Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(6): 16697–16711.
DOI:10.3934/math.2026685
Received: 27 December 2025
Revised: 25 May 2026
Accepted: 03 June 2026
Published: 11 June 2026

Theory article

On the characteristic polynomials of Dutch windmill graphs and their applications

Wenjing Li, Yiwei Zhang and Ying Wang*

School of Mathematics and Statistics, North China University of Water Resources and Electric Power, Zhengzhou 450046, China

* **Correspondence:** Email: wangying2019@ncwu.edu.cn.

Abstract: The Dutch windmill graph D_p^q is formed by q cycles of length p sharing a common vertex v_0. In this paper, we derive closed-form expressions for the characteristic polynomials—specifically, the adjacency polynomial $\Phi_A(D_p^q, \lambda)$, the Laplacian polynomial $\Phi_L(D_p^q, \mu)$, and the signless Laplacian polynomial $\Phi_{L^+}(D_p^q, \nu)$ —of this family of graphs. As a direct consequence, we compute the exact values of the graph energy, Laplacian energy, and signless Laplacian energy of Dutch windmill graphs.

Keywords: adjacency spectrum; Laplacian spectrum; signless Laplacian spectrum; graph energy; Laplacian energy; signless Laplacian energy; Dutch windmill graph
Mathematics Subject Classification: 05C50

1. Introduction

All graphs discussed in this paper are simple, finite, undirected, and connected. Terminology and notation not explicitly defined herein follow those of Bondy and Murty [1].

Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, and let $A(G)$ denote its adjacency matrix. The degree of vertex v_i is denoted by d_i, and $D(G) = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ represents the diagonal degree matrix. The Laplacian matrix and signless Laplacian matrix are defined as $L(G) = D(G) - A(G)$ and $L^+(G) = D(G) + A(G)$, respectively. The characteristic polynomial $\Phi_M(G, x) = |xI_n - M|$ is referred to as the A-polynomial, L-polynomial, or L^+-polynomial of G when $M = A(G)$, $L(G)$, or $L^+(G)$, respectively. Throughout the paper, the variables λ, μ, and ν are used for the adjacency, Laplacian, and signless Laplacian characteristic polynomials, respectively; that is, we write $\Phi_A(G, \lambda)$, $\Phi_L(G, \mu)$, and $\Phi_{L^+}(G, \nu)$. The corresponding eigenvalues are denoted by λ_i, μ_i, and ν_i. For further details on spectral properties, see [2–4].

Graph energy is a significant concept in spectral graph theory, originally introduced by Gutman [5] in the context of mathematical chemistry. For a simple graph G with adjacency eigenvalues

$\lambda_1, \lambda_2, \ldots, \lambda_n$, the energy is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

More recently, Gutman and Zhou [6] defined the Laplacian energy of a graph G with n vertices and m edges. Let $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of $L(G)$. The Laplacian energy is given by

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|.$$

Similarly, let $\nu_1, \nu_2, \ldots, \nu_n$ denote the eigenvalues of $L^+(G)$. The signless Laplacian energy [7] is defined as

$$LE^+(G) = \sum_{i=1}^{n} \left| \nu_i - \frac{2m}{n} \right|.$$

We use the standard Gutman–Zhou definition of Laplacian energy, in which the Laplacian eigenvalues are centered at the average degree $2m/n$. This centered form measures the deviation of the Laplacian spectrum from the average degree and is different from the nonstandard variant that sums the absolute Laplacian eigenvalues without centering, which equals $2m$, according to the fact that the Laplacian matrix is a positive semi-definite matrix.

For additional background on graph energies and their chemical applications, we refer to [8–10].

A Dutch windmill graph D_p^q consists of q cycles of length p sharing a common vertex v_0. Its vertex set is

$$V = \{v_0\} \cup \{u_{k,\ell} \mid 1 \le k \le q,\ 1 \le \ell \le p-1\},$$

and its edge set is

$$E = \{v_0 u_{k,1},\ u_{k,\ell} u_{k,\ell+1},\ u_{k,p-1} v_0 \mid 1 \le k \le q,\ 1 \le \ell \le p-2\}.$$

Thus, D_p^q contains $(p-1)q+1$ vertices and pq edges. Note that D_3^q corresponds to the friendship graph, and D_p^1 is the cycle C_p.

Farahani et al. [11] and Wu et al. [12] computed the energies of several families of Dutch windmill graphs, as summarized below.

Theorem 1. *[11, 12] The energies of the Dutch windmill graphs D_3^q, D_4^q, D_5^q, and D_6^q are given by:*

(i) $E(D_3^q) = 2q - 1 + \sqrt{1+8q}$*;*
(ii) $E(D_4^q) = 2(\sqrt{2}q - \sqrt{2} + \sqrt{2q+2})$*;*
(iii) $E(D_5^q) = 2\sqrt{5}q - \sqrt{5} + \chi$*, where* $\chi = |\lambda_5| + |\lambda_6| + |\lambda_7|$*, and* $\lambda_5, \lambda_6, \lambda_7$ *are the roots of* $\lambda^3 - \lambda^2 - (2q+1)\lambda + 2q = 0$*;*
(iv) $E(D_6^q) = 2q + 2\sqrt{3}(q-1) + 2\sqrt{\frac{2q+3+\sqrt{4q^2-4q+9}}{2}} + 2\sqrt{\frac{2q+3-\sqrt{4q^2-4q+9}}{2}}$.

These earlier works motivate a systematic treatment of the spectra of Dutch windmill graphs. In particular, the energy formulas for several small values of p are available in the literature, while the corresponding characteristic polynomials and the centered Laplacian energy formulas require a unified presentation. Zhao and Wang [13] also obtained an upper bound for the Laplacian energy of the windmill graph denoted by D_{m,C_n}.

Dutch windmill graphs and related windmill-type graphs also occur in studies of topological indices, resistance distances, and Kirchhoff indices. For example, vertex-based topological indices of double and strong double graphs of Dutch windmill graphs were studied in [14], topological indices of line graphs of Dutch windmill graphs were considered in [15], and resistance distance and Kirchhoff index formulas for windmill graphs were investigated in [16].

The contributions of this paper are as follows. First, we derive closed-form expressions for the A-, L-, and L^+-polynomials of D_p^q for general p and q. Second, we recover the known energy formulas for D_3^q, D_4^q, D_5^q, and D_6^q from the general A-polynomial. Third, using the standard Gutman–Zhou definition, we obtain explicit Laplacian and signless Laplacian energy formulas for these graphs. Finally, we record additional factors of the characteristic polynomials, which provide further spectral information for general p.

2. Preliminaries

Throughout the paper, we use $\Phi_N(x) = |xI_n - N|$ to denote the characteristic polynomial of an $n \times n$ matrix N.

Now we state some useful results.

Theorem 2. *[2] Let C_n, $n \geq 3$ be a cycle with n vertices. The eigenvalues of C_n are* $2\cos\frac{2k\pi}{n}$, $k = 1, 2, \cdots, n$.

Theorem 3. *[2] Let P_n, $n \geq 1$ be a path with n vertices. The eigenvalues of P_n are* $2\cos\frac{k\pi}{n+1}$, $k = 1, \ldots, n$.

Corollary 1. *Let P_n, $n \geq 1$ be a path with n vertices. The A-polynomial of P_n is*

$$\Phi_A(P_n, \lambda) = \prod_{k=1}^{n}(\lambda - 2cos\frac{k\pi}{n+1}).$$

The next result plays a key role in our proofs.

Lemma 1. *[17] Let the graph of order n have a special kind of symmetry so that its associated matrix is written in the form*

$$M = \begin{pmatrix} X & \beta & \beta & \cdots & \beta \\ \beta^T & B & C & \cdots & C \\ \beta^T & C & B & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & C \\ \beta^T & C & \cdots & C & B \end{pmatrix},$$

where $X \in \mathbb{R}^{t\times t}$, $\beta \in \mathbb{R}^{t\times s}$, $B, C \in \mathbb{R}^{s\times s}$, and $n = t + cs$, where $c \geq 1$ is the number of copies of the block B. Let $\sigma(X)$ denote the spectrum of the matrix X, and let $\sigma^k(X)$ denote k copies of the spectrum of X. Then
(1) $\sigma(B - C) \subseteq \sigma(M)$ with multiplicity $c - 1$.
(2) $\sigma(M)\backslash\sigma^{(c-1)}(B - C) = \sigma(M')$ is the set of the remaining $t + s$ eigenvalues of M', where

$$M' = \begin{pmatrix} X & \sqrt{c}\beta \\ \sqrt{c}\beta^T & B + (c-1)C \end{pmatrix}.$$

3. A-polynomial of Dutch windmill graph D_p^q

In this section, we mainly consider the A-polynomial of the Dutch windmill graph D_p^q and derive a general closed-form expression for it.

Theorem 4. *Let $p \geq 3$, $q \geq 1$ be integers. Then the A-polynomial of the Dutch windmill graph D_p^q is*

$$\Phi_A(D_p^q, \lambda) = \prod_{k=1}^{p-1}(\lambda - 2\cos\frac{k\pi}{p})^{q-1} \cdot \left[\lambda \prod_{k=1}^{p-1}(\lambda - 2\cos\frac{k\pi}{p}) - 2q \prod_{k=1}^{p-2}(\lambda - 2\cos\frac{k\pi}{p-1}) - 2q\right].$$

Proof. Let $V(D_p^q) = \{v_0, u_{1,1}, u_{1,2}, \ldots, u_{1,p-1}, \ldots, u_{q,1}, u_{q,2}, \ldots, u_{q,p-1}\}$. Then the adjacency matrix of the Dutch windmill graph D_p^q can be written as

$$A = \begin{pmatrix} 0 & \beta & \beta & \cdots & \beta \\ \beta^T & B & C & \cdots & C \\ \beta^T & C & B & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & C \\ \beta^T & C & \cdots & C & B \end{pmatrix}_{(p-1)q+1},$$

where $X = (0) \in \mathbb{R}^{1\times 1}, \beta \in \mathbb{R}^{1\times(p-1)}$,

$$\beta = \begin{pmatrix} 1, & 0, & \cdots, & 0, & 1 \end{pmatrix}_{1\times(p-1)}, \; B = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 1 & 0 \end{pmatrix}_{(p-1)\times(p-1)},$$

and $C \in \mathbb{R}^{(p-1)\times(p-1)}$ is the zero matrix.

Here, $B - C = B$ is the adjacency matrix of the path P_{p-1}. So, the A-polynomial of D_p^q obtained by Lemma 1 and Corollary 1 is

$$\begin{aligned}\Phi_A(D_p^q, \lambda) &= \Phi_{B-C}^{q-1}(\lambda) \cdot \Phi_{A'}(\lambda) \\ &= \Phi_B^{q-1}(\lambda) \cdot \Phi_{A'}(\lambda) \\ &= \left[\prod_{k=1}^{p-1}(\lambda - 2\cos\frac{k\pi}{p})\right]^{q-1} \cdot \Phi_{A'}(\lambda),\end{aligned}$$

where by Lemma 1(2), we can obtain that A' is a $p \times p$ matrix as follows:

$$A' = \begin{pmatrix} 0 & \sqrt{q} & 0 & \cdots & \sqrt{q} \\ \sqrt{q} & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ \sqrt{q} & 0 & \cdots & 1 & 0 \end{pmatrix}_p.$$

The characteristic polynomial of A' is

$$\Phi_{A'}(\lambda) = |\lambda I_p - A'| = \begin{vmatrix} \lambda & -\sqrt{q} & 0 & \cdots & -\sqrt{q} \\ -\sqrt{q} & \lambda & -1 & \cdots & 0 \\ 0 & -1 & \lambda & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 \\ -\sqrt{q} & 0 & \cdots & -1 & \lambda \end{vmatrix}_p .$$

Expand the first column of the above determinant, and get three parts. Then expand the first rows of the last two parts, respectively, and we get

$$\Phi_{A'}(\lambda) = \lambda \cdot \Phi_A(P_{p-1}, \lambda) - \Phi_A(P_{p-2}, \lambda) - 2q.$$

From Corollary 1, we further obtain

$$\Phi_{A'}(\lambda) = \lambda \cdot \prod_{k=1}^{p-1}(\lambda - 2\cos\frac{k\pi}{p}) - 2q\prod_{k=1}^{p-2}(\lambda - 2\cos\frac{k\pi}{p-1}) - 2q.$$

The proof is thus complete. □

By setting $p = 3, 4, 5, 6$ in Theorem 4, the known formulas in Theorem 1 are recovered. Since Theorem 1 has already been cited from the literature, we omit the detailed verification.

While the eigenvalues of $A(D_p^q)$ can, in principle, be determined for specific values of p and q, the explicit form of the A-polynomial is highly complex, making direct factorization impractical for general eigenvalue computation. Although symbolic computation tools such as Maple allow us to calculate graph energies for a broader range of parameters, we refrain from including these results due to the excessively cumbersome and intricate nature of the resulting expressions. Nonetheless, alternative methods enable the derivation of additional eigenvalues beyond those obtained by direct factorization.

Theorem 5. *Let $p \geq 3$, $q \geq 1$ be integers. Then the A-polynomial of the Dutch windmill graph D_p^q has the following factorizations:*

(i) if p is odd, then

$$\Phi_A(D_p^q, \lambda) = \prod_{k=1}^{\frac{p-1}{2}}\left(\lambda - 2\cos\frac{2k\pi}{p}\right)^q \cdot \prod_{k=1}^{\frac{p-1}{2}}\left(\lambda - 2\cos\frac{(2k-1)\pi}{p}\right)^{q-1} \cdot \left(\lambda^{\frac{p+1}{2}} + C_{\frac{p-1}{2}}\lambda^{\frac{p-1}{2}} + \cdots + C_0\right);$$

(ii) if $p \equiv 2 \pmod 4$, then

$$\Phi_A(D_p^q, \lambda) = \prod_{k=1}^{\frac{p}{2}-1}\left(\lambda - 2\cos\frac{2k\pi}{p}\right)^q \cdot \prod_{k=1}^{\frac{p}{2}}\left(\lambda - 2\cos\frac{(2k-1)\pi}{p}\right)^{q-1} \cdot \left(\lambda^{\frac{p}{2}+1} + C_{\frac{p}{2}}\lambda^{\frac{p}{2}} + \cdots + C_0\right);$$

(iii) if $p \equiv 0 \pmod 4$, then

$$\Phi_A(D_p^q, \lambda) = \lambda \cdot \prod_{k=1}^{\frac{p}{2}-1}\left(\lambda - 2\cos\frac{2k\pi}{p}\right)^q \cdot \prod_{k=1}^{\frac{p}{2}}\left(\lambda - 2\cos\frac{(2k-1)\pi}{p}\right)^{q-1} \cdot \left(\lambda^{\frac{p}{2}} + C_{\frac{p}{2}-1}\lambda^{\frac{p}{2}-1} + \cdots + C_0\right).$$

Proof. Following the idea of the proof of Theorem 4, we only need to prove $\Phi_{A'}(\lambda)$ is
$\prod_{k=1}^{\frac{p-1}{2}}(\lambda - 2\cos\frac{2k\pi}{p}) \cdot (\lambda^{\frac{p+1}{2}} + C_{\frac{p-1}{2}}\lambda^{\frac{p-1}{2}} + \ldots + C_0)$, if p is odd;
$\prod_{k=1}^{\frac{p}{2}-1}(\lambda - 2\cos\frac{2k\pi}{p}) \cdot (\lambda^{\frac{p}{2}+1} + C_{\frac{p}{2}}\lambda^{\frac{p}{2}} + \ldots + C_0)$, if $p \equiv 2 \pmod 4$;
$\lambda \cdot \prod_{k=1}^{\frac{p}{2}-1}(\lambda - 2\cos\frac{2k\pi}{p}) \cdot (\lambda^{\frac{p}{2}} + C_{\frac{p}{2}-1}\lambda^{\frac{p}{2}-1} + \ldots + C_0)$, if $p \equiv 0 \pmod 4$.

Note that $D_p^1 = C_p$. It follows from Theorems 2 and 4 that

$$\lambda \cdot \prod_{k=1}^{p-1}(\lambda - 2\cos\frac{k\pi}{p}) - 2\prod_{k=1}^{p-2}(\lambda - 2\cos\frac{k\pi}{p-1}) - 2 = \prod_{k=1}^{p}(\lambda - 2\cos\frac{2k\pi}{p}).$$

So, we get

$$2\prod_{k=1}^{p-2}(\lambda - 2\cos\frac{k\pi}{p-1}) + 2 = \lambda \cdot \prod_{k=1}^{p-1}(\lambda - 2\cos\frac{k\pi}{p}) - \prod_{k=1}^{p}(\lambda - 2\cos\frac{2k\pi}{p}),$$

and thus

$$\begin{aligned}\Phi_{A'}(\lambda) &= \lambda \cdot \prod_{k=1}^{p-1}(\lambda - 2\cos\frac{k\pi}{p}) - 2q\prod_{k=1}^{p-2}(\lambda - 2\cos\frac{k\pi}{p-1}) - 2q \\ &= (1-q)\lambda \cdot \prod_{k=1}^{p-1}(\lambda - 2\cos\frac{k\pi}{p}) + q\prod_{k=1}^{p}(\lambda - 2\cos\frac{2k\pi}{p}).\end{aligned}$$

Factoring out the common parts from the two terms on the right-hand side, we get:
If p is odd, then

$$\begin{aligned}\Phi_{A'}(\lambda) &= (1-q)\lambda \cdot \prod_{k=1}^{\frac{p-1}{2}}(\lambda - 2\cos\frac{2k\pi}{p}) \cdot \prod_{k=1}^{\frac{p-1}{2}}(\lambda - 2\cos\frac{(2k-1)\pi}{p}) \\ &\quad + q\prod_{k=1}^{\frac{p-1}{2}}(\lambda - 2\cos\frac{2k\pi}{p}) \cdot \prod_{k=\frac{p+1}{2}}^{p-1}(\lambda - 2\cos\frac{2k\pi}{p}) \\ &= \prod_{k=1}^{\frac{p-1}{2}}(\lambda - 2\cos\frac{2k\pi}{p}) \cdot (\lambda^{\frac{p+1}{2}} + C_{\frac{p-1}{2}}\lambda^{\frac{p-1}{2}} + \ldots + C_0).\end{aligned}$$

If $p \equiv 2 \pmod 4$, then

$$\begin{aligned}\Phi_{A'}(\lambda) &= (1-q)\lambda \cdot \prod_{k=1}^{\frac{p}{2}-1}(\lambda - 2\cos\frac{2k\pi}{p}) \cdot \prod_{k=1}^{\frac{p}{2}}(\lambda - 2\cos\frac{(2k-1)\pi}{p}) \\ &\quad + q\prod_{k=1}^{\frac{p}{2}-1}(\lambda - 2\cos\frac{2k\pi}{p}) \cdot \prod_{k=\frac{p}{2}}^{p-1}(\lambda - 2\cos\frac{2k\pi}{p}) \\ &= \prod_{k=1}^{\frac{p}{2}-1}(\lambda - 2\cos\frac{2k\pi}{p}) \cdot (\lambda^{\frac{p}{2}+1} + C_{\frac{p}{2}}\lambda^{\frac{p}{2}} + \ldots + C_0).\end{aligned}$$

If $p \equiv 0 \pmod 4$, then

$$\begin{aligned}\Phi_{A'}(\lambda) =&(1-q)\lambda\cdot\prod_{k=1}^{\frac{p}{2}-1}(\lambda-2\cos\frac{2k\pi}{p})\cdot\prod_{k=1}^{\frac{p}{2}}(\lambda-2\cos\frac{(2k-1)\pi}{p})\\&+q\prod_{k=1}^{\frac{p}{2}-1}(\lambda-2\cos\frac{2k\pi}{p})\cdot\prod_{k=\frac{p}{2}}^{\frac{3p}{4}-1}(\lambda-2\cos\frac{2k\pi}{p})\cdot(\lambda-2\cos\frac{2\cdot\frac{3p}{4}\pi}{p})\cdot\prod_{k=\frac{3p}{4}+1}^{p-1}(\lambda-2\cos\frac{2k\pi}{p})\\=&\lambda\cdot\prod_{k=1}^{\frac{p}{2}-1}(\lambda-2\cos\frac{2k\pi}{p})\cdot(\lambda^{\frac{p}{2}}+C_{\frac{p}{2}-1}\lambda^{\frac{p}{2}-1}+\ldots+C_0).\end{aligned}$$

Thus, we complete our proof. □

4. L-polynomial of the Dutch windmill graph D_p^q

In this section, we mainly consider the L-polynomial of Dutch windmill graph D_p^q, and obtain Laplacian energies of D_p^q, following the ideas of Section 3.

Lemma 2. *Let $B'_n = 2I_n - A(P_n)$, $n \geq 1$, where P_n is a path of n vertices. Then the eigenvalues of B'_n are*

$$2-2cos\frac{\pi j}{n+1},\quad j=1,\ldots,n.$$

The lemma follows immediately from Theorem 3 and the linear-algebraic fact that if $A(P_n)\vec{x} = \lambda\vec{x}$, then $(2I_n - A(P_n))\vec{x} = (2-\lambda)\vec{x}$.

Theorem 6. *Let $p \geq 3$, $q \geq 1$ be integers. Then the L-polynomial of the Dutch windmill graph D_p^q is*

$$\begin{aligned}\Phi_L(D_p^q,\mu)=\prod_{j=1}^{p-1}(\mu-2+2\cos\frac{j\pi}{p})^{q-1}\cdot\Bigg[&(\mu-2q)\prod_{j=1}^{p-1}(\mu-2+2\cos\frac{j\pi}{p})\\&-2q\prod_{j=1}^{p-2}(\mu-2+2\cos\frac{j\pi}{p-1})+(-1)^{p+1}\cdot 2q\Bigg].\end{aligned}$$

Proof. From Theorem 4, we know the adjacency matrix of D_p^q is

$$A(D_p^q)=\begin{pmatrix}0&\beta&\beta&\cdots&\beta\\ \beta^T&B&C&\cdots&C\\ \beta^T&C&B&\ddots&\vdots\\ \vdots&\vdots&\ddots&\ddots&C\\ \beta^T&C&\cdots&C&B\end{pmatrix}_{(p-1)q+1},$$

where $X=(0)\in\mathbb{R}^{1\times1}, \beta\in\mathbb{R}^{1\times(p-1)}$,

$$\beta=\begin{pmatrix}1,&0,&\cdots,&0,&1\end{pmatrix}_{1\times(p-1)},\ B=\begin{pmatrix}0&1&\cdots&0\\1&0&\ddots&\vdots\\ \vdots&\ddots&\ddots&1\\0&\cdots&1&0\end{pmatrix}_{(p-1)\times(p-1)},$$

and $C \in \mathbb{R}^{(p-1)\times(p-1)}$ is the zero matrix. The degree matrix of D_p^q is

$$D(D_p^q) = \begin{pmatrix} 2q & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 2 \end{pmatrix}_{(p-1)q+1}.$$

Thus, the Laplacian matrix of D_p^q is

$$\begin{aligned} L(D_p^q) =& D(D_p^q) - A(D_p^q) \\ =& \begin{pmatrix} 2q & -\beta & -\beta & \cdots & -\beta \\ -\beta^T & 2I_{p-1} - B & -C & \cdots & -C \\ -\beta^T & -C & 2I_{p-1} - B & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -C \\ -\beta^T & -C & \cdots & -C & 2I_{p-1} - B \end{pmatrix}_{(p-1)q+1}. \end{aligned}$$

Here, $(2I_{p-1} - B) - (-C) = 2I_{p-1} - B$ is the matrix B'_{p-1} in Lemma 2. So, it follows from Lemmas 1 and 2 that the L-polynomial of D_p^q is

$$\begin{aligned} \Phi_L(D_p^q, \mu) &= |\mu I_{(p-1)q+1} - L(D_p^q)| \\ &= \Phi_{B'_{p-1}}^{q-1}(\mu) \cdot \Phi_{L'}(\mu) \\ &= \left[\prod_{j=1}^{p-1} (\mu - 2 + 2\cos\frac{j\pi}{p})\right]^{q-1} \cdot \Phi_{L'}(\mu), \end{aligned}$$

where

$$L' = \begin{pmatrix} 2q & -\sqrt{q} & 0 & \cdots & -\sqrt{q} \\ -\sqrt{q} & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 \\ -\sqrt{q} & 0 & \cdots & -1 & 2 \end{pmatrix}_p.$$

Expanding the determinant as in the proof of Theorem 4 and using Lemma 2, we get

$$\begin{aligned} \Phi_{L'}(\mu) =& (\mu - 2q) \cdot \Phi_{B'_{p-1}}(\mu) - 2q\Phi_{B'_{p-2}}(\mu) + (-1)^{p+1} \cdot 2q \\ =& (\mu - 2q) \cdot \prod_{j=1}^{p-1} (\mu - 2 + 2\cos\frac{j\pi}{p}) - 2q \prod_{j=1}^{p-2} (\mu - 2 + 2\cos\frac{j\pi}{p-1}) + (-1)^{p+1} \cdot 2q. \end{aligned}$$

Thus, the proof is complete. □

Now we can get the Laplacian energies of the Dutch windmill graphs.

Theorem 7. *The Laplacian energies of the Dutch windmill graphs D_3^q, D_4^q, D_5^q, and D_6^q are, respectively,* *(i)* $LE(D_3^q) = \frac{8q^2+2q+2}{2q+1}$;

(ii) $LE(D_4^q) = 2\sqrt{2}(q-1) + \frac{8q^2+4q+4}{3q+1}$;

(iii) $LE(D_5^q) = \sqrt{4q^2-4q+5} + (2q-1)\sqrt{5} + \frac{10q}{4q+1}$;

(iv) $LE(D_6^q) = 2\sqrt{3}(q-1) + \frac{12q^2+10q+2}{5q+1} + \gamma$, *where* $\gamma = \sum_{i=7}^{9}\left|\mu_i - \frac{12q}{5q+1}\right|$, *and* μ_7, μ_8, μ_9 *are the roots of* $\mu^3 - (2q+6)\mu^2 + (10q+9)\mu - 10q - 2 = 0$.

Proof. Note that D_p^q contains $n = (p-1)q+1$ vertices and $m = pq$ edges.

(i) By Theorem 6, we can obtain that

$$\begin{aligned}\Phi_L(D_3^q, \mu) =&(\mu-1)^{q-1}\cdot(\mu-3)^{q-1}\cdot[(\mu-2q)(\mu-1)(\mu-3)-2q(\mu-2)+2q]\\ =&\mu(\mu-1)^{q-1}(\mu-3)^q(\mu-2q-1).\end{aligned}$$

The eigenvalues of $L(D_3^q)$ are $\mu_1 = 0$, $\mu_2 = 1$ ($q-1$ times), $\mu_3 = 3$ (q times), and $\mu_4 = 2q+1$, respectively. So

$$\begin{aligned}LE(D_3^q) =&|\mu_1 - \frac{2m}{n}| + (q-1)\cdot|\mu_2 - \frac{2m}{n}| + q\cdot|\mu_3 - \frac{2m}{n}| + |\mu_4 - \frac{2m}{n}|\\ =&|0 - \frac{6q}{2q+1}| + (q-1)\cdot|1 - \frac{6q}{2q+1}| + q\cdot|3 - \frac{6q}{2q+1}| + |2q+1 - \frac{6q}{2q+1}|\\ =&\frac{8q^2+2q+2}{2q+1}.\end{aligned}$$

(ii) Similarly, we have

$$\begin{aligned}\Phi_L(D_4^q, \mu) =&\mu\cdot(\mu-2)^q\cdot(\mu^2-4\mu+2)^{q-1}\cdot[\mu^2-(2q+4)\mu+6q+2],\\ \Phi_L(D_5^q, \mu) =&\mu\cdot(\mu^2-3\mu+1)^{q-1}\cdot(\mu^2-5\mu+5)^q\cdot[\mu^2-(2q+3)\mu+4q+1],\\ \Phi_L(D_6^q, \mu) =&\mu\cdot(\mu-2)^{q-1}\cdot(\mu^2-4\mu+1)^{q-1}\cdot(\mu-1)^q\cdot(\mu-3)^q\cdot[\mu^3-(2q+6)\\ &\mu^2+(10q+9)\mu-10q-2].\end{aligned}$$

The following factorizations and absolute-value simplifications were obtained by direct symbolic expansion; Maple was used only to check the algebra:

$$\begin{aligned}LE(D_4^q) =&2\sqrt{2}(q-1) + \frac{8q^2+4q+4}{3q+1},\\ LE(D_5^q) =& \sqrt{4q^2-4q+5} + (2q-1)\sqrt{5} + \frac{10q}{4q+1},\\ LE(D_6^q) =&2\sqrt{3}(q-1) + \frac{12q^2+10q+2}{5q+1} + \gamma,\end{aligned}$$

where $\gamma = |\mu_7 - \frac{12q}{5q+1}| + |\mu_8 - \frac{12q}{5q+1}| + |\mu_9 - \frac{12q}{5q+1}|$, and μ_7, μ_8, μ_9 are roots of $\mu^3 - (2q+6)\mu^2 + (10q+9)\mu - 10q - 2 = 0$. □

Similarly to Theorem 5, we can get more eigenvalues of $L(D_p^q)$.

Theorem 8. *Let* $p \geq 3$, $q \geq 1$ *be integers. Then the L-polynomial of the Dutch windmill graph* D_p^q *has the following factorizations.*

(i) If p is odd, then

$$\Phi_L(D_p^q,\mu) = \mu\cdot\prod_{j=1}^{\frac{p-1}{2}}\left(\mu-2+2\cos\frac{2j\pi}{p}\right)^q\cdot\prod_{j=1}^{\frac{p-1}{2}}\left(\mu-2+2\cos\frac{(2j-1)\pi}{p}\right)^{q-1}$$
$$\cdot\left(\mu^{\frac{p-1}{2}}+C'_{\frac{p-3}{2}}\mu^{\frac{p-3}{2}}+\cdots+C'_0\right).$$

(ii) If p is even, then

$$\Phi_L(D_p^q,\mu) = \mu\cdot\prod_{j=1}^{\frac{p}{2}-1}\left(\mu-2+2\cos\frac{2j\pi}{p}\right)^q\cdot\prod_{j=1}^{\frac{p}{2}}\left(\mu-2+2\cos\frac{(2j-1)\pi}{p}\right)^{q-1}$$
$$\cdot\left(\mu^{\frac{p}{2}}+C'_{\frac{p}{2}-1}\mu^{\frac{p}{2}-1}+\cdots+C'_0\right).$$

Proof. Following the proof of Theorem 6, it remains to prove $\Phi_{L'}(\mu)$ is
$\mu\cdot\prod_{j=1}^{\frac{p-1}{2}}(\mu-2+2\cos\frac{2j\pi}{p})\cdot(\mu^{\frac{p-1}{2}}+C'_{\frac{p-3}{2}}\mu^{\frac{p-3}{2}}+\ldots+C'_0)$, if p is odd;
$\mu\cdot\prod_{j=1}^{\frac{p}{2}-1}(\mu-2+2\cos\frac{2j\pi}{p})\cdot(\mu^{\frac{p}{2}}+C'_{\frac{p}{2}-1}\mu^{\frac{p}{2}-1}+\ldots+C'_0)$, if p is even.

The graph D_p^q is a connected graph, and the row sum of its Laplacian matrix is 0. Therefore, 0 is an eigenvalue of $L(D_p^q)$, and the multiplicity is 1. Next, we discuss two cases according to the parity of p.

First, consider the case when p is odd. Let $\mu_j = 2-2\cos\frac{2j\pi}{p}$ $(1\le j\le\frac{p-1}{2})$, and let the vector $\vec{x_j}=(x_1,x_2,x_3,\cdots,x_p)$, where

$$x_i=\begin{cases}0, & i=1;\\ 1, & i=2;\\ \mu_j x_{i-1}-x_{i-2}, & 3\le i\le\frac{p+1}{2};\\ -x_{p-i+2}, & \frac{p+3}{2}\le i\le p.\end{cases}$$

Then we have

$$L'\vec{x_j}=\begin{pmatrix}2q & -\sqrt{q} & 0 & \cdots & -\sqrt{q}\\ -\sqrt{q} & 2 & -1 & \cdots & 0\\ 0 & -1 & 2 & \ddots & \vdots\\ \vdots & \vdots & \ddots & \ddots & -1\\ -\sqrt{q} & 0 & \cdots & -1 & 2\end{pmatrix}_p\begin{pmatrix}0\\ 1\\ \mu_j x_2-x_1\\ \vdots\\ -1\end{pmatrix}_p$$
$$=\begin{pmatrix}0\\ \mu_j x_2\\ \mu_j x_3\\ \vdots\\ \mu_j x_p\end{pmatrix}_p=\mu_j\vec{x_j}.$$

So, μ_j is an eigenvalue of the matrix L'. Thus,

$$\Phi_{L'}(\mu)=\mu\cdot\prod_{j=1}^{\frac{p-1}{2}}(\mu-2+2\cos\frac{2j\pi}{p})\cdot(\mu^{\frac{p-1}{2}}+C'_{\frac{p-3}{2}}\mu^{\frac{p-3}{2}}+\ldots+C'_0).$$

Next, consider the case when p is even. Let $\mu_j = 2 - 2\cos\frac{2j\pi}{p}$ $(1 \le j \le \frac{p}{2})$, and let the vector $\vec{x_j} = (x_1, x_2, x_3, \cdots, x_p)$, where

$$x_i = \begin{cases} 0, & i = 1 \quad \text{or} \quad i = \frac{p}{2} + 1; \\ 1, & i = 2; \\ \mu_j x_{i-1} - x_{i-2}, & 3 \le i \le \frac{p}{2}; \\ -x_{p-i+2}, & \frac{p}{2} + 2 \le i \le p. \end{cases}$$

Similarly, we have

$$L'\vec{x_j} = \mu_j \vec{x_j}.$$

So, μ_j is an eigenvalue of the matrix L'. Thus,

$$\Phi_{L'}(\mu) = \mu \cdot \prod_{j=1}^{\frac{p}{2}-1} (\mu - 2 + 2\cos\frac{2j\pi}{p}) \cdot (\mu^{\frac{p}{2}} + C'_{\frac{p}{2}-1}\mu^{\frac{p}{2}-1} + \ldots + C'_0).$$

The proof is now complete. □

5. L^+-polynomial of the Dutch windmill graph D_p^q

In this section, we focus on the L^+-polynomial of the Dutch windmill graph D_p^q and derive corresponding signless Laplacian energy formulas. The proof is parallel to that of the Laplacian case, but we keep the main reduction explicit in order to fix the notation.

For the path P_n, the matrix $B_n^+ = 2I_n + A(P_n)$ has eigenvalues

$$2 + 2\cos\frac{\pi j}{n+1}, \quad j = 1, \ldots, n.$$

This follows immediately from Theorem 3.

Theorem 9. *Let $p \ge 3, q \ge 1$ be integers. Then the L^+-polynomial of the Dutch windmill graph D_p^q is*

$$\Phi_{L^+}(D_p^q, \nu) = \prod_{j=1}^{p-1}\left(\nu - 2 - 2\cos\frac{j\pi}{p}\right)^{q-1} \cdot \left[(\nu - 2q) \cdot \prod_{j=1}^{p-1}\left(\nu - 2 - 2\cos\frac{j\pi}{p}\right) - 2q\prod_{j=1}^{p-2}\left(\nu - 2 - 2\cos\frac{j\pi}{p-1}\right) - 2q\right].$$

Proof. With the same block decomposition as in the proof of Theorem 6, the signless Laplacian matrix is obtained from the Laplacian matrix by replacing the off-diagonal entries $-\beta$, $-\beta^T$, and $-B$ with β, β^T, and B, respectively. In the notation of Lemma 1, we have $X = (2q) \in \mathbb{R}^{1\times 1}$, $\beta \in \mathbb{R}^{1\times(p-1)}$, $B = 2I_{p-1} + A(P_{p-1}) \in \mathbb{R}^{(P-1)\times(p-1)}$, $C = 0 \in \mathbb{R}^{(p-1)\times(p-1)}$, $s = p - 1$, and $c = q$. Hence,

$$\Phi_{L^+}(D_p^q, \nu) = \Phi_{B_{p-1}^+}^{q-1}(\nu)\Phi_{Q'}(\nu),$$

where

$$Q' = \begin{pmatrix} 2q & \sqrt{q} & 0 & \cdots & \sqrt{q} \\ \sqrt{q} & 2 & 1 & \cdots & 0 \\ 0 & 1 & 2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ \sqrt{q} & 0 & \cdots & 1 & 2 \end{pmatrix}_{p\times p}.$$

Expanding $|\nu I_p - Q'|$ along the first row and first column, and using the above eigenvalues of B_n^+, gives the displayed formula. □

Theorem 10. *The signless Laplacian energies of the Dutch windmill graphs D_3^q, D_4^q, D_5^q, and D_6^q are as follows:*

(i) $LE^+(D_3^q) = \frac{4q^2+2q-3}{2q+1} + \sqrt{4q^2-4q+9}$;
(ii) $LE^+(D_4^q) = 2\sqrt{2}(q-1) + \frac{8q^2+4q+4}{3q+1}$;
(iii) $LE^+(D_5^q) = (2q-1)\sqrt{5} + \delta$, *where* $\delta = \sum_{i=1}^{3} \left|\nu_i - \frac{10q}{4q+1}\right|$, *and* ν_1, ν_2, ν_3 *are roots of* $\nu^3 - (2q+5)\nu^2 + (8q+5)\nu - 4q = 0$;
(iv) $LE^+(D_6^q) = 2\sqrt{3}(q-1) + \frac{12q^2+10q+2}{5q+1} + \gamma$, *where* $\gamma = \sum_{i=4}^{6} \left|\nu_i - \frac{12q}{5q+1}\right|$, *and* ν_4, ν_5, ν_6 *are the roots of* $\nu^3 - (2q+6)\nu^2 + (10q+9)\nu - 10q - 2 = 0$.

Proof. The graph D_p^q has $n = (p-1)q + 1$ vertices and $m = pq$ edges. For $p = 3$, Theorem 9 gives

$$\begin{aligned}\Phi_{L^+}(D_3^q, \nu) &= (\nu-3)^{q-1} \cdot (\nu-1)^{q-1} \cdot [(\nu-2q)(\nu-3)(\nu-1) - 2q(\nu-2) - 2q] \\ &= (\nu-3)^{q-1} \cdot (\nu-1)^{q} \cdot [\nu^2 - (2q+3)\nu + 4q].\end{aligned}$$

Thus, the signless Laplacian eigenvalues are 1 with multiplicity q, 3 with multiplicity $q-1$, and

$$q + \frac{3 + \sqrt{4q^2-4q+9}}{2}, \qquad q + \frac{3 - \sqrt{4q^2-4q+9}}{2}.$$

Substituting these eigenvalues into the definition of LE^+ yields (i). For $p = 4, 5, 6$, direct substitution in Theorem 9 gives

$$\begin{aligned}\Phi_{L^+}(D_4^q, \nu) =& \nu \cdot (\nu-2)^q \cdot (\nu-2-\sqrt{2})^{q-1} \cdot (\nu-2+\sqrt{2})^{q-1} \cdot [\nu^2 - (2q+4)\nu + 6q + 2], \\ \Phi_{L^+}(D_5^q, \nu) =& (\nu^2-3\nu+1)^q \cdot (\nu^2-5\nu+5)^{q-1} \cdot [\nu^3 - (2q+5)\nu^2 + (8q+5\nu-4q)], \\ \Phi_{L^+}(D_6^q, \nu) =& \nu \cdot (\nu-2)^{q-1} \cdot (\nu^2-4\nu+1)^{q-1} \cdot (\nu-1)^q(\nu-3)^q \\ & \cdot [\nu^3 - (2q+6)\nu^2 + (10q+9)\nu - 10q - 2].\end{aligned}$$

The stated formulas follow by applying the definition of LE^+ to these factors. The polynomial expansions and factorizations displayed above were checked by direct symbolic calculation; Maple was used only to verify the algebra. □

We note that $LE(D_4^q) = LE^+(D_4^q)$ and $LE(D_6^q) = LE^+(D_6^q)$. This is consistent with the standard fact [18] that $\Phi_L(G, x) = \Phi_{L^+}(G, x)$ for every bipartite graph G; here, D_4^q and D_6^q are bipartite.

Similarly to Theorem 8, we can obtain additional factors of the L^+-polynomial.

Theorem 11. *Let $p \geq 3$ and $q \geq 1$ be integers. Then the L^+-polynomial of the Dutch windmill graph D_p^q has the following factorizations:*

(i) If p is odd, then

$$\Phi_{L^+}(D_p^q, \nu) = \prod_{j=1}^{\frac{p-1}{2}} \left(\nu - 2 - 2\cos\frac{2j\pi}{p}\right)^q \cdot \prod_{j=1}^{\frac{p-1}{2}} \left(\nu - 2 - 2\cos\frac{(2j-1)\pi}{p}\right)^{q-1} \cdot \left(\nu^{\frac{p+1}{2}} + C''_{\frac{p-1}{2}}\nu^{\frac{p-1}{2}} + \cdots + C''_0\right).$$

(ii) If p is even, then

$$\Phi_{L^+}(D_p^q, \nu) = \nu \cdot \prod_{j=1}^{\frac{p}{2}-1} \left(\nu - 2 - 2\cos\frac{2j\pi}{p}\right)^q \cdot [\prod_{j=1}^{\frac{p}{2}} \left(\nu - 2 - 2\cos\frac{(2j-1)\pi}{p}\right)^{q-1} \cdot \left(\nu^{\frac{p}{2}} + C''_{\frac{p}{2}-1}\nu^{\frac{p}{2}-1} + \cdots + C''_0\right).$$

6. Conclusions

In this paper, we obtained unified closed-form formulas for the adjacency, Laplacian, and signless Laplacian characteristic polynomials of Dutch windmill graphs D_p^q. These formulas recover the known graph energy formulas for D_3^q, D_4^q, D_5^q, and D_6^q and lead to explicit expressions for the Laplacian energy and signless Laplacian energy under the standard centered definitions. We also derived additional factors of the characteristic polynomials, which provide further information about the spectra of D_p^q for general p.

Several directions remain open. It would be natural to extend the present approach to weighted Dutch windmill graphs, generalized windmill graphs, line graphs of windmill graphs, and other windmill-type constructions. Related spectral invariants, such as the Laplacian Estrada index, Kirchhoff index, and distance-based energies, may also be studied by combining the block-decomposition method with graph operations.

Author contributions

Wenjing Li: Conceptualization, Supervision, Writing–review & editing; Yiwei Zhang: Formal analysis, Writing–original draft; Ying Wang: Software, Validation.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

Wenjing Li is supported by Natural Science Foundation of China (No. 11901196). Ying Wang is supported by the Natural Science Foundation of Henan Province No. 252300420912 and No. 262300422649.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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