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Research article

A unified framework for fixed point results of three-point contractions in multiple perturbed metric spaces with applications to fractional differential equations

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Abstract: In many practical situations, distance measurements are affected by unavoidable inaccuracies due to instrumental limitations and external factors. Although such errors are often small, their accumulation may significantly impact the validity of mathematical models. This motivates the use of perturbed metric spaces as a natural framework to incorporate such imperfections into fixed point theory. In the current study, many theorems are established for three-point mappings contracting the perimeters of triangles under the structure of triple perturbed metric spaces. The findings presented here extend and consolidate numerous known results in fixed point theory by combining three-point contraction techniques with different perturbed metric structures. The use of three distinct perturbed metrics provides a more flexible and generalized contractive framework. Some examples are given showing that these satisfy the proposed conditions, while existing results cannot be applied to these examples, highlighting the wider applicability of the present results. Finally, we derived rigorous existence and uniqueness conditions that guarantee solutions for fractional differential equations and illustrated their relevance in modeling population dynamics, including factors such as memory effects and mortality in rabbit growth. A complementary numerical example further validates the rigorous results and demonstrates the practical applications of the iterative approximation scheme.

Keywords: fixed points; three-point contractions; perturbed metric spaces; mappings contracting perimeters of triangles
Mathematics Subject Classification: 47H10, 54H25

1. Introduction and preliminaries

A simple yet powerful criterion for guaranteeing the existence and uniqueness of solutions to a wide class of problems is provided by one of the most fundamental results in nonlinear analysis, namely, Banach's fixed point principle [1]. Its effectiveness stems from the fact that any mapping on a space to itself, which is in fact a complete metric space, admits a unique fixed point whenever it satisfies the contraction condition given by

$$d(\mathcal{T}\mathfrak{x}, \mathcal{T}\mathfrak{y}) \leq \alpha\, d(\mathfrak{x}, \mathfrak{y}) \quad \forall \mathfrak{x}, \mathfrak{y} \in X,$$

where $\alpha \in [0, 1)$. In addition to guaranteeing existence and uniqueness, the theorem is constructive in nature, as the Picard iterative process converges to the fixed point, thereby providing a practical method for approximation. Due to its clarity and minimal assumptions, this principle has become an essential tool in the study of differential equations (DEs), integral and integro-differential equations, fractional differential equations (FDEs), as well as matrix and operator equations in functional analysis. It also finds wide applications in optimization theory, control theory, dynamical systems, mathematical physics, engineering, economic equilibrium problems, and biological models, where many real-life problems can naturally be reformulated as fixed point equations. This blend of simplicity, applicability, and strong conclusions continues to make Banach's contraction principle a central result in both theoretical frameworks and practical applications. Fixed point theory has wide applications in differential and fractional differential equations. Li et al. [2] studied nonlinear integral impulsive differential equations using monotone iterative techniques, while Hu and Liao [3] established convergence conditions for impulsive differential systems. Reunsumrit et al. [4] investigated fractional symmetric Hahn difference equations with nonlocal boundary conditions. Fixed point results for nonlinear contractions were established in [5], while fixed point theorems for multivalued mappings were obtained in [6]. These works further demonstrate the effectiveness of fixed point methods in the analysis of differential and fractional problems.

Over the years, Banach's fixed point theorem has motivated extensive research, resulting in a wide range of extensions and refinements that have broadened its scope and strengthened its applicability, giving rise to a rich and evolving literature. These developments can broadly be classified into three main categories. The first direction is devoted to relaxing the contractive condition by introducing broader classes of mappings. Boyd and Wong [7] studied nonlinear contractions, Ćirić [8] introduced a generalized contraction principle, Khojasteh et al. [9] developed the simulation function approach, Kirk [10] investigated asymptotic contractions, Popescu [11] discussed generalized contractive mappings, Proinov [12] established fixed point results for generalized contractive mappings in metric spaces, and Rakotch [13] considered a class of contractive mappings extending the Banach framework. These contributions significantly broaden the scope of the classical contraction theory. The second direction concerns the extension of fixed point theory to generalized metric settings. Branciari [14] established a Banach-type fixed point theorem in generalized metric spaces, Czerwik [15] introduced contraction mappings in b-metric spaces, Jleli and Samet [16] proposed a new generalization of metric spaces, Mustafa and Sims [17] developed the theory of G-metric spaces, and Oltra and Valero [18] proved a version of Banach's fixed point theorem in partial metric spaces. These developments significantly broadened the range of spaces in which fixed point results can be obtained. A third important direction focuses on the incorporation of auxiliary control functions into contractive conditions. Wardowski [19] introduced the notion of F-contractions, Udo-utun [20] investigated the relationship between F-contractions and weak contractions, Jleli and Samet [21] proposed a generalized Banach contraction principle via introducing θ-mappings, Ahmad et al. [22] established fixed point results for generalized θ-contractions, Akram et al. [23] introduced A-contractions, and Kumam et al. [24] studied Suzuki-type Z-contraction mappings. These developments have substantially broadened the scope and applicability of fixed point theory and continue to stimulate ongoing research in nonlinear analysis.

In practical situations, the evaluation of the distance function between two objects is often subject to unavoidable inaccuracies originating from various sources, such as limitations in measurement instruments or external disturbances. Although these discrepancies are typically small, their cumulative effect may become non-negligible and influence the reliability of the results. This observation naturally raises an important question: What happens to the classical fixed point theory if the exact metric is replaced by an experimentally observed distance? More precisely, if the standard contractive condition is formulated in terms of a measured function $\mathcal{P}$, which approximates the true metric d, that is,

$$\mathcal{P}(\mathcal{T}\mathfrak{x}, \mathcal{T}\mathfrak{y}) \leq \alpha\, \mathcal{P}(\mathfrak{x}, \mathfrak{y}), \qquad \forall\ \mathfrak{x}, \mathfrak{y} \in X,$$

where $\alpha \in (0, 1)$ and $\mathcal{P}$ may fail to be a metric, it becomes natural to examine whether the Banach's fixed point theorem remains valid under such perturbations. Motivated by this perspective, the concept of perturbed metric spaces (PMSs was given by Jleli and Samet in [25], together with corresponding extensions of the classical fixed point principle established within this more flexible and realistic framework.

Throughout this paper, X denotes a nonempty set, $\mathbb{R}^+$ represents the interval $[0, \infty)$, and $\mathbb{N}$ stands for the set of nonnegative integers. To account for possible inaccuracies in distance measurements, the concept of PMSs was given by Jleli and Samet in [25].

Definition 1.1 ([25])**.** *Let us consider the two mappings $\mathcal{P}, \Lambda : X \times X \to [0, \infty)$. The mapping $\mathcal{P}$ is known as a* perturbed metric *on X with respect to (w.r.t.) the mapping Λ if the function $d : X \times X \to [0, \infty)$ defined by*

$$d(\mathfrak{x}, \mathfrak{y}) := \mathcal{P}(\mathfrak{x}, \mathfrak{y}) - \Lambda(\mathfrak{x}, \mathfrak{y}), \qquad \forall\ \mathfrak{x}, \mathfrak{y} \in X,$$

is a metric on X. In this case, d is called the exact metric and $(X, \mathcal{P}, \Lambda)$ is called a PMS.

Note that $\mathcal{P}$ need not be a metric on X, as illustrated by the following counterexample.

Example 1.2. *Take $X = \mathbb{R}$ and consider $\mathcal{P}, \Lambda : X \times X \to [0, \infty)$ defined by*

$$\mathcal{P}(\mathfrak{x}, \mathfrak{y}) = |\mathfrak{x} - \mathfrak{y}| + \mathfrak{x}^2\mathfrak{y}^4, \qquad \forall\ \mathfrak{x}, \mathfrak{y} \in \mathbb{R},$$

and

$$\Lambda(\mathfrak{x}, \mathfrak{y}) = \mathfrak{x}^2\mathfrak{y}^4, \qquad \forall\ \mathfrak{x}, \mathfrak{y} \in \mathbb{R}.$$

Then the mapping $d(\mathfrak{x}, \mathfrak{y}) := \mathcal{P}(\mathfrak{x}, \mathfrak{y}) - \Lambda(\mathfrak{x}, \mathfrak{y}) = |\mathfrak{x} - \mathfrak{y}|$, $\forall\ \mathfrak{x}, \mathfrak{y} \in \mathbb{R}$, is an exact metric on $\mathbb{R}$. Hence, $\mathcal{P}$ is a perturbed metric with respect to (w.r.t.) Λ. However, $\mathcal{P}$ is not a metric on X since

$$\mathcal{P}(1, 1) = 1 \neq 0.$$

In order to develop a suitable analytical framework in PMSs, it is essential to extend the fundamental topological notions to this setting. Since the mapping $\mathcal{P}$ is not necessarily a metric, these concepts cannot be defined directly in terms of $\mathcal{P}$. Instead, they are naturally formulated via the associated exact metric $d := \mathcal{P} - \Lambda$. This approach ensures consistency with classical metric space theory while accommodating the perturbation structure. The following definitions, given by Jleli and Samet in [25], provide the basis for convergence, the Cauchy sequence, completeness, and continuity in PMSs.

Remark 1.3 ([25]). *Let $(X, \mathcal{P}, \Lambda)$ be a PMS. A sequence, the space, and a mapping are called perturbed convergent (or Cauchy), complete, and perturbed continuous, respectively if they are convergent (or Cauchy), complete, and continuous w.r.t. the associated exact metric $d := \mathcal{P} - \Lambda$ on X.*

Within this framework, Jleli and Samet in [25] obtained an extension of the well-known Banach fixed point principle.

Theorem 1.4 ([25]). *Let $(X, \mathcal{P}, \Lambda)$ be a complete PMS with a mapping $\mathcal{T} : X \to X$ such that (s.t.)*

(i) $\mathcal{T}$ is perturbed continuous;
(ii) $\mathcal{P}(\mathcal{T}\mathfrak{x}, \mathcal{T}\mathfrak{y}) \leq \alpha\, \mathcal{P}(\mathfrak{x}, \mathfrak{y}), \quad \forall\ \mathfrak{x}, \mathfrak{y} \in X$, where $\alpha \in (0, 1)$.

Then, $\mathcal{T}$ admits a unique fixed point in X.

The following example, adapted from [25], illustrates that the mapping $\mathcal{T}$ fails to be a Banach contraction w.r.t. the exact metric d, while it satisfies a Banach-type contractive condition in terms of the perturbed mapping $\mathcal{P}$.

Example 1.5. *Let $X = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\} \subset \mathbb{R}^3$, where $\varsigma_1, \varsigma_2, \varsigma_3$, and ς_4 are the vertices of a regular tetrahedron, i.e., $\|\varsigma_p - \varsigma_q\| = 1$ for $p \neq q$, where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^3$. Define the mapping $\mathcal{T} : X \to X$ by*

$$\mathcal{T}\varsigma_1 = \varsigma_1, \quad \mathcal{T}\varsigma_2 = \varsigma_3, \quad \mathcal{T}\varsigma_3 = \varsigma_4, \quad \mathcal{T}\varsigma_4 = \varsigma_1.$$

Let $\Lambda : X \times X \to [0, \infty)$ be defined as follows:

$$\Lambda(\varsigma_1, \varsigma_2) = 4,\ \Lambda(\varsigma_1, \varsigma_3) = 3,\ \Lambda(\varsigma_1, \varsigma_4) = 2,$$

$$\Lambda(\varsigma_2, \varsigma_3) = 4,\ \Lambda(\varsigma_2, \varsigma_4) = 9,\ \Lambda(\varsigma_3, \varsigma_4) = 3,$$

with symmetry $\Lambda(\varsigma_p, \varsigma_q) = \Lambda(\varsigma_q, \varsigma_p)$ and $\Lambda(\varsigma_p, \varsigma_p) = 0,\ \forall\ p, q \in \{1, 2, 3, 4\}$. Define $\mathcal{P} : X \times X \to [0, \infty)$ by

$$\mathcal{P}(\varsigma_p, \varsigma_q) = \|\varsigma_p - \varsigma_q\| + \Lambda(\varsigma_p, \varsigma_q), \qquad \forall\ p, q \in \{1, 2, 3, 4\}.$$

Then, $(X, \mathcal{P}, \Lambda)$ is a PMS. The exact metric $d := \mathcal{P} - \Lambda$ is the discrete metric on X, that is,

$$d(\varsigma_p, \varsigma_q) = \begin{cases} 1, & p \neq q, \\ 0, & p = q. \end{cases}$$

It is clear that $\mathcal{T}$ is perturbed continuous. Moreover, one can verify that $\mathcal{T}$ satisfies the contractive condition

$$\mathcal{P}(\mathcal{T}\mathfrak{x}, \mathcal{T}\mathfrak{y}) \leq \frac{4}{5}\mathcal{P}(\mathfrak{x}, \mathfrak{y})$$

for all $\mathfrak{x}, \mathfrak{y} \in X$, and hence $\mathcal{T}$ is a Banach-type contraction in the sense of the perturbed metric $\mathcal{P}$. However, $\mathcal{T}$ is not a Banach contraction w.r.t. the exact metric d, since $d(\mathcal{T}\varsigma_1, \mathcal{T}\varsigma_2) = d(\varsigma_1, \varsigma_2)$. This example highlights the distinction between the perturbed and classical metric frameworks.

Recently, Bisht and Petrov introduced a novel extension of Banach's fixed point theorem in [26] by defining a class of single-valued mappings that contract the perimeters of triangles, referred to as three-point contractions. This framework replaces the classical pairwise contractive structure with a condition involving three variables, thereby capturing a more general form of contractive behavior. Within this setting, the existence of fixed points is established under suitable assumptions, demonstrating that the classical theory can be effectively extended to multi-point interaction schemes while preserving the essential convergence properties.

Definition 1.6 ([26]). *A self-mapping $\mathcal{T}$ on a metric space (X, d) with $|X| \geq 3$ is said to contract the perimeters of triangles on X, if*

$$d(\mathcal{T}\mathfrak{x}, \mathcal{T}\mathfrak{y}) + d(\mathcal{T}\mathfrak{y}, \mathcal{T}z) + d(\mathcal{T}z, \mathcal{T}\mathfrak{x}) \leq \alpha\big[d(\mathfrak{x}, \mathfrak{y}) + d(\mathfrak{y}, z) + d(z, \mathfrak{x})\big] \tag{1.1}$$

for all mutually distinct points $\mathfrak{x}, \mathfrak{y}, z \in X$, where $\alpha \in [0, 1)$ is a fixed constant, and $|X|$ represents the number of elements present in the set X.

A corresponding fixed point result established in [26] is stated as follows.

Theorem 1.7 ([26]). *Consider a self-mapping $\mathcal{T}$ on a complete metric space (X, d) with $|X| \geq 3$ s.t.:*

(i) $\mathcal{T}(\mathcal{T}\mathfrak{x}) \neq \mathfrak{x}$ whenever $\mathcal{T}\mathfrak{x} \neq \mathfrak{x}$ for each $\mathfrak{x} \in X$;
(ii) $\mathcal{T}$ fulfills condition (1.1).

Then, $\mathrm{Fix}(\mathcal{T}) \neq \emptyset$ *and* $|\mathrm{Fix}(\mathcal{T})| \leq 2$.

Several studies have examined fixed point results for mappings defined by contraction conditions involving triangle perimeters; see [27–29]. Furthermore, the corresponding results for multi-point generalizations of the classical Kannan and Chatterjea fixed point theorems have been established in [30, 31] and [32], respectively.

Furthermore, Jleli et al. in [33], introduced a more general framework for studying fixed points of single-valued mappings based on three distinct metrics. This approach provides a novel and effective generalization of the idea of mappings contracting the perimeters of triangles by replacing a single metric with three possibly different metrics, thereby offering a more flexible and refined structure.

Definition 1.8 ([33]). *Consider the three metric spaces (X, d_1, d_2, d_3) with $|X| \geq 3$, and $\Phi \in \mathfrak{I}$. We denote the collection of mappings $\mathcal{T} : X \to X$ by $\mu(X, d_1, d_2, d_3, \Phi)$, if*

$$d_1(\mathcal{T}\mathfrak{x}, \mathcal{T}\mathfrak{y}) + d_2(\mathcal{T}\mathfrak{y}, \mathcal{T}z) + d_3(\mathcal{T}z, \mathcal{T}\mathfrak{x}) \leq \Phi\big(d_1(\mathfrak{x}, \mathfrak{y}) + d_2(\mathfrak{y}, z) + d_3(z, \mathfrak{x})\big) \tag{1.2}$$

holds for all mutually distinct points $\mathfrak{x}, \mathfrak{y}, z \in X$. Here, $\mathfrak{I}$ denotes the set of all functions $\Phi : [0, \infty) \to [0, \infty)$, which are nondecreasing, and $\sum_{q=1}^{\infty} \Phi^q(\zeta) < \infty$, $\forall\, \zeta > 0$.

Remark 1.9. *It is worth noting that if $\mathcal{T} : (X, d) \to (X, d)$ is a mapping that contracts triangle perimeters in the sense of Definition 1.6, then condition (1.2) reduces to this case by letting $d_1 = d_2 = d_3 = d$ and defining $\Phi(\zeta) = \alpha\zeta$, $\forall\; \zeta \geq 0$, where $\alpha \in [0, 1)$. Consequently, $\mathcal{T} \in \mu(X, d, d, d, \Phi)$.*

Moreover, it has been shown in [33] that mappings belonging to the class $\mu(X, d_1, d_2, d_3, \Phi)$ need not be continuous on X w.r.t. at least one of the metrics d_i, $i \in \{1, 2, 3\}$. This fact is illustrated by the following example. Let $X = [0, 1]$, let d_1 be the usual Euclidean metric on X, and define d_2, d_3 by

$$d_2(\mathfrak{x}, \mathfrak{y}) = d_3(\mathfrak{x}, \mathfrak{y}) = \begin{cases} d_1(\mathfrak{x}, \mathfrak{y}), & \text{if } \mathfrak{x}, \mathfrak{y} \in [0, \frac{1}{2}] \text{ or } \mathfrak{x}, \mathfrak{y} \in (\frac{1}{2}, 1], \\ 1, & \text{otherwise.} \end{cases}$$

It can be verified that d_2 and d_3 are both metrics on X. Consider the mapping $\mathcal{T} : X \to X$ defined by

$$\mathcal{T}\mathfrak{x} = \begin{cases} \frac{1}{3}\mathfrak{x}, & \mathfrak{x} \in [0, \frac{1}{2}], \\ \frac{1}{2}\mathfrak{x}, & \mathfrak{x} \in (\frac{1}{2}, 1]. \end{cases}$$

Then, $\mathcal{T}$ is discontinuous at $\mathfrak{x} = \frac{1}{2}$ w.r.t. d_1. Furthermore, $\mathcal{T}$ is not a Banach contraction and also not a contraction mapping contracting the perimeters of triangles in the classical sense. However, $\mathcal{T}$ belongs to the class $\mu(X, d_1, d_2, d_3, \Phi)$ for $\Phi(\eta) = \frac{\eta}{2}$, $\forall\, \eta \in \mathbb{R}^+$, which highlights the broader applicability of this generalized framework.

The central result in this context, due to Jleli et al. [33], is stated in the subsequent fixed point theorem.

Theorem 1.10 ([33]). *Let (X, d_i), for $i = 1, 2, 3$, be three metric spaces on X with $|X| \geq 3$, and assume that (X, d_1) is complete and $\mathcal{T} : X \to X$ be s.t.:*

(i) $\forall\ \mathfrak{x} \in X$, $\mathcal{T}(\mathcal{T}\mathfrak{x}) \neq \mathfrak{x}$, provided that $\mathcal{T}\mathfrak{x} \neq \mathfrak{x}$;
(ii) $\mathcal{T} \in \mu(X, d_1, d_2, d_3, \Phi)$ for some $\Phi \in \mathfrak{I}$ and $\mathcal{T}$ is continuous on (X, d_1).

Then, $\mathrm{Fix}(\mathcal{T}) \neq \emptyset$ *and* $|\mathrm{Fix}(\mathcal{T})| \leq 2$.

Remark 1.11. *Note that if $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function s.t. $\sum_{n=1}^{\infty} \Phi^n(\mathring{\zeta}) < \infty$, $\forall\, \mathring{\zeta} > 0$, then the following properties hold:*

(C_1) $\lim_{n\to\infty} \Phi^n(\mathring{\zeta}) = 0$, $\forall\, \mathring{\zeta} > 0$;
(C_2) $\Phi(\mathring{\zeta}) < \mathring{\zeta}$, $\forall\, \mathring{\zeta} > 0$;
(C_3) the series $\sum_{n=0}^{\infty} \Phi^n(\mathring{\zeta})$ *is convergent;*
(C_4) $\lim_{\mathring{\zeta}\to 0^+} \Phi(\mathring{\zeta}) = 0$.

Below are some commonly used examples of functions $\Phi \in \mathfrak{I}$.

Example 1.12. *A well-known example is given by $\Phi(\eta) = \alpha\eta$, $\eta \in \mathbb{R}^+$, where $\alpha \in (0, 1)$ is a constant.*

Example 1.13. *Define $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ by*

$$\Phi(\eta) = \frac{1}{2}\ln(\eta + 1), \quad \forall \eta \in \mathbb{R}^+.$$

Then $\Phi \in \mathfrak{I}$.

Example 1.14. *Let $0 < \varsigma_1 < \varsigma_2 < 1$. Define $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ by*

$$\Phi(\eta) = \begin{cases} \arctan(\varsigma_1\eta), & 0 \leq \eta \leq \dfrac{1}{\varsigma_1}, \\ \arctan(\varsigma_2\eta), & \eta > \dfrac{1}{\varsigma_1}. \end{cases}$$

Then $\Phi \in \mathfrak{I}$.

Motivated by the work of Jleli and Samet in [25] on PMSs, along with subsequent developments by Petrov [26] and Jleli et al. [33], the current research further explores the potential of this framework in fixed point theory. In particular, the flexibility offered by PMSs provides a natural setting to extend and unify various contraction principles. Building on these ideas, this work investigates fixed point theorems for multi-point contraction mappings within a more generalized structure involving multiple perturbed metrics. The results obtained herein significantly extend and generalize several existing fixed point theorems in the literature. Furthermore, some examples are presented to demonstrate the effectiveness of the proposed approach, showing that these satisfy the given conditions, while previously established results are not applicable in this setting. Finally, we have derived existence and uniqueness conditions that guarantee solutions for fractional differential equations and illustrated their relevance to population dynamics, such as modeling rabbit growth with memory and mortality effects. The numerical example further validates the theoretical results and demonstrates the usefulness of the iterative scheme in approximating the solutions.

2. Main results

This section is devoted to introduce a new collection of mappings contracting the perimeters of triangles in the framework of triple PMSs. Based on this concept, several fixed point results are established, which develop a more general extension of a number of well-known results in the existing literature. Finally, an illustrative example is presented to demonstrate the applicability of the proposed approach, showing that the proposed results apply, whereas the existing results fail to do so.

We begin with the definition given below, which serves as the foundation of the subsequent analysis.

Definition 2.1. *Consider $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ to be three perturbed mappings on X with $|X| \geq 3$, and let $\Phi \in \mathfrak{I}$. Denote by $\mathcal{M}_{\mathcal{P}}(X, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \Phi)$ the class of mappings $\mathcal{T} : X \to X$ satisfying*

$$\mathcal{P}_1(\mathcal{T}\mathfrak{x}, \mathcal{T}\mathfrak{y}) + \mathcal{P}_2(\mathcal{T}\mathfrak{y}, \mathcal{T}z) + \mathcal{P}_3(\mathcal{T}z, \mathcal{T}\mathfrak{x}) \leq \Phi(\mathcal{P}_1(\mathfrak{x}, \mathfrak{y}) + \mathcal{P}_2(\mathfrak{y}, z) + \mathcal{P}_3(z, \mathfrak{x})) \tag{2.1}$$

for all pairwise distinct points $\mathfrak{x}, \mathfrak{y}, z \in X$, where $\mathfrak{I}$ represents the set of all nondecreasing functions $\Phi : [0, \infty) \to [0, \infty)$ s.t. $\sum_{n=1}^{\infty} \Phi^n(\zeta) < \infty$ for every $\zeta > 0$.

We now present the main fixed point result corresponding to the above definition.

Theorem 2.2. *Let $(X, \mathcal{P}_i, \Lambda_i)$, for $i = 1, 2, 3$, be three PMSs on X with $|X| \geq 3$, and assume that $(X, \mathcal{P}_1, \Lambda_1)$ is a complete PMS with $\mathcal{T} : X \to X$ s.t.:*

(i) $\forall\ \mathfrak{x} \in X$, $\mathcal{T}(\mathcal{T}\mathfrak{x}) \neq \mathfrak{x}$, provided that $\mathcal{T}\mathfrak{x} \neq \mathfrak{x}$;
(ii) $\mathcal{T}$ is a perturbed continuous mapping on $(X, \mathcal{P}_1, \Lambda_1)$;
(iii) $\mathcal{T} \in \mathcal{M}_{\mathcal{P}}(X, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \Phi)$ for some $\Phi \in \mathfrak{I}$.

Then, $\mathcal{T}$ has a fixed point and $|\mathrm{Fix}(\mathcal{T})| \leq 2$.

Proof. We begin by showing that $\mathrm{Fix}(\mathcal{T}) \neq \emptyset$. Choose an arbitrary element $\mathfrak{x}_0 \in X$ and generate the iterative Picard sequence $\{\mathfrak{x}_n\}$ by

$$\mathfrak{x}_n = \mathcal{T}^n \mathfrak{x}_0, \quad \forall\ n \geq 0.$$

If, for some $n \geq 1$, one has $\mathfrak{x}_n = \mathfrak{x}_{n-1}$, then $\mathfrak{x}_{n-1}$ is a fixed point of $\mathcal{T}$. Therefore, we consider

$$\mathfrak{x}_n \neq \mathfrak{x}_{n-1}, \quad \forall\ n \geq 1.$$

In view of condition (i), it can be concluded that $\mathcal{T}(\mathcal{T}\mathfrak{x}_{n-1}) \neq \mathfrak{x}_{n-1}$, which yields

$$\mathfrak{x}_{n+1} \neq \mathfrak{x}_{n-1}, \quad \forall\ n \geq 1.$$

For this reason, $\forall\ n \geq 1$, the points $\mathfrak{x}_{n-1}, \mathfrak{x}_n$, and $\mathfrak{x}_{n+1}$ are mutually distinct. Applying condition (iii) to the triple $(\mathfrak{x}_0, \mathfrak{x}_1, \mathfrak{x}_2)$, we obtain

$$\begin{aligned}\mathcal{P}_1(\mathfrak{x}_1, \mathfrak{x}_2) + \mathcal{P}_2(\mathfrak{x}_2, \mathfrak{x}_3) + \mathcal{P}_3(\mathfrak{x}_3, \mathfrak{x}_1) &= \mathcal{P}_1(\mathcal{T}\mathfrak{x}_0, \mathcal{T}\mathfrak{x}_1) + \mathcal{P}_2(\mathcal{T}\mathfrak{x}_1, \mathcal{T}\mathfrak{x}_2) + \mathcal{P}_3(\mathcal{T}\mathfrak{x}_2, \mathcal{T}\mathfrak{x}_0)\\ &\overset{(2.1)}{\leq} \Phi(\mathcal{P}_1(\mathfrak{x}_0, \mathfrak{x}_1) + \mathcal{P}_2(\mathfrak{x}_1, \mathfrak{x}_2) + \mathcal{P}_3(\mathfrak{x}_2, \mathfrak{x}_0)).\end{aligned}$$

Due to the nondecreasing behavior of Φ, it follows that

$$\Phi(\mathcal{P}_1(\mathfrak{x}_1, \mathfrak{x}_2) + \mathcal{P}_2(\mathfrak{x}_2, \mathfrak{x}_3) + \mathcal{P}_3(\mathfrak{x}_3, \mathfrak{x}_1)) \leq \Phi^2(\mathring{k}_0), \tag{2.2}$$

where $\mathring{k}_0 = \mathcal{P}_1(\mathfrak{x}_0, \mathfrak{x}_1) + \mathcal{P}_2(\mathfrak{x}_1, \mathfrak{x}_2) + \mathcal{P}_3(\mathfrak{x}_2, \mathfrak{x}_0) > 0$. Similarly, taking $(\mathfrak{x}, \mathfrak{y}, z) = (\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3)$ in (2.1), we obtain

$$\begin{aligned}\mathcal{P}_1(\mathfrak{x}_2, \mathfrak{x}_3) + \mathcal{P}_2(\mathfrak{x}_3, \mathfrak{x}_4) + \mathcal{P}_3(\mathfrak{x}_4, \mathfrak{x}_2) &= \mathcal{P}_1(\mathcal{T}\mathfrak{x}_1, \mathcal{T}\mathfrak{x}_2) + \mathcal{P}_2(\mathcal{T}\mathfrak{x}_2, \mathcal{T}\mathfrak{x}_3) + \mathcal{P}_3(\mathcal{T}\mathfrak{x}_3, \mathcal{T}\mathfrak{x}_1)\\ &\overset{(2.1)}{\leq} \Phi(\mathcal{P}_1(\mathfrak{x}_1, \mathfrak{x}_2) + \mathcal{P}_2(\mathfrak{x}_2, \mathfrak{x}_3) + \mathcal{P}_3(\mathfrak{x}_3, \mathfrak{x}_1)),\end{aligned}$$

which implies, by the aid of (2.2), that

$$\mathcal{P}_1(\mathfrak{x}_2, \mathfrak{x}_3) + \mathcal{P}_2(\mathfrak{x}_3, \mathfrak{x}_4) + \mathcal{P}_3(\mathfrak{x}_4, \mathfrak{x}_2) \leq \Phi^2(\mathring{k}_0).$$

Proceeding inductively, $\forall\ n \geq 1$, we obtain

$$\mathcal{P}_1(\mathfrak{x}_n, \mathfrak{x}_{n+1}) + \mathcal{P}_2(\mathfrak{x}_{n+1}, \mathfrak{x}_{n+2}) + \mathcal{P}_3(\mathfrak{x}_{n+2}, \mathfrak{x}_n) \leq \Phi^n(\mathring{k}_0),$$

which yields, in particular,

$$\mathcal{P}_1(\mathfrak{x}_n, \mathfrak{x}_{n+1}) \leq \Phi^n(\mathring{k}_0), \quad \forall\ n \geq 1. \tag{2.3}$$

Since the exact metric $d_1 := \mathcal{P}_1 - \Lambda_1$, therefore (2.3) implies that

$$d_1(\mathfrak{x}_n, \mathfrak{x}_{n+1}) \leq d_1(\mathfrak{x}_n, \mathfrak{x}_{n+1}) + \Lambda_1(\mathfrak{x}_n, \mathfrak{x}_{n+1}) = \mathcal{P}_1(\mathfrak{x}_n, \mathfrak{x}_{n+1}) \leq \Phi^n(\mathring{k}_0), \quad \forall\ n \geq 1. \tag{2.4}$$

We now show that $\{\mathfrak{x}_n\}$ is a perturbed Cauchy sequence in $(X, \mathcal{P}_1, \Lambda_1)$. For all $r, q \geq 1$, applying the triangle inequality associated with the exact metric $d_1 := \mathcal{P}_1 - \Lambda_1$, together with (2.4), yields

$$\begin{aligned}d_1(\mathfrak{x}_r, \mathfrak{x}_{r+q}) &\leq d_1(\mathfrak{x}_r, \mathfrak{x}_{r+1}) + d_1(\mathfrak{x}_{r+1}, \mathfrak{x}_{r+2}) + \cdots + d_1(\mathfrak{x}_{r+q-1}, \mathfrak{x}_{r+q})\\ &\leq \Phi^r(\mathring{k}_0) + \Phi^{r+1}(\mathring{k}_0) + \cdots + \Phi^{r+q-1}(\mathring{k}_0)\\ &= \sum_{m=r}^{r+q-1} \Phi^m(\mathring{k}_0)\end{aligned}$$

$$= \sum_{m=0}^{r+q-1} \Phi^m(\mathring{k}_0) - \sum_{m=0}^{r-1} \Phi^m(\mathring{k}_0).$$

By the property of the mapping $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$, it follows that

$$d_1(\mathfrak{x}_r, \mathfrak{x}_{r+q}) \leq \Big(\sum_{m=0}^{\infty} \Phi^m(\mathring{k}_0) - \sum_{m=0}^{r-1} \Phi^m(\mathring{k}_0) \Big) \to 0 \quad \text{as } r \to \infty.$$

For this reason, it shows that $\{\mathfrak{x}_n\}$ is a perturbed Cauchy sequence. Since $(X, \mathcal{P}_1, \Lambda_1)$ is a complete PMS, $\exists\ \mathfrak{x}^* \in X$ s.t.

$$\lim_{n\to\infty} d_1(\mathfrak{x}_n, \mathfrak{x}^*) = 0. \tag{2.5}$$

Next, we establish that $\mathfrak{x}^* \in X$ is s.t. $\mathcal{T}\mathfrak{x}^* = \mathfrak{x}^*$. Indeed, by the perturbed continuity of $\mathcal{T}$, we deduce from (2.5) that

$$\lim_{n\to\infty} d_1(\mathfrak{x}_{n+1}, \mathcal{T}\mathfrak{x}^*) = 0.$$

Hence, by the uniqueness of the limit in the exact metric d_1, where $d_1 := \mathcal{P}_1 - \Lambda_1$, we obtain $\mathfrak{x}^* = \mathcal{T}\mathfrak{x}^*$, which proves our claim.

As a final step, we show that $|\mathrm{Fix}(\mathcal{T})| \leq 2$. Suppose, by contradiction, that there exist three fixed points $\mathfrak{x}, \mathfrak{y}, z \in X$, which are mutually distinct. Then,

$$\begin{aligned} \mathcal{P}_1(\mathfrak{x}, \mathfrak{y}) + \mathcal{P}_2(\mathfrak{y}, z) + \mathcal{P}_3(z, \mathfrak{x}) &= \mathcal{P}_1(\mathcal{T}\mathfrak{x}, \mathcal{T}\mathfrak{y}) + \mathcal{P}_2(\mathcal{T}\mathfrak{y}, \mathcal{T}z) + \mathcal{P}_3(\mathcal{T}z, \mathcal{T}\mathfrak{x}) \\ &\overset{(2.1)}{\leq} \Phi(\mathcal{P}_1(\mathfrak{x}, \mathfrak{y}) + \mathcal{P}_2(\mathfrak{y}, z) + \mathcal{P}_3(z, \mathfrak{x})) \\ &< \mathcal{P}_1(\mathfrak{x}, \mathfrak{y}) + \mathcal{P}_2(\mathfrak{y}, z) + \mathcal{P}_3(z, \mathfrak{x}), \end{aligned}$$

as $\mathcal{P}_1(\mathfrak{x}, \mathfrak{y}) + \mathcal{P}_2(\mathfrak{y}, z) + \mathcal{P}_3(z, \mathfrak{x}) > 0$ and $\Phi(\eta) < \eta,\ \forall\ \eta > 0$. Then, a contradiction is obtained. Hence, $|\mathrm{Fix}(\mathcal{T})| \leq 2$ and this completes the proof. □

Notice that Theorem 2.2 is also true if any of the perturbed metrics $\mathcal{P}_1$, $\mathcal{P}_2$, or $\mathcal{P}_3$ is perturbed complete. Thus, a refined version can be stated as follows.

Theorem 2.3. *Let $(X, \mathcal{P}_i, \Lambda_i)$, for $i = 1, 2, 3$, be three PMSs on X with $|X| \geq 3$. Suppose that at least one of these spaces is perturbed complete, and that $\mathcal{T} : X \to X$ is a perturbed continuous mapping w.r.t. the exact metric corresponding to the complete PMS. Assume further that:*

(i) $\forall\ \mathfrak{x} \in X$, $\mathcal{T}(\mathcal{T}\mathfrak{x}) \neq \mathfrak{x}$ whenever $\mathcal{T}\mathfrak{x} \neq \mathfrak{x}$;
(ii) $\mathcal{T} \in \mathcal{M}_{\mathcal{P}}(X, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \Phi)$ for some $\Phi \in \mathfrak{I}$.

Then, $|\mathrm{Fix}(\mathcal{T})| \in \{1, 2\}$.

We next derive several consequences of the main results. In particular, by setting $\mathcal{P}_i = \mathcal{P}$ for $i = 1, 2, 3$, we obtain the following fixed point theorem for mappings that contract triangle perimeters in PMSs, which extends the corresponding result in [25].

Corollary 2.4. *Let $(X, \mathcal{P}, \Lambda)$ be a complete PMS with $|X| \geq 3$, and let $\mathcal{T} : X \to X$ be a mapping s.t.:*

(i) $\forall\ \mathfrak{x} \in X$, $\mathcal{T}(\mathcal{T}\mathfrak{x}) \neq \mathfrak{x}$, provided that $\mathcal{T}\mathfrak{x} \neq \mathfrak{x}$;
(ii) $\mathcal{T}$ is a perturbed continuous mapping on $(X, \mathcal{P}, \Lambda)$;

(iii) $\mathcal{T} \in \mathcal{M}_{\mathcal{P}}(X, \mathcal{P}, \mathcal{P}, \mathcal{P}, \Phi)$ *for some* $\Phi \in \Im$*, that is, the following holds:*

$$\mathcal{P}(\mathcal{T}\mathfrak{x}, \mathcal{T}\mathfrak{y}) + \mathcal{P}(\mathcal{T}\mathfrak{y}, \mathcal{T}z) + \mathcal{P}(\mathcal{T}z, \mathcal{T}\mathfrak{x}) \leq \Phi\big(\mathcal{P}(\mathfrak{x}, \mathfrak{y}) + \mathcal{P}(\mathfrak{y}, z) + \mathcal{P}(z, \mathfrak{x})\big) \tag{2.6}$$

for all pairwise distinct points $\mathfrak{x}, \mathfrak{y}, z \in X$*.*

Then, $|\mathrm{Fix}(\mathcal{T})| \in \{1, 2\}$*.*

Next, we present the following result, which is in fact due to the Banach-type contraction mapping within perturbed metric spaces, stated below.

Corollary 2.5 ([25]). *Let* $(X, \mathcal{P}, \Lambda)$ *be a complete PMS with* $|X| \geq 3$*, and* $\mathcal{T} : X \to X$ *be a mapping s.t.:*

(i) $\forall\ \mathfrak{x} \in X$, $\mathcal{T}(\mathcal{T}\mathfrak{x}) \neq \mathfrak{x}$*, provided that* $\mathcal{T}\mathfrak{x} \neq \mathfrak{x}$*;*
(ii) $\mathcal{T}$ *is a perturbed continuous mapping on* $(X, \mathcal{P}, \Lambda)$*;*
(iii) $\mathcal{T} \in \mathcal{M}_{\mathcal{P}}(X, \mathcal{P}, \mathcal{P}, \mathcal{P}, \Phi)$*, where* $\Phi(s) := \alpha s,\ \forall\ s \geq 0$*, and* $\alpha \in (0, \frac{1}{3})$ *is a constant, that is, the following holds:*

$$\mathcal{P}(\mathcal{T}\mathfrak{x}, \mathcal{T}\mathfrak{y}) \leq \alpha\mathcal{P}(\mathfrak{x}, \mathfrak{y}) \tag{2.7}$$

for all distinct points $\mathfrak{x}, \mathfrak{y} \in X$*.*

Then, $|\mathrm{Fix}(\mathcal{T})| = 1$*.*

Proof. Due to assumption (iii), we obtain $\mathcal{P}(\mathcal{T}\mathfrak{y}, \mathcal{T}z) \leq \alpha\mathcal{P}(\mathfrak{x}, z)$ and $\mathcal{P}(\mathcal{T}z, \mathcal{T}\mathfrak{x}) \leq \alpha\mathcal{P}(z, \mathfrak{x})$ for all points $\mathfrak{x}, \mathfrak{y}, z \in X$. Thus for all pairwise disjoint points $\mathfrak{x}, \mathfrak{y}, z \in X$, we obtain

$$\mathcal{P}(\mathcal{T}\mathfrak{x}, \mathcal{T}\mathfrak{y}) + \mathcal{P}(\mathcal{T}\mathfrak{y}, \mathcal{T}z) + \mathcal{P}(\mathcal{T}z, \mathcal{T}\mathfrak{x}) \leq \alpha\big[\mathcal{P}(\mathfrak{x}, \mathfrak{y}) + \mathcal{P}(\mathfrak{y}, z) + \mathcal{P}(z, \mathfrak{x})\big].$$

Thus all assumptions of Corollary 2.4 are satisfied, so it has a fixed point, and uniqueness follows directly from (2.7). This completes the proof. □

By setting $\Lambda_i = 0$ for $i = 1, 2, 3$, we obtain $\mathcal{P}_i = d_i$ for $i = 1, 2, 3$, and hence recover the following fixed point result, which coincides with the main result established by Jleli et al. [33].

Corollary 2.6 ([33]). *Let* (X, d_i) *be three metric spaces on* X *with* $|X| \geq 3$*, and* (X, d_1) *is complete together with* $\mathcal{T} : X \to X$ *s.t.:*

(i) $\forall\ \mathfrak{x} \in X$, $\mathcal{T}(\mathcal{T}\mathfrak{x}) \neq \mathfrak{x}$*, provided that* $\mathcal{T}\mathfrak{x} \neq \mathfrak{x}$*;*
(ii) $\mathcal{T}$ *is a continuous mapping on* (X, d_1)*;*
(iii) $\mathcal{T} \in \mu(X, d_1, d_2, d_3, \Phi)$ *for some* $\Phi \in \Im$*, that is, the following holds:*

$$d_1(\mathcal{T}\mathfrak{x}, \mathcal{T}\mathfrak{y}) + d_2(\mathcal{T}\mathfrak{y}, \mathcal{T}z) + d_3(\mathcal{T}z, \mathcal{T}\mathfrak{x}) \leq \Phi\big(d_1(\mathfrak{x}, \mathfrak{y}) + d_2(\mathfrak{y}, z) + d_3(z, \mathfrak{x})\big) \tag{2.8}$$

for all mutually distinct points $\mathfrak{x}, \mathfrak{y}, z \in X$*.*

Then, $|\mathrm{Fix}(\mathcal{T})| \in \{1, 2\}$*.*

In Corollary 2.6, if we assume $\mathcal{P}_i := d,\ \forall\ i = 1, 2, 3$, then we obtain a generalized version of the main result of Petrov [26]. Moreover, the original result of Petrov [26] can be recovered by introducing $\Phi(\eta) := \alpha\eta,\ \forall\ \eta \geq 0$, where $\alpha \in (0, 1)$.

Corollary 2.7. *Let (X,d) be a complete metric space on X with $|X| \geq 3$ and $\mathcal{T} : X \to X$ be a mapping s.t.:*

(i) $\forall\ \mathfrak{x} \in X$, $\mathcal{T}(\mathcal{T}\mathfrak{x}) \neq \mathfrak{x}$, *provided that* $\mathcal{T}\mathfrak{x} \neq \mathfrak{x}$;

(ii) $\mathcal{T} \in \mathcal{M}_{\mathcal{P}}(X,d,d,d,\Phi)$ *for some* $\Phi \in \mathfrak{I}$, *that is, the following holds:*

$$d(\mathcal{T}\mathfrak{x},\mathcal{T}\mathfrak{y}) + d(\mathcal{T}\mathfrak{y},\mathcal{T}z) + d(\mathcal{T}z,\mathcal{T}\mathfrak{x}) \leq \Phi\big(d(\mathfrak{x},\mathfrak{y}) + d(\mathfrak{y},z) + d(z,\mathfrak{x})\big) \tag{2.9}$$

for all mutually distinct points $\mathfrak{x},\mathfrak{y},z \in X$.

Then, $|\mathrm{Fix}(\mathcal{T})| \in \{1,2\}$.

Corollary 2.8 ([26])**.** *Let (X,d) be a complete metric space on X with $|X| \geq 3$ and $\mathcal{T} : X \to X$ be a mapping s.t.:*

(i) $\forall\ \mathfrak{x} \in X$, $\mathcal{T}(\mathcal{T}\mathfrak{x}) \neq \mathfrak{x}$, *provided that* $\mathcal{T}\mathfrak{x} \neq \mathfrak{x}$;

(ii) $\mathcal{T} \in \mathcal{M}_{\mathcal{P}}(X,d,d,d,\Phi)$, *where* $\Phi(s) := \alpha s$, $\forall\ s \geq 0$, *and* $\alpha \in (0,1)$ *is a constant, that is, the following holds:*

$$d(\mathcal{T}\mathfrak{x},\mathcal{T}\mathfrak{y}) + d(\mathcal{T}\mathfrak{y},\mathcal{T}z) + d(\mathcal{T}z,\mathcal{T}\mathfrak{x}) \leq \alpha\big[d(\mathfrak{x},\mathfrak{y}) + d(\mathfrak{y},z) + d(z,\mathfrak{x})\big] \tag{2.10}$$

for all mutually distinct points $\mathfrak{x},\mathfrak{y},z \in X$.

Then, $|\mathrm{Fix}(\mathcal{T})| \in \{1,2\}$.

Finally, we provide an illustrative example which highlights the novelty and effectiveness of our results. In this example, only our fixed point theorems are applicable, while the existing results in the literature, such as those of Banach [1], Petrov [26], Jleli and Samet [25], and Jleli et al. [33], fail to apply. This demonstrates that our approach extends and improves upon previous fixed point results, providing new applicability in cases where classical and recent results cannot be used.

Example 2.9. *Let* $X = \{1,2,3,4\}$, *and define the self-mapping* $\mathcal{T}$ *on* X *as*

$$\mathcal{T}(1) = 1,\ \mathcal{T}(2) = 2,\ \mathcal{T}(3) = 4,\ \mathcal{T}(4) = 1.$$

We now define three mappings $\Lambda_1,\Lambda_2,\Lambda_3 : X \times X \to [0,\infty)$ *as given in Table 1.*

Table 1. Values of the mappings $\Lambda_1,\Lambda_2,\Lambda_3$.

$(\mathfrak{x},\mathfrak{y})$	$\Lambda_1(\mathfrak{x},\mathfrak{y})$	$\Lambda_2(\mathfrak{x},\mathfrak{y})$	$\Lambda_3(\mathfrak{x},\mathfrak{y})$
$(1,2)$	7	3	1
$(2,1)$	5	3	1
$(1,3)$	6	4	5
$(3,1)$	6	4	0
$(1,4)$	2	2	2
$(4,1)$	2	2	2
$(2,3)$	8	6	4
$(3,2)$	8	6	4
$(2,4)$	6	3	5
$(4,2)$	6	3	5
$(3,4)$	9	7	8
$(4,3)$	9	3	8
$(\mathfrak{x},\mathfrak{x})$	0	0	0

We also consider three mappings $\mathcal{P}_i : X \times X \to [0, \infty)$ for $i = 1, 2, 3$, defined in Table 2.

Observe that $(X, \mathcal{P}_i, \Lambda_i)$ for $i = 1, 2, 3$ are PMSs. In this setting, the corresponding exact metrics $d_i : X \times X \to [0, \infty)$ are defined as discrete-type metrics by

$$d_i(\mathfrak{x}, \mathfrak{y}) = \begin{cases} 1 + i, & \text{if } \mathfrak{x} \neq \mathfrak{y}, \\ 0, & \text{if } \mathfrak{x} = \mathfrak{y}, \end{cases}$$

$\forall\ i = 1, 2, 3$. *Since each d_i is of discrete type, it is complete. Consequently, the spaces $(X, \mathcal{P}_i, \Lambda_i)$ are perturbed complete for all $i = 1, 2, 3$. Notice that none of the mappings $\mathcal{P}_i$, for $i = 1, 2, 3$, are the metrics on X. Indeed, we have*

$$\mathcal{P}_1(1, 2) \neq \mathcal{P}_1(2, 1), \quad \mathcal{P}_2(3, 4) \neq \mathcal{P}_2(4, 3), \quad \mathcal{P}_3(1, 3) \neq \mathcal{P}_3(3, 1),$$

which shows that each $\mathcal{P}_i$ fails to satisfy the symmetry condition. Consequently, $\mathcal{P}_i$ is not a metric on X for any $i = 1, 2, 3$.

Table 2. Values of the mappings $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ before and after applying $\mathcal{T}$.

$(\mathfrak{x}, \mathfrak{y})$	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$	$(\mathcal{T}\mathfrak{x}, \mathcal{T}\mathfrak{y})$	$\mathcal{P}_1(\mathcal{T}\mathfrak{x}, \mathcal{T}\mathfrak{y})$	$\mathcal{P}_2(\mathcal{T}\mathfrak{x}, \mathcal{T}\mathfrak{y})$	$\mathcal{P}_3(\mathcal{T}\mathfrak{x}, \mathcal{T}\mathfrak{y})$
(1,2)	9	6	5	(1,2)	9	6	5
(2,1)	7	6	5	(2,1)	7	6	5
(1,3)	8	7	9	(1,4)	4	5	6
(3,1)	8	7	4	(4,1)	4	5	6
(1,4)	4	5	6	(1,1)	0	0	0
(4,1)	4	5	6	(1,1)	0	0	0
(2,3)	10	9	8	(2,4)	8	6	9
(3,2)	10	9	8	(4,2)	8	6	9
(2,4)	8	6	9	(2,1)	7	6	5
(4,2)	8	6	9	(1,2)	9	6	5
(3,4)	11	10	12	(4,1)	4	5	6
(4,3)	11	6	12	(1,4)	4	5	6
$(\mathfrak{x}, \mathfrak{x})$	0	0	0	$(\mathcal{T}\mathfrak{x}, \mathcal{T}\mathfrak{x})$	0	0	0

We now show that all the assumptions of Theorem 2.2 are satisfied. For all $\mathfrak{x} \in X$, we have $\mathcal{T}(\mathcal{T}\mathfrak{x}) \neq \mathfrak{x}$ whenever $\mathcal{T}\mathfrak{x} \neq \mathfrak{x}$, and thus condition (i) holds. Moreover, since $\mathcal{T}$ is a perturbed continuous mapping, condition (ii) is also satisfied. Further, define $\Phi : [0, \infty) \to [0, \infty)$ by $\Phi(\eta) := \alpha\eta$, $\forall\ \eta \geq 0$, where $\alpha \in \left[\frac{21}{22}, 1\right)$. Then, the inequality (2.1) *is satisfied, as illustrated in Table 3. Hence, condition (iii) is also fulfilled. Therefore, all the assumptions of Theorem 2.2 are satisfied. Thus, $\mathcal{T}$ has a fixed point. Moreover, $\mathcal{T}$ has at most two fixed points, that is, $|\text{Fix}(\mathcal{T})| \leq 2$, and in fact* $\text{Fix}(\mathcal{T}) = \{1, 2\}$.

Since $\mathcal{T}$ has more than one fixed point, it is evident that $\mathcal{T}$ is not a Banach contraction as shown in [1] w.r.t. any classical metric. Moreover, the result of Jleli and Samet [25], which ensures the uniqueness of the fixed point, is not applicable in this case. In fact, $\mathcal{T}$ is not a Banach-type contraction w.r.t. any of the perturbed mappings.

Moreover, $\mathcal{T}$ is not a three-point contraction mapping in the sense of Jleli et al. [33]. Indeed, we have

$$d_1(\mathcal{T}2, \mathcal{T}3) + d_2(\mathcal{T}3, \mathcal{T}4) + d_3(\mathcal{T}4, \mathcal{T}2) \quad = \quad d_1(2, 4) + d_2(4, 1) + d_3(1, 2)$$

$$= \ 2 + 3 + 4$$
$$= \ d_1(2,3) + d_2(3,4) + d_3(4,2),$$

which shows that there does not exist any $\Phi \in \mathfrak{I}$ *s.t. the inequality* (1.2) *is satisfied. Therefore, the results in [33] are not applicable in this setting.*

However, our main result can still be applied, which demonstrates that our approach properly extends and strengthens the existing results in the literature.

Table 3. Values of $\mathcal{P}_1(\mathcal{T}\mathfrak{x}, \mathcal{T}\mathfrak{y}) + \mathcal{P}_2(\mathcal{T}\mathfrak{y}, \mathcal{T}z) + \mathcal{P}_3(\mathcal{T}z, \mathcal{T}\mathfrak{x})$ and $\mathcal{P}_1(\mathfrak{x}, \mathfrak{y}) + \mathcal{P}_2(\mathfrak{y}, z) + \mathcal{P}_3(z, \mathfrak{x})$, $\forall$ pairwise distinct elements $\mathfrak{x}, \mathfrak{y}, z \in X$.

$(\mathfrak{x}, \mathfrak{y}, z)$	$\mathcal{P}_1(\mathcal{T}\mathfrak{x}, \mathcal{T}\mathfrak{y}) + \mathcal{P}_2(\mathcal{T}\mathfrak{y}, \mathcal{T}z) + \mathcal{P}_3(\mathcal{T}z, \mathcal{T}\mathfrak{x})$	$\mathcal{P}_1(\mathfrak{x}, \mathfrak{y}) + \mathcal{P}_2(\mathfrak{y}, z) + \mathcal{P}_3(z, \mathfrak{x})$
(1,2,3)	9 + 6 + 6 = 21	9 + 9 + 4 = 22
(1,2,4)	9 + 6 + 0 = 15	9 + 6 + 6 = 21
(1,3,2)	4 + 6 + 5 = 15	8 + 9 + 5 = 22
(1,3,4)	4 + 5 + 0 = 9	8 + 10 + 6 = 24
(1,4,2)	0 + 6 + 5 = 11	4 + 6 + 5 = 15
(1,4,3)	0 + 5 + 6 = 11	4 + 6 + 9 = 19
(2,1,3)	7 + 5 + 9 = 21	7 + 7 + 8 = 22
(2,1,4)	7 + 0 + 5 = 12	7 + 5 + 9 = 21
(2,3,1)	8 + 5 + 5 = 18	10 + 7 + 5 = 22
(2,3,4)	8 + 5 + 5 = 18	10 + 10 + 9 = 29
(2,4,1)	7 + 0 + 5 = 12	8 + 5 + 5 = 18
(2,4,3)	7 + 5 + 9 = 21	8 + 6 + 8 = 22
(3,1,2)	4 + 6 + 9 = 19	8 + 6 + 8 = 22
(3,1,4)	4 + 0 + 6 = 10	8 + 5 + 12 = 25
(3,2,1)	8 + 6 + 6 = 20	10 + 6 + 9 = 25
(3,2,4)	8 + 6 + 6 = 20	10 + 6 + 12 = 28
(3,4,1)	4 + 0 + 6 = 10	11 + 5 + 9 = 25
(3,4,2)	4 + 6 + 9 = 19	11 + 6 + 8 = 25
(4,1,2)	0 + 6 + 5 = 11	4 + 6 + 9 = 19
(4,1,3)	0 + 5 + 6 = 11	4 + 7 + 12 = 23
(4,2,1)	9 + 6 + 0 = 15	8 + 6 + 6 = 20
(4,2,3)	9 + 6 + 6 = 21	8 + 9 + 12 = 29
(4,3,1)	4 + 5 + 0 = 9	11 + 7 + 6 = 24
(4,3,2)	4 + 6 + 5 = 15	11 + 9 + 9 = 29

Example 2.10. *Let* $X = [0,1]$ *and define the self-mapping* $\mathcal{T} : X \to X$ *by*

$$\mathcal{T}(\mathfrak{x}) = \begin{cases} \mathfrak{x}, & \mathfrak{x} \in \{0,1\}, \\ 0, & \mathfrak{x} \in (0,1). \end{cases}$$

Clearly, the mapping $\mathcal{T}$ *is discontinuous at* $\mathfrak{x} = 1$ *and so it is not a Banach contraction mapping for any classical metric.*

Define the functions $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3 : X \times X \to [0, \infty)$ *as follows:*

$$\mathcal{P}_1(\mathfrak{x}, \mathfrak{y}) = \begin{cases} 1, & \mathfrak{x}, \mathfrak{y} \in \{0, 1\},\ \mathfrak{x} \neq \mathfrak{y}, \\ 20, & \mathfrak{x} \neq \mathfrak{y} \text{ and at least one of } \mathfrak{x}, \mathfrak{y} \notin \{0, 1\}, \\ 0, & \mathfrak{x} = \mathfrak{y}, \end{cases}$$

$$\mathcal{P}_2(\mathfrak{x}, \mathfrak{y}) = \begin{cases} 2, & \mathfrak{x}, \mathfrak{y} \in \{0, 1\},\ \mathfrak{x} \neq \mathfrak{y}, \\ 25, & \mathfrak{x} \neq \mathfrak{y} \text{ and at least one of } \mathfrak{x}, \mathfrak{y} \notin \{0, 1\}, \\ 0, & \mathfrak{x} = \mathfrak{y}, \end{cases}$$

and

$$\mathcal{P}_3(\mathfrak{x}, \mathfrak{y}) = \begin{cases} 3, & \mathfrak{x}, \mathfrak{y} \in \{0, 1\},\ \mathfrak{x} \neq \mathfrak{y}, \\ 30, & \mathfrak{x} \neq \mathfrak{y} \text{ and at least one of } \mathfrak{x}, \mathfrak{y} \notin \{0, 1\}, \\ 0, & \mathfrak{x} = \mathfrak{y}. \end{cases}$$

Now, let $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in X$ *be pairwise distinct. Since* $\mathcal{T}\mathfrak{x}, \mathcal{T}\mathfrak{y}, \mathcal{T}\mathfrak{z} \in \{0, 1\}$, *we have* $\mathcal{P}_1(\mathcal{T}\mathfrak{x}, \mathcal{T}\mathfrak{y}) \leq 1$, $\mathcal{P}_2(\mathcal{T}\mathfrak{y}, \mathcal{T}\mathfrak{z}) \leq 2$, *and* $\mathcal{P}_3(\mathcal{T}\mathfrak{z}, \mathcal{T}\mathfrak{x}) \leq 3$. *Hence,*

$$\mathcal{P}_1(\mathcal{T}\mathfrak{x}, \mathcal{T}\mathfrak{y}) + \mathcal{P}_2(\mathcal{T}\mathfrak{y}, \mathcal{T}\mathfrak{z}) + \mathcal{P}_3(\mathcal{T}\mathfrak{z}, \mathcal{T}\mathfrak{x}) \leq 1 + 2 + 3 = 6.$$

On the other hand, since $\mathfrak{x}, \mathfrak{y}, \mathfrak{z}$ *are pairwise distinct, then at least one of these points is outside the set* $\{0, 1\}$. *Consequently, two of the following pairs*

$$(\mathfrak{x}, \mathfrak{y}), \quad (\mathfrak{y}, \mathfrak{z}), \quad (\mathfrak{z}, \mathfrak{x})$$

are disjoint with pair $(0, 1)$. *Therefore, we must choose two of the following values* $\mathcal{P}_1(\mathfrak{x}, \mathfrak{y}) = 20$, $\mathcal{P}_2(\mathfrak{y}, \mathfrak{z}) = 25$, *or* $\mathcal{P}_3(\mathfrak{z}, \mathfrak{x}) = 30$. *For the safety of the bounds, we choose the smallest values, and we obtain*

$$\frac{1}{2}\Big(\mathcal{P}_1(\mathfrak{x}, \mathfrak{y}) + \mathcal{P}_2(\mathfrak{y}, \mathfrak{z}) + \mathcal{P}_3(\mathfrak{z}, \mathfrak{x})\Big) \geq \frac{1}{2}(20 + 25 + 3) = 24.$$

Consequently,

$$\mathcal{P}_1(\mathcal{T}\mathfrak{x}, \mathcal{T}\mathfrak{y}) + \mathcal{P}_2(\mathcal{T}\mathfrak{y}, \mathcal{T}\mathfrak{z}) + \mathcal{P}_3(\mathcal{T}\mathfrak{z}, \mathcal{T}\mathfrak{x}) \leq 6 < 24 \leq \frac{1}{2}\Big(\mathcal{P}_1(\mathfrak{x}, \mathfrak{y}) + \mathcal{P}_2(\mathfrak{y}, \mathfrak{z}) + \mathcal{P}_3(\mathfrak{z}, \mathfrak{x})\Big)$$

for all pairwise distinct $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in X$, *showing that* $\mathcal{T}$ *satisfies inequality* (2.1) *for a function* $\Phi : [0, \infty) \to [0, \infty)$ *defined by* $\Phi(\eta) := \alpha\eta,\ \forall\ \eta \geq 0$, *where* $\alpha \in \left[\frac{1}{2}, 1\right)$.

Next, by taking the three functions Λ_i *for* $i = 1, 2, 3$ *as*

$$\Lambda_1(\mathfrak{x}, \mathfrak{y}) = \begin{cases} 0, & \mathfrak{x}, \mathfrak{y} \in \{0, 1\},\ \mathfrak{x} \neq \mathfrak{y}, \\ 19, & \mathfrak{x} \neq \mathfrak{y} \text{ and at least one of } \mathfrak{x}, \mathfrak{y} \notin \{0, 1\}, \\ 0, & \mathfrak{x} = \mathfrak{y}, \end{cases}$$

$$\Lambda_2(\mathfrak{x}, \mathfrak{y}) = \begin{cases} 1, & \mathfrak{x}, \mathfrak{y} \in \{0, 1\},\ \mathfrak{x} \neq \mathfrak{y}, \\ 24, & \mathfrak{x} \neq \mathfrak{y} \text{ and at least one of } \mathfrak{x}, \mathfrak{y} \notin \{0, 1\}, \\ 0, & \mathfrak{x} = \mathfrak{y}, \end{cases}$$

and

$$\Lambda_3(\mathfrak{x}, \mathfrak{y}) = \begin{cases} 2, & \mathfrak{x}, \mathfrak{y} \in \{0, 1\},\ \mathfrak{x} \neq \mathfrak{y}, \\ 29, & \mathfrak{x} \neq \mathfrak{y} \text{ and at least one of } \mathfrak{x}, \mathfrak{y} \notin \{0, 1\}, \\ 0, & \mathfrak{x} = \mathfrak{y}, \end{cases}$$

we get the following corresponding exact metrics $d_i : X \times X \to [0, \infty)$ defined as discrete-type metrics by

$$d_i(\mathfrak{x}, \mathfrak{y}) = \begin{cases} 1, & \text{if } \mathfrak{x} \neq \mathfrak{y}, \\ 0, & \text{if } \mathfrak{x} = \mathfrak{y}, \end{cases}$$

for all $i = 1, 2, 3$. Since each d_i is discrete, it is complete. Consequently, the spaces $(X, \mathcal{P}_i, \Lambda_i)$ are perturbed complete for all $i = 1, 2, 3$. We now show that all the assumptions of Theorem 2.2 are satisfied. For all $\mathfrak{x} \in X$, we have $\mathcal{T}(\mathcal{T}\mathfrak{x}) \neq \mathfrak{x}$ whenever $\mathcal{T}\mathfrak{x} \neq \mathfrak{x}$, and thus condition (i) holds. Moreover, since $\mathcal{T}$ is a perturbed continuous mapping, condition (ii) is also satisfied.

Therefore, all the assumptions of Theorem 2.2 are satisfied. Thus, $\mathcal{T}$ has a fixed point. Moreover, $\mathcal{T}$ has at most two fixed points, that is, $|\mathrm{Fix}(\mathcal{T})| \leq 2$, and in fact $\mathrm{Fix}(\mathcal{T}) = \{0, 1\}$.

Notice that $\mathcal{T}$ has more than one fixed point, which means that $\mathcal{T}$ is not a Banach contraction as in [1] w.r.t. any classical metric. Moreover, the result of Jleli and Samet [25], which ensures the uniqueness of the fixed point, is not applicable in this case. In fact, $\mathcal{T}$ is not a Banach-type contraction w.r.t. any of the perturbed mappings.

3. Applications to fractional differential equations

In recent decades, fractional differential equations (FDEs) have become important tools for modeling complex phenomena that cannot be properly explained by classical integer-order models, due to their ability to capture memory effects, non-local behavior, and anomalous dynamics. Consequently, they have been widely applied in fields such as physics, biology, control systems, electromagnetism, porous media, viscoelasticity, and electrochemistry, effectively describing processes like subdiffusion, superdiffusion, and molecular transport in cellular environments, which are relevant to biomolecular interactions and drug delivery. At the same time, fixed point theory has emerged as a key technique in nonlinear analysis, providing robust methods to establish the existence and uniqueness of solutions. Its application to fractional differential equations has significantly advanced the understanding of their solvability, and fixed point techniques have increasingly become central tools for addressing fractional differential and integral equations. Readers interested in a deeper exploration of these developments are referred to comprehensive sources such as [34–36], which provide detailed explanations and perspectives on this rapidly evolving area. Motivated by these advances, this work aims to prove the existence and uniqueness of a solution to the fractional-order differential equation.

To begin, we recall essential definitions from fractional calculus. Let $\mathfrak{x} : [0, 1] \to \mathbb{R}$ be a continuous function. The Caputo derivative of order $\Delta > 0$ for $\mathfrak{x}$ is defined as

$${}^{C}D^{\Delta}(\mathfrak{x}(\beta)) := \frac{1}{\Gamma(n - \Delta)} \int_0^{\beta} (\beta - W)^{n-\Delta-1} \mathfrak{x}^{(n)}(W)\, dW,$$

where $n = [\Delta] + 1$, $[\Delta]$ denotes the integer part of Δ, and Γ is the gamma function.

The objective here is to apply a fixed point approach to verify the existence of solutions for the following boundary value problem involving a fractional differential equation:

$$\begin{cases} {}^{C}D^{\Delta}(\mathfrak{x}(\beta)) + \mathring{\chi}(\beta, \mathfrak{x}(\beta)) = 0, & 0 \leq \beta \leq 1, \ 1 < \Delta < 2, \\ \mathfrak{x}(0) = \mathfrak{x}(1) = 0, \end{cases} \tag{3.1}$$

where $\mathfrak{x} : [0, 1] \to \mathbb{R}$ is the unknown function and $\mathring{\chi} : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous. The corresponding Green function $\mathcal{G}$ for this problem is expressed as

$$\mathcal{G}(\beta, W) = \begin{cases} \beta(1 - W)^{\Delta-1} - (\beta - W)^{\Delta-1}, & 0 \leq W \leq \beta \leq 1, \\ \frac{\beta(1-W)^{\Delta-1}}{\Gamma(\Delta)}, & 0 \leq \beta \leq W \leq 1. \end{cases} \tag{3.2}$$

Notice that

$$\sup_{\beta \in [0,1]} \int_0^1 \mathcal{G}(\beta, W)\, dW = \sup_{\beta \in [0,1]} \int_0^{\beta} \left[\frac{\beta(1 - W)^{\Delta-1} - (\beta - W)^{\Delta-1}}{\Gamma(\Delta)} \right] dW + \int_{\beta}^1 \frac{\beta(1 - W)^{\Delta-1}}{\Gamma(\Delta)}\, dW.$$

After a direct computation of integration, we obtain

$$\int_0^1 \mathcal{G}(\beta, W)\, dW = \frac{\beta - \beta^{\Delta}}{\Gamma(\Delta + 1)} \leq 1,$$

for all $\beta \in [0, 1]$. Hence,

$$\sup_{\beta \in [0,1]} \int_0^1 \mathcal{G}(\beta, W)\, dW \leq 1.$$

In this section, we consider X as the space of all continuous real-valued functions defined on $[0, 1]$. To quantify differences between functions, we introduce a perturbed metric $\mathcal{P} : X \times X \to [0, \infty)$, defined for any $\mathfrak{x}, \mathfrak{p} \in X$ by

$$\mathcal{P}(\mathfrak{x}, \mathfrak{p}) := \max_{\beta \in [0,1]} |\mathfrak{x}(\beta) - \mathfrak{p}(\beta)| \ + \ \max_{\beta \in [0,1]} |\mathfrak{x}(\beta) - \mathfrak{p}(\beta)|^2,$$

with the associated perturbation mapping

$$\Lambda := \max_{\beta \in [0,1]} |\mathfrak{x}(\beta) - \mathfrak{p}(\beta)|^2.$$

Equipped with these definitions, the triplet $(X, \mathcal{P}, \Lambda)$ forms a complete perturbed metric space, since the exact metric

$$d := \mathcal{P} - \Lambda$$

is itself a complete metric on X. This framework provides a robust setting for analyzing operators and guarantees that the fixed point techniques we employ will yield strong, reliable results.

We now establish a result derived from Theorem 2.2, offering a robust framework to guarantee the existence of solutions for the boundary value fractional differential equation (3.1). A simplified, more practical form will be discussed afterward.

Theorem 3.1. *Let us consider the boundary fractional differential equation* (3.1) *with Green's function* $\mathcal{G}$ *as defined in* (3.2). *Assume there exists a function* $\Phi \in \mathfrak{I}$ *s.t. for any three distinct elements* $\mathfrak{x}, \mathfrak{p}, z \in X$, *the following inequality holds:*

$$\mathcal{P}(\mathcal{T}\mathfrak{x}, \mathcal{T}\mathfrak{p}) + \mathcal{P}(\mathcal{T}\mathfrak{p}, \mathcal{T}z) + \mathcal{P}(\mathcal{T}z, \mathcal{T}\mathfrak{x}) \leq \Phi\Big(\mathcal{P}(\mathfrak{x}, \mathfrak{p}) + \mathcal{P}(\mathfrak{p}, z) + \mathcal{P}(z, \mathfrak{x})\Big). \tag{3.3}$$

The operator $\mathcal{T} : X \to X$ *is defined for each* $\mathfrak{x} \in X$ *as*

$$(\mathcal{T}\mathfrak{x})(\beta) := \int_0^1 \mathcal{G}(\beta, W)\, \mathring{\chi}(W, \mathfrak{x}(W))\, dW, \quad \forall\ \beta \in [0, 1]. \tag{3.4}$$

Under these conditions, the boundary problem (3.1) *admits at least one solution, and moreover, the number of distinct solutions is at most two.*

Proof. Let us consider the subspace C of X such that for all $\mathfrak{x} \in C$, we have $\mathcal{T}(\mathcal{T}\mathfrak{x}) \neq \mathfrak{x}$ whenever $\mathcal{T}\mathfrak{x} \neq \mathfrak{x}$. This shows that assumption (i) of Corollary 2.5 is satisfied.

We begin by noting that solving the boundary fractional differential equation (3.1) is equivalent to solving the integral equation

$$\mathfrak{x}(\beta) = \int_0^1 \mathcal{G}(\beta, W)\, \mathring{\chi}(W, \mathfrak{x}(W))\, dW, \quad \beta \in [0, 1], \tag{3.5}$$

where $\mathfrak{x} : [0, 1] \to [0, 1]$ is the unknown function. This integral formulation corresponds precisely to seeking a fixed point of the operator $\mathcal{T}$ defined in (3.4).

Within the framework of the complete perturbed metric space $(X, \mathcal{P}, \Lambda)$, the operator $\mathcal{T}$ satisfies the generalized contractive condition given by (3.3), which aligns with the setting of inequality (2.1). Therefore, by applying Corollary 2.5, we deduce that $\mathcal{T}$ possesses a unique fixed point. It follows immediately that the original fractional boundary value problem (3.1) has a unique solution in C. □

While Theorem 3.1 provides a broad framework emphasizing the role of the control function $\Phi \in \mathfrak{I}$, verifying its conditions in practice can be challenging. To make the theorem easier to apply, we now present a simpler version that maintains full mathematical rigor.

Theorem 3.2. *Consider the boundary fractional differential equation* (3.1) *with Green's function* $\mathcal{G}$ *as defined in* (3.2). *Assume there exists a nonnegative constant* θ *satisfying* $\theta + \theta^2 \in [0, \frac{1}{3})$ *s.t.*

$$|\, \mathring{\chi}(\beta, z_1) - \mathring{\chi}(\beta, z_2)| \leq \theta\, |z_1 - z_2| \tag{3.6}$$

for all $\beta \in [0, 1]$ *and* $z_1, z_2 \in \mathbb{R}$.

Under this Lipschitz-type condition, the boundary problem (3.1) *admits at least one solution, and the total number of distinct solutions is at most two. Furthermore, the Picard iterative scheme corresponding to the operator* $\mathcal{T}$ *in* (3.4) *converges to a solution, with the particular limit depending on the initial guess chosen for iteration.*

Proof. Let us consider the subspace C of X such that for all $\mathfrak{x} \in C$, we have $\mathcal{T}(\mathcal{T}\mathfrak{x}) \neq \mathfrak{x}$ whenever $\mathcal{T}\mathfrak{x} \neq \mathfrak{x}$. This shows that assumption (i) of Corollary 2.5 is satisfied. Setting the framework within the exact metric space (X, d) and considering the operator $\mathcal{T}$ as defined in (3.4), we observe that the Lipschitz condition (3.6) aligns with the general form of (2.1).

For any distinct $\mathfrak{x}, \mathfrak{p} \in C$ and $\beta \in [0,1]$, we have

$$
\begin{aligned}
&|(\mathcal{T}\mathfrak{x})(\beta) - (\mathcal{T}\mathfrak{p})(\beta)| + |(\mathcal{T}\mathfrak{x})(\beta) - (\mathcal{T}\mathfrak{p})(\beta)|^2 \\
&= \left| \int_0^1 \mathcal{G}(\beta, W)(\mathring{\chi}(W, \mathfrak{x}(W)) - \mathring{\chi}(W, \mathfrak{p}(W)))\, dW \right| + \left| \int_0^1 \mathcal{G}(\beta, W)(\mathring{\chi}(W, \mathfrak{x}(W)) - \mathring{\chi}(W, \mathfrak{p}(W)))\, dW \right|^2 \\
&\le \int_0^1 \mathcal{G}(\beta, W)\left|\mathring{\chi}(W, \mathfrak{x}(W)) - \mathring{\chi}(W, \mathfrak{p}(W))\right| dW + \left(\int_0^1 \mathcal{G}(\beta, W)\left|\mathring{\chi}(W, \mathfrak{x}(W)) - \mathring{\chi}(W, \mathfrak{p}(W))\right| dW \right)^2 \\
&\le \theta \int_0^1 \mathcal{G}(\beta, W)|\mathfrak{x}(W) - \mathfrak{p}(W)|\, dW + \theta^2 \left(\int_0^1 \mathcal{G}(\beta, W)|\mathfrak{x}(W) - \mathfrak{p}(W)|\, dW \right)^2 \\
&\le \theta \sup_{\beta\in[0,1]} |\mathfrak{x}(\beta) - \mathfrak{p}(\beta)| \sup_{\beta\in[0,1]} \int_0^1 \mathcal{G}(\beta, W)\, dW \\
&\quad + \theta^2 \sup_{\beta\in[0,1]} |\mathfrak{x}(\beta) - \mathfrak{p}(\beta)|^2 \sup_{\beta\in[0,1]} \left(\int_0^1 \mathcal{G}(\beta, W)\, dW \right)^2 \\
&\le (\theta + \theta^2)\, \mathcal{P}(\mathfrak{x}, \mathfrak{p}).
\end{aligned}
$$

Therefore, it follows that $\mathcal{P}(\mathcal{T}\mathfrak{x}, \mathcal{T}\mathfrak{p}) \le (\theta + \theta^2)\,\mathcal{P}(\mathfrak{x}, \mathfrak{p})$ for any distinct $\mathfrak{x}, \mathfrak{p} \in C$. Hence, all the assumptions required by Corollary 2.5 are met, which guarantees that the operator $\mathcal{T}$ admits a unique fixed point. As a direct consequence, the boundary value problem (3.1) possesses a unique solution in C. □

By following an argument analogous to the previous result, but considering the case $\Lambda \equiv 0$ (so that the exact metric reduces to $d = \mathcal{P}$), we arrive at the following theorem.

Theorem 3.3. *Consider the fractional differential equation*

$$
\begin{cases}
{}^{C}D^{\Delta}(\mathfrak{x}(\beta)) + \mathring{\chi}(\beta, \mathfrak{x}(\beta)) = 0, & \forall\ \beta \in [0,1], \\
\mathfrak{x}(0) = \sigma \in [0,1],
\end{cases}
\tag{3.7}
$$

where $\Delta \in (0,1)$, $\mathfrak{x} : [0,1] \to \mathbb{R}$ *is an unknown function, and* $\mathring{\chi} : [0,1] \times \mathbb{R} \to \mathbb{R}$ *is continuous. Assume that there exists a constant* $\theta \in [0, \frac{1}{3})$ *s.t.*

$$
|\,\mathring{\chi}(\beta, z_1) - \mathring{\chi}(\beta, z_2)| \le \theta\, |z_1 - z_2|
\tag{3.8}
$$

for all $\beta \in [0,1]$ *and* $z_1, z_2 \in \mathbb{R}$. *Then the problem* (3.7) *admits a unique solution. In addition, the Picard iteration generated by the operator* $\mathcal{T} : X \to X$, *defined for each* $\mathfrak{x} \in X$ *by*

$$
(\mathcal{T}\mathfrak{x})(\beta) := c - \frac{1}{\Gamma(\Delta)} \int_0^\beta (\beta - W)^{\Delta - 1}\, \mathring{\chi}(W, \mathfrak{x}(W))\, dW, \quad \forall\ \beta \in [0,1],
\tag{3.9}
$$

converges to the unique solution of problem (3.7).

4. Numerical illustration with a scaling parameter and population growth modeling

To illustrate the effectiveness of Theorem 3.3, we examine the following fractional differential equation:

$$\begin{cases} {}^{C}D^{0.5}(\mathfrak{x}(\beta)) = -\theta\,\mathfrak{x}(\beta) + \beta, \quad \forall\ \beta \in [0,1], \\ \mathfrak{x}(0) = 0, \end{cases} \tag{4.1}$$

where $\mathfrak{x} : [0,1] \to \mathbb{R}$ is the unknown function and $\theta \in [0, \frac{1}{3})$. By transforming (4.1) into its equivalent integral representation, we obtain

$$\mathfrak{x}(\beta) = \frac{1}{\Gamma(0.5)} \int_0^{\beta} (\beta - W)^{-0.5}\big(- \theta\,\mathfrak{x}(W) + W\big)\, dW.$$

Define the operator $\mathcal{T} : X \to X$, where X denotes the space of all continuous real-valued functions on $[0,1]$, by

$$(\mathcal{T}\mathfrak{x})(\beta) = \frac{1}{\Gamma(0.5)} \int_0^{\beta} (\beta - W)^{-0.5}\big(- \theta\,\mathfrak{x}(W) + W\big)\, dW, \quad \forall\ \beta \in [0,1].$$

In this setting, the solution of (4.1) is equivalent to the fixed point of the operator $\mathcal{T}$. Invoking Theorem 3.3, we conclude that a unique solution exists and can be approximated through the iterative sequence $\{\mathfrak{x}_n\} \subseteq X$ defined by

$$\mathfrak{x}_{n+1}(\beta) = (\mathcal{T}\mathfrak{x}_n)(\beta), \quad n = 0, 1, 2, \ldots,$$

with initial choice $\mathfrak{x}_0 \in X$. For $\theta = 0.5$, numerical approximations obtained via a trapezoidal quadrature scheme are presented in Table 4, and the corresponding solution profile is illustrated in Figure 1.

Table 4. Numerical approximation values of $\mathfrak{x}(\beta)$ at different iterations (up to Iteration 4).

β	Iteration 0	Iteration 1	Iteration 2	Iteration 3	Iteration 4	...
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	...
0.1	0.0000	0.0180	0.0167	0.0168	0.0168	...
0.2	0.0000	0.0557	0.0492	0.0499	0.0500	...
0.3	0.0000	0.1062	0.0903	0.0924	0.0927	...
0.4	0.0000	0.1670	0.1374	0.1420	0.1428	...
0.5	0.0000	0.2369	0.1890	0.1975	0.1986	...
0.6	0.0000	0.3147	0.2440	0.2580	0.2596	...
0.7	0.0000	0.3998	0.3018	0.3230	0.3251	...
0.8	0.0000	0.4917	0.3617	0.3921	0.3948	...
0.9	0.0000	0.5899	0.4232	0.4650	0.4682	...
1.0	0.0000	0.6940	0.4861	0.5414	0.5451	...

The numerical patterns presented in Table 4 and Figure 1 can be naturally interpreted within the framework of population dynamics. Let $\mathfrak{x}(\beta)$ represent the size of a rabbit population at time $\beta \in [0,1]$. In this model, the term $-\theta\,\mathfrak{x}(\beta)$ captures the effect of mortality, where the parameter $\theta \in [0,1)$ regulates the intensity of decay. Smaller values of θ correspond to weaker mortality effects, whereas values closer to 1 indicate stronger population suppression. The term β acts as an external input, which may be interpreted as a source of population increase due to reproduction or migration. Moreover, the presence of a fractional derivative of order 0.5 introduces a memory effect into the system, implying that the population evolution is influenced not only by its current state but also by its past history.

Consequently, the convergence behavior observed in the numerical approximations reflects the long-term stabilization of the population under the combined influence of memory and scaled mortality. This example highlights how the abstract fixed point methodology can effectively capture and explain realistic phenomena arising in ecological models.

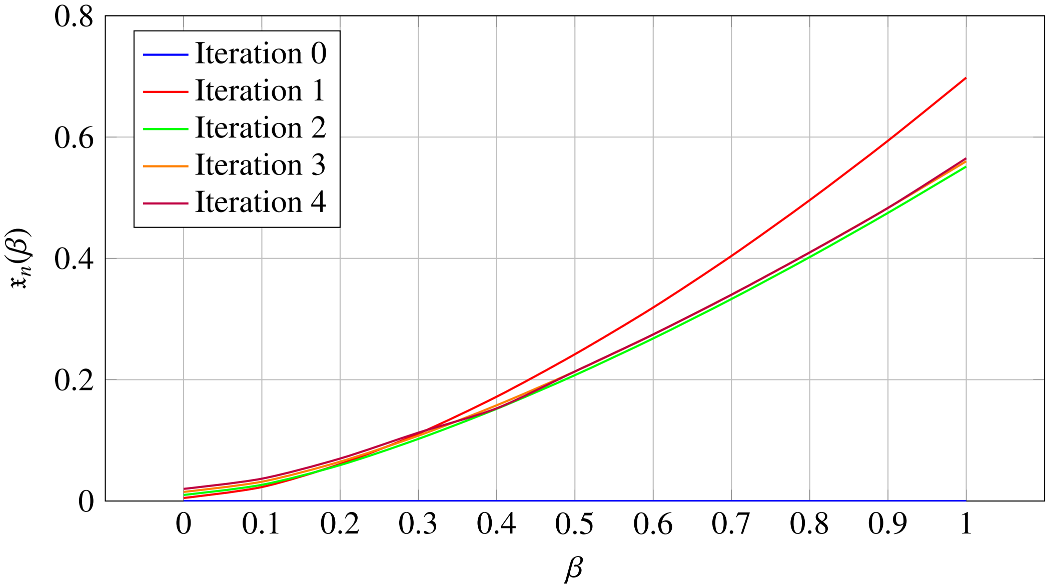

Figure 1. Numerical approximation of the solution $\mathfrak{x}(\beta)$ for the model with $\theta = 0.5$.

5. Conclusions and future directions

In this paper, fixed point results for three-point contraction mappings were established in the setting of triple P.M.S's. The obtained results provide a unified and generalized framework that extends several existing results in the literature. By incorporating three perturbed metrics, a more flexible structure was achieved, allowing the treatment of cases not covered by classical approaches. Some example are presented to demonstrate the applicability of the results, showing that the proposed conditions are satisfied while previously known results fail to apply. These findings highlight the effectiveness and broader applicability of the developed framework and suggest further potential for extensions in more generalized settings. Finally, we have derived existence and uniqueness conditions that guarantee solutions for fractional differential equations and illustrated their relevance to population dynamics, such as modeling rabbit growth with memory and mortality effects. The numerical example further validates the theoretical results and demonstrates the effectiveness of the iterative scheme in approximating the solutions.

Furthermore, it would be of interest to extend the present results to other classes of mappings, such as cyclic contraction mappings and admissible mappings. In addition, the proposed framework may find applications in the study of differential, fractional, and integral equations, as well as in optimization theory and fractal geometry. Moreover, the results may also be generalized to other types of distance structures, including settings based on generalized metrics and related spaces, as well as in perturbed frameworks such as the Kannan- and Chatterjea-type three-point contractions.

Author contributions

M.W., M.D., M.A., and M.Z. conceived the presented idea and wrote the initial draft of the manuscript. M.W. and M.D. further developed and edited the draft. All authors read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that no Artificial Intelligence (AI) tools were used in the preparation of this manuscript.

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Conflict of interest

The authors declare that they have no conflict of interest in this paper.

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