^{(r)}(\vartheta)\Psi^{-1}\\ &= Q\,[\Lambda Z(\vartheta)\Omega - Z(\vartheta) + S\,]\,\Psi^{-1}\\ &= Q\Lambda Q^{-1}\left[QZ(\vartheta)\Psi^{-1}\right]\Psi\Omega\Psi^{-1} - QZ(\vartheta)\Psi^{-1} + QS\,\Psi^{-1}\\ &= HP(\vartheta)L - P(\vartheta) + M.\end{aligned}$$

Thus, Eq (3.20) satisfies both the fractional differential equation and the initial condition. Since the solution of Eq (3.19) is unique, Eq (3.20) is the unique exact solution. □

Remark 4 (Structure preservation). *If H and L are symmetric, then Q and Ψ can be chosen orthogonal ($Q^{-1} = Q^T$, $\Psi^{-1} = \Psi^T$), and the solution formula simplifies to:*

$$P(\vartheta) = Q\Big[\mathbb{E}^{\Omega,\Lambda}_{r,1}(\vartheta^r) \odot (Q^T P_0\Psi) + \vartheta^r\mathbb{E}^{\Omega,\Lambda}_{r,r+1}(\vartheta^r) \odot (Q^T M\Psi)\Big]\Psi^T.$$

For the integer-order case, define the rectangular matrix

$$\Delta = [\delta_{ij}]_{m_1\times m_2}, \qquad \delta_{ij} = \lambda_i\omega_j - 1,$$

and define the element-wise functions

$$\exp^{\odot}(\Delta\vartheta) = \left[e^{\delta_{ij}\vartheta}\right]_{m_1\times m_2},$$

$$\Phi_\vartheta(\Delta) = \left[\phi_\vartheta(\delta_{ij})\right]_{m_1\times m_2}, \qquad \phi_\vartheta(x) = \begin{cases}\dfrac{e^{x\vartheta} - 1}{x}, & x \neq 0,\\ \vartheta, & x = 0.\end{cases}$$

Corollary 1. *For $r = 1$, the solution of the classical Stein equation (1.2) is*

$$P(\vartheta) = Q\left[\exp^{\odot}(\Delta\vartheta) \odot (Q^{-1}P_0\Psi) + \Phi_\vartheta(\Delta) \odot (Q^{-1}M\Psi)\right]\Psi^{-1}. \tag{3.26}$$

Proof. The result follows from Theorem 5, $E_{1,1}(x) = e^x$, and $\vartheta E_{1,2}(x\vartheta) = (e^{x\vartheta} - 1)/x$ for $x \neq 0$. The value at $x = 0$ is obtained by continuity. □

Remark 5. *Degenerate cases:*

(1) When $\lambda_i\omega_j - 1 = 0$ in the fractional case,

$$\lim_{\alpha\to 0} \vartheta^r E_{r,r+1}(\alpha\vartheta^r) = \frac{\vartheta^r}{\Gamma(r+1)}.$$

(2) When $\lambda_i\omega_j - 1 = 0$ in the classical case,

$$\lim_{\alpha\to 0} \frac{e^{\alpha\vartheta} - 1}{\alpha} = \vartheta.$$

These limits ensure continuity of the solution formulas. When multiple index pairs (i, j) simultaneously satisfy $\lambda_i\omega_j - 1 = 0$, the solution remains well-defined and continuous as each component is treated independently.

Perturbation of a defective Jordan block

The separate diagonalizability assumptions on the coefficient matrices H and L are only required to derive the compact Hadamard-product representation involving scalar Mittag–Leffler functions. They are not necessary for the existence and uniqueness of the solution. To illustrate this distinction, we consider a case in which both coefficient matrices are equal to the same defective Jordan block.

Let

$$H = L = J, \qquad J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad P_0 = I_2, \qquad M = 0.$$

The FDSME becomes

$$P^{(r)}(\vartheta) = JP(\vartheta)J - P(\vartheta), \qquad P(0) = I_2, \qquad 0 < r \leq 1. \tag{3.27}$$

Since J has a unique eigenvalue of algebraic multiplicity two and only one linearly independent eigenvector, it is not diagonalizable. Let

$$P(\vartheta) = \begin{pmatrix} p_{11}(\vartheta) & p_{12}(\vartheta) \\ p_{21}(\vartheta) & p_{22}(\vartheta) \end{pmatrix}.$$

A direct calculation gives

$$JP(\vartheta)J - P(\vartheta) = \begin{pmatrix} p_{21}(\vartheta) & p_{11}(\vartheta) + p_{21}(\vartheta) + p_{22}(\vartheta) \\ 0 & p_{21}(\vartheta) \end{pmatrix}.$$

Therefore, Eq (3.27) is equivalent to

$$\begin{cases} p_{11}^{(r)}(\vartheta) = p_{21}(\vartheta), \\ p_{12}^{(r)}(\vartheta) = p_{11}(\vartheta) + p_{21}(\vartheta) + p_{22}(\vartheta), \\ p_{21}^{(r)}(\vartheta) = 0, \\ p_{22}^{(r)}(\vartheta) = p_{21}(\vartheta), \end{cases}$$

subject to

$$p_{11}(0) = 1, \qquad p_{12}(0) = 0, \qquad p_{21}(0) = 0, \qquad p_{22}(0) = 1.$$

It follows that

$$p_{21}(\vartheta) = 0, \qquad p_{11}(\vartheta) = p_{22}(\vartheta) = 1.$$

Consequently,

$$p_{12}^{(r)}(\vartheta) = 2, \qquad p_{12}(0) = 0.$$

Using

$${}^{C}D_{0+}^{r}\left(\frac{\vartheta^{r}}{\Gamma(r+1)}\right) = 1,$$

we obtain

$$p_{12}(\vartheta) = \frac{2\vartheta^{r}}{\Gamma(r+1)}.$$

Hence, the exact solution of Eq (3.27) is

$$P(\vartheta) = \begin{pmatrix} 1 & \dfrac{2\vartheta^{r}}{\Gamma(r+1)} \\ 0 & 1 \end{pmatrix}. \tag{3.28}$$

We now recover this solution through a family of diagonalizable perturbations. For $\varepsilon \neq 0$, define

$$H_{\varepsilon} = L_{\varepsilon} = J_{\varepsilon} = \begin{pmatrix} 1-\varepsilon & 1 \\ 0 & 1+\varepsilon \end{pmatrix}.$$

The two eigenvalues of J_{ε} are distinct:

$$\lambda_{1}^{(\varepsilon)} = 1-\varepsilon, \qquad \lambda_{2}^{(\varepsilon)} = 1+\varepsilon.$$

Thus, J_{ε} is diagonalizable and can be written as

$$J_{\varepsilon} = Q_{\varepsilon}\Lambda_{\varepsilon}Q_{\varepsilon}^{-1}, \tag{3.29}$$

where

$$Q_{\varepsilon} = \begin{pmatrix} 1 & 1 \\ 0 & 2\varepsilon \end{pmatrix}, \qquad \Lambda_{\varepsilon} = \begin{pmatrix} 1-\varepsilon & 0 \\ 0 & 1+\varepsilon \end{pmatrix},$$

and

$$Q_{\varepsilon}^{-1} = \begin{pmatrix} 1 & -\dfrac{1}{2\varepsilon} \\ 0 & \dfrac{1}{2\varepsilon} \end{pmatrix}.$$

Consider the perturbed FDSME

$$P_{\varepsilon}^{(r)}(\vartheta) = J_{\varepsilon}P_{\varepsilon}(\vartheta)J_{\varepsilon} - P_{\varepsilon}(\vartheta), \qquad P_{\varepsilon}(0) = I_{2}. \tag{3.30}$$

Using the transformation

$$P_{\varepsilon}(\vartheta) = Q_{\varepsilon}Z_{\varepsilon}(\vartheta)Q_{\varepsilon}^{-1},$$

we obtain

$$Z_{\varepsilon}^{(r)}(\vartheta) = \Lambda_{\varepsilon}Z_{\varepsilon}(\vartheta)\Lambda_{\varepsilon} - Z_{\varepsilon}(\vartheta), \qquad Z_{\varepsilon}(0) = I_{2}.$$

Since the initial matrix is diagonal, the transformed solution remains diagonal. Therefore,

$$Z_\varepsilon(\vartheta) = \begin{pmatrix} A_\varepsilon(\vartheta) & 0 \\ 0 & B_\varepsilon(\vartheta) \end{pmatrix},$$

where

$$A_\varepsilon(\vartheta) = E_{r,1}\left(\left[(1-\varepsilon)^2 - 1\right]\vartheta^r\right) = E_{r,1}\left((-2\varepsilon + \varepsilon^2)\vartheta^r\right),$$

and

$$B_\varepsilon(\vartheta) = E_{r,1}\left(\left[(1+\varepsilon)^2 - 1\right]\vartheta^r\right) = E_{r,1}\left((2\varepsilon + \varepsilon^2)\vartheta^r\right).$$

Consequently,

$$P_\varepsilon(\vartheta) = \begin{pmatrix} A_\varepsilon(\vartheta) & \dfrac{B_\varepsilon(\vartheta) - A_\varepsilon(\vartheta)}{2\varepsilon} \\ 0 & B_\varepsilon(\vartheta) \end{pmatrix}. \tag{3.31}$$

To evaluate the limit as $\varepsilon \to 0$, we use the local expansion

$$E_{r,1}(z) = 1 + \frac{z}{\Gamma(r+1)} + O(z^2), \qquad z \to 0.$$

It follows that

$$A_\varepsilon(\vartheta) \longrightarrow 1, \qquad B_\varepsilon(\vartheta) \longrightarrow 1.$$

Moreover,

$$\lim_{\varepsilon\to 0} \frac{B_\varepsilon(\vartheta) - A_\varepsilon(\vartheta)}{2\varepsilon} = \lim_{\varepsilon\to 0} \frac{E_{r,1}\Big((2\varepsilon + \varepsilon^2)\vartheta^r\Big) - E_{r,1}\Big((-2\varepsilon + \varepsilon^2)\vartheta^r\Big)}{2\varepsilon} = \frac{2\vartheta^r}{\Gamma(r+1)}.$$

Hence,

$$\lim_{\varepsilon\to 0} P_\varepsilon(\vartheta) = \begin{pmatrix} 1 & \dfrac{2\vartheta^r}{\Gamma(r+1)} \\ 0 & 1 \end{pmatrix} = P(\vartheta).$$

Thus, the exact solution associated with the defective Jordan block is recovered as the limit of solutions corresponding to diagonalizable perturbations.

For the classical case $r = 1$, the Mittag–Leffler function reduces to the exponential function. In this case,

$$A_\varepsilon(\vartheta) = e^{(-2\varepsilon+\varepsilon^2)\vartheta}, \qquad B_\varepsilon(\vartheta) = e^{(2\varepsilon+\varepsilon^2)\vartheta},$$

and

$$\lim_{\varepsilon\to 0} P_\varepsilon(\vartheta) = \begin{pmatrix} 1 & 2\vartheta \\ 0 & 1 \end{pmatrix}.$$

3.2. *Equal-coefficient case: L = H*

An important special case arises when $L = H$. If, in addition, H, P_0, and M are symmetric, the solution preserves symmetry. The equal-coefficient FDSME is

$$\begin{cases} P^{(r)}(\vartheta) = HP(\vartheta)H - P(\vartheta) + M, & \vartheta \geq 0, \\ P(0) = P_0. \end{cases} \tag{3.32}$$

Proposition 1. *If H, P_0, and M are symmetric, then the solution $P(\vartheta)$ is symmetric for all $\vartheta \geq 0$.*

Proof. We have

$$\begin{aligned} P^{(r)}(\vartheta)^T &= (HP(\vartheta)H - P(\vartheta) + M)^T \\ &= (HP(\vartheta)H)^T - P(\vartheta)^T + M^T \\ &= HP(\vartheta)^T H - P(\vartheta)^T + M, \end{aligned}$$

and $P(0)^T = P_0$ since Eq (3.32) admits a unique solution, then $P^T(\vartheta) = P(\vartheta)$. □

Corollary 2. *For H with diagonalization $H = Q\Lambda Q^{-1}$, the solution of Eq (3.32) is given by*

$$P(\vartheta) = Q\Big[\mathbb{E}_{r,1}^{\Lambda,\Lambda}(\vartheta^r) \odot (Q^{-1}P_0Q) + \vartheta^r \mathbb{E}_{r,r+1}^{\Lambda,\Lambda}(\vartheta^r) \odot (Q^{-1}MQ)\Big]Q^{-1}, \tag{3.33}$$

where $\mathbb{E}_{\gamma,r}^{\Lambda,\Lambda}(\mu)$ has entries $E_{\gamma,r}((\lambda_i\lambda_j - 1)\mu)$.

Proof. The symmetric case $L = H$ is obtained from Theorem 5 by setting

$$L = H, \quad \Psi = Q, \quad \Omega = \Lambda.$$

Substituting these into the general solution formula (3.20), we have

$$\begin{aligned} P(\vartheta) &= Q\Big[\mathbb{E}_{r,1}^{\Omega,\Lambda}(\vartheta^r) \odot (Q^{-1}P_0\Psi) + \vartheta^r \mathbb{E}_{r,r+1}^{\Omega,\Lambda}(\vartheta^r) \odot (Q^{-1}M\Psi)\Big]\Psi^{-1} \\ &= Q\Big[\mathbb{E}_{r,1}^{\Lambda,\Lambda}(\vartheta^r) \odot (Q^{-1}P_0Q) + \vartheta^r \mathbb{E}_{r,r+1}^{\Lambda,\Lambda}(\vartheta^r) \odot (Q^{-1}MQ)\Big]Q^{-1}. \end{aligned}$$

This establishes the formula (3.33). □

Remark 6. *For $r = 1$, define*

$$\Delta_H = [\lambda_i\lambda_j - 1]_{i,j=1}^{m_1}.$$

Then, the equal-coefficient solution is

$$P(\vartheta) = Q\left[\exp^{\odot}(\Delta_H\vartheta) \odot (Q^{-1}P_0Q) + \Phi_\vartheta(\Delta_H) \odot (Q^{-1}MQ)\right]Q^{-1}, \tag{3.34}$$

where all functions applied to Δ_H are interpreted element-wise. In general, Δ_H is not the diagonal matrix $\Lambda^2 - I$ because its (i, j)-th entry is $\lambda_i\lambda_j - 1$.

4. Illustrative examples

This section provides detailed examples demonstrating the application of our method. We present the solutions as functions of the fractional order $r \in (0, 1]$.

Example 1. *Consider the equal-coefficient FDSME. Note that M is not symmetric, so this example illustrates the case $L = H$ but not the symmetry-preservation proposition:*

$$\begin{cases} P^{(r)}(\vartheta) = HP(\vartheta)H - P(\vartheta) + M, \quad \vartheta \geq 0, \\ P(0) = P_0, \end{cases}$$

where

$$H = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \quad P_0 = I_2, \quad M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Step 1: Diagonalization of H

The eigenvalues of H are $\lambda_1 = -1$ *and* $\lambda_2 = -3$ *with eigenvectors* $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *and* $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$*. Thus,*

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad Q^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}.$$

Step 2: Transformed matrices

$$Q^{-1}P_0Q = Q^{-1}I_2Q = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$Q^{-1}MQ = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

A direct multiplication gives

$$MQ = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}, \qquad Q^{-1}(MQ) = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}.$$

Thus,

$$S = Q^{-1}MQ = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}.$$

Step 3: Arguments for Mittag-Leffler functions

Since $\Lambda = \operatorname{diag}(-1, -3)$*, we compute* $\lambda_i\lambda_j - 1$*:*

$$\begin{aligned} \lambda_1\lambda_1 - 1 &= (-1)(-1) - 1 = 1 - 1 = 0, \\ \lambda_1\lambda_2 - 1 &= (-1)(-3) - 1 = 3 - 1 = 2, \\ \lambda_2\lambda_1 - 1 &= (-3)(-1) - 1 = 3 - 1 = 2, \\ \lambda_2\lambda_2 - 1 &= (-3)(-3) - 1 = 9 - 1 = 8. \end{aligned}$$

Therefore,

$$\mathbb{E}_{r,1}^{\Lambda,\Lambda}(\vartheta^r) = \begin{pmatrix} E_{r,1}(0) & E_{r,1}(2\vartheta^r) \\ E_{r,1}(2\vartheta^r) & E_{r,1}(8\vartheta^r) \end{pmatrix},$$

$$\mathbb{E}_{r,r+1}^{\Lambda,\Lambda}(\vartheta^r) = \begin{pmatrix} E_{r,r+1}(0) & E_{r,r+1}(2\vartheta^r) \\ E_{r,r+1}(2\vartheta^r) & E_{r,r+1}(8\vartheta^r) \end{pmatrix}.$$

Step 4: Exact solution

According to Corollary 2, we have

$$P(\vartheta) = Q\Big[\mathbb{E}_{r,1}^{\Lambda,\Lambda}(\vartheta^r) \odot I_2 + \vartheta^r\mathbb{E}_{r,r+1}^{\Lambda,\Lambda}(\vartheta^r) \odot S\Big]Q^{-1}.$$

Let

$$Z(\vartheta) = \mathbb{E}_{r,1}^{\Lambda,\Lambda}(\vartheta^r) \odot I_2 + \vartheta^r\mathbb{E}_{r,r+1}^{\Lambda,\Lambda}(\vartheta^r) \odot S.$$

Then,

$$\begin{aligned}
Z(\vartheta) &= \mathbb{E}^{\Lambda,\Lambda}_{r,1}(\vartheta^r) \odot I_2 + \vartheta^r \mathbb{E}^{\Lambda,\Lambda}_{r,r+1}(\vartheta^r) \odot S \\
&= \begin{pmatrix} E_{r,1}(0) & E_{r,1}(2\vartheta^r) \\ E_{r,1}(2\vartheta^r) & E_{r,1}(8\vartheta^r) \end{pmatrix} \odot I_2 + \vartheta^r \begin{pmatrix} E_{r,r+1}(0) & E_{r,r+1}(2\vartheta^r) \\ E_{r,r+1}(2\vartheta^r) & E_{r,r+1}(8\vartheta^r) \end{pmatrix} \odot S \\
&= \begin{pmatrix} E_{r,1}(0) & 0 \\ 0 & E_{r,1}(8\vartheta^r) \end{pmatrix} + \frac{1}{2}\vartheta^r \begin{pmatrix} E_{r,r+1}(0) & E_{r,r+1}(2\vartheta^r) \\ E_{r,r+1}(2\vartheta^r) & E_{r,r+1}(8\vartheta^r) \end{pmatrix} \odot \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} E_{r,1}(0) + \frac{3}{2}\vartheta^r E_{r,r+1}(0) & -\frac{1}{2}\vartheta^r E_{r,r+1}(2\vartheta^r) \\ \frac{1}{2}\vartheta^r E_{r,r+1}(2\vartheta^r) & E_{r,1}(8\vartheta^r) + \frac{1}{2}\vartheta^r E_{r,r+1}(8\vartheta^r) \end{pmatrix} \\
&= \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
Z_{11} &= E_{r,1}(0) + \frac{3}{2}\vartheta^r E_{r,r+1}(0), \\
Z_{12} &= -\frac{1}{2}\vartheta^r E_{r,r+1}(2\vartheta^r), \\
Z_{21} &= \frac{1}{2}\vartheta^r E_{r,r+1}(2\vartheta^r), \\
Z_{22} &= E_{r,1}(8\vartheta^r) + \frac{1}{2}\vartheta^r E_{r,r+1}(8\vartheta^r).
\end{aligned}$$

Finally,

$$P(\vartheta) = Q\,Z(\vartheta)\,Q^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

For $r = 1$*, the entries of* Z *become*

$$Z_{11} = 1 + \frac{3}{2}\vartheta, \qquad Z_{12} = -\frac{e^{2\vartheta} - 1}{4}, \qquad Z_{21} = \frac{e^{2\vartheta} - 1}{4}, \qquad Z_{22} = e^{8\vartheta} + \frac{e^{8\vartheta} - 1}{16}.$$

Therefore,

$$P(\vartheta) = \begin{pmatrix} \dfrac{3\vartheta}{4} + \dfrac{17e^{8\vartheta}}{32} + \dfrac{15}{32} & \dfrac{3\vartheta}{4} - \dfrac{17e^{8\vartheta}}{32} + \dfrac{e^{2\vartheta}}{4} + \dfrac{9}{32} \\ \dfrac{3\vartheta}{4} - \dfrac{17e^{8\vartheta}}{32} - \dfrac{e^{2\vartheta}}{4} + \dfrac{25}{32} & \dfrac{3\vartheta}{4} + \dfrac{17e^{8\vartheta}}{32} + \dfrac{15}{32} \end{pmatrix}. \tag{4.1}$$

One directly verifies that $P_1(0) = I_2$*.*

We compare them with the numerical solutions obtained by the method reported in [13]. The numerical approximation is computed on a fine time mesh over the whole interval. The figures below provide a qualitative visual comparison between the Frobenius norm of analytical solution and the Frobenius norm of numerical approximation. In the present validation, we use $\vartheta \in [0, 1]$ *with* $N = 1000$ *for different* $r = 0.5, 0.7, 0.9, 1$*. Figure 1 compares all entries of the analytical and numerical solutions on a fine time mesh over* $[0, 1]$*. Figure 2 compares* $P_{11}(\vartheta)$ *of the analytical and numerical solutions on a fine time mesh over* $[0, 1]$*.*

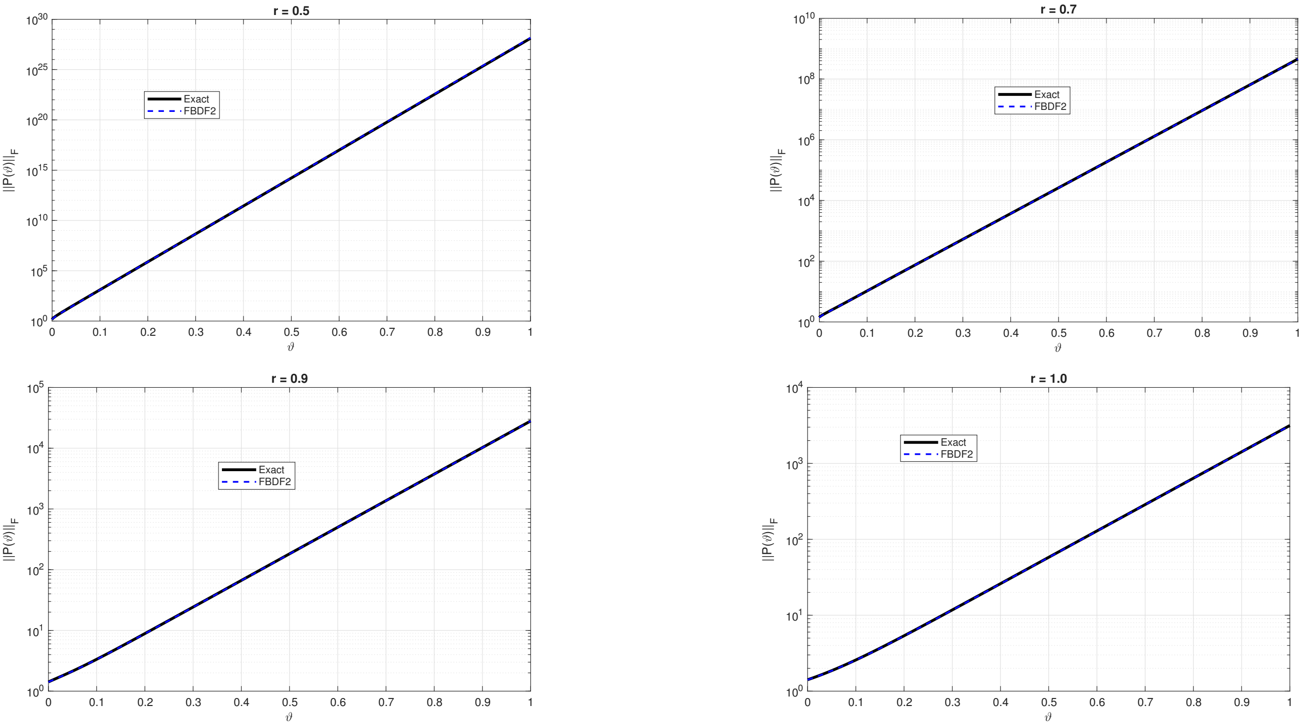

Figure 1. Comparison between the Frobenius norm of analytical solution and the Frobenius norm of numerical approximation for Example 1 on a fine time mesh over $[0, 1]$.

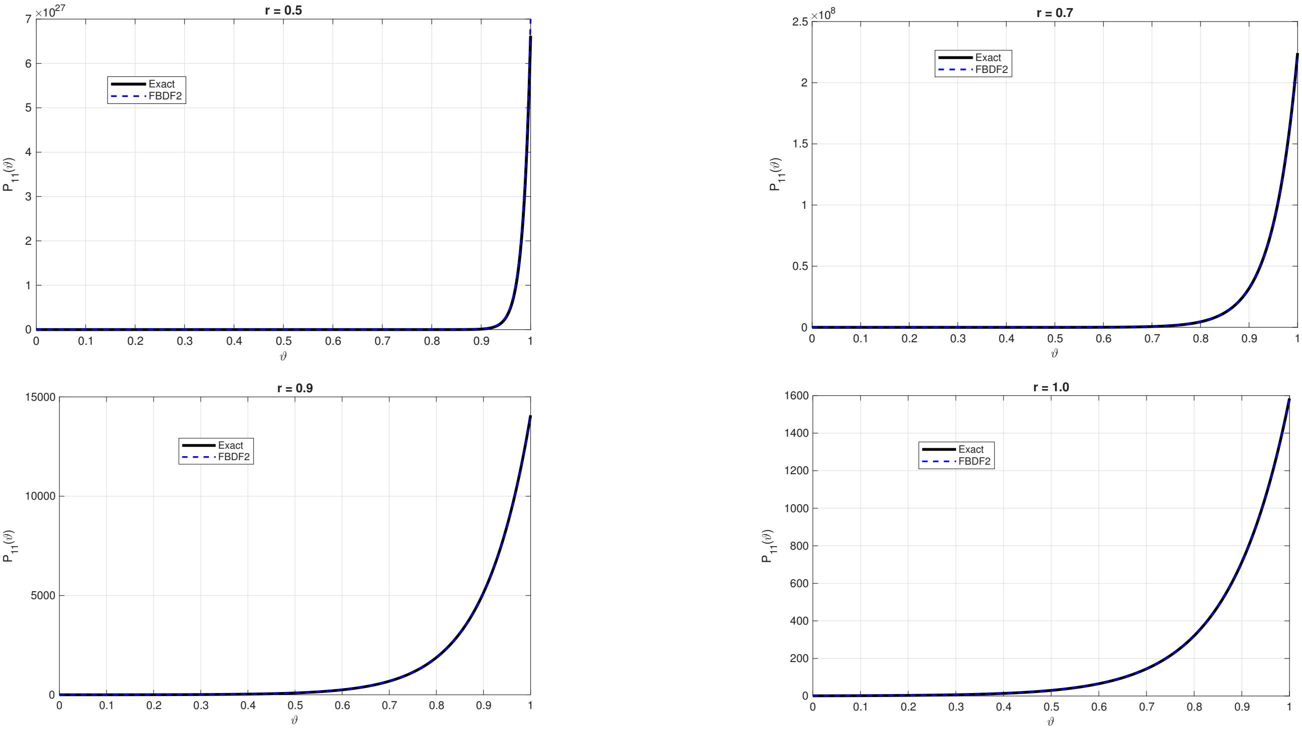

Figure 2. Comparison $P_{11}(\vartheta)$ of the analytical and numerical solutions for Example 1 on a fine time mesh over $[0, 1]$.

Example 2. *Consider the nonsymmetric FDSME:*

$$\begin{cases} P^{(r)}(\vartheta) = HP(\vartheta)L - P(\vartheta) + M, \quad \vartheta \geq 0, \ r \in (0,1], \\ P(0) = P_0, \end{cases}$$

where

$$H = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \quad L = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad P_0 = I_2, \quad M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(1) ***Diagonalization of*** H***:***

$$\lambda_1 = 1, \quad \lambda_2 = 3,$$

$$Q_H = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad Q_H^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

(2) ***Diagonalization of*** L***:***

$$\omega_1 = 1, \quad \omega_2 = 3,$$

$$Q_L = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad Q_L^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

(3) ***Transformed matrices:***

$$Z_0 = Q_H^{-1} P_0 Q_L = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix},$$

$$S = Q_H^{-1} M Q_L = \begin{pmatrix} -2 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then, according to Theorem 5, the solution is

$$P_r(\vartheta) = Q_H \Big[\mathbb{E}_{r,1}^{\Omega,\Lambda}(\vartheta^r) \odot Z_0 + \vartheta^r \mathbb{E}_{r,r+1}^{\Omega,\Lambda}(\vartheta^r) \odot S \Big] Q_L^{-1},$$

where

$$\mathbb{E}_{r,1}^{\Omega,\Lambda}(\vartheta^r) = \begin{pmatrix} E_{r,1}(0) & E_{r,1}(2\vartheta^r) \\ E_{r,1}(2\vartheta^r) & E_{r,1}(8\vartheta^r) \end{pmatrix}, \qquad \mathbb{E}_{r,r+1}^{\Omega,\Lambda}(\vartheta^r) = \begin{pmatrix} E_{r,r+1}(0) & E_{r,r+1}(2\vartheta^r) \\ E_{r,r+1}(2\vartheta^r) & E_{r,r+1}(8\vartheta^r) \end{pmatrix}.$$

Thus,

$$P_r(\vartheta) = Q_H \, Z(\vartheta) \, Q_L^{-1},$$

where

$$Z(\vartheta) = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}$$

with

$$Z_{11} = 2E_{r,1}(0) - 2\vartheta^r E_{r,r+1}(0),$$

$$Z_{12} = 0,$$

$$Z_{21} = -E_{r,1}(2\vartheta^r) + \vartheta^r E_{r,r+1}(2\vartheta^r),$$
$$Z_{22} = E_{r,1}(8\vartheta^r) + \vartheta^r E_{r,r+1}(8\vartheta^r).$$

Explicitly,

$$P_r(\vartheta) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

We compare them with the numerical solutions obtained by the method reported in [13]. The numerical approximation is computed on a fine time mesh over the whole interval. The figures below provide a qualitative visual comparison between the Frobenius norm of analytical solution and the Frobenius norm of numerical approximation. In the present validation, we use $\vartheta \in [0, 2]$ *with* $N = 1000$ *for different* $r = 0.9, 1$. *Figure 3 compares all entries of the analytical and numerical solutions on a fine time mesh over* $[0, 2]$. *Figure 4 compares* $P_{11}(\vartheta)$ *of the analytical and numerical solutions on a fine time mesh over* $[0, 2]$.

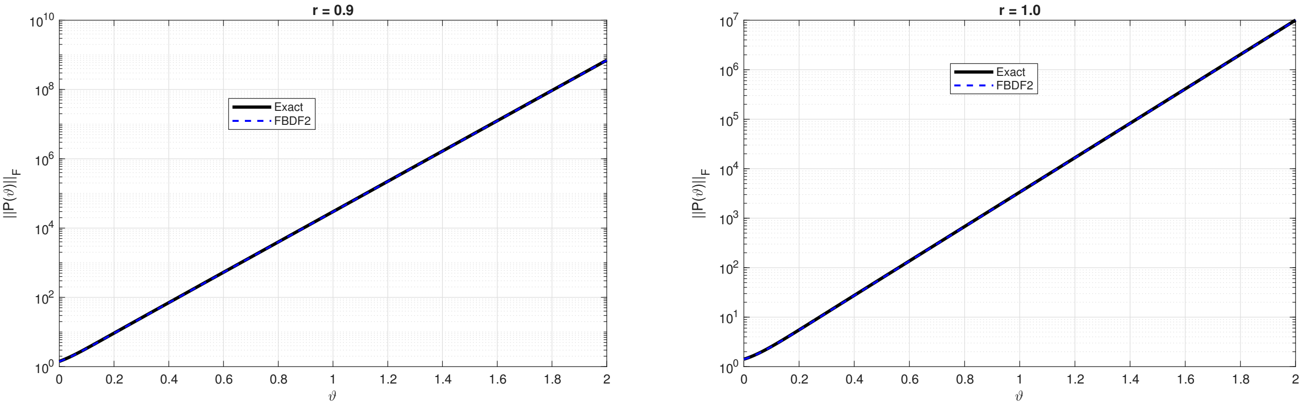

Figure 3. Comparison between the Frobenius norm of analytical solution and the Frobenius norm of numerical approximation for Example 2 on a fine time mesh over $[0, 2]$.

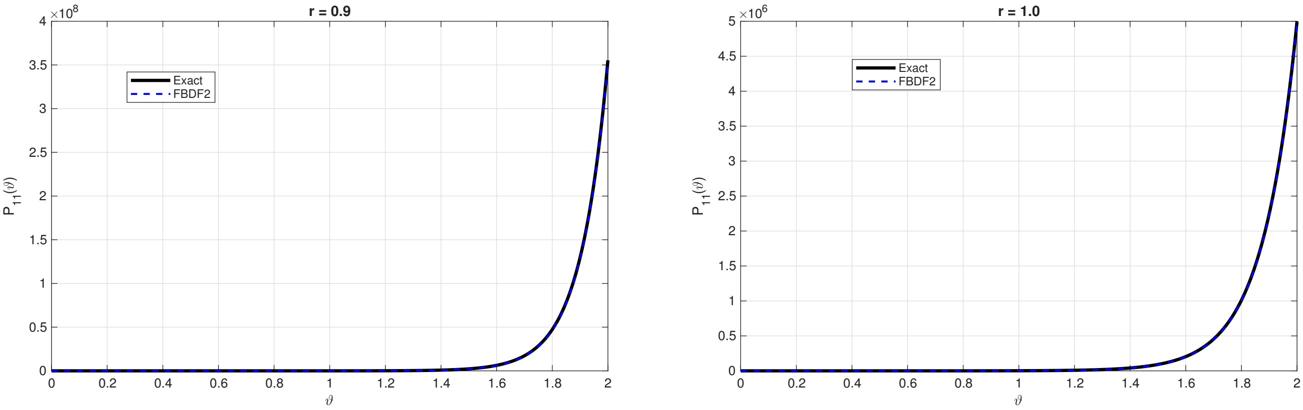

Figure 4. Comparison $P_{11}(\vartheta)$ of the analytical and numerical solutions for Example 2 on a fine time mesh over $[0, 2]$.

Example 3. *Consider a* 3×3 *symmetric FDSME:*

$$H = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad P_0 = I_3, \quad M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

The eigenvalues are

$$\begin{aligned} \lambda_1 &= 3 + \sqrt{3} \approx 4.732, \\ \lambda_2 &= 3, \\ \lambda_3 &= 3 - \sqrt{3} \approx 1.268. \end{aligned}$$

Since H is symmetric, we choose an orthogonal eigenvector matrix:

$$Q = \begin{pmatrix} \dfrac{1+\sqrt{3}}{2\sqrt{3}} & \dfrac{1}{\sqrt{3}} & \dfrac{1-\sqrt{3}}{2\sqrt{3}} \\ \dfrac{1}{\sqrt{3}} & -\dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} \\ \dfrac{\sqrt{3}-1}{2\sqrt{3}} & -\dfrac{1}{\sqrt{3}} & -\dfrac{1+\sqrt{3}}{2\sqrt{3}} \end{pmatrix}, \qquad Q^{-1} = Q^T.$$

A direct verification gives

$$Q^T H Q = \mathrm{diag}(3+\sqrt{3}, 3, 3-\sqrt{3}),$$

and

$$Q^T M Q = \begin{pmatrix} \frac{5}{3} & -\frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{1}{3} & -\frac{2}{3} & \frac{5}{3} \end{pmatrix}.$$

The solution is given by

$$P_r(\vartheta) = Q\Big[\mathbb{E}_{r,1}^{\Lambda,\Lambda}(\vartheta^r) \odot I_3 + \vartheta^r \mathbb{E}_{r,r+1}^{\Lambda,\Lambda}(\vartheta^r) \odot (Q^T M Q)\Big] Q^T,$$

where the (i,j)*-th element of* $\mathbb{E}_{r,1}^{\Lambda,\Lambda}(\vartheta^r)$ *is* $E_{r,1}((\lambda_i\lambda_j - 1)\vartheta^r)$ *and the* (i,j)*-th element of* $\mathbb{E}_{r,r+1}^{\Lambda,\Lambda}(\vartheta^r)$ *is* $E_{r,r+1}((\lambda_i\lambda_j - 1)\vartheta^r)$.

We compare them with the numerical solutions obtained by the method reported in [13]. The numerical approximation is computed on a fine time mesh over the whole interval. The figures below provide a qualitative visual comparison between the Frobenius norm of analytical solution and the Frobenius norm of numerical approximation. In the present validation, we use $\vartheta \in [0,2]$ *with* $N = 1000$ *for different* $r = 0.9, 1$. *Figure 5 compares all entries of the analytical and numerical solutions on a fine time mesh over* $[0,2]$. *Figure 6 compares* $P_{11}(\vartheta)$ *of the analytical and numerical solutions on a fine time mesh over* $[0,2]$.

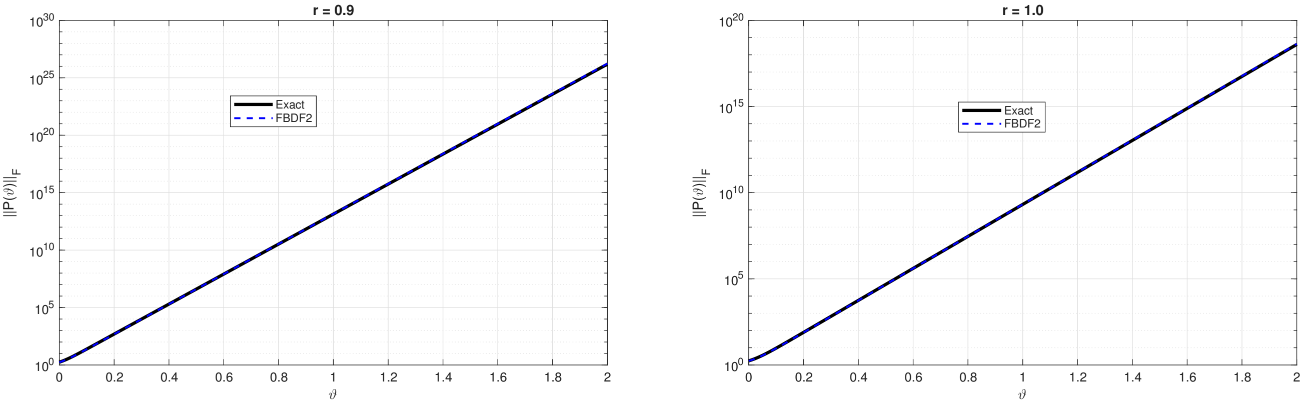

Figure 5. Comparison between the Frobenius norm of analytical solution and the Frobenius norm of numerical approximation for Example 3 on a fine time mesh over $[0, 2]$.

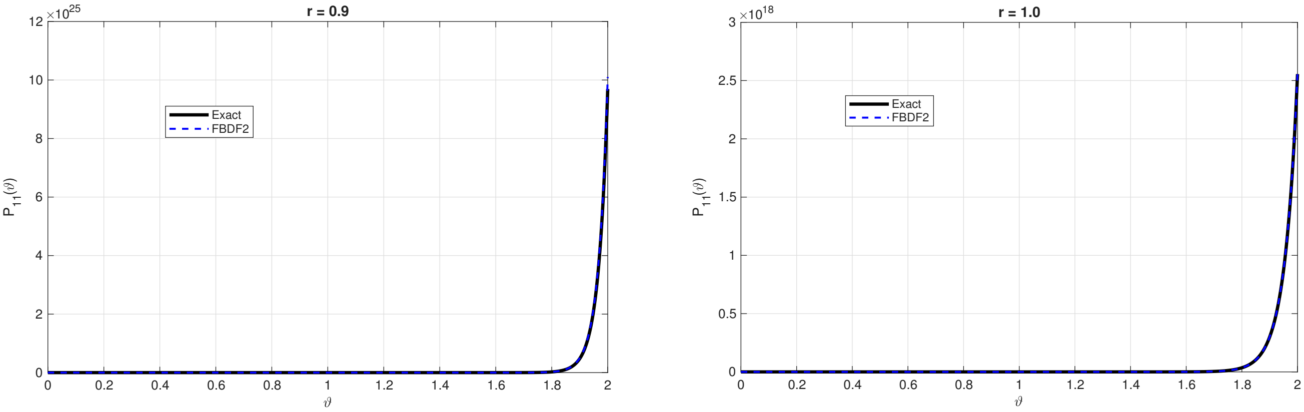

Figure 6. Comparison $P_{11}(\vartheta)$ of the analytical and numerical solutions for Example 3 on a fine time mesh over $[0, 2]$.

5. Conclusions

An explicit Hadamard product form of FDSMEs with diagonalizable matrices has been presented in this paper. First, the Hadamard product formulation of the solutions has been derived where no assumption on diagonalizability has been made. With the use of diagonalizability, the solution form can be evaluated in terms of the matrix Mittag–Leffler function from a decoupled equation of the form of scalar Caputo equations. The standard result for the ordinary equation is retrieved. The results have been illustrated using both equal-coefficients and nonsymmetric cases. Illustrative numerical examples compare our explicit forms with those obtained using the numerical method of Ref. [13] over a fine mesh of time intervals. Future works include Schur or Jordan formulations of the block matrix form, perturbation analysis, Krylov type algorithm for large-scale equations, and time-dependent coefficient matrices.

Author contributions

Lakhlifa Sadek and Ali Algefary: Conceptualization, Formal analysis, Funding acquisition, Investigation, Methodology, Writing–original draft, Writing–review and editing. All authors of this article have contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The Researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University (www.qu.edu.sa) for financial support (QU-APC-2026).

Funding

The Researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University for financial support (QU-APC-2026).

Conflict of interest

The authors declare that they have no competing interests.

References

1. L. Sadek, D. Baleanu, M. S. Abdo, W. Shatanawi, Introducing novel Θ-fractional operators: advances in fractional calculus, *J. King Saud Univ.-Sci.*, **36** (2024), 103352. https://doi.org/10.1016/j.jksus.2024.103352
2. L. Sadek, A cotangent fractional derivative with the application, *Fractal Fract.*, **7** (2023), 444. https://doi.org/10.3390/fractalfract7060444
3. W. M. Ahmad, R. El-Khazali, Fractional-order dynamical models of love, *Chaos Soliton. Fract.*, **33** (2007), 1367–1375. https://doi.org/10.1016/j.chaos.2006.01.098
4. A. D. Freed, K. Diethelm, Fractional calculus in biomechanics: a 3D viscoelastic model using regularized fractional derivative kernels with application to the human calcaneal fat pad, *Biomech. Model. Mechanobiol.*, **5** (2006), 203–215. https://doi.org/10.1007/s10237-005-0011-0
5. K. B. Oldham, J. Spanier, *The fractional calculus: theory and applications of differentiation and integration to arbitrary order*, 1 Ed., Vol. 111, Elsevier, 1974.
6. J. Sabatier, O. P. Agrawal, J. A. Tenreiro Machado, *Advances in fractional calculus*, Theoretical Developments and Applications in Physics and Engineering, Springer, 2007. https://doi.org/10.1007/978-1-4020-6042-7

7. O. Sadek, L. Sadek, S. Touhtouh, A. Hajjaji, The mathematical fractional modeling of TiO_2 nanopowder synthesis by sol–gel method at low temperature, *Math. Model. Comput.*, **9** (2022), 616–626. https://doi.org/10.23939/mmc2022.03.616

8. P. Kumar, V. S. Erturk, S. Tyagi, J. Banas, A. Manickam, A generalized Caputo-type fractional-order neuron model under the electromagnetic field, *Int. J. Dyn. Control*, **11** (2023), 2179–2192. https://doi.org/10.1007/s40435-023-01134-4

9. M. C. Malar, M. Gayathri, A. Manickam, A novel study on the maize streak virus epidemic model using Caputo–Fabrizio fractional derivative, *Contemp. Math.*, **4** (2023), 379–619. https://doi.org/10.37256/cm.4320232383

10. A. L. Rose, A. Manickam, M. Agrawal, A mathematical model for the special effects of phosphatidylserine on endocrine reaction to reasonable concentration exercise in healthy male subjects, *Turk. J. Comput. Math. Educ.*, **12** (2021), 3555–3559.

11. R. Garrappa, M. Popolizio, Computing the matrix Mittag–Leffler function with applications to fractional calculus, *J. Sci. Comput.*, **77** (2018), 129–153. https://doi.org/10.1007/s10915-018-0699-5

12. J. R. Cardoso, Computing the Mittag–Leffler function of a matrix argument, *Fract. Calcu. Appl. Anal.*, **27** (2024), 2248–2274. https://doi.org/10.1007/s13540-024-00326-9

13. L. Sadek, Fractional BDF methods for solving fractional differential matrix equations, *Int. J. Appl. Comput. Math.*, **8** (2022), 238. https://doi.org/10.1007/s40819-022-01455-6

14. H. Abou-Kandil, G. Freiling, V. Ionescu, G. Jank, *Matrix Riccati equations in control and systems theory*, Birkhäuser, 2003. https://doi.org/10.1007/978-3-0348-8081-7

15. L. Sadek, H. Talibi Alaoui, The extended block Arnoldi method for solving generalized differential Sylvester equations, *J. Math. Model.*, **8** (2020), 189–206. https://doi.org/10.22124/jmm.2020.15871.1388

16. E. M. Sadek, A. H. Bentbib, L. Sadek, H. Talibi Alaoui, Global extended Krylov subspace methods for large-scale differential Sylvester matrix equations, *J. Appl. Math. Comput.*, **62** (2020), 157–177. https://doi.org/10.1007/s12190-019-01278-7

17. L. Sadek, E. M. Sadek, H. Talibi Alaoui, On some numerical methods for solving large differential nonsymmetric stein matrix equations, *Math. Comput. Appl.*, **27** (2022), 69. https://doi.org/10.3390/mca27040069

18. I. Podlubny, *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, 1 Ed., Vol. 198, Elsevier, 1998.

19. H. Neudecker, S. Liu, W. Polasek, The Hadamard product and some of its applications in statistics, *Statistics*, **26** (1995), 365–373. https://doi.org/10.1080/02331889508802503

AIMS Press