AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(6): 16735–16762.
DOI: 10.3934/math.2026687
Received: 28 March 2026
Revised: 15 May 2026
Accepted: 02 June 2026
Published: 11 June 2026

Research article

Lyapunov-type inequalities for the modified discrete Helmholtz operator on $\mathbb{Z}$

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Abstract: We establish Lyapunov-type inequalities for Dirichlet boundary value problems driven by the modified discrete Helmholtz operator on finite balls of the integer lattice $\mathbb{Z}$. This yields weighted inequalities for scalar problems and a spectral-radius criterion for coupled systems, as well as sharpness results. As an application, we study a weighted eigenvalue problem and obtain explicit two-sided bounds for the first eigenvalue by combining a Lyapunov-type lower estimate with a variational upper bound. Numerical illustrations are provided for localized weights.

Keywords: Lyapunov-type inequality; difference equations; discrete Helmholtz operator; normalized discrete Laplacian; Dirichlet boundary value problem; eigenvalue bounds
Mathematics Subject Classification: 39A06, 39A27, 05C50, 05A20

1. Introduction

Inequalities of Lyapunov type provide a quantitative obstruction to the existence of nontrivial solutions for boundary value problems. Rather than describing the qualitative behavior of solutions, they impose explicit integral constraints on the coefficients of the equation, revealing how the geometry of the domain and the strength of the potential interact. Introduced in Lyapunov's foundational study of stability [1], such inequalities have become a central tool in the analysis of oscillatory behavior, eigenvalue localization, and nonexistence phenomena for both linear and nonlinear problems.

In its classical continuous form, the Lyapunov inequality asserts that if $\mu \in L^1(a,b)$, and the Dirichlet problem

$$u''(x) + \mu(x)u(x) = 0, \qquad x \in (a,b), \qquad u(a) = u(b) = 0$$

admits a nontrivial solution, then μ necessarily satisfies

$$\int_a^b |\mu(x)|\, dx \geq \frac{4}{b-a}.$$

This estimate reflects a robust mechanism: Dirichlet confinement enforces a minimal integrated strength of the potential needed to support a nonzero solution.

Numerous extensions and variants of this result have been developed, including weighted inequalities [2], higher-order differential equations [3–5], nonlinear differential equations involving quasilinear operators [6], relativistic and curvature operators [7–9], ψ-Laplacian operators [10], and fractional differential equations [11–13]. We also refer to [14, 15] for local fractional differential equations and to [16] for fractional partial differential equations.

Difference equations arise naturally in discrete-time models and in numerical discretizations of differential problems. Finite-dimensional formulations also appear in numerical and mechanical models, including finite element approaches for dynamical systems [17]. They provide a setting in which boundary value problems, existence criteria, and spectral quantities can often be analyzed explicitly. In this context, Lyapunov-type inequalities give quantitative restrictions on the coefficients of difference equations and yield summation conditions that are necessary for the existence of nontrivial solutions under boundary constraints.

A first sharp Lyapunov-type inequality for second-order difference equations was obtained by Cheng [18]. He showed that, under discrete Dirichlet boundary conditions, the existence of a nontrivial solution forces the non-negative coefficient to satisfy a sharp summation lower bound. Several extensions of this result to higher-order difference equations and related discrete problems can be found in [19–22].

Motivated by these developments, we study Dirichlet boundary value problems involving the modified discrete Helmholtz operator on finite balls of the integer lattice $\mathbb{Z}$. More precisely, for $\alpha > 0$, we consider $\mathcal{L}_\alpha u = -\Delta u + \alpha u$, where Δ denotes the normalized discrete Laplacian on $\mathbb{Z}$ [23]. Our main objective is to establish Lyapunov-type inequalities for this operator. These inequalities provide explicit necessary lower bounds on the size of the coefficient appearing in the equation, ensuring that the corresponding Dirichlet problem can admit a nontrivial solution. As an application, we investigate an associated weighted eigenvalue problem and derive two-sided estimates for its first eigenvalue.

The operator $\mathcal{L}_\alpha$ arises naturally when elliptic equations with a positive mass term are discretized on graphs or lattices [24, 25]. The term $-\Delta u$ describes discrete diffusion, or nearest-neighbor interaction. The term αu represents a restoring, absorption, or mass effect.

The paper is organized as follows. Section 2 introduces the geometric and analytic framework on the integer lattice $\mathbb{Z}$, including the normalized discrete Laplacian and several properties used throughout the paper. In Section 3, we construct the Dirichlet Green function associated with the modified discrete Helmholtz operator and derive its main positivity and supremum estimates. Section 4 is devoted to Lyapunov-type inequalities for scalar Dirichlet problems and for coupled modified discrete Helmholtz systems, together with sharpness results. Finally, in Section 5, we apply these inequalities to a weighted eigenvalue problem, obtain explicit two-sided bounds for the first eigenvalue, and present numerical illustrations.

2. Preliminaries on the integer lattice $\mathbb{Z}$

In this section, we recall the basic graph-theoretic and analytic framework used throughout the paper.

We denote by $\mathbb{N}$ the set of positive integers.

2.1. Graph Laplacian and boundary notation

Let $G = (V, E)$ be a locally finite simple graph. For $x, y \in V$, we write $x \sim y$ if x and y are adjacent. The degree of a vertex $x \in V$ is denoted by

$$d_x = \#\{y \in V : y \sim x\}.$$

For a real-valued function $u\colon V \to \mathbb{R}$, the normalized graph Laplacian is defined by

$$\Delta_G u(x) = \frac{1}{d_x} \sum_{y \sim x} (u(y) - u(x)).$$

If $\Omega \subset V$ is a finite set, its vertex boundary is given by

$$\partial\Omega = \{x \in V \setminus \Omega : \text{ there exists } y \in \Omega \text{ such that } x \sim y\}. \tag{2.1}$$

For $x \in \partial\Omega$, the discrete outward normal derivative is defined by

$$\partial_\nu u(x) = u(x) - \frac{1}{\#\{y \in \Omega : y \sim x\}} \sum_{\substack{y \in \Omega \\ y \sim x}} u(y), \tag{2.2}$$

whenever the denominator is nonzero.

For more details on graph Laplacians and analysis on graphs, we refer to [26].

2.2. The integer lattice and its geometry

We now specialize the above graph-theoretic notions to the integer lattice $\mathbb{Z}$.

We regard $\mathbb{Z}$ as a graph with vertex set $\mathbb{Z}$ and adjacency relation

$$m \sim n \quad \Longleftrightarrow \quad |m - n| = 1.$$

Thus, two integers are adjacent precisely when they differ by one.

The associated graph distance is the length of the shortest path connecting two vertices. The set $\mathbb{Z}$ is an infinite path graph; hence, this distance is given by

$$d(m, n) = |m - n|, \qquad m, n \in \mathbb{Z}.$$

Therefore, for $N \in \mathbb{N}$, the ball of radius N centered at the origin is

$$B(0, N) = \{\, n \in \mathbb{Z} : d(n, 0) \le N \,\} = \{\, n \in \mathbb{Z} : |n| \le N \,\}.$$

The corresponding sphere is

$$S(0, N) = \{\, n \in \mathbb{Z} : d(n, 0) = N \,\} = \{\, n \in \mathbb{Z} : |n| = N \,\} = \{-N, N\}.$$

Remark 2.1. *With the above adjacency relation, the integer lattice $\mathbb{Z}$ is a connected acyclic graph in which every vertex has degree 2. Thus, $\mathbb{Z}$ may be viewed as the infinite 2-regular tree, or equivalently as an infinite path graph; see Figure 1.*

For general background on $(q+1)$-regular trees T_{q+1}, usually considered for $q \geq 2$, we refer to [27, 28].

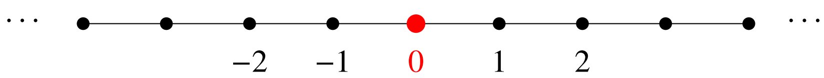

Figure 1. The graph $\mathbb{Z}$ with adjacency $n \sim n \pm 1$, viewed as an infinite path graph.

2.3. *Discrete differential operators on $\mathbb{Z}$*

Let $u\colon \mathbb{Z} \to \mathbb{R}$ be a real-valued function. We apply the normalized graph Laplacian introduced above to the graph $\mathbb{Z}$. For every $n \in \mathbb{Z}$, the vertex n has degree

$$d_n = 2,$$

and its neighbors are precisely $n-1$ and $n+1$. Hence,

$$\Delta u(n) = \frac{1}{2}\sum_{m\sim n}(u(m) - u(n)) = \frac{1}{2}((u(n+1) - u(n)) + (u(n-1) - u(n))).$$

Therefore,

$$\Delta u(n) = \frac{1}{2}(u(n+1) + u(n-1) - 2u(n)), \qquad n \in \mathbb{Z}.$$

Let $\alpha > 0$. The modified discrete Helmholtz operator on $\mathbb{Z}$ is defined by

$$\mathcal{L}_\alpha u(n) = -\Delta u(n) + \alpha\, u(n), \qquad n \in \mathbb{Z}.$$

An equivalent expression is

$$\mathcal{L}_\alpha u(n) = (1+\alpha)\, u(n) - \frac{1}{2}(u(n+1) + u(n-1)), \qquad n \in \mathbb{Z}.$$

We note that, in the literature on difference equations, the second-difference operator $x(k+1)+x(k-1)-2x(k)$ is often denoted by $\Delta^2 x(k-1)$. In the present work, we use the notation Δ for the normalized discrete Laplacian in order to emphasize the graph-theoretic interpretation of $\mathbb{Z}$ as a 2-regular graph.

Let $N \in \mathbb{N}$ with $N \geq 2$, and set

$$\Omega = B(0, N-1).$$

Using the general definition (2.1) with $V = \mathbb{Z}$, we obtain

$$\partial\Omega = \{x \in \mathbb{Z} \setminus B(0, N-1) : \text{ there exists } y \in B(0, N-1) \text{ such that } |x-y| = 1\}.$$

By

$$B(0, N-1) = \{-N+1, \ldots, N-1\},$$

the only vertices outside $B(0, N-1)$ adjacent to vertices of $B(0, N-1)$ are $-N$ and N. Hence,

$$\partial\Omega = \{-N, N\} = S(0, N).$$

Moreover, each boundary vertex has exactly one neighbor inside $B(0, N-1)$. More precisely,

$$\{y \in B(0, N-1) : y \sim -N\} = \{-N+1\}, \qquad \{y \in B(0, N-1) : y \sim N\} = \{N-1\}.$$

Therefore, applying the general definition of the discrete outward normal derivative (2.2) with $\Omega = B(0, N-1)$, we obtain

$$\partial_\nu u(-N) = u(-N) - u(-N+1), \qquad \partial_\nu u(N) = u(N) - u(N-1).$$

Lemma 2.2 (Discrete Green identity on $B(0, N-1)$)**.** *Let $N \in \mathbb{N}$ with $N \geq 2$, and let $f \colon B(0, N) \to \mathbb{R}$. Then*

$$\sum_{n \in B(0,N-1)} f(n)\,\Delta f(n) = -\frac{1}{2} \sum_{n=-N+1}^{N-2} \left(f(n+1) - f(n)\right)^2 + \frac{1}{2}\Big[f(N-1)\,\partial_\nu f(N) + f(-N+1)\,\partial_\nu f(-N)\Big].$$

Proof. Using the definition of the normalized discrete Laplacian, we have

$$\sum_{n \in B(0,N-1)} f(n)\,\Delta f(n) = \frac{1}{2} \sum_{n=-N+1}^{N-1} f(n)(f(n+1) + f(n-1) - 2f(n)).$$

Expanding the right-hand side yields

$$\sum_{n \in B(0,N-1)} f(n)\,\Delta f(n) = \frac{1}{2} \sum_{n=-N+1}^{N-1} f(n)f(n+1) + \frac{1}{2} \sum_{n=-N+1}^{N-1} f(n)f(n-1) - \sum_{n=-N+1}^{N-1} f(n)^2.$$

We now isolate the boundary contributions and reindex the second mixed term. First,

$$\sum_{n=-N+1}^{N-1} f(n)f(n+1) = \sum_{n=-N+1}^{N-2} f(n)f(n+1) + f(N-1)f(N).$$

Next,

$$\begin{aligned} \sum_{n=-N+1}^{N-1} f(n)f(n-1) &= f(-N+1)f(-N) + \sum_{n=-N+2}^{N-1} f(n)f(n-1) \\ &= f(-N+1)f(-N) + \sum_{n=-N+1}^{N-2} f(n)f(n+1). \end{aligned}$$

Consequently,

$$\begin{aligned} \sum_{n \in B(0,N-1)} f(n)\,\Delta f(n) = &\sum_{n=-N+1}^{N-2} f(n)f(n+1) \\ &- \sum_{n=-N+1}^{N-1} f(n)^2 + \frac{1}{2}\Big[f(N-1)f(N) + f(-N+1)f(-N)\Big]. \end{aligned}$$

We use the identity

$$\sum_{n=-N+1}^{N-1} f(n)^2 = \frac{1}{2}\sum_{n=-N+1}^{N-2} (f(n)^2 + f(n+1)^2) + \frac{1}{2}(f(-N+1)^2 + f(N-1)^2),$$

which gives

$$\sum_{n\in B(0,N-1)} f(n)\,\Delta f(n) = \sum_{n=-N+1}^{N-2} \Big[f(n)f(n+1) - \frac{1}{2}(f(n)^2 + f(n+1)^2)\Big] + \frac{1}{2}\Big[f(N-1)\,\partial_\nu f(N) + f(-N+1)\,\partial_\nu f(-N)\Big].$$

Indeed,

$$f(N-1)f(N) - f(N-1)^2 = f(N-1)(f(N) - f(N-1)) = f(N-1)\,\partial_\nu f(N),$$

and

$$f(-N+1)f(-N) - f(-N+1)^2 = f(-N+1)(f(-N) - f(-N+1)) = f(-N+1)\,\partial_\nu f(-N).$$

Finally, for every $n \in \mathbb{Z}$,

$$f(n)f(n+1) - \frac{1}{2}(f(n)^2 + f(n+1)^2) = -\frac{1}{2}(f(n+1) - f(n))^2.$$

Summing over $n = -N+1, \ldots, N-2$ concludes the proof. □

3. Green function

In this section, we study the Green function associated with the modified discrete Helmholtz operator $\mathcal{L}_\alpha$ on finite balls under Dirichlet boundary conditions. Throughout the paper, we fix $\alpha > 0$ and an integer $N \in \mathbb{N}$ with $N \geq 2$.

For every $s > 0$, we introduce the parameter $\theta_s > 0$ defined by

$$\cosh\theta_s = 1 + s.$$

We denote by

$$G^D_{N,\alpha} \colon B(0,N) \times B(0,N-1) \to \mathbb{R}$$

the Dirichlet Green function associated with the operator $\mathcal{L}_\alpha$ on $B(0,N-1)$. For each fixed $m \in B(0,N-1)$, the function $n \mapsto G^D_{N,\alpha}(n,m)$ is defined as the unique solution of

$$\begin{cases} (\mathcal{L}_\alpha)_n\, G^D_{N,\alpha}(n,m) = \delta_{n,m}, & n \in B(0,N-1), \\ G^D_{N,\alpha}(-N,m) = 0, \qquad G^D_{N,\alpha}(N,m) = 0. \end{cases} \tag{3.1}$$

Here $(\mathcal{L}_\alpha)_n$ indicates that the operator $\mathcal{L}_\alpha$ acts with respect to the n-variable. More precisely, for any function $F\colon \mathbb{Z}\times\mathbb{Z} \to \mathbb{R}$ and any fixed $m \in \mathbb{Z}$, we set

$$(\mathcal{L}_\alpha)_n F(n,m) = -\Delta_n F(n,m) + \alpha F(n,m) = (1+\alpha)\,F(n,m) - \frac{1}{2}(F(n+1,m) + F(n-1,m)),$$

where

$$\Delta_n F(n,m) = \frac{1}{2}(F(n+1,m) + F(n-1,m) - 2F(n,m)).$$

Moreover, $\delta_{n,m}$ denotes the Kronecker delta, that is,

$$\delta_{n,m} = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$

The Green function $G^D_{N,\alpha}$ defined by (3.1) is introduced in order to represent solutions of inhomogeneous Dirichlet problems driven by $\mathcal{L}_\alpha$. This representation is the main tool used later to derive the Lyapunov-type inequalities.

Fix $m \in B(0, N-1)$. In order to solve (3.1), we look for a function $u\colon B(0,N) \to \mathbb{R}$, given by

$$u(n) = G^D_{N,\alpha}(n,m),$$

such that

$$(\mathcal{L}_\alpha)_n u(n) = \delta_{n,m}, \qquad n \in B(0, N-1),$$

together with the Dirichlet boundary conditions

$$u(-N) = 0, \qquad u(N) = 0.$$

Using the explicit form of $\mathcal{L}_\alpha$, this is equivalent to

$$u(n+1) - 2(1+\alpha)u(n) + u(n-1) = -2\,\delta_{n,m}, \qquad n \in B(0, N-1).$$

For $n \in B(0, N-1) \setminus \{m\}$, the right-hand side vanishes and u satisfies the homogeneous recurrence relation

$$u(n+1) - 2(1+\alpha)u(n) + u(n-1) = 0. \tag{3.2}$$

Let v_1 and v_2 be two linearly independent solutions of the homogeneous recurrence (3.2). We define the discrete Wronskian by

$$W(n) = v_1(n+1)\,v_2(n) - v_1(n)\,v_2(n+1).$$

On the other hand, v_1 and v_2 satisfy

$$v_i(n+2) = 2(1+\alpha)v_i(n+1) - v_i(n), \qquad i = 1, 2.$$

Then,

$$\begin{aligned} W(n+1) &= v_1(n+2)v_2(n+1) - v_1(n+1)v_2(n+2) \\ &= (2(1+\alpha)v_1(n+1) - v_1(n))v_2(n+1) \\ &\quad - v_1(n+1)(2(1+\alpha)v_2(n+1) - v_2(n)) \\ &= v_1(n+1)v_2(n) - v_1(n)v_2(n+1) \\ &= W(n). \end{aligned}$$

Hence, $W(n)$ is independent of n, and we denote this constant by W. Moreover, using that v_1 and v_2 are linearly independent, we obtain $W \neq 0$.

We now choose two particular solutions of (3.2) adapted to the Dirichlet boundary conditions. We choose v_1 such that

$$v_1(-N) = 0,$$

and v_2 such that

$$v_2(N) = 0.$$

For $m \in B(0, N-1)$, we define $G^D_{N,\alpha}(\cdot, m)$ on $B(0, N)$ by

$$G^D_{N,\alpha}(n, m) = \begin{cases} A_m\, v_1(n), & -N \le n \le m, \\ B_m\, v_2(n), & m \le n \le N, \end{cases}$$

for some constants A_m and B_m.

The matching condition at $n = m$ gives

$$A_m\, v_1(m) = B_m\, v_2(m). \tag{3.3}$$

We next impose the equation at $n = m$. Using that $\delta_{m,m} = 1$, we obtain

$$(\mathcal{L}_\alpha)_n G^D_{N,\alpha}(n, m)\Big|_{n=m} = 1,$$

that is,

$$(1+\alpha) G^D_{N,\alpha}(m, m) - \frac{1}{2}\left(G^D_{N,\alpha}(m+1, m) + G^D_{N,\alpha}(m-1, m)\right) = 1.$$

Using the piecewise definition of $G^D_{N,\alpha}(\cdot, m)$, we obtain

$$(1+\alpha) A_m v_1(m) - \frac{1}{2}\left(B_m v_2(m+1) + A_m v_1(m-1)\right) = 1.$$

Multiplying by 2 and using the recurrence relation for v_1,

$$v_1(m+1) = 2(1+\alpha) v_1(m) - v_1(m-1),$$

we rewrite the previous identity as

$$A_m v_1(m+1) - B_m v_2(m+1) = 2.$$

Together with (3.3), this gives the linear system

$$\begin{cases} A_m\, v_1(m) = B_m\, v_2(m), \\ A_m\, v_1(m+1) - B_m\, v_2(m+1) = 2, \end{cases}$$

whose unique solution is

$$A_m = \frac{2\, v_2(m)}{W}, \qquad B_m = \frac{2\, v_1(m)}{W}.$$

Substituting these expressions into the definition of $G^D_{N,\alpha}$, we obtain

$$G^D_{N,\alpha}(n, m) = \frac{2}{W} \begin{cases} v_1(n)\, v_2(m), & -N \le n \le m, \\ v_1(m)\, v_2(n), & m \le n \le N. \end{cases} \tag{3.4}$$

We now choose explicit solutions of the homogeneous recurrence adapted to the Dirichlet boundary conditions. For $n \in B(0, N)$, we set

$$v_1(n) = \sinh(\theta_\alpha(n + N)), \qquad v_2(n) = \sinh(\theta_\alpha(N - n)),$$

so that $v_1(-N) = 0$ and $v_2(N) = 0$. Moreover, by $\cosh\theta_\alpha = 1 + \alpha$, the identity

$$\sinh(x + \theta_\alpha) + \sinh(x - \theta_\alpha) = 2\cosh(\theta_\alpha)\sinh x$$

shows that both v_1 and v_2 satisfy

$$u(n + 1) - 2(1 + \alpha)u(n) + u(n - 1) = 0.$$

With the discrete Wronskian

$$W = v_1(n + 1)\, v_2(n) - v_1(n)\, v_2(n + 1),$$

we compute it at $n = -N$. Using that $v_1(-N) = 0$, we get

$$W = v_1(-N + 1)v_2(-N) = \sinh(\theta_\alpha)\sinh(2N\theta_\alpha).$$

Thus,

$$W = \sinh(\theta_\alpha)\ \sinh(2N\theta_\alpha).$$

Substituting the expressions of v_1, v_2, and W into (3.4), we obtain

$$G^D_{N,\alpha}(n, m) = \begin{cases} \dfrac{2\ \sinh(\theta_\alpha(n + N))\sinh(\theta_\alpha(N - m))}{\sinh(\theta_\alpha)\sinh(2N\theta_\alpha)}, & -N \le n \le m, \\ \dfrac{2\ \sinh(\theta_\alpha(m + N))\sinh(\theta_\alpha(N - n))}{\sinh(\theta_\alpha)\sinh(2N\theta_\alpha)}, & m \le n \le N. \end{cases}$$

In the first case, $n \le m$, so $\min\{n, m\} = n$, and $\max\{n, m\} = m$. In the second case, $m \le n$, so $\min\{n, m\} = m$, and $\max\{n, m\} = n$. This yields the following result.

Lemma 3.1. *For each $m \in B(0, N - 1)$, let $G^D_{N,\alpha}(\cdot, m)$ be the unique function on $B(0, N)$ satisfying* (3.1). *Then, for all $(n, m) \in B(0, N) \times B(0, N - 1)$,*

$$G^D_{N,\alpha}(n, m) = \frac{2\ \sinh(\theta_\alpha(\min\{n, m\} + N))\ \sinh(\theta_\alpha(N - \max\{n, m\}))}{\sinh(\theta_\alpha)\ \sinh(2N\theta_\alpha)}. \tag{3.5}$$

In particular, $G^D_{N,\alpha}(n, m) \ge 0$ for all $(n, m) \in B(0, N) \times B(0, N - 1)$.

Remark 3.2. *The Green function $G^D_{N,\alpha}$ constructed above plays the same qualitative role as the Dirichlet Green function in the continuous theory. Indeed, in the continuous one-dimensional counterpart, one considers the operator*

$$-\frac{d^2}{dx^2} + \alpha$$

on an interval, for instance $(-N, N)$, and the associated Dirichlet Green function $G_\alpha(x,\xi)$ is characterized by

$$\left(-\frac{d^2}{dx^2} + \alpha\right) G_\alpha(x,\xi) = \delta_\xi(x), \qquad G_\alpha(-N,\xi) = G_\alpha(N,\xi) = 0.$$

In that setting, G_α is obtained by solving the homogeneous differential equation

$$-u'' + \alpha u = 0$$

on the two subintervals separated by the pole ξ, together with continuity at $x = \xi$ and the usual jump condition on the derivative.

In the present discrete setting, the corresponding homogeneous equation is the recurrence

$$u(n+1) - 2(1+\alpha)u(n) + u(n-1) = 0.$$

Thus, the discrete Green function is obtained by solving this recurrence on the two discrete intervals separated by the pole m, together with a matching condition at $n = m$ and the discrete equation at the pole. Hence, while the continuous Green function is governed by differential equations and derivative jump conditions, the discrete Green function is governed by recurrence relations and algebraic matching conditions.

Despite this difference, both Green functions serve the same purpose: they represent solutions of inhomogeneous Dirichlet problems and provide positivity and maximum estimates. These estimates are then used to derive Lyapunov-type inequalities.

Lemma 3.3. *Fix $m \in B(0, N-1)$. Then the function $n \mapsto G^D_{N,\alpha}(n,m)$ attains its maximum at $n = m$. In particular,*

$$\sup_{n \in B(0,N)} G^D_{N,\alpha}(n,m) = G^D_{N,\alpha}(m,m) = \frac{2\,\sinh\!\big(\theta_\alpha(m+N)\big)\,\sinh\!\big(\theta_\alpha(N-m)\big)}{\sinh(\theta_\alpha)\,\sinh(2N\theta_\alpha)}.$$

Proof. Assume first that $n \le m$. Then $\min\{n,m\} = n$ and $\max\{n,m\} = m$. Hence, by (3.5),

$$G^D_{N,\alpha}(n,m) = C(m)\,\sinh\!\big(\theta_\alpha(n+N)\big), \qquad C(m) = \frac{2\,\sinh\!\big(\theta_\alpha(N-m)\big)}{\sinh(\theta_\alpha)\,\sinh(2N\theta_\alpha)}.$$

Since $C(m) > 0$ and $\theta_\alpha > 0$, the map $n \mapsto \sinh\!\big(\theta_\alpha(n+N)\big)$ is strictly increasing. Therefore, $G^D_{N,\alpha}(n,m)$ is maximized at $n = m$ on the set $\{n \in B(0,N) : n \le m\}$.

Assume next that $n \ge m$. Then $\min\{n,m\} = m$ and $\max\{n,m\} = n$. Hence,

$$G^D_{N,\alpha}(n,m) = D(m)\,\sinh\!\big(\theta_\alpha(N-n)\big), \qquad D(m) = \frac{2\,\sinh\!\big(\theta_\alpha(m+N)\big)}{\sinh(\theta_\alpha)\,\sinh(2N\theta_\alpha)}.$$

Since $D(m) > 0$ and $\theta_\alpha > 0$, the map $n \mapsto \sinh\!\big(\theta_\alpha(N-n)\big)$ is strictly decreasing. Therefore, $G^D_{N,\alpha}(n,m)$ is maximized at $n = m$ on the set $\{n \in B(0,N) : n \ge m\}$.

Combining the two cases, the global maximum is attained at $n = m$, which gives the stated value. □

Lemma 3.4. *One has*

$$\sup_{m \in B(0,N-1)} G^D_{N,\alpha}(m,m) = G^D_{N,\alpha}(0,0) = \frac{\tanh(N\theta_\alpha)}{\sinh(\theta_\alpha)}.$$

Proof. By Lemma 3.3, for each $m \in B(0, N-1)$,

$$G^D_{N,\alpha}(m,m) = \frac{2\,\sinh\!\big(\theta_\alpha(m+N)\big)\,\sinh\!\big(\theta_\alpha(N-m)\big)}{\sinh(\theta_\alpha)\,\sinh(2N\theta_\alpha)}.$$

Set

$$H(m) = \sinh\!\big(\theta_\alpha(m+N)\big)\,\sinh\!\big(\theta_\alpha(N-m)\big), \qquad m \in B(0, N-1).$$

Using the identity

$$2\sinh x\,\sinh y = \cosh(x+y) - \cosh(x-y),$$

with $x = \theta_\alpha(m+N)$ and $y = \theta_\alpha(N-m)$, we obtain

$$2H(m) = \cosh(2N\theta_\alpha) - \cosh(2m\theta_\alpha).$$

Since $\theta_\alpha > 0$ and cosh is strictly increasing on $[0,\infty)$, the map $m \mapsto \cosh(2m\theta_\alpha)$ is minimized at $m = 0$ on $B(0, N-1)$. Hence $H(m)$ is maximized at $m = 0$, and therefore

$$\sup_{m\in B(0,N-1)} G^D_{N,\alpha}(m,m) = G^D_{N,\alpha}(0,0) = \frac{2\,\sinh(N\theta_\alpha)^2}{\sinh(\theta_\alpha)\,\sinh(2N\theta_\alpha)}.$$

Finally, using $\sinh(2x) = 2\sinh x\cosh x$, we obtain

$$\frac{2\,\sinh(N\theta_\alpha)^2}{\sinh(\theta_\alpha)\,\sinh(2N\theta_\alpha)} = \frac{2\,\sinh(N\theta_\alpha)^2}{\sinh(\theta_\alpha)\,2\sinh(N\theta_\alpha)\cosh(N\theta_\alpha)} = \frac{\tanh(N\theta_\alpha)}{\sinh(\theta_\alpha)}.$$

□

4. Lyapunov-type inequalities

In this section, we establish Lyapunov-type inequalities for boundary value problems involving the modified discrete Helmholtz operator on finite balls of the integer lattice. We first deal with a scalar discrete Helmholtz equation subject to Dirichlet boundary conditions. Relying on the explicit representation and sharp estimates of the Dirichlet Green function derived in Section 3, we obtain necessary conditions for the existence of nontrivial solutions. We next extend the analysis to a coupled discrete Helmholtz system with Dirichlet boundary conditions.

For a function $u\colon B(0,N) \to \mathbb{R}$, we set

$$\|u\|_\infty = \max_{n\in B(0,N)} |u(n)|.$$

4.1. The scalar Dirichlet problem

We first consider the scalar Dirichlet problem associated with the modified discrete Helmholtz operator. This problem is motivated by discrete spectral theory. It describes a finite-dimensional Dirichlet equation in which the potential μ competes with the coercive operator $\mathcal{L}_\alpha$. The existence of a nontrivial solution imposes a quantitative restriction on the size of μ, and Lyapunov-type inequalities provide an explicit form of this restriction. This also prepares the eigenvalue estimates developed in Section 5.

Let μ be a real-valued function defined on $B(0, N-1)$. We investigate the existence of nontrivial solutions $u\colon B(0,N) \to \mathbb{R}$ to the following Dirichlet problem:

$$\begin{cases} \mathcal{L}_\alpha u(n) = \mu(n)\, u(n), & n \in B(0, N-1), \\ u(-N) = 0, \qquad u(N) = 0. \end{cases} \tag{4.1}$$

Theorem 4.1. *Assume that the Dirichlet problem* (4.1) *admits a nontrivial solution u. Then*

$$\sum_{m \in B(0,N-1)} \sinh\big(\theta_\alpha(m+N)\big)\, \sinh\big(\theta_\alpha(N-m)\big)\, |\mu(m)| \geq \frac{1}{2}\, \sinh(\theta_\alpha)\, \sinh(2N\theta_\alpha). \tag{4.2}$$

Proof. We first establish the Green representation formula

$$u(n) = \sum_{m \in B(0,N-1)} G^D_{N,\alpha}(n,m)\, \mu(m)\, u(m), \qquad n \in B(0,N), \tag{4.3}$$

where $G^D_{N,\alpha}$ is the Dirichlet Green function associated with $\mathcal{L}_\alpha$ on $B(0, N-1)$, defined in (3.1).

Define

$$v(n) = \sum_{m \in B(0,N-1)} G^D_{N,\alpha}(n,m)\, \mu(m)\, u(m), \qquad n \in B(0,N).$$

By (3.1), for every $n \in B(0, N-1)$,

$$\mathcal{L}_\alpha v(n) = \sum_{m \in B(0,N-1)} (\mathcal{L}_\alpha)_n G^D_{N,\alpha}(n,m)\, \mu(m)\, u(m) = \sum_{m \in B(0,N-1)} \delta_{n,m}\, \mu(m)\, u(m) = \mu(n)\, u(n),$$

and $v(-N) = v(N) = 0$. Hence $w = u - v$ satisfies

$$\mathcal{L}_\alpha w(n) = 0 \quad \text{for } n \in B(0, N-1), \qquad w(-N) = w(N) = 0.$$

Multiplying the equation by $w(n)$ and summing over $B(0, N-1)$ gives

$$-\sum_{n \in B(0,N-1)} w(n)\, \Delta w(n) + \alpha \sum_{n \in B(0,N-1)} w(n)^2 = 0.$$

Applying Lemma 2.2 with $f = w$, we obtain

$$\sum_{n \in B(0,N-1)} w(n)\, \Delta w(n) = -\frac{1}{2} \sum_{n=-N+1}^{N-2} \big(w(n+1) - w(n)\big)^2 + \frac{1}{2}\Big[w(N-1)\, \partial_\nu w(N) + w(-N+1)\, \partial_\nu w(-N)\Big].$$

Therefore,

$$-\sum_{n \in B(0,N-1)} w(n)\, \Delta w(n) = \frac{1}{2} \sum_{n=-N+1}^{N-2} \big(w(n+1) - w(n)\big)^2 - \frac{1}{2}\Big[w(N-1)\, \partial_\nu w(N) + w(-N+1)\, \partial_\nu w(-N)\Big].$$

Since $w(N) = w(-N) = 0$, we have

$$\partial_\nu w(N) = w(N) - w(N-1) = -w(N-1), \qquad \partial_\nu w(-N) = w(-N) - w(-N+1) = -w(-N+1),$$

and hence

$$-\frac{1}{2}\Big[w(N-1)\,\partial_\nu w(N) + w(-N+1)\,\partial_\nu w(-N)\Big] = \frac{1}{2}\big(w(N-1)^2 + w(-N+1)^2\big).$$

Consequently,

$$-\sum_{n\in B(0,N-1)} w(n)\,\Delta w(n) = \frac{1}{2}\sum_{n=-N+1}^{N-2}\big(w(n+1)-w(n)\big)^2 + \frac{1}{2}\big(w(N-1)^2 + w(-N+1)^2\big).$$

Using again $w(N) = w(-N) = 0$, we rewrite

$$w(-N+1)^2 = \big(w(-N+1) - w(-N)\big)^2, \qquad w(N-1)^2 = \big(w(N) - w(N-1)\big)^2,$$

and thus

$$-\sum_{n\in B(0,N-1)} w(n)\,\Delta w(n) = \frac{1}{2}\sum_{n=-N}^{N-1}\big(w(n+1)-w(n)\big)^2.$$

Therefore,

$$\frac{1}{2}\sum_{n=-N}^{N-1}\big(w(n+1)-w(n)\big)^2 + \alpha\sum_{n\in B(0,N-1)} w(n)^2 = 0.$$

Since $\alpha > 0$, both sums are non-negative, hence they must both be zero. In particular, $w(n) = 0$ for all $n \in B(0, N-1)$. Together with $w(\pm N) = 0$, this implies $w \equiv 0$ on $B(0, N)$, and (4.3) follows.

We now estimate u. From (4.3) and the positivity of $G^D_{N,\alpha}$ (Lemma 3.1), for any $n \in B(0, N)$,

$$|u(n)| \le \sum_{m\in B(0,N-1)} G^D_{N,\alpha}(n,m)\,|\mu(m)|\,|u(m)|.$$

Let $n_0 \in B(0, N)$ satisfy $|u(n_0)| = \|u\|_\infty > 0$. Evaluating at $n = n_0$ and dividing by $\|u\|_\infty$ yields

$$1 \le \sum_{m\in B(0,N-1)} G^D_{N,\alpha}(n_0,m)\,|\mu(m)|.$$

Using Lemma 3.3, we have for each $m \in B(0, N-1)$,

$$G^D_{N,\alpha}(n_0,m) \le \sup_{n\in B(0,N)} G^D_{N,\alpha}(n,m) = G^D_{N,\alpha}(m,m).$$

Hence

$$1 \le \sum_{m\in B(0,N-1)} G^D_{N,\alpha}(m,m)\,|\mu(m)|.$$

Finally, using the explicit diagonal formula,

$$G^D_{N,\alpha}(m,m) = \frac{2\,\sinh\big(\theta_\alpha(m+N)\big)\,\sinh\big(\theta_\alpha(N-m)\big)}{\sinh(\theta_\alpha)\,\sinh(2N\theta_\alpha)},$$

we obtain (4.2). □

We next study the sharpness of the Lyapunov-type inequality (4.2).

We define the class

$$\mathcal{A} = \Big\{\mu\colon\ B(0,N-1) \to \mathbb{R} :\ \text{the Dirichlet problem (4.1) admits a nontrivial solution}\Big\}.$$

For $\mu \in \mathcal{A}$, we set

$$M(\mu) = \frac{\displaystyle\sum_{m\in B(0,N-1)} \sinh\big(\theta_\alpha(m+N)\big)\ \sinh\big(\theta_\alpha(N-m)\big)\,|\mu(m)|}{\sinh(\theta_\alpha)\ \sinh(2N\theta_\alpha)}.$$

Theorem 4.2. *One has*

$$\min_{\mu\in\mathcal{A}} M(\mu) = \frac{1}{2}.$$

Proof. By Theorem 4.1, one has

$$M(\mu) \geq \frac{1}{2} \qquad \text{for all } \mu \in \mathcal{A}.$$

We now show that the value $\frac{1}{2}$ is attained. Fix $m_0 \in B(0,N-1)$ and define

$$\mu_0(m) = \begin{cases} \dfrac{1}{G^D_{N,\alpha}(m_0,m_0)}, & m = m_0, \\ 0, & m \neq m_0, \end{cases} \qquad m \in B(0,N-1).$$

Let $u_0(n) = G^D_{N,\alpha}(n,m_0)$ for $n \in B(0,N)$. By (3.1), we have

$$\mathcal{L}_\alpha u_0(n) = \delta_{n,m_0}, \qquad n \in B(0,N-1),$$

and $u_0(\pm N) = 0$. On the other hand, for $n \in B(0,N-1)$,

$$\mu_0(n)\,u_0(n) = \frac{1}{G^D_{N,\alpha}(m_0,m_0)}\, G^D_{N,\alpha}(n,m_0)\,\mathbf{1}_{\{n=m_0\}} = \delta_{n,m_0}.$$

Therefore,

$$\mathcal{L}_\alpha u_0(n) = \mu_0(n)\,u_0(n), \qquad n \in B(0,N-1),$$

with $u_0(\pm N) = 0$. Hence u_0 is a nontrivial solution of (4.1) with $\mu = \mu_0$, and thus $\mu_0 \in \mathcal{A}$.

Finally, by the definition of $M(\mu)$ and the fact that μ_0 is supported at m_0, we have

$$M(\mu_0) = \frac{\sinh\big(\theta_\alpha(m_0+N)\big)\ \sinh\big(\theta_\alpha(N-m_0)\big)\,|\mu_0(m_0)|}{\sinh(\theta_\alpha)\ \sinh(2N\theta_\alpha)} = \frac{\sinh\big(\theta_\alpha(m_0+N)\big)\ \sinh\big(\theta_\alpha(N-m_0)\big)}{G^D_{N,\alpha}(m_0,m_0)\ \sinh(\theta_\alpha)\ \sinh(2N\theta_\alpha)}.$$

Using the diagonal identity from (3.5),

$$G^D_{N,\alpha}(m_0,m_0) = \frac{2\ \sinh\big(\theta_\alpha(m_0+N)\big)\ \sinh\big(\theta_\alpha(N-m_0)\big)}{\sinh(\theta_\alpha)\ \sinh(2N\theta_\alpha)},$$

we obtain $M(\mu_0) = \frac{1}{2}$. Consequently, $\min_{\mu\in\mathcal{A}} M(\mu) = \frac{1}{2}$. □

Corollary 4.3. *Assume that the Dirichlet problem* (4.1) *admits a nontrivial solution. Then*

$$\sum_{m\in B(0,N-1)} |\mu(m)| \geq \frac{\sinh(\theta_\alpha)}{\tanh(N\theta_\alpha)}. \tag{4.4}$$

Proof. Let u be a nontrivial solution. As in the proof of Theorem 4.1, we obtain

$$1 \leq \sum_{m\in B(0,N-1)} G^D_{N,\alpha}(m,m)\,|\mu(m)|.$$

Using $\sup_{m\in B(0,N-1)} G^D_{N,\alpha}(m,m) = \tanh(N\theta_\alpha)/\sinh(\theta_\alpha)$ (Lemma 3.4), we get

$$1 \leq \Big(\sup_{m\in B(0,N-1)} G^D_{N,\alpha}(m,m)\Big) \sum_{m\in B(0,N-1)} |\mu(m)| = \frac{\tanh(N\theta_\alpha)}{\sinh(\theta_\alpha)} \sum_{m\in B(0,N-1)} |\mu(m)|.$$

This yields (4.4). □

Remark 4.4. *As $\alpha \to 0^+$, the operator $\mathcal{L}_\alpha$ reduces to $-\Delta$, and the interior equation in* (4.1) *becomes a standard second-difference equation. Indeed, when $\alpha = 0$ the relation*

$$-\Delta u(n) = \mu(n)\,u(n)$$

is equivalent to

$$u(n+1) - 2u(n) + u(n-1) + p(n)\,u(n) = 0, \qquad n \in B(0,N-1),$$

after setting $p(n) = 2\mu(n)$ (in particular, $p \geq 0$ corresponds to $\mu \geq 0$).

With the shift $k = n + N$, the Dirichlet problem on $B(0,N-1) = \{-N+1,\dots,N-1\}$ is equivalent to the classical Dirichlet problem on $\{1,\dots,2N-1\}$ with boundary values at 0 and $2N$.

Moreover, since $\cosh\theta_\alpha = 1+\alpha$, one has $\theta_\alpha \to 0$ as $\alpha \to 0^+$ and

$$\frac{\sinh(\theta_\alpha)}{\tanh(N\theta_\alpha)} \longrightarrow \frac{1}{N}.$$

Consequently, when $\mu \geq 0$ the Lyapunov-type inequality (4.4) *yields, in the limit,*

$$\sum_{k=1}^{2N-1} p(k) \geq \frac{2}{N},$$

which agrees with Cheng's sharp constant for length $2N-1$ (see [18]).

4.2. The Dirichlet system

We next consider a coupled Dirichlet problem on $B(0,N-1)$ involving modified discrete Helmholtz operators. Such systems arise when several discrete states or components interact through coupling potentials. In this setting, the scalar coefficient μ is replaced by a matrix-valued potential (μ_{ij}), and the existence of a nontrivial vector solution is controlled by a spectral condition involving the size of the coupling terms. This motivates the extension of the scalar Lyapunov-type inequality to a system criterion based on the spectral radius of an associated non-negative matrix.

Let $\mu_{ij}\colon B(0,N-1)\to\mathbb{R}$ for $i,j\in\{1,2\}$. We investigate the existence of nontrivial solutions $(u,v)\colon B(0,N)\to\mathbb{R}^2$ to the system

$$\begin{cases}\mathcal{L}_\alpha u(n)=\mu_{11}(n)\,u(n)+\mu_{12}(n)\,v(n), & n\in B(0,N-1),\\ \mathcal{L}_\beta v(n)=\mu_{21}(n)\,u(n)+\mu_{22}(n)\,v(n), & n\in B(0,N-1),\\ u(\pm N)=0, \qquad v(\pm N)=0.\end{cases} \tag{4.5}$$

Here $\alpha,\beta>0$, $\mathcal{L}_\alpha w=-\Delta w+\alpha w$, and $\mathcal{L}_\beta w=-\Delta w+\beta w$.

By a nontrivial solution of (4.5), we mean a pair $(u,v)\colon B(0,N)\to\mathbb{R}^2$ satisfying the system and the Dirichlet boundary conditions $u(\pm N)=v(\pm N)=0$, and such that $(u,v)\not\equiv(0,0)$ on $B(0,N)$.

For a real 2×2 matrix $A=(a_{ij})_{i,j=1}^2$, we denote by $\rho(A)$ its spectral radius, that is,

$$\rho(A)=\max\{|\lambda|:\ \lambda \text{ is an eigenvalue of } A\}.$$

Theorem 4.5. *Assume that the coupled Dirichlet system* (4.5) *admits a nontrivial solution* $(u,v)\colon B(0,N)\to\mathbb{R}^2$*. Set*

$$a(N,\alpha)=\frac{\tanh(N\theta_\alpha)}{\sinh(\theta_\alpha)},\qquad b(N,\beta)=\frac{\tanh(N\theta_\beta)}{\sinh(\theta_\beta)},\qquad M_{ij}=\sum_{m\in B(0,N-1)}|\mu_{ij}(m)|,\quad i,j\in\{1,2\}.$$

Then

$$\rho(K)\ge 1,\qquad K=\begin{pmatrix}a(N,\alpha)M_{11} & a(N,\alpha)M_{12}\\ b(N,\beta)M_{21} & b(N,\beta)M_{22}\end{pmatrix}. \tag{4.6}$$

Proof. Let (u,v) be a nontrivial solution. Using the Dirichlet Green functions associated with $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$, we have, for every $n\in B(0,N)$,

$$u(n)=\sum_{m\in B(0,N-1)}G^D_{N,\alpha}(n,m)\big(\mu_{11}(m)u(m)+\mu_{12}(m)v(m)\big),$$

$$v(n)=\sum_{m\in B(0,N-1)}G^D_{N,\beta}(n,m)\big(\mu_{21}(m)u(m)+\mu_{22}(m)v(m)\big).$$

Since $G^D_{N,\alpha}\ge 0$ and $G^D_{N,\beta}\ge 0$ (see Lemma 3.1), it follows that

$$|u(n)|\le\sum_{m\in B(0,N-1)}G^D_{N,\alpha}(n,m)\Big(|\mu_{11}(m)|\,|u(m)|+|\mu_{12}(m)|\,|v(m)|\Big),$$

$$|v(n)|\le\sum_{m\in B(0,N-1)}G^D_{N,\beta}(n,m)\Big(|\mu_{21}(m)|\,|u(m)|+|\mu_{22}(m)|\,|v(m)|\Big).$$

Taking the maximum in n and using (see Lemmas 3.3 and 3.4)

$$\sup_{n\in B(0,N)}G^D_{N,\alpha}(n,m)=G^D_{N,\alpha}(m,m)\le\sup_{m\in B(0,N-1)}G^D_{N,\alpha}(m,m)=a(N,\alpha),$$

$$\sup_{n\in B(0,N)}G^D_{N,\beta}(n,m)=G^D_{N,\beta}(m,m)\le\sup_{m\in B(0,N-1)}G^D_{N,\beta}(m,m)=b(N,\beta),$$

we obtain

$$\|u\|_\infty \le a(N,\alpha)\Big(M_{11}\|u\|_\infty + M_{12}\|v\|_\infty\Big),$$

$$\|v\|_\infty \le b(N,\beta)\Big(M_{21}\|u\|_\infty + M_{22}\|v\|_\infty\Big).$$

Set

$$X = \begin{pmatrix} \|u\|_\infty \\ \|v\|_\infty \end{pmatrix} \in \mathbb{R}^2.$$

Using the componentwise partial order $\le$ on $\mathbb{R}^2$, the above inequalities can be written as

$$X \le KX.$$

Since (u, v) is nontrivial, we have $X \ne 0$ and $X \ge 0$. Moreover, K has non-negative entries.

Assume by contradiction that $\rho(K) < 1$. Then the Neumann series converges and

$$(I-K)^{-1} = \sum_{n=0}^{\infty} K^n$$

is well-defined and has non-negative entries. Multiplying $X \le KX$ by $(I-K)^{-1}$ yields

$$(I-K)^{-1}X \le (I-K)^{-1}KX = ((I-K)^{-1} - I)X,$$

and hence $X \le 0$. Since also $X \ge 0$, we obtain $X = 0$, a contradiction. Therefore, $\rho(K) \ge 1$, which proves (4.6). □

We next study the sharpness of the Lyapunov-type inequality (4.6).

We introduce the class

$$\mathcal{M} = \Big\{\mu = (\mu_{ij})_{1\le i,j\le 2} : \ \mu_{ij}\colon B(0,N-1) \to [0,\infty) \ \text{ and the system (4.5) has a nontrivial solution}\Big\}.$$

For $\mu \in \mathcal{M}$, let $K = K(\mu)$ be the 2×2 matrix given by (4.6).

Theorem 4.6. *One has*

$$\min_{\mu\in\mathcal{M}} \rho(K(\mu)) = 1.$$

Proof. By Theorem 4.5, on has

$$\rho(K(\mu)) \ge 1 \qquad \text{for all } \mu \in \mathcal{M}.$$

It remains to show that the above lower bound is attained. Fix $m_\alpha, m_\beta \in B(0, N-1)$. For all $m \in B(0, N-1)$, define

$$\mu_{11}(m) = \frac{\delta_{m,m_\alpha}}{G^D_{N,\alpha}(m_\alpha, m_\alpha)}, \qquad \mu_{22}(m) = \frac{\delta_{m,m_\beta}}{G^D_{N,\beta}(m_\beta, m_\beta)}, \qquad \mu_{12}(m) = \mu_{21}(m) = 0.$$

Set

$$u(n) = G^D_{N,\alpha}(n, m_\alpha), \qquad v(n) = G^D_{N,\beta}(n, m_\beta), \qquad n \in B(0,N).$$

Then $u(\pm N) = v(\pm N) = 0$ and $(u, v) \not\equiv (0, 0)$. Moreover,

$$\mathcal{L}_\alpha u(n) = \delta_{n,m_\alpha}, \qquad \mathcal{L}_\beta v(n) = \delta_{n,m_\beta}, \qquad n \in B(0, N-1).$$

On the other hand,

$$\mu_{11}(n)u(n) = \delta_{n,m_\alpha}, \qquad \mu_{22}(n)v(n) = \delta_{n,m_\beta}, \qquad n \in B(0, N-1),$$

and $\mu_{12} = \mu_{21} = 0$. Hence (u, v) is a nontrivial solution of (4.5), so $\mu \in \mathcal{M}$.

For this choice,

$$M_{11} = \sum_{m\in B(0,N-1)} \mu_{11}(m) = \frac{1}{G^D_{N,\alpha}(m_\alpha, m_\alpha)}, \qquad M_{22} = \sum_{m\in B(0,N-1)} \mu_{22}(m) = \frac{1}{G^D_{N,\beta}(m_\beta, m_\beta)},$$

and $M_{12} = M_{21} = 0$. Therefore

$$K(\mu) = \begin{pmatrix} \dfrac{a(N,\alpha)}{G^D_{N,\alpha}(m_\alpha, m_\alpha)} & 0 \\ 0 & \dfrac{b(N,\beta)}{G^D_{N,\beta}(m_\beta, m_\beta)} \end{pmatrix}.$$

Choose $m_\alpha = m_\beta = 0$. By Lemma 3.4,

$$G^D_{N,\alpha}(0, 0) = \sup_{m\in B(0,N-1)} G^D_{N,\alpha}(m, m) = a(N, \alpha)$$

and

$$G^D_{N,\beta}(0, 0) = \sup_{m\in B(0,N-1)} G^D_{N,\beta}(m, m) = b(N, \beta).$$

Hence $K(\mu) = I$, so $\rho(K(\mu)) = 1$. Consequently, $\min_{\mu\in\mathcal{M}} \rho(K(\mu)) = 1$. □

We now consider the special triangular case: $\mu_{11} = \mu_{22} = 0$. In this case, the system (4.5) reduces to

$$\begin{cases} \mathcal{L}_\alpha u(n) = \mu_{12}(n)\, v(n), & n \in B(0, N-1), \\ \mathcal{L}_\beta v(n) = \mu_{21}(n)\, u(n), & n \in B(0, N-1), \\ u(\pm N) = 0, \qquad v(\pm N) = 0. \end{cases} \tag{4.7}$$

Corollary 4.7. *Assume that the system* (4.7) *admits a nontrivial solution. Then*

$$\Big(\sum_{m\in B(0,N-1)} \mu_{12}(m)\Big)\Big(\sum_{m\in B(0,N-1)} \mu_{21}(m)\Big) \geq \frac{\sinh(\theta_\alpha)\,\sinh(\theta_\beta)}{\tanh(N\theta_\alpha)\,\tanh(N\theta_\beta)}.$$

Proof. Since $\mu_{11} = \mu_{22} = 0$, we have $M_{11} = M_{22} = 0$, and the matrix K given by (4.6) reduces to

$$K = \begin{pmatrix} 0 & a(N,\alpha)\, M_{12} \\ b(N,\beta)\, M_{21} & 0 \end{pmatrix}.$$

Hence,

$$\rho(K) = \sqrt{a(N,\alpha)b(N,\beta)\, M_{12}M_{21}}.$$

By Theorem 4.5, the existence of a nontrivial solution implies $\rho(K) \geq 1$. Therefore,

$$a(N,\alpha)b(N,\beta)\, M_{12}M_{21} \geq 1,$$

which is exactly the desired inequality. □

5. Applications to a weighted eigenvalue problem

In this section, we apply the Lyapunov-type inequalities obtained in Section 4 to a weighted eigenvalue problem. The bounds obtained below have a concrete spectral utility. They provide explicit and easily computable estimates for the first eigenvalue of a weighted discrete Helmholtz problem without requiring the exact solution of the generalized eigenvalue equation. Such information is useful when one wants to locate the spectrum, compare the effect of different weights, or estimate how the parameters N, α, and μ influence the threshold for the existence of nontrivial solutions.

More precisely, let $\mu\colon B(0,N-1)\to(0,\infty)$ be a given weight function, and consider the Dirichlet problem

$$\begin{cases}\mathcal{L}_\alpha u(n)=\lambda\,\mu(n)\,u(n), & n\in B(0,N-1),\\ u(-N)=0, \qquad u(N)=0.\end{cases} \tag{5.1}$$

A real number λ is called an eigenvalue of (5.1) if the problem admits a nontrivial solution $u\not\equiv 0$.

5.1. Matrix representation of $\mathcal{L}_\alpha$

We identify a function $u\colon B(0,N)\to\mathbb{R}$ with the vector

$$U=\big(u(-N+1),u(-N+2),\dots,u(N-1)\big)^\top=\big(U_1,U_2,\dots,U_{2N-1}\big)^\top\in\mathbb{R}^{2N-1},$$

and impose the Dirichlet boundary conditions by setting $u(-N)=u(N)=0$.

Recall that, for $n\in B(0,N-1)$,

$$\mathcal{L}_\alpha u(n)=(1+\alpha)u(n)-\frac{1}{2}\big(u(n+1)+u(n-1)\big).$$

Using this identification and the boundary conditions, the mapping

$$u\mapsto\big(\mathcal{L}_\alpha u(n)\big)_{n\in B(0,N-1)}$$

is represented by a $(2N-1)\times(2N-1)$ real matrix, denoted by L_α, defined by

$$(L_\alpha U)_j=(1+\alpha)U_j-\frac{1}{2}(U_{j-1}+U_{j+1}),\qquad j=1,\dots,2N-1,$$

with the convention $U_0=U_{2N}=0$, corresponding to the Dirichlet conditions.

Equivalently, L_α is the tridiagonal symmetric matrix

$$L_\alpha=\begin{pmatrix}1+\alpha & -\frac12 & 0 & \cdots & 0\\ -\frac12 & 1+\alpha & -\frac12 & \ddots & \vdots\\ 0 & -\frac12 & 1+\alpha & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & -\frac12\\ 0 & \cdots & 0 & -\frac12 & 1+\alpha\end{pmatrix}\in\mathbb{R}^{(2N-1)\times(2N-1)}.$$

In particular, L_α is symmetric positive definite and therefore invertible. Since L_α is real symmetric, all its eigenvalues are real. We denote by

$$0<\Lambda_{1,\alpha}\le\Lambda_{2,\alpha}\le\cdots\le\Lambda_{2N-1,\alpha}$$

the eigenvalues of L_α, counted with multiplicity and arranged in nondecreasing order. In particular, $\Lambda_{1,\alpha}$ denotes the smallest Dirichlet eigenvalue of L_α. More precisely, we have the following result.

Lemma 5.1. *The eigenvalues of L_α are given by*

$$\Lambda_{k,\alpha} = (1+\alpha) - \cos\Big(\frac{k\pi}{2N}\Big), \qquad k = 1, \ldots, 2N-1.$$

In particular,

$$\Lambda_{1,\alpha} = (1+\alpha) - \cos\Big(\frac{\pi}{2N}\Big).$$

Proof. Fix $k \in \{1, \ldots, 2N-1\}$ and set $t = \frac{k\pi}{2N}$. Define $V \in \mathbb{R}^{2N-1}$ by

$$V_j = \sin(jt), \qquad j = 1, \ldots, 2N-1.$$

Using the identity $\sin((j-1)t) + \sin((j+1)t) = 2\sin(jt)\cos t$, we obtain for $j = 1, \ldots, 2N-1$, with the convention $V_0 = V_{2N} = 0$,

$$(L_\alpha V)_j = (1+\alpha)V_j - \frac{1}{2}(V_{j-1} + V_{j+1}) = (1+\alpha - \cos t)\, V_j.$$

Hence $V \neq 0$ is an eigenvector and the associated eigenvalue is $\Lambda_{k,\alpha} = (1+\alpha) - \cos t = (1+\alpha) - \cos\big(\frac{k\pi}{2N}\big)$. Taking $k = 1$ yields the formula for $\Lambda_{1,\alpha}$. □

5.2. Matrix formulation of the weighted eigenvalue problem

We now rewrite (5.1) in matrix form. As before, we identify a function $u\colon B(0,N) \to \mathbb{R}$ satisfying $u(-N) = u(N) = 0$ with the vector

$$U = \big(u(-N+1), u(-N+2), \ldots, u(N-1)\big)^\top \in \mathbb{R}^{2N-1}.$$

With this identification, the operator $\mathcal{L}_\alpha$ is represented by the matrix L_α.

Define the diagonal matrix

$$M_\mu = \operatorname{diag}\big(\mu(-N+1), \mu(-N+2), \ldots, \mu(N-1)\big) \in \mathbb{R}^{(2N-1)\times(2N-1)}.$$

Then problem (5.1) is equivalent to the generalized eigenvalue problem

$$L_\alpha U = \lambda\, M_\mu U. \tag{5.2}$$

Since L_α is real symmetric positive definite and M_μ is real symmetric positive definite, problem (5.2) is a symmetric definite generalized eigenvalue problem. In particular, all its eigenvalues are real, and the smallest eigenvalue admits the variational characterization

$$0 < \lambda_{1,\alpha} = \min_{U \neq 0} \frac{\langle L_\alpha U, U\rangle}{\langle M_\mu U, U\rangle},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product on $\mathbb{R}^{2N-1}$.

5.3. Two-sided bounds for the first eigenvalue

In this subsection, we derive two-sided estimates for the first eigenvalue $\lambda_{1,\alpha}$ of the weighted problem (5.1). We first establish a Lyapunov-type lower bound and then obtain an upper bound using the variational characterization. Combining these two estimates yields an explicit bracket for $\lambda_{1,\alpha}$.

Theorem 5.2. *Let $\mu\colon B(0,N-1) \to (0,\infty)$, and let $\lambda_{1,\alpha}$ be the smallest eigenvalue of* (5.1). *Then*

$$\lambda_{1,\alpha} \geq \frac{\sinh(\theta_\alpha)}{\tanh(N\theta_\alpha)} \frac{1}{\sum\limits_{m\in B(0,N-1)} \mu(m)}.$$

Proof. By the definition of $\lambda_{1,\alpha}$, there exists a nontrivial function $u \not\equiv 0$ such that

$$\mathcal{L}_\alpha u(n) = \lambda_{1,\alpha}\, \mu(n)\, u(n), \qquad n \in B(0,N-1), \qquad u(\pm N) = 0.$$

Applying Corollary 4.3 to $q(n) = \lambda_{1,\alpha}\mu(n)$ yields

$$\sum_{m\in B(0,N-1)} \lambda_{1,\alpha}\mu(m) \geq \frac{\sinh(\theta_\alpha)}{\tanh(N\theta_\alpha)}.$$

Since $\sum_{m\in B(0,N-1)} \mu(m) > 0$, dividing both sides by $\sum_{m\in B(0,N-1)} \mu(m)$ gives the claim. □

Theorem 5.3. *Let $\mu\colon B(0,N-1) \to (0,\infty)$, and let $\lambda_{1,\alpha}$ be the smallest eigenvalue of* (5.1). *Then*

$$\lambda_{1,\alpha} \leq \frac{\Lambda_{1,\alpha}}{\min\limits_{n\in B(0,N-1)} \mu(n)},$$

where $\Lambda_{1,\alpha}$ is the smallest Dirichlet eigenvalue of L_α.

Proof. Let $M_\mu = \operatorname{diag}(\mu(-N+1),\ldots,\mu(N-1))$. By the variational characterization of the smallest generalized eigenvalue of (5.2), we have

$$\lambda_{1,\alpha} = \min_{U\in\mathbb{R}^{2N-1}\setminus\{0\}} \frac{\langle L_\alpha U, U\rangle}{\langle M_\mu U, U\rangle}.$$

Let $U_{1,\alpha}$ be an eigenvector of L_α associated with $\Lambda_{1,\alpha}$. Then

$$\langle L_\alpha U_{1,\alpha}, U_{1,\alpha}\rangle = \Lambda_{1,\alpha}\langle U_{1,\alpha}, U_{1,\alpha}\rangle.$$

Moreover,

$$\begin{aligned}\langle M_\mu U_{1,\alpha}, U_{1,\alpha}\rangle &= \sum_{n\in B(0,N-1)} \mu(n)\, U_{1,\alpha}(n)^2 \\ &\geq \Big(\min_{n\in B(0,N-1)} \mu(n)\Big) \sum_{n\in B(0,N-1)} U_{1,\alpha}(n)^2 \\ &= \Big(\min_{n\in B(0,N-1)} \mu(n)\Big)\langle U_{1,\alpha}, U_{1,\alpha}\rangle.\end{aligned}$$

Therefore,

$$\lambda_{1,\alpha} \leq \frac{\langle L_\alpha U_{1,\alpha}, U_{1,\alpha}\rangle}{\langle M_\mu U_{1,\alpha}, U_{1,\alpha}\rangle} \leq \frac{\Lambda_{1,\alpha}}{\min_{n\in B(0,N-1)} \mu(n)}.$$

□

Corollary 5.4. *Let* $\mu\colon B(0,N-1)\to(0,\infty)$. *Then the first eigenvalue* $\lambda_{1,\alpha}$ *of* (5.1) *satisfies*

$$\frac{\sinh(\theta_\alpha)}{\tanh(N\theta_\alpha)}\frac{1}{\displaystyle\sum_{m\in B(0,N-1)}\mu(m)} \;\le\; \lambda_{1,\alpha} \;\le\; \frac{(1+\alpha)-\cos\!\left(\frac{\pi}{2N}\right)}{\displaystyle\min_{n\in B(0,N-1)}\mu(n)}.$$

Proof. The lower bound follows from Theorem 5.2, while the upper bound follows from Theorem 5.3 and the expression of $\Lambda_{1,\alpha}$ given in Lemma 5.1. □

5.4. Numerical illustration of the eigenvalue bounds

In this subsection, we illustrate numerically the two-sided estimate obtained in Corollary 5.4. For a given positive weight function μ, we compute the first eigenvalue $\lambda_{1,\alpha}$ of the weighted problem (5.1) for several values of the parameters N and α, and we compare it with the explicit lower and upper bounds provided by the Lyapunov-type inequality and the variational estimate.

More precisely, for each choice of (N,α) we evaluate

$$\frac{\sinh(\theta_\alpha)}{\tanh(N\theta_\alpha)}\frac{1}{\sum_{m\in B(0,N-1)}\mu(m)},\qquad \lambda_{1,\alpha},\qquad \frac{(1+\alpha)-\cos\!\left(\frac{\pi}{2N}\right)}{\min_{n\in B(0,N-1)}\mu(n)},$$

and verify numerically that the first eigenvalue lies between these two quantities. We also examine how the sharpness of the bracket depends on N, α, and on the choice of μ.

All numerical computations were performed using Matlab.

Baseline weight $\mu\equiv 1$. When $\mu(n)\equiv 1$ on $B(0,N-1)$, we have $M_\mu = I$, where I denotes the identity matrix in $\mathbb{R}^{(2N-1)\times(2N-1)}$. Hence the generalized eigenvalue problem (5.2) reduces to the standard eigenvalue problem

$$L_\alpha U = \lambda U.$$

In particular,

$$\lambda_{1,\alpha} = \Lambda_{1,\alpha} = (1+\alpha)-\cos\left(\frac{\pi}{2N}\right),$$

and the double inequality in Corollary 5.4 reduces to

$$\frac{1}{2N-1}\frac{\sinh(\theta_\alpha)}{\tanh(N\theta_\alpha)} \le (1+\alpha)-\cos\left(\frac{\pi}{2N}\right).$$

Equivalently,

$$f(N,\alpha)\le g(N,\alpha), \tag{5.3}$$

where

$$f(N,\alpha)=\frac{\sinh(\theta_\alpha)}{\tanh(N\theta_\alpha)},\qquad g(N,\alpha)=(2N-1)\Big((1+\alpha)-\cos\left(\tfrac{\pi}{2N}\right)\Big).$$

Figure 2 provides a three-dimensional numerical illustration of inequality (5.3). The surface corresponds to the function $f(N,\alpha)$, while the wireframe represents $g(N,\alpha)$. The figure shows that $f(N,\alpha)\le g(N,\alpha)$ for all $N=2,\dots,30$ and $\alpha\in(0,50)$, confirming that the Lyapunov-type lower bound holds throughout the displayed range of parameters. Moreover, the increasing separation between the two graphs as N or α grows illustrates the explicit dependence of the bound on both the size of the discrete domain and the parameter α.

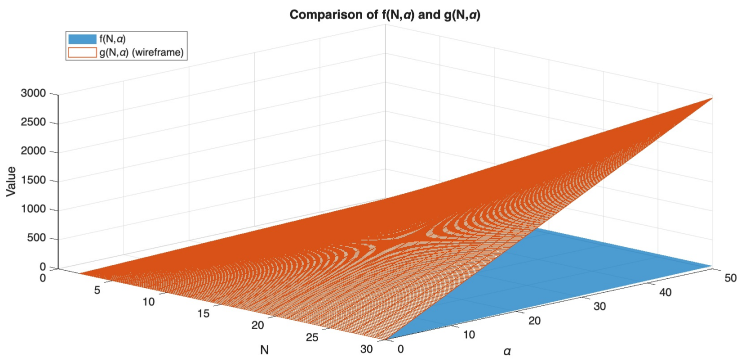

Figure 2. Three-dimensional representation of the functions $f(N,\alpha)$ (surface) and $g(N,\alpha)$ (wireframe) for $N = 2,\ldots,30$ and $\alpha \in (0,50)$.

The data used in Figure 2 were generated by evaluating the closed-form functions $f(N,\alpha)$ and $g(N,\alpha)$ on the grid

$$N = 2,\ldots,30, \qquad \alpha \in (0,50).$$

A localized weight depending on a parameter $c > 0$. We now consider a nonconstant weight function depending on a parameter $c > 0$, defined by

$$\mu_c(n) = \begin{cases} c, & n = 0, \\ 1, & n \in B(0, N-1) \setminus \{0\}. \end{cases}$$

Clearly, $\mu_c(n) > 0$ for all $n \in B(0, N-1)$, so the weighted eigenvalue problem (5.1) is well defined for every $c > 0$. This choice represents a localized perturbation of the constant weight and allows us to examine the influence of a single parameter on the bounds for the first eigenvalue.

In this case, one has

$$\sum_{m \in B(0,N-1)} \mu_c(m) = c + 2N - 2, \qquad \min_{n \in B(0,N-1)} \mu_c(n) = \min\{c, 1\}.$$

Therefore, Corollary 5.4 yields

$$\frac{1}{c + 2N - 2} \frac{\sinh(\theta_\alpha)}{\tanh(N\theta_\alpha)} \leq \lambda_{1,\alpha}(c) \leq \frac{(1+\alpha) - \cos(\frac{\pi}{2N})}{\min\{c, 1\}}. \tag{5.4}$$

To compute $\lambda_{1,\alpha}(c)$, we use the matrix formulation introduced in Subsection 5.1. In the present localized case, the weight matrix M_{μ_c} is diagonal with entries 1, except at the site $n = 0$, where the entry is c. Hence,

$$M_{\mu_c} = \mathrm{diag}(1,\ldots,1,c,1,\ldots,1) \in \mathbb{R}^{(2N-1)\times(2N-1)},$$

where the entry c corresponds to the node $n = 0$.

Thus, $\lambda_{1,\alpha}(c)$ is the smallest generalized eigenvalue of

$$L_\alpha U = \lambda\, M_{\mu_c} U.$$

Equivalently, since M_{μ_c} is positive definite, setting

$$A_c = M_{\mu_c}^{-1/2} L_\alpha M_{\mu_c}^{-1/2},$$

one has

$$\lambda_{1,\alpha}(c) = \min \sigma(A_c).$$

Therefore, $\lambda_{1,\alpha}(c)$ is computed as the smallest eigenvalue of the real symmetric matrix A_c of size $2N-1$. Here $\sigma(A_c)$ denotes the spectrum of A_c.

To illustrate (5.4) numerically, we compute $\lambda_{1,\alpha}(c)$ by evaluating the spectrum of the symmetric matrix A_c. Namely, for each fixed (N, α, c) we compute $\sigma(A_c)$ in Matlab and extract its smallest element. We then compare this value with the explicit bounds in (5.4).

Tables 1–3 provide numerical values of the first eigenvalue $\lambda_{1,\alpha}(c)$ for $N = 4$ and for three representative choices of α.

c	$\lambda_{1,\alpha}(c)$
0.1	0.218300
0.5	0.199147
1	0.176120
2	0.137725
3	0.110402
4	0.091236
5	0.077408
6	0.067079
7	0.059115
8	0.052807
9	0.047696
10	0.043476

Table 1. $\lambda_{1,\alpha}(c)$ for $N = 4$ and $\alpha = 0.1$.

c	$\lambda_{1,\alpha}(c)$
0.1	1.208429
0.5	1.171647
1	1.076120
2	0.752246
3	0.535557
4	0.411581
5	0.333411
6	0.279958
7	0.241188
8	0.211809
9	0.188791
10	0.170275

Table 2. $\lambda_{1,\alpha}(c)$ for $N = 4$ and $\alpha = 1$.

c	$\lambda_{1,\alpha}(c)$
0.1	20.286149
0.5	20.280902
1	20.076120
2	10.476190
3	6.988090
4	5.242059
5	4.194044
6	3.495236
7	2.996030
8	2.621597
9	2.330355
10	2.097353

Table 3. $\lambda_{1,\alpha}(c)$ for $N = 4$ and $\alpha = 20$.

A comparison of these tables reveals two systematic effects. For each fixed value of α, the first eigenvalue $\lambda_{1,\alpha}(c)$ decreases as c increases, showing that strengthening the localized weight at the origin consistently lowers the first eigenvalue of the weighted problem. On the other hand, for each fixed c, the values of $\lambda_{1,\alpha}(c)$ increase with α, and the sensitivity with respect to c becomes markedly stronger as α grows. In particular, the range of variation of $\lambda_{1,\alpha}(c)$ over $c \in \{0.1, 0.5, 1, 2, \ldots, 10\}$ is relatively small for $\alpha = 0.1$, more pronounced for $\alpha = 1$, and very large for $\alpha = 20$. This comparison highlights the combined influence of the parameter α and the localized parameter c on the spectral behavior of the weighted discrete Helmholtz problem.

Figures 3–5 are generated from the numerical values of $\lambda_{1,\alpha}(c) = \min \sigma(A_c)$ and from the explicit lower and upper bounds in (5.4), for $N = 4$, $\alpha \in \{0.1, 1, 20\}$, and $c \in [0.1, 10]$.

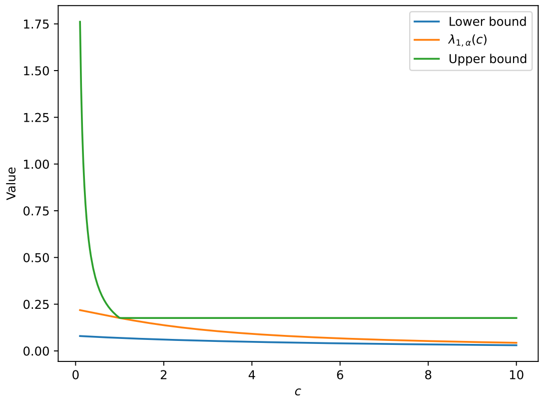

Figure 3. Graphical comparison, for $N = 4$ and $\alpha = 0.1$, of the lower bound $\frac{1}{c+2N-2}\frac{\sinh(\theta_\alpha)}{\tanh(N\theta_\alpha)}$, the first eigenvalue $\lambda_{1,\alpha}(c)$, and the upper bound $\frac{(1+\alpha)-\cos\left(\frac{\pi}{2N}\right)}{\min\{c,1\}}$ as functions of $c > 0$.

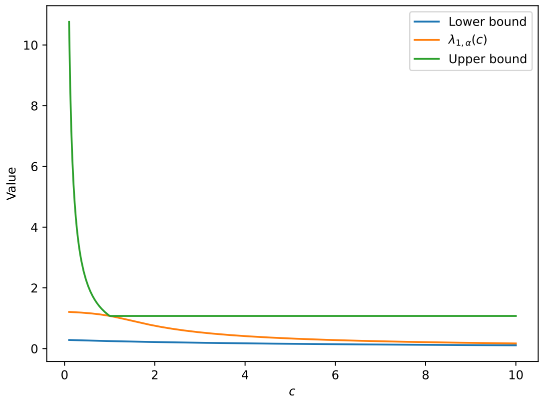

Figure 4. Graphical comparison, for $N = 4$ and $\alpha = 1$, of the lower bound $\frac{1}{c+2N-2}\frac{\sinh(\theta_\alpha)}{\tanh(N\theta_\alpha)}$, the first eigenvalue $\lambda_{1,\alpha}(c)$, and the upper bound $\frac{(1+\alpha)-\cos\left(\frac{\pi}{2N}\right)}{\min\{c,1\}}$ as functions of $c > 0$.

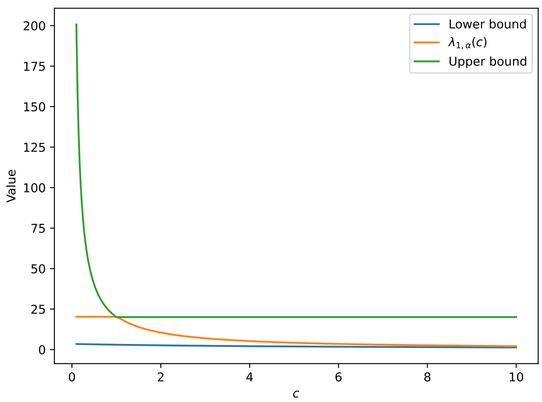

Figure 5. Graphical comparison, for $N = 4$ and $\alpha = 20$, of the lower bound $\frac{1}{c+2N-2}\frac{\sinh(\theta_\alpha)}{\tanh(N\theta_\alpha)}$, the first eigenvalue $\lambda_{1,\alpha}(c)$, and the upper bound $\frac{(1+\alpha)-\cos\left(\frac{\pi}{2N}\right)}{\min\{c,1\}}$ as functions of $c > 0$.

Similarly, Figures 4 and 5 compare, for $N = 4$ and $\alpha = 1$ and $\alpha = 20$, respectively, the numerically computed first eigenvalue $\lambda_{1,\alpha}(c) = \min \sigma(A_c)$ with the explicit lower and upper bounds in (5.4). In both cases, the curve of $\lambda_{1,\alpha}(c)$ remains between the two bounds for all displayed values of c, illustrating the behavior predicted by (5.4) for these choices of parameters.

6. Conclusions

In this work, we investigated Lyapunov-type inequalities for boundary value problems involving the modified discrete Helmholtz operator on finite balls of the integer lattice. Our approach relies on an explicit construction of the Dirichlet Green function associated with $\mathcal{L}_\alpha$ on $B(0, N-1)$, together with sharp positivity and supremum estimates. These estimates yield a weighted Lyapunov-type inequality for the scalar Dirichlet problem, with a sharpness result, and they also lead to a spectral-radius criterion for the corresponding coupled Dirichlet system.

As an application, we considered a weighted eigenvalue problem and derived explicit two-sided bounds for the first eigenvalue by combining the Lyapunov-type lower estimate with a variational upper bound, using the explicit eigenvalues of the associated tridiagonal matrix. Numerical experiments for localized weights confirm the validity of the bounds and illustrate their dependence on N, α, and the weight.

Finally, motivated by Remark 2.1 and the identification of $\mathbb{Z}$ with the infinite 2-regular tree, it would be natural to extend the present framework to $(q+1)$-regular trees T_{q+1} with $q \geq 2$. In particular, developing sharp Green function estimates on finite balls of T_{q+1} for discrete Helmholtz-type operators and deriving corresponding Lyapunov-type inequalities and eigenvalue bounds constitute an interesting direction for future work.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author was supported by Ongoing Research Funding Program, (ORF-2026-4), King Saud University, Riyadh, Saudi Arabia.

Conflict of interest

The author declares no conflict of interest in this work.

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