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Research article

Bayesian inference of the common standardized mean difference in meta-analysis

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Abstract: We develop an objective Bayesian framework for estimating the common standardized mean difference in meta-analysis. To construct noninformative priors, we derive both probability matching and reference priors. Our analysis reveals that although general second-order matching prior does not exist, a valid version can be obtained when the sample sizes of the two arms in each study are equal. Among the reference priors evaluated, we find that both one-at-a-time and two-group reference priors satisfy the first-order matching criterion, whereas Jeffreys' prior does not. Simulation studies demonstrate that the proposed matching prior and the one-at-a-time reference prior yield accurate frequentist coverage probabilities, consistently outperforming Jeffreys' prior. Finally, the practical utility of this framework is validated through two real-world meta-analytic applications, underscoring its effectiveness for robust objective Bayesian inference in standardized mean difference models.

Keywords: Bayesian inference; matching prior; meta analysis; reference prior; standardized mean difference
Mathematics Subject Classification: 62F10, 62N01, 62N02

1. Introduction

Suppose that each of k experiments or studies consists of two independent random samples, $(X_{i1}, \dots, X_{in_i})$ and $(Y_{i1}, \dots, Y_{im_i})$, where the X_{ij} are independently and identically distributed according to $N(\mu_{i1}, \sigma_i^2)$, and the Y_{ij} are independently and identically distributed according to $N(\mu_{i2}, \sigma_i^2)$. The effect size measured by the standardized mean difference in the ith study is defined as

$$\eta_i = \frac{\mu_{i1} - \mu_{i2}}{\sigma_i}.$$

Standardizing the mean difference η_i allows the comparison of effects across studies on a scale-free basis, making it a widely used measure of intervention effects in various fields, including psychology, clinical trials, sociology, and education.

Meta-analysis provides a systematic approach to combining results from multiple studies or experiments to improve the precision of statistical inference and decision-making. It is common to perform a test of homogeneity to determine whether the study-specific effects are sufficiently similar to be combined into a single overall effect. Effect sizes can be assessed using various statistics such as mean differences, standardized mean differences, or odds ratios. If the null hypothesis of homogeneity is not rejected, one can proceed to estimate the common effect. Among available tests, Cochran's Q test (Cochran, 1937) [1] is the most widely used. It computes a weighted sum of squared deviations of observed effects from the null hypothesis of equal effects (see Hedges, 1982 [2] and Rosenthal and Rubin, 1982 [3], for their work with standardized mean differences). However, the asymptotic distribution of Q is known to perform poorly with moderate sample sizes. To address this limitation, Kulinskaya et al. (2011) [4] proposed an improved test in which the null distribution of Q is approximated by a chi-squared distribution with fractional degrees of freedom, estimated via an expansion of the first moment of Q under standardized mean differences. This modified test has proven effective in a variety of situations, including those with unbalanced sample sizes between study arms or differing total sample sizes across studies.

The estimation of point and interval estimates for standardized mean differences has been studied extensively. Glass (1976) [5], Hedges (1981 [6], 1982 [2]), and Hedges and Olkin (1985) [7] examined asymptotic confidence interval estimation for standardized mean differences. Hedges and Vevea (1998) [8] introduced a meta-analytic confidence interval for the average standardized mean difference, and Bond et al. (2003) [9] proposed a similar method for unstandardized mean differences. Bonett (2009) [10] further refined these approaches by developing meta-analytic confidence intervals for both standardized and unstandardized mean differences without assuming equal population variances. Recent investigations have revisited these issues, emphasizing both the strengths and limitations of standardized mean differences as summary measures. In particular, methodological studies have highlighted potential biases of standardized metrics in small-sample settings and questioned whether standardized mean differences are truly "standardized," given their dependence on within-study variance estimates (see Lin and Aloe, 2021 [11]). Another line of research has focused on prior specification for between-study variance (Veroniki et al., 2016 [12]), recommending weakly informative priors or empirical hyperpriors, especially when the number of studies is small, and prior choice can substantially affect posterior inference (Friede and Röver, 2024 [13]). These developments align with the growing use of meta-analytic predictive priors, which leverage accumulated evidence to construct informative priors for future analyses. Such advances reflect a broader methodological shift from purely objective or noninformative prior formulations toward data-driven and context-sensitive approaches. Additionally, new estimators of heterogeneity variance have been proposed to improve performance in small-sample conditions, underscoring ongoing efforts to enhance the accuracy and robustness of meta-analytic inference when standardized effect sizes are employed.

When the standardized mean differences are assumed to be equal across studies, the parameter of interest is the common standardized mean difference,

$$\eta = \frac{\mu_{i1} - \mu_{i2}}{\sigma_i}, \quad i = 1, \ldots, k.$$

We consider a Bayesian framework for estimating this common standardized mean difference η. In the absence of prior information, it is often desirable to use noninformative priors that yield objective Bayesian inferences. Among various approaches, the probability matching prior, first introduced by Welch and Peers (1963) [14], has gained prominence for providing posterior credible intervals with accurate frequentist coverage properties. This idea was further developed by Stein (1985) [15], Tibshirani (1989) [16], Mukerjee and Dey (1993) [17], DiCiccio and Stern (1994) [18], Datta and Ghosh (1995 [19], 1996 [20]), and Mukerjee and Ghosh (1997) [21]. A comprehensive account of probability matching criteria can be found in the monograph by Datta and Mukerjee (2004) [22]. Another widely used noninformative prior is the reference prior, introduced by Bernardo (1979) [23], which maximizes the Kullback-Leibler divergence between the prior and posterior distributions. Berger and Bernardo (1989 [24], 1992 [25]) and Ghosh and Mukerjee (1992) [26] developed general algorithms for deriving reference priors by grouping parameters according to their inferential importance. In many cases, reference priors coincide with probability matching priors. Recent studies have also highlighted the growing role of Bayesian methodologies in reliability analysis, degradation modeling, and survival inference. For example, Xu and Wang (2026) [27] developed a recursive Bayesian framework for predicting remaining useful life under gamma degradation processes, and Xu et al. (2026) [28] proposed a hierarchical Bayesian multivariate Wiener process model incorporating dependent degradation rates and volatilities. Zhu et al. (2026) [29] introduced an online Bayesian framework for identifying latent degradation states, and Zhuang et al. (2026) [30] studied heterogeneous two-scale degradation modeling using a unified random-effects inverse Gaussian framework. These recent developments demonstrate the broad applicability and continuing advancement of Bayesian inferential methods in complex reliability and survival settings.

Despite their wide applicability, these Bayesian frameworks have not been systematically developed for the common standardized mean difference across multiple studies. Ghosh and Yang (1996) [31] developed a first-order matching prior for standardized mean differences in a single experiment, and Mukerjee and Ghosh (1997) [21] further established a second-order matching prior distribution for the equal sample size. This study aims to extend this issue from single study to multiple studies and present a systematic approach to combining the results of multiple studies. When the effects of individual studies are homogeneous, meta-analysis integrates these results to obtain a consolidated conclusion. Whereas the previous study by Mukerjee and Ghosh (1997) [21] assumed equal sample sizes, this study addresses a generalized case where sample sizes are unequal across studies. Therefore, because existing single-study results cannot be directly applied to a multiple-study context and represent an inherently different issue, this paper provides a comprehensive objective Bayesian framework for common standardized mean differences in meta-analysis.

We establish theoretical limitations on higher-order matching priors in this context and derive valid second-order matching priors when sample sizes are equal across study arms. Furthermore, by comparing matching priors, reference priors, and Jeffreys' prior through simulation studies and real meta-analytic applications, we demonstrate the practical advantages of probability matching and reference-based approaches. Our results extend the scope of objective Bayesian inference in effect size estimation and offer more reliable tools for researchers conducting meta-analytic studies.

2. Noninformative priors for the common standardized mean difference

2.1. Matching priors

In this section, we introduce the concept of matching priors, which are constructed to ensure the approximate frequentist validity of one-sided Bayesian credible intervals derived from posterior quantiles of a one-dimensional parameter of interest. More precisely, our goal is to identify priors π that satisfy, asymptotically as the sample size n approaches ∞,

$$P_\theta[\theta_1 \le \theta_1^{1-\alpha}(\pi; \mathbf{X})] = 1 - \alpha + o(n^{-u}). \tag{2.1}$$

Here, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_p)^T$, θ_1 denotes the parameter of interest, and $\theta_1^{1-\alpha}(\pi; \mathbf{X})$ represents the $(1-\alpha)$th posterior quantile of θ_1 given the prior π and data $\mathbf{X}$. A prior π satisfying (2.1) is called a *matching prior*. When $u = 1/2$, the prior is said to be *first-order matching*; when $u = 1$, it is referred to as a *second-order matching* prior.

Consider k experiments or studies, each consisting of two independent random samples $(X_{i1}, \ldots, X_{in_i})$ and $(Y_{i1}, \ldots, Y_{im_i})$, where $X_{ij} \sim N(\mu_{i1}, \sigma_i^2)$ and $Y_{ij} \sim N(\mu_{i2}, \sigma_i^2)$ independently. The likelihood function is given by

$$L(\mu_{11}, \mu_{12}, \sigma_1, \ldots, \mu_{k1}, \mu_{k2}, \sigma_k) \propto \prod_{i=1}^{k} \sigma_i^{-(n_i+m_i)} \exp\left\{-\sum_{i=1}^{k} \frac{1}{2\sigma_i^2}\left[\sum_{j=1}^{n_i}(x_{ij}-\mu_{i1})^2 + \sum_{j=1}^{m_i}(y_{ij}-\mu_{i2})^2\right]\right\}.$$

To simplify the likelihood, we adopt an orthogonal reparameterization for the nuisance parameters. Let

$$\theta_1 = \frac{\mu_{i1}-\mu_{i2}}{\sigma_i}, \quad \theta_{2i} = \left[2(n_i+m_i)^2\sigma_i^2 + n_i m_i(\mu_{i1}-\mu_{i2})^2\right]^{1/2}, \quad i = 1, \ldots, k,$$

and

$$\theta_{3i} = \frac{n_i\mu_{i1} + m_i\mu_{i2}}{n_i+m_i}, \quad i = 1, \ldots, k.$$

Let $\boldsymbol{\theta}_2 = (\theta_{21}, \ldots, \theta_{2k})$, and $\boldsymbol{\theta}_3 = (\theta_{31}, \ldots, \theta_{3k})$. Then, because the transformation is one-to-one, the likelihood function can be expressed as

$$\begin{aligned} L(\theta_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3) \propto & \left(\prod_{i=1}^{k} \theta_{2i}^{-(n_i+m_i)} g_i(\theta_1)^{(n_i+m_i)/2}\right) \\ \times \ \exp & \left\{-\sum_{i=1}^{k}\frac{g_i(\theta_1)}{2\theta_{2i}^2}\left[\sum_{j=1}^{n_i}\left(x_{ij}-\theta_{3i}-\frac{m_i}{n_i+m_i}\theta_1\theta_{2i}g_i(\theta_1)^{-1/2}\right)^2\right.\right. \\ & \left.\left. + \sum_{j=1}^{m_i}\left(y_{ij}-\theta_{3i}+\frac{n_i}{n_i+m_i}\theta_1\theta_{2i}g_i(\theta_1)^{-1/2}\right)^2\right]\right\}, \end{aligned} \tag{2.2}$$

where $g_i(\theta_1) = 2(n_i+m_i)^2 + n_i m_i \theta_1^2$.

The Fisher information matrix corresponding to (2.2) is given by

$$\mathbf{I}(\theta_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3) = \begin{pmatrix} \mathbf{D}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^T & \mathbf{D}_2 & \mathbf{0} \\ \mathbf{0}^T & \mathbf{0}^T & \mathbf{D}_3 \end{pmatrix}, \tag{2.3}$$

which indicates that θ_1 is orthogonal to $\boldsymbol{\theta}_2$ and $\boldsymbol{\theta}_3$ in the sense of Cox and Reid (1987) [32]. Here, $\mathbf{D}_1 = \sum_{i=1}^k \frac{2n_i m_i(n_i+m_i)}{g_i(\theta_1)}$, $\mathbf{D}_2 = \text{Diag}\{\frac{g_1(\theta_1)}{(n_1+m_1)\theta_{21}^2}, \ldots, \frac{g_k(\theta_1)}{(n_k+m_k)\theta_{2k}^2}\}$, $\mathbf{D}_3 = \text{Diag}\{\frac{(n_1+m_1)g_1(\theta_1)}{\theta_{21}^2}, \ldots, \frac{(n_k+m_k)g_k(\theta_1)}{\theta_{2k}^2}\}$, and $\mathbf{0}^T = (0, \ldots, 0)^T$ is a $k \times 1$ vector.

Following Tibshirani (1989) [16], the class of the first-order probability matching prior can be characterized as

$$\pi_m^{(1)}(\theta_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3) \propto \left[\sum_{i=1}^k \frac{n_i m_i(n_i + m_i)}{g_i(\theta_1)}\right]^{1/2} d(\boldsymbol{\theta}_2, \boldsymbol{\theta}_3), \tag{2.4}$$

where $d(\boldsymbol{\theta}_2, \boldsymbol{\theta}_3) > 0$ is a differentiable function of its arguments.

The class of priors given in (2.4) can be further reduced to the second-order probability matching priors as defined by Mukerjee and Ghosh (1997) [21]. A second-order probability matching prior takes the same form as (2.4) but must also satisfy the differential equation (2.10) of Mukerjee and Ghosh (1997) [21]:

$$\begin{aligned} &\frac{1}{6} d(\boldsymbol{\theta}_2, \boldsymbol{\theta}_3) \frac{\partial}{\partial \theta_1}\{I_{11}^{-3/2} L_{1,1,1}\} \\ &+ \sum_{i=1}^k \frac{\partial}{\partial \theta_{2i}}\{I_{11}^{-1/2} L_{11(2i)} I^{(2i)(2i)} d(\boldsymbol{\theta}_2, \boldsymbol{\theta}_3)\} \\ &+ \sum_{i=1}^k \frac{\partial}{\partial \theta_{3i}}\{I_{11}^{-1/2} L_{11(3i)} I^{(3i)(3i)} d(\boldsymbol{\theta}_2, \boldsymbol{\theta}_3)\} = 0, \end{aligned} \tag{2.5}$$

where $\mathbf{I}^{-1} = (I^{ij})_{(2k+1)\times(2k+1)}$ is the inverse Fisher information matrix, and

$$L_{1,1,1} = E\left[\left(\frac{\partial \log L}{\partial \theta_1}\right)^3\right] = -\sum_{i=1}^k \frac{8n_i^2 m_i^2 (n_i + m_i)\theta_1[g_i(\theta_1) + (n_i + m_i)^2]}{g_i(\theta_1)^3}, \tag{2.6}$$

$$L_{11(2i)} = E\left[\frac{\partial^3 \log L}{\partial \theta_1^2 \partial \theta_{2i}}\right] = \frac{4n_i m_i (n_i + m_i)^3}{g_i(\theta_1)^2 \theta_{2i}}, \quad i = 1, \ldots, k, \tag{2.7}$$

$$L_{11(3i)} = 0, \quad i = 1, \ldots, k, \tag{2.8}$$

$$I_{11} = \sum_{i=1}^k \frac{2n_i m_i (n_i + m_i)}{g_i(\theta_1)}, I^{(2i)(2i)} = \frac{(n_i + m_i)\theta_{2i}^2}{g_i(\theta_1)}, I^{(3i)(3i)} = \frac{\theta_{2i}^2}{(n_i + m_i) g_i(\theta_1)}. \tag{2.9}$$

Here, $I^{(2i)(2i)}$ and $I^{(3i)(3i)}$ denote the diagonal elements of the inverse Fisher information matrix associated with the ith components of $\boldsymbol{\theta}_2$ and $\boldsymbol{\theta}_3$, respectively.

Utilizing the Fisher information matrix (2.3) and the elements (2.6)–(2.9), Eq (2.5) simplifies to

$$\left\{\left[\sum_{i=1}^{k}\frac{2n_im_i(n_i+m_i)}{g_i(\theta_1)}\right]^{-2}\left[\sum_{i=1}^{k}\frac{n_i^2m_i^2(n_i+m_i)}{g_i(\theta_1)^2}\right]\left[\sum_{i=1}^{k}\frac{8n_i^2m_i^2(n_i+m_i)\theta_1[g_i(\theta_1)+(n_i+m_i)^2]}{g_i(\theta_1)^3}\right]\right.$$
$$\left.+\left[\sum_{i=1}^{k}\frac{2n_im_i(n_i+m_i)}{g_i(\theta_1)}\right]^{-1}\left[\sum_{i=1}^{k}\frac{4n_i^2m_i^2(n_i+m_i)\{2(n_i+m_i)^4-n_im_i\theta_1^2[g_i(\theta_1)+(n_i+m_i)^2]\}}{g_i(\theta_1)^4}\right]\right\}$$
$$-\frac{1}{d(\boldsymbol{\theta}_2,\boldsymbol{\theta}_3)}\sum_{i=1}^{k}\frac{4n_im_i(n_i+m_i)^4}{g_i(\theta_1)^3}\frac{\partial}{\partial\theta_{2i}}\{\theta_{2i}d(\boldsymbol{\theta}_2,\boldsymbol{\theta}_3)\}=0. \tag{2.10}$$

Note that the first term on the left-hand side of (2.10) depends only on the model, whereas the second term involves the prior. Because the first term depends exclusively on θ_1, whereas the second term involves θ_1, $\boldsymbol{\theta}_2$, and $\boldsymbol{\theta}_3$, the equation admits no general solution for $d(\boldsymbol{\theta}_2,\boldsymbol{\theta}_3)$. For a solution to exist, $d(\boldsymbol{\theta}_2,\boldsymbol{\theta}_3)$ depends only on θ_1, as well, or the first term vanishes. This observation implies that a general second-order matching prior does not exist.

Remark 1. *(Existence of a second-order matching prior)* Suppose the sample sizes of the two arms in each study are equal, that is, $n_i = m_i$ for $i = 1,\dots,k$. In this case, Eq (2.10) simplifies to

$$\left[\frac{32(\sum_{i=1}^{k}n_i)^{-1/2}}{(8+\theta_1^2)^{5/2}}\right]d(\boldsymbol{\theta}_2,\boldsymbol{\theta}_3)=\left[\frac{32(\sum_{i=1}^{k}n_i)^{-1/2}}{(8+\theta_1^2)^{5/2}}\right]\sum_{i=1}^{k}\frac{\partial}{\partial\theta_{2i}}\{\theta_{2i}d(\boldsymbol{\theta}_2,\boldsymbol{\theta}_3)\}. \tag{2.11}$$

The general solution of (2.11) takes the form

$$d(\boldsymbol{\theta}_2,\boldsymbol{\theta}_3)=\theta_{21}^{-(k-1)}h_1\left(\frac{\theta_{22}}{\theta_{21}},\dots,\frac{\theta_{2k}}{\theta_{21}}\right)h_2(\boldsymbol{\theta}_3),$$

where $h_1(\cdot,\dots,\cdot)>0$ and $h_2(\cdot)>0$ are arbitrary differentiable functions. Consequently, the resulting second-order probability matching prior is

$$\pi_m^{(2)}(\theta_1,\boldsymbol{\theta}_2,\boldsymbol{\theta}_3)\propto(\theta_1^2+8)^{-1/2}\,\theta_{21}^{-(k-1)}\,h_1\left(\frac{\theta_{22}}{\theta_{21}},\dots,\frac{\theta_{2k}}{\theta_{21}}\right)h_2(\boldsymbol{\theta}_3). \tag{2.12}$$

2.2. Reference priors

Reference priors, introduced by Bernardo (1979) [23] and further developed by Berger and Bernardo (1992) [25], have become a standard device for constructing noninformative priors. In this section, we derive reference priors for different group orderings of $(\theta_1,\boldsymbol{\theta}_2,\boldsymbol{\theta}_3)$. Owing to the orthogonality of the parameters, when θ_1 is the parameter of interest, the reference priors can be obtained by choosing rectangular compacts for each of θ_1, $\boldsymbol{\theta}_2$, and $\boldsymbol{\theta}_3$; see Datta and Ghosh (1995) [18]. The resulting reference prior distributions for the model (2.2) are as follows.

For the ordering $\{(\theta_1,\boldsymbol{\theta}_2,\boldsymbol{\theta}_3)\}$ with θ_1 as the parameter of interest, the reference prior is

$$\pi_1(\theta_1,\boldsymbol{\theta}_2,\boldsymbol{\theta}_3)\quad\propto\quad\left[\sum_{i=1}^{k}\frac{n_im_i(n_i+m_i)}{2(n_i+m_i)^2+n_im_i\theta_1^2}\right]^{1/2} \tag{2.13}$$

$$\times \prod_{i=1}^{k} \theta_{2i}^{-2} \{2(n_i + m_i)^2 + n_i m_i \theta_1^2\}.$$

For the group ordering $\{\theta_1, (\boldsymbol{\theta}_2, \boldsymbol{\theta}_3)\}$, the two-group reference prior is

$$\pi_2(\theta_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3) \propto \left[\sum_{i=1}^{k} \frac{n_i m_i (n_i + m_i)}{2(n_i + m_i)^2 + n_i m_i \theta_1^2}\right]^{1/2} \prod_{i=1}^{k} \theta_{2i}^{-2}. \tag{2.14}$$

Finally, for the group orderings $\{\theta_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3\}$ and $\{\theta_1, \boldsymbol{\theta}_3, \boldsymbol{\theta}_2\}$, the one-at-a-time reference prior is

$$\pi_3(\theta_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3) \propto \left[\sum_{i=1}^{k} \frac{n_i m_i (n_i + m_i)}{2(n_i + m_i)^2 + n_i m_i \theta_1^2}\right]^{1/2} \prod_{i=1}^{k} \theta_{2i}^{-1}. \tag{2.15}$$

Remark 2. Among the above, the two-group reference prior π_2 and the one-at-a-time reference prior π_3 satisfy a first-order matching criterion, whereas Jeffreys' prior π_1 does not satisfy a first-order matching criterion.

Remark 3. Recall that the general first-order matching class can be written as

$$\pi_m^{(1)}(\theta_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3) \propto \left[\sum_{i=1}^{k} \frac{n_i m_i (n_i + m_i)}{g_i(\theta_1)}\right]^{1/2} d(\boldsymbol{\theta}_2, \boldsymbol{\theta}_3),$$

where $d(\boldsymbol{\theta}_2, \boldsymbol{\theta}_3) > 0$ is any smooth positive function, and $g_i(\theta_1) = 2(n_i + m_i)^2 + n_i m_i \theta_1^2$. Notice that the above matching prior includes many different matching priors because of the arbitrary selection of the function d. Also note that for some functions, there does not seem to be any improvement in the coverage probabilities with these posteriors. As a specific choice, let $d(\boldsymbol{\theta}_2, \boldsymbol{\theta}_3) = \prod_{i=2}^{k} \theta_{2i}^{-1}$. Then,

$$\pi_m(\theta_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3) \propto \left[\sum_{i=1}^{k} \frac{n_i m_i (n_i + m_i)}{2(n_i + m_i)^2 + n_i m_i \theta_1^2}\right]^{1/2} \prod_{i=2}^{k} \theta_{2i}^{-1}. \tag{2.16}$$

This matching prior is selected on the grounds that it satisfies the second-order matching criterion when the sample sizes of the two arms are balanced across studies. Furthermore, in that case, it improves the frequentist coverage probability of the resulting posteriors while maintaining a functional form similar to that of the reference prior.

3. Posterior distributions

We study the propriety of the posterior distribution for a general class of priors that includes the reference priors (2.13)–(2.15). Consider the prior family

$$\begin{aligned} \pi(\theta_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3) \propto & \left[\sum_{i=1}^{k} \frac{n_i m_i (n_i + m_i)}{2(n_i + m_i)^2 + n_i m_i \theta_1^2}\right]^{a} \\ & \times \prod_{i=1}^{k} \theta_{2i}^{-b} [2(n_i + m_i)^2 + n_i m_i \theta_1^2]^{c}, \end{aligned} \tag{3.1}$$

where $a > 0$, $b > 0$, and $c \geq 0$. We first establish the following result.

Theorem 1. The posterior distribution of $(\theta_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3)$ under the prior (3.1) is proper if $n_i + m_i + b - 2 > 0$ for $i = 1, \ldots, k$ and $b - 2c - 1 \geq 0$.

Proof. Under (3.1), the joint posterior for $(\theta_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3)$ given $\mathbf{x}$ and $\mathbf{y}$ is

$$\begin{aligned}\pi(\theta_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3|\mathbf{x}, \mathbf{y}) &\propto \left[\sum i = 1^k \frac{n_i m_i (n_i + m_i)}{2(n_i + m_i)^2 + n_i m_i \theta_1^2}\right]^a \prod_{i=1}^k \theta_{2i}^{-(n_i+m_i)-b} \prod_{i=1}^k g_i(\theta_1)^{\frac{n_i+m_i+2c}{2}} \\ &\times \exp\left\{-\sum_{i=1}^k \frac{g_i(\theta_1)}{2\theta_2^2}\left[\sum_{j=1}^{n_i}\left(x_{ij} - \theta_{3i} - \frac{m_i}{n_i + m_i}\theta_1\theta_{2i} g_i(\theta_1)^{-\frac{1}{2}}\right)^2\right.\right. \\ &\left.\left. + \sum_{j=1}^{m_i}\left(y_{ij} - \theta_{3i} + \frac{n_i}{n_i + m_i}\theta_1\theta_{2i} g_i(\theta_1)^{-\frac{1}{2}}\right)^2\right]\right\}, \end{aligned} \tag{3.2}$$

where $g_i(\theta_1) = 2(n_i + m_i)^2 + n_i m_i \theta_1^2$.

Integrating (3.2) with respect to θ_{3i}, $i = 1, \ldots, k$, we obtain

$$\begin{aligned}\pi(\theta_1, \boldsymbol{\theta}_2|\mathbf{x}, \mathbf{y}) &\propto \left[\sum_{i=1}^k \frac{n_i m_i (n_i + m_i)}{2(n_i + m_i)^2 + n_i m_i \theta_1^2}\right]^a \prod_{i=1}^k \theta_{2i}^{-(n_i+m_i+b-1)} \prod_{i=1}^k g_i(\theta_1)^{\frac{n_i+m_i+2c-1}{2}} \\ &\times \exp\left\{-\sum_{i=1}^k \frac{g_i(\theta_1)}{2\theta_{2i}^2}\left(S_{xi}^2 + S_{yi}^2 + \frac{n_i m_i}{n_i + m_i}[\bar{x}_i - \bar{y}_i - \theta_1\theta_{2i} g_i(\theta_1)^{-\frac{1}{2}}]^2\right)\right\}, \end{aligned} \tag{3.3}$$

where $\bar{x}_i = n_i^{-1}\sum_{j=1}^{n_i} x_{ij}$, $\bar{y}_i = m_i^{-1}\sum_{j=1}^{m_i} y_{ij}$, $S_{xi}^2 = \sum_{j=1}^{n_i}(x_{ij} - \bar{x}_i)^2$, and $S_{yi}^2 = \sum_{j=1}^{m_i}(y_{ij} - \bar{y}_i)^2$. Let $\omega_i = \theta_{2i}\, g_i(\theta_1)^{-1/2}$, $i = 1, \ldots, k$. Then,

$$\begin{aligned}\pi(\theta_1, \boldsymbol{\omega}|\mathbf{x}, \mathbf{y}) &\propto \left[\sum_{i=1}^k \frac{n_i m_i (n_i + m_i)}{2(n_i + m_i)^2 + n_i m_i \theta_1^2}\right]^a \prod_{i=1}^k \omega_i^{-(n_i+m_i+b-1)} \prod_{i=1}^k g_i(\theta_1)^{\frac{-b+2c+1}{2}} \\ &\times \exp\left\{-\sum_{i=1}^k \frac{1}{2\omega_i^2}(S_{xi}^2 + S_{yi}^2) - \sum_{i=1}^k \frac{n_i m_i}{2(n_i + m_i)}\left(\frac{\bar{x}_i - \bar{y}_i}{\omega_i} - \theta_1\right)^2\right\}, \end{aligned} \tag{3.4}$$

where $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_k)$. As in the posterior distribution (3.4), it follows that for a positive constant c_1,

$$\left[\sum_{i=1}^k \frac{n_i m_i (n_i + m_i)}{2(n_i + m_i)^2 + n_i m_i \theta_1^2}\right]^a \leq c_1. \tag{3.5}$$

If $b - 2c - 1 \geq 0$, we have

$$\begin{aligned}\prod_{i=1}^k g_i(\theta_1)^{\frac{-b+2c+1}{2}} &= \prod_{i=1}^k [2(n_i + m_i)^2 + n_i m_i \theta_1^2]^{\frac{-b+2c+1}{2}} \\ &\leq \prod_{i=1}^k 2(n_i + m_i)^2. \end{aligned} \tag{3.6}$$

Furthermore, by letting $u_i = (\bar{x}_i - \bar{y}_i)/\omega_i$, we obtain

$$\begin{aligned} -\sum_{i=1}^{k}(u_i-\theta_1)^2 &= -k\left(\frac{\sum_{i=1}^{k}u_i}{k}-\theta_1\right)^2 - \sum_{i=1}^{k}\left(u_i - \frac{\sum_{j=1}^{k}u_j}{k}\right)^2 \\ &\le -k\left(\frac{\sum_{i=1}^{k}u_i}{k}-\theta_1\right)^2, \end{aligned} \tag{3.7}$$

and then,

$$\begin{aligned} \exp\left\{-\sum_{i=1}^{k}\frac{n_im_i}{2(n_i+m_i)}\left(\frac{\bar{x}_i-\bar{y}_i}{\omega_i}-\theta_1\right)^2\right\} &\le \exp\left\{-\frac{n_1m_1}{2(n_k+m_k)}\sum_{i=1}^{k}\left(\frac{\bar{x}_i-\bar{y}_i}{\omega_i}-\theta_1\right)^2\right\} \\ &\le \exp\left\{-\frac{kn_1m_1}{2(n_k+m_k)}\left(f(\omega_1,\ldots,\omega_k)-\theta_1\right)^2\right\}, \end{aligned} \tag{3.8}$$

where $n_1 = \min\{n_1,\ldots,n_k\}$, $m_1 = \min\{m_1,\ldots,m_k\}$, $n_k = \max\{n_1,\ldots,n_k\}$, $m_k = \max\{m_1,\ldots,m_k\}$, and $f(\omega_1,\ldots,\omega_k) = k^{-1}\sum_{i=1}^{k}(\bar{x}_i-\bar{y}_i)/\omega_i$. Combining (3.5)–(3.8) yields

$$\begin{aligned} \pi(\theta_1,\boldsymbol{\omega}|\mathbf{x},\mathbf{y}) &\le c_2\prod_{i=1}^{k}\omega_i^{-(n_i+m_i+b-1)}\times\exp\left\{-\sum_{i=1}^{k}\frac{1}{2\omega_i^2}(S_{xi}^2+S_{yi}^2)-\frac{kn_1m_1}{2(n_k+m_k)}\left(f(\omega_1,\ldots,\omega_k)-\theta_1\right)^2\right\} \\ &\equiv \pi'(\theta_1,\boldsymbol{\omega}|\mathbf{x},\mathbf{y}) \end{aligned} \tag{3.9}$$

for a constant $c_2 > 0$. Therefore,

$$\begin{aligned} &\int_0^{\infty}\cdots\int_0^{\infty}\int_{-\infty}^{\infty}\pi'(\theta_1,\boldsymbol{\omega}\mid\mathbf{x},\mathbf{y})\,d\theta_1\,d\omega_1\cdots d\omega_k \\ &= \int_0^{\infty}\cdots\int_0^{\infty}c_3\prod_{i=1}^{k}\omega_i^{-(n_i+m_i+b-1)}\times\exp\left\{-\sum_{i=1}^{k}\frac{1}{2\omega_i^2}(S_{xi}^2+S_{yi}^2)\right\}d\omega_1\cdots d\omega_k<\infty, \end{aligned} \tag{3.10}$$

provided that $n_i + m_i + b - 2 > 0$ for $i = 1,\ldots,k$, where c_3 is a positive constant. This completes the proof. □

Theorem 2. The posterior distribution of $(\theta_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3)$ under the matching prior (2.16) is proper.

Proof. From (3.3) in Theorem 1,

$$\begin{aligned} \pi(\theta_1,\boldsymbol{\theta}_2|\mathbf{x},\mathbf{y}) &\propto \left[\sum_{i=1}^{k}\frac{n_im_i(n_i+m_i)}{2(n_i+m_i)^2+n_im_i\theta_1^2}\right]^{\frac{1}{2}}\theta_{21}^{-(n_1+m_1-1)}\prod_{i=2}^{k}\theta_{2i}^{-(n_i+m_i)}\prod_{i=1}^{k}g_i(\theta_1)^{\frac{n_i+m_i-1}{2}} \\ &\times\ \exp\left\{-\sum_{i=1}^{k}\frac{g_i(\theta_1)}{2\theta_{2i}^2}\left(S_{xi}^2+S_{yi}^2+\frac{n_im_i}{n_i+m_i}[\bar{x}_i-\bar{y}_i-\theta_1\theta_{2i}g_i(\theta_1)^{-\frac{1}{2}}]^2\right)\right\}. \end{aligned}$$

Letting $\omega_i = \theta_{2i}\,g_i(\theta_1)^{-1/2}$, $i = 1,\ldots,k$, we have

$$\pi(\theta_1,\boldsymbol{\omega}|\mathbf{x},\mathbf{y}) \propto \left[\sum_{i=1}^{k}\frac{n_im_i(n_i+m_i)}{2(n_i+m_i)^2+n_im_i\theta_1^2}\right]^{\frac{1}{2}}\omega_1^{-(n_1+m_1-1)}g_1(\theta_1)^{\frac{1}{2}}\prod_{i=2}^{k}\omega_i^{-(n_i+m_i)} \tag{3.11}$$

$$\times \quad \exp\left\{-\sum_{i=1}^{k}\frac{1}{2\omega_i^2}(S_{xi}^2+S_{yi}^2)-\sum_{i=1}^{k}\frac{n_im_i}{2(n_i+m_i)}\left(\frac{\bar{x}_i-\bar{y}_i}{\omega_i}-\theta_1\right)^2\right\}.$$

As in (3.11), for a constant $c_1 > 0$,

$$\left[\sum_{i=1}^{k}\frac{n_im_i(n_i+m_i)}{2(n_i+m_i)^2+n_im_i\theta_1^2}\right]^{1/2}\{2(n_1+m_1)^2+n_1m_1\theta_1^2\}^{1/2}\le c_1, \tag{3.12}$$

and from (3.8) in Theorem 1,

$$\exp\left\{-\sum_{i=1}^{k}\frac{n_im_i}{2(n_i+m_i)}\left(\frac{\bar{x}_i-\bar{y}_i}{\omega_i}-\theta_1\right)^2\right\}\le \exp\left\{-\frac{kn_1m_1}{2(n_k+m_k)}\left(f(\omega_1,\ldots,\omega_k)-\theta_1\right)^2\right\}. \tag{3.13}$$

Hence, combining (3.12) and (3.13), we obtain

$$\begin{aligned}\pi(\theta_1,\boldsymbol{\omega}|\mathbf{x},\mathbf{y}) \quad &\le \quad c_1\omega_1^{-(n_1+m_1-1)}\prod_{i=2}^{k}\omega_i^{-(n_i+m_i)} \qquad (3.14)\\ &\times \quad \exp\left\{-\sum_{i=1}^{k}\frac{1}{2\omega_i^2}(S_{xi}^2+S_{yi}^2)-\frac{kn_1m_1}{2(n_k+m_k)}\left(f(\omega_1,\ldots,\omega_k)-\theta_1\right)^2\right\}\\ &\equiv \quad \pi'(\theta_1,\omega|\mathbf{x},\mathbf{y})\end{aligned}$$

and

$$\int_0^{\infty}\cdots\int_0^{\infty}\int_{-\infty}^{\infty}\pi'(\theta_1,\boldsymbol{\omega}|\mathbf{x},\mathbf{y})d\theta_1 d\omega_1\cdots d\omega_k<\infty, \tag{3.15}$$

whenever $n_1 + m_1 - 2 > 0$ and $n_i + m_i - 1 > 0$ for $i = 2, \ldots, k$. This completes the proof. □

Theorem 3. The posterior distribution of $(\theta_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3)$ under Jeffreys' prior is proper.

Proof. From (3.4) in Theorem 1,

$$\begin{aligned}\pi(\theta_1,\boldsymbol{\omega}|\mathbf{x},\mathbf{y}) \quad &\propto \quad \left[\sum_{i=1}^{k}\frac{n_im_i(n_i+m_i)}{2(n_i+m_i)^2+n_im_i\theta_1^2}\right]^{\frac{1}{2}}\prod_{i=1}^{k}\omega_i^{-(n_i+m_i+1)}\prod_{i=1}^{k}g_i(\theta_1)^{\frac{1}{2}}\\ &\times \quad \exp\left\{-\sum_{i=1}^{k}\frac{1}{2\omega_i^2}(S_{xi}^2+S_{yi}^2)-\sum_{i=1}^{k}\frac{n_im_i}{2(n_i+m_i)}\left(\frac{\bar{x}_i-\bar{y}_i}{\omega_i}-\theta_1\right)^2\right\}. \qquad (3.16)\end{aligned}$$

Moreover, as in (3.16), for some positive constants c_1 and c_2, we obtain

$$\left[\sum_{i=1}^{k}\frac{n_im_i(n_i+m_i)}{2(n_i+m_i)^2+n_im_i\theta_1^2}\right]^{1/2}\le c_1, \tag{3.17}$$

$$\prod_{i=1}^{k}g_i(\theta_1)^{\frac{1}{2}} \quad = \quad \prod_{i=1}^{k}[2(n_i+m_i)^2+n_im_i\theta_1^2]^{\frac{1}{2}}$$

$$\begin{aligned}&\le [2(n_k+m_k)^2+n_km_k\theta_1^2]^{\frac{k}{2}}\\&\le c_2([2(n_k+m_k)^2]^{\frac{k}{2}}+(n_km_k)^{\frac{k}{2}}|\theta_1|^k),\end{aligned}\tag{3.18}$$

and from (3.8) in Theorem 1,

$$\exp\left\{-\sum_{i=1}^{k}\frac{n_im_i}{2(n_i+m_i)}\left(\frac{\bar{x}_i-\bar{y}_i}{\omega_i}-\theta_1\right)^2\right\}\le\exp\left\{-\frac{kn_1m_1}{2(n_k+m_k)}\left(f(\omega_1,\ldots,\omega_k)-\theta_1\right)^2\right\}.\tag{3.19}$$

Thus, from (3.17)–(3.19),

$$\begin{aligned}\pi(\theta_1,\boldsymbol{\omega}|\mathbf{x},\mathbf{y}) &\le c_3([2(n_k+m_k)^2]^{\frac{k}{2}}+(n_km_k)^{\frac{k}{2}}|\theta_1|^k)\prod_{i=1}^{k}\omega_i^{-(n_i+m_i+1)}\\&\times \exp\left\{-\sum_{i=1}^{k}\frac{1}{2\omega_i^2}(S_{xi}^2+S_{yi}^2)-\frac{kn_1m_1}{2(n_k+m_k)}\left(f(\omega_1,\ldots,\omega_k)-\theta_1\right)^2\right\}\\&\equiv \pi'(\theta_1,\omega|\mathbf{x},\mathbf{y})\end{aligned}\tag{3.20}$$

for a constant $c_3>0$. As in (3.20), we have

$$\int_{-\infty}^{\infty}\exp\left\{-\frac{kn_1m_1}{2(n_k+m_k)}(f(\boldsymbol{\omega})-\theta_1)^2\right\}d\theta_1=\int_{-\infty}^{\infty}e^{-c_4\eta_1^2}\,d\eta_1,\tag{3.21}$$

$$\left|f(\boldsymbol{\omega})-\eta_1\right|^k\le c_5(\left|f(\boldsymbol{\omega})\right|^k+\left|\eta_1\right|^k)\tag{3.22}$$

for some positive constants $c_4=\frac{kn_1m_1}{2(n_k+m_k)}$ and c_5. Hence, using (3.21) and (3.22),

$$\begin{aligned}&\int_0^{\infty}\int_{-\infty}^{\infty}\pi'(\theta_1,\boldsymbol{\omega}\mid\mathbf{x},\mathbf{y})\,d\theta_1\,d\boldsymbol{\omega}\\&\le\int_0^{\infty}\int_{-\infty}^{\infty}c_3\Big([2(n_k+m_k)^2]^{k/2}+c_5(n_km_k)^{k/2}(|f(\boldsymbol{\omega})|^k+|\eta_1|^k)\Big)\prod_{i=1}^{k}\omega_i^{-(n_i+m_i+1)}\\&\quad\times\exp\left\{-\sum_{i=1}^{k}\frac{S_{xi}^2+S_{yi}^2}{2\omega_i^2}-c_4\eta_1^2\right\}d\eta_1\,d\boldsymbol{\omega}<\infty,\end{aligned}$$

whenever $n_i>0$ and $m_i>0$ for $i=1,\ldots,k$. This completes the proof. □

Theorem 4. Under the general prior (3.1), the marginal posterior density of θ_1 is

$$\begin{aligned}\pi(\theta_1|\mathbf{x},\mathbf{y}) &\propto \int_0^{\infty}\left[\sum_{i=1}^{k}\frac{n_im_i(n_i+m_i)}{2(n_i+m_i)^2+n_im_i\theta_1^2}\right]^a\prod_{i=1}^{k}\theta_{2i}^{-(n_i+m_i+b-1)}\prod_{i=1}^{k}g_i(\theta_1)^{\frac{n_i+m_i+2c-1}{2}}\\&\times \exp\left\{-\sum_{i=1}^{k}\frac{g_i(\theta_1)}{2\theta_{2i}^2}\left(S_{xi}^2+S_{yi}^2+\frac{n_im_i}{n_i+m_i}[\bar{x}_i-\bar{y}_i-\theta_1\theta_{2i}g_i(\theta_1)^{-\frac{1}{2}}]^2\right)\right\}d\boldsymbol{\theta}_2.\end{aligned}\tag{3.23}$$

Under the matching prior (2.16), the marginal posterior density of θ_1 is

$$\pi(\theta_1|\mathbf{x},\mathbf{y}) \propto \int_0^{\infty}\left[\sum_{i=1}^{k}\frac{n_im_i(n_i+m_i)}{2(n_i+m_i)^2+n_im_i\theta_1^2}\right]^{\frac{1}{2}}\theta_{21}^{n_1+m_1-1}\prod_{i=2}^{k}\theta_{2i}^{-(n_i+m_i)}\prod_{i=1}^{k}g_i(\theta_1)^{\frac{n_i+m_i-1}{2}}\tag{3.24}$$

$$\times \quad \exp\left\{-\sum_{i=1}^{k}\frac{g_i(\theta_1)}{2\theta_{2i}^2}\left(S_{xi}^2+S_{yi}^2+\frac{n_i m_i}{n_i+m_i}[\bar{x}_i-\bar{y}_i-\theta_1\theta_{2i}g_i(\theta_1)^{-\frac{1}{2}}]^2\right)\right\}d\boldsymbol{\theta}_2,$$

where $g_i(\theta_1) = 2(n_i+m_i)^2 + n_i m_i \theta_1^2$, $\bar{x}_i = n_i^{-1}\sum_{j=1}^{n_i} x_{ij}$, $\bar{y}_i = m_i^{-1}\sum_{j=1}^{m_i} y_{ij}$, $S_{xi}^2 = \sum_{j=1}^{n_i}(x_{ij}-\bar{x}_i)^2$, and $S_{yi}^2 = \sum_{j=1}^{m_i}(y_{ij}-\bar{y}_i)^2$.

The normalizing constant for the marginal density of θ_1 involves a two-dimensional integration, which enables computation of the marginal posterior density and moments of θ_1. In Section 4, we investigate frequentist coverage probabilities for Jeffreys' prior π_1, the two-group reference prior π_2, the one-at-a-time reference prior π_3, and the matching prior π_m.

4. Numerical study

We evaluate frequentist coverage by numerically computing credible intervals for the marginal posterior density of θ_1 across several configurations of $(\mu_{i1}, \mu_{i2}, \sigma_i)$, (n_i, m_i), and k under the noninformative prior π introduced in Section 2. Specifically, the frequentist coverage of the $(1-\alpha)$th posterior quantile should be close to $1-\alpha$. Tables 1–6 report coverage probabilities for the $\alpha \in \{0.05, 0.95\}$ posterior quantiles under our prior.

To compute these values, fix the true $(\mu_{i1}, \mu_{i2}, \sigma_i)$ and a target probability $\alpha \in \{0.05, 0.95\}$. Let $\theta_1^\pi(\alpha \mid \mathbf{X}, \mathbf{Y})$ denote the posterior α-quantile of θ_1 given data $(\mathbf{X}, \mathbf{Y})$, that is, $F(\theta_1^\pi(\alpha \mid \mathbf{X}, \mathbf{Y}) \mid \mathbf{X}, \mathbf{Y}) = \alpha$, where $F(\cdot \mid \mathbf{X}, \mathbf{Y})$ is the marginal posterior distribution of θ_1. The frequentist coverage probability of the one-sided credible interval is

$$P(\alpha;\theta_1) = \Pr(-\infty < \theta_1 \le \theta_1^\pi(\alpha \mid \mathbf{X}, \mathbf{Y})).$$

Estimated values of $P(\alpha;\theta_1)$ for $\alpha = 0.05$ and 0.95 are presented in Tables 1–3, together with the frequentist coverage of the 90% and 95% credible intervals (Tables 4–6). Moreover, for comparison with the frequentist methods, the results using the methods of Hedges 1982 [2] and Bonett (2009) [10] are provided in the tables.

For each configuration, we generated 10,000 independent samples $(\mathbf{X}, \mathbf{Y})$ from the normal model $N(\mu_{i1}, \sigma_i^2)$ and $N(\mu_{i2}, \sigma_i^2)$, respectively, with $\mu_{i2} = 0$ and $(\mu_{i1}, \mu_{i2}, \sigma_i)$ held fixed. In this simulation, the common standardized mean difference ranges from 0 to 2, the standard deviations between studies are set to three levels (identical, moderately different, and significantly different), and the number of studies varies from small ($k = 3$) to large ($k = 10$). Furthermore, the sample sizes are configured to reflect small-sample conditions. A numerical study for computing posterior quantiles was performed using Gaussian quadrature numerical integration in Fortran 90 with IMSL.

Across the scenarios in Tables 1–6, the matching prior π_m achieves coverage closest to the nominal targets and generally outperforms both the reference priors and Jeffreys' prior. The one-at-a-time reference prior π_3, which satisfies the first-order matching criterion, exhibits coverage comparable to that of π_m. By contrast, Jeffreys' prior π_1 tends to underperform, reflecting its failure to meet the matching criterion. We also note that the results for π_m and π_3 are largely robust to changes in $(\mu_{i1}, \mu_{i2}, \sigma_i)$ and k. These findings support the use of π_m or π_3 for Bayesian inference in practice. On the other hand, the results by Hedges (1982) [2]) and Bonett (2009) [10] show that Bonett's method outperforms Hedges' and aligns more closely with the results of reference or matching priors. However, Bonett's method tends to underdeliver on the target coverage probability for quantiles as the common standardized mean difference (θ_1) increases. In particular, Hedges' method becomes significantly less effective when both the number of studies (k) and the common standardized mean difference (θ_1) are large. This poor performance is fundamentally due to the small sample sizes.

Example 1. This example, taken from Kulinskaya et al. (2011) [4], illustrates estimation of the common standardized mean difference. We consider the meta-analysis by Hróbjartsson and Gøtzsche (2004) [33] comprising 17 randomized clinical trials that compared placebo for pain against no treatment. Because the studies used different pain scales, standardized mean differences were employed to place all effects on a common, scale-free metric.

Table 7 reports the summary statistics (sample means ($\bar{X}_T, \bar{X}_C$), standard deviations (S_T, S_C), and and sample sizes (n_T, n_C)), where the subscripts T and C denote the treatment and control arms, respectively (see Kulinskaya et al., 2011 [4]). Kulinskaya et al. (2011) [4] developed homogeneity tests for standardized mean differences and, for this dataset, reported p-values of 0.141 for the classical Q statistic and 0.113 for the improved Q statistic. Moreover, except for three of the 17 studies, the F-tests for equal variances yielded p-values greater than 0.116; thus, we proceed with the analysis assuming equal variances. Accordingly, we proceed to estimate the common standardized mean difference.

Table 8 presents Bayes estimates and 90% and 95% Bayesian credible intervals based on Jeffreys' prior (JP), the two-group reference prior (TR), the one-at-a-time reference prior (OR), and the matching prior (MP). For comparison, we also include the estimates and approximate confidence intervals of Hedges (1982) [2] and Bonett (2009) [10]. The results in Table 8 indicate broadly similar point and interval estimates across methods. In particular, the JP, TR, OR, and MP estimates are nearly identical and differ only slightly from Bonett's (2009) [10] estimate, whereas the Hedges (1982) [2] estimate is somewhat smaller. For interval estimation, JP, TR, OR, and MP again yield very similar intervals, which differ modestly from Bonett's intervals; the Hedges (1982) [2] interval appears slightly right-skewed relative to the others.

Example 2. This example, from Bonett (2009) [10], involves four eyewitness identification studies that assessed participants' certainty when selecting a target individual from a photo lineup. Across studies, various treatments were used, but two instructions were common: Participants were told the target individual "will be"or "might be" in the lineup. Two studies used five photos, and the other two used seven. All samples consisted of volunteer introductory psychology students. Table 9 lists the sample means and standard deviations (see Bonett, 2009 [10]).

For these data, the p-value of the Kulinskaya et al. (2011) [2] Q statistic for homogeneity of standardized mean differences is 0.991. We therefore estimate the common standardized mean difference. Table 10 reports Bayes estimates and 90% and 95% Bayesian credible intervals based on JP, TR, OR, and MP, together with the estimates and approximate confidence intervals of Hedges (1982) [2] and Bonett (2009) [10].

The point and interval estimates are again quite similar. In particular, the TR, OR, and MP point estimates coincide with Bonett's (2009) [10] estimate and differ only slightly from JP and Hedges (1982) [2]. For interval estimates, TR, OR, and MP produce essentially identical intervals, which are only modestly different from those of JP, Hedges (1982) [2], and Bonett (2009) [10].

Across all identified examples, the differences in results among the developed priors, excluding Jeffreys' prior, were not significant. This lack of significance is attributed to the large sample sizes in both examples, a finding further supported by simulation results (e.g., when the sample size is (10, 15)).

Table 1. Frequentist coverage probability of 0.05 (0.95) quantiles of θ_1 when $k = 3$.

θ_1	$(\sigma_1, \ldots, \sigma_k)$	$(n_i, m_i))$	π_1	π_2	π_3	π_m	Hedges	Bonett
0.0	(0.5,0.5,0.5)	(5,5)	0.068 (0.931)	0.060 (0.942)	0.053 (0.948)	0.050 (0.950)	0.041 (0.960)	0.045 (0.955)
		(5,10)	0.062 (0.940)	0.056 (0.947)	0.052 (0.950)	0.051 (0.952)	0.043 (0.960)	0.047 (0.955)
		(10,10)	0.058 (0.941)	0.053 (0.946)	0.051 (0.947)	0.050 (0.948)	0.045 (0.954)	0.047 (0.953)
		(10,15)	0.054 (0.941)	0.051 (0.944)	0.048 (0.947)	0.048 (0.947)	0.043 (0.952)	0.046 (0.949)
	(0.1,0.5,1.0)	(5,5)	0.067 (0.930)	0.057 (0.940)	0.051 (0.947)	0.050 (0.950)	0.039 (0.960)	0.043 (0.956)
		(5,10)	0.063 (0.937)	0.057 (0.943)	0.054 (0.946)	0.053 (0.947)	0.046 (0.955)	0.048 (0.951)
		(10,10)	0.059 (0.939)	0.055 (0.943)	0.052 (0.945)	0.051 (0.947)	0.046 (0.953)	0.047 (0.952)
		(10,15)	0.056 (0.944)	0.052 (0.947)	0.051 (0.950)	0.050 (0.950)	0.046 (0.954)	0.048 (0.952)
	(1,1,1)	(5,5)	0.064 (0.932)	0.056 (0.942)	0.049 (0.947)	0.048 (0.950)	0.038 (0.961)	0.042 (0.956)
		(5,10)	0.061 (0.937)	0.055 (0.944)	0.052 (0.947)	0.052 (0.949)	0.044 (0.957)	0.046 (0.952)
		(10,10)	0.059 (0.942)	0.054 (0.946)	0.052 (0.949)	0.051 (0.950)	0.045 (0.955)	0.047 (0.953)
		(10,15)	0.059 (0.947)	0.056 (0.950)	0.053 (0.952)	0.053 (0.953)	0.049 (0.957)	0.049 (0.957)
	(1,5,10)	(5,5)	0.063 (0.932)	0.054 (0.940)	0.049 (0.946)	0.047 (0.949)	0.037 (0.958)	0.042 (0.953)
		(5,10)	0.060 (0.938)	0.054 (0.945)	0.049 (0.948)	0.049 (0.949)	0.042 (0.956)	0.047 (0.954)
		(10,10)	0.057 (0.941)	0.053 (0.945)	0.051 (0.947)	0.050 (0.948)	0.044 (0.954)	0.047 (0.953)
		(10,15)	0.057 (0.944)	0.054 (0.949)	0.051 (0.951)	0.051 (0.952)	0.047 (0.956)	0.048 (0.953)
0.2	(0.5,0.5,0.5)	(5,5)	0.077 (0.939)	0.060 (0.943)	0.053 (0.948)	0.051 (0.950)	0.037 (0.955)	0.045 (0.953)
		(5,10)	0.065 (0.948)	0.054 (0.949)	0.052 (0.952)	0.050 (0.953)	0.040 (0.956)	0.046 (0.957)
		(10,10)	0.064 (0.949)	0.055 (0.949)	0.052 (0.952)	0.051 (0.952)	0.043 (0.954)	0.046 (0.954)
		(10,15)	0.059 (0.952)	0.051 (0.951)	0.050 (0.953)	0.049 (0.953)	0.042 (0.955)	0.047 (0.956)
	(0.1,0.5,1.0)	(5,5)	0.067 (0.938)	0.052 (0.942)	0.047 (0.948)	0.045 (0.950)	0.031 (0.956)	0.036 (0.953)
		(5,10)	0.063 (0.941)	0.052 (0.942)	0.049 (0.945)	0.047 (0.946)	0.038 (0.951)	0.045 (0.950)
		(10,10)	0.059 (0.945)	0.049 (0.946)	0.048 (0.948)	0.047 (0.948)	0.038 (0.950)	0.043 (0.951)
		(10,15)	0.059 (0.943)	0.051 (0.943)	0.049 (0.945)	0.048 (0.946)	0.040 (0.946)	0.046 (0.948)
	(1,1,1)	(5,5)	0.071 (0.937)	0.056 (0.941)	0.050 (0.946)	0.049 (0.948)	0.034 (0.955)	0.040 (0.953)
		(5,10)	0.063 (0.943)	0.053 (0.944)	0.050 (0.946)	0.048 (0.948)	0.039 (0.952)	0.045 (0.951)
		(10,10)	0.058 (0.947)	0.050 (0.947)	0.047 (0.950)	0.047 (0.950)	0.040 (0.954)	0.043 (0.954)
		(10,15)	0.065 (0.947)	0.057 (0.947)	0.055 (0.949)	0.054 (0.950)	0.046 (0.951)	0.050 (0.952)
	(1,5,10)	(5,5)	0.080 (0.939)	0.062 (0.942)	0.055 (0.947)	0.052 (0.949)	0.038 (0.955)	0.046 (0.953)
		(5,10)	0.068 (0.942)	0.056 (0.943)	0.052 (0.947)	0.051 (0.947)	0.042 (0.953)	0.047 (0.953)
		(10,10)	0.064 (0.947)	0.056 (0.947)	0.053 (0.950)	0.052 (0.951)	0.043 (0.953)	0.046 (0.953)
		(10,15)	0.065 (0.951)	0.056 (0.951)	0.055 (0.953)	0.054 (0.954)	0.046 (0.955)	0.049 (0.956)
0.5	(0.5,0.5,0.5)	(5,5)	0.088 (0.948)	0.061 (0.942)	0.054 (0.946)	0.051 (0.947)	0.033 (0.946)	0.043 (0.950)
		(5,10)	0.071 (0.953)	0.053 (0.947)	0.049 (0.950)	0.048 (0.952)	0.034 (0.949)	0.043 (0.954)
		(10,10)	0.066 (0.955)	0.050 (0.949)	0.047 (0.951)	0.046 (0.952)	0.036 (0.948)	0.043 (0.953)
		(10,15)	0.067 (0.958)	0.053 (0.953)	0.051 (0.955)	0.051 (0.955)	0.041 (0.952)	0.048 (0.956)
	(0.1,0.5,1.0)	(5,5)	0.086 (0.951)	0.059 (0.946)	0.053 (0.950)	0.051 (0.952)	0.033 (0.951)	0.040 (0.953)
		(5,10)	0.069 (0.952)	0.051 (0.946)	0.048 (0.950)	0.047 (0.951)	0.033 (0.949)	0.042 (0.954)
		(10,10)	0.067 (0.953)	0.051 (0.949)	0.048 (0.951)	0.048 (0.952)	0.035 (0.949)	0.043 (0.952)
		(10,15)	0.065 (0.952)	0.052 (0.947)	0.051 (0.948)	0.050 (0.949)	0.039 (0.946)	0.045 (0.950)
	(1,1,1)	(5,5)	0.087 (0.951)	0.061 (0.946)	0.055 (0.949)	0.053 (0.951)	0.034 (0.951)	0.044 (0.954)
		(5,10)	0.075 (0.954)	0.057 (0.950)	0.054 (0.952)	0.052 (0.953)	0.039 (0.952)	0.050 (0.957)
		(10,10)	0.066 (0.952)	0.052 (0.946)	0.049 (0.948)	0.048 (0.949)	0.038 (0.946)	0.044 (0.950)
		(10,15)	0.062 (0.952)	0.048 (0.947)	0.047 (0.949)	0.047 (0.949)	0.036 (0.945)	0.044 (0.950)
	(1,5,10)	(5,5)	0.088 (0.952)	0.059 (0.947)	0.053 (0.951)	0.051 (0.953)	0.033 (0.952)	0.043 (0.955)
		(5,10)	0.076 (0.951)	0.057 (0.946)	0.054 (0.948)	0.052 (0.950)	0.037 (0.948)	0.049 (0.951)
		(10,10)	0.065 (0.951)	0.049 (0.946)	0.047 (0.948)	0.047 (0.949)	0.035 (0.946)	0.042 (0.951)
		(10,15)	0.067 (0.953)	0.054 (0.946)	0.052 (0.948)	0.051 (0.949)	0.042 (0.945)	0.048 (0.951)
1.0	(0.5,0.5,0.5)	(5,5)	0.103 (0.960)	0.057 (0.943)	0.050 (0.946)	0.048 (0.948)	0.025 (0.932)	0.038 (0.947)
		(5,10)	0.087 (0.962)	0.054 (0.948)	0.049 (0.950)	0.048 (0.951)	0.030 (0.937)	0.044 (0.953)
		(10,10)	0.080 (0.961)	0.051 (0.948)	0.049 (0.950)	0.048 (0.950)	0.032 (0.936)	0.042 (0.949)
		(10,15)	0.077 (0.959)	0.055 (0.946)	0.052 (0.947)	0.051 (0.948)	0.037 (0.935)	0.048 (0.949)
	(0.1,0.5,1.0)	(5,5)	0.110 (0.963)	0.062 (0.944)	0.056 (0.947)	0.054 (0.950)	0.029 (0.932)	0.041 (0.947)
		(5,10)	0.086 (0.957)	0.054 (0.943)	0.052 (0.945)	0.050 (0.945)	0.030 (0.933)	0.047 (0.949)
		(10,10)	0.080 (0.961)	0.053 (0.947)	0.052 (0.948)	0.052 (0.949)	0.035 (0.937)	0.044 (0.947)
		(10,15)	0.076 (0.959)	0.053 (0.949)	0.052 (0.951)	0.052 (0.951)	0.034 (0.939)	0.047 (0.951)
	(1,1,1)	(5,5)	0.109 (0.965)	0.061 (0.948)	0.056 (0.952)	0.052 (0.952)	0.028 (0.936)	0.040 (0.951)
		(5,10)	0.092 (0.960)	0.060 (0.949)	0.055 (0.950)	0.054 (0.951)	0.035 (0.939)	0.047 (0.953)
		(10,10)	0.081 (0.963)	0.053 (0.951)	0.050 (0.952)	0.049 (0.952)	0.035 (0.940)	0.044 (0.951)
		(10,15)	0.076 (0.961)	0.053 (0.948)	0.051 (0.949)	0.051 (0.950)	0.036 (0.938)	0.048 (0.950)
	(1,5,10)	(5,5)	0.106 (0.962)	0.059 (0.945)	0.053 (0.948)	0.050 (0.949)	0.028 (0.935)	0.040 (0.948)
		(5,10)	0.086 (0.957)	0.055 (0.941)	0.052 (0.943)	0.050 (0.944)	0.031 (0.931)	0.043 (0.946)
		(10,10)	0.083 (0.962)	0.052 (0.949)	0.050 (0.950)	0.049 (0.951)	0.032 (0.938)	0.043 (0.949)
		(10,15)	0.078 (0.961)	0.053 (0.951)	0.051 (0.952)	0.051 (0.952)	0.036 (0.943)	0.045 (0.952)
2.0	(0.5,0.5,0.5)	(5,5)	0.148 (0.977)	0.056 (0.952)	0.052 (0.952)	0.050 (0.953)	0.020 (0.900)	0.034 (0.943)
		(5,10)	0.116 (0.974)	0.057 (0.951)	0.053 (0.952)	0.053 (0.951)	0.027 (0.913)	0.044 (0.952)
		(10,10)	0.102 (0.973)	0.048 (0.949)	0.045 (0.949)	0.045 (0.949)	0.027 (0.912)	0.040 (0.943)
		(10,15)	0.099 (0.970)	0.053 (0.951)	0.052 (0.951)	0.051 (0.952)	0.032 (0.922)	0.046 (0.947)
	(0.1,0.5,1.0)	(5,5)	0.147 (0.977)	0.059 (0.948)	0.054 (0.947)	0.051 (0.947)	0.022 (0.894)	0.036 (0.939)
		(5,10)	0.110 (0.971)	0.052 (0.948)	0.049 (0.949)	0.050 (0.949)	0.025 (0.911)	0.043 (0.952)
		(10,10)	0.099 (0.971)	0.052 (0.948)	0.051 (0.950)	0.050 (0.950)	0.027 (0.919)	0.040 (0.944)
		(10,15)	0.101 (0.969)	0.058 (0.951)	0.058 (0.951)	0.057 (0.952)	0.032 (0.922)	0.044 (0.948)
	(1,1,1)	(5,5)	0.141 (0.974)	0.053 (0.945)	0.049 (0.946)	0.047 (0.945)	0.021 (0.887)	0.034 (0.933)
		(5,10)	0.114 (0.972)	0.056 (0.946)	0.052 (0.947)	0.051 (0.947)	0.027 (0.908)	0.046 (0.950)
		(10,10)	0.106 (0.973)	0.052 (0.951)	0.050 (0.951)	0.049 (0.951)	0.026 (0.915)	0.041 (0.944)
		(10,15)	0.091 (0.967)	0.049 (0.946)	0.046 (0.946)	0.045 (0.947)	0.029 (0.918)	0.042 (0.947)
	(1,5,10)	(5,5)	0.143 (0.978)	0.055 (0.952)	0.051 (0.952)	0.048 (0.951)	0.022 (0.895)	0.034 (0.942)
		(5,10)	0.113 (0.972)	0.054 (0.946)	0.051 (0.946)	0.050 (0.947)	0.024 (0.909)	0.044 (0.948)
		(10,10)	0.103 (0.973)	0.054 (0.949)	0.052 (0.950)	0.052 (0.950)	0.030 (0.916)	0.043 (0.945)
		(10,15)	0.096 (0.966)	0.055 (0.947)	0.054 (0.947)	0.053 (0.947)	0.031 (0.915)	0.046 (0.945)

Table 2. Frequentist coverage probability of 0.05 (0.95) quantiles of θ_1 when $k = 5$.

θ_1	$(\sigma_1, \ldots, \sigma_k)$	(n_i, m_i))	π_1	π_2	π_3	π_m	Hedges	Bonett
0.0	(0.5,0.5,0.5,0.5,0.5)	(5,5)	0.066 (0.934)	0.057 (0.942)	0.051 (0.947)	0.050 (0.949)	0.039 (0.961)	0.046 (0.953)
		(5,10)	0.059 (0.942)	0.053 (0.948)	0.049 (0.951)	0.048 (0.952)	0.041 (0.959)	0.046 (0.953)
		(10,10)	0.056 (0.948)	0.052 (0.952)	0.049 (0.954)	0.049 (0.954)	0.043 (0.959)	0.046 (0.956)
		(10,15)	0.058 (0.943)	0.053 (0.948)	0.052 (0.949)	0.051 (0.950)	0.046 (0.954)	0.048 (0.951)
	(0.1,0.3,0.5,0.7,1.0)	(5,5)	0.066 (0.935)	0.056 (0.944)	0.051 (0.951)	0.050 (0.952)	0.038 (0.962)	0.044 (0.954)
		(5,10)	0.062 (0.940)	0.056 (0.946)	0.052 (0.949)	0.051 (0.949)	0.043 (0.959)	0.047 (0.955)
		(10,10)	0.058 (0.945)	0.054 (0.949)	0.052 (0.952)	0.051 (0.953)	0.045 (0.957)	0.049 (0.955)
		(10,15)	0.053 (0.944)	0.050 (0.948)	0.048 (0.950)	0.048 (0.950)	0.044 (0.955)	0.047 (0.953)
	(1,1,1,1,1)	(5,5)	0.067 (0.937)	0.058 (0.946)	0.051 (0.951)	0.051 (0.951)	0.040 (0.963)	0.047 (0.954)
		(5,10)	0.058 (0.939)	0.053 (0.948)	0.049 (0.952)	0.049 (0.952)	0.041 (0.961)	0.045 (0.955)
		(10,10)	0.059 (0.942)	0.054 (0.947)	0.051 (0.949)	0.051 (0.950)	0.044 (0.956)	0.048 (0.952)
		(10,15)	0.059 (0.943)	0.055 (0.947)	0.054 (0.949)	0.054 (0.949)	0.050 (0.954)	0.052 (0.952)
	(1,3,5,7,10)	(5,5)	0.063 (0.940)	0.053 (0.948)	0.048 (0.954)	0.047 (0.955)	0.035 (0.965)	0.043 (0.959)
		(5,10)	0.057 (0.939)	0.050 (0.944)	0.047 (0.947)	0.046 (0.948)	0.040 (0.956)	0.045 (0.951)
		(10,10)	0.055 (0.943)	0.051 (0.947)	0.049 (0.949)	0.049 (0.950)	0.043 (0.954)	0.046 (0.951)
		(10,15)	0.056 (0.948)	0.054 (0.950)	0.053 (0.951)	0.052 (0.951)	0.049 (0.957)	0.050 (0.954)
0.2	(0.5,0.5,0.5,0.5,0.5)	(5,5)	0.070 (0.941)	0.054 (0.942)	0.048 (0.948)	0.047 (0.949)	0.032 (0.954)	0.041 (0.952)
		(5,10)	0.066 (0.947)	0.054 (0.947)	0.050 (0.950)	0.050 (0.950)	0.038 (0.953)	0.046 (0.953)
		(10,10)	0.069 (0.951)	0.059 (0.951)	0.056 (0.953)	0.055 (0.953)	0.044 (0.955)	0.051 (0.956)
		(10,15)	0.062 (0.952)	0.054 (0.952)	0.052 (0.953)	0.051 (0.953)	0.041 (0.954)	0.048 (0.955)
	(0.1,0.3,0.5,0.7,1.0)	(5,5)	0.080 (0.945)	0.059 (0.946)	0.054 (0.951)	0.054 (0.951)	0.036 (0.956)	0.049 (0.954)
		(5,10)	0.064 (0.947)	0.054 (0.947)	0.050 (0.949)	0.049 (0.949)	0.038 (0.953)	0.046 (0.954)
		(10,10)	0.061 (0.951)	0.051 (0.949)	0.048 (0.952)	0.048 (0.953)	0.038 (0.953)	0.044 (0.955)
		(10,15)	0.060 (0.949)	0.051 (0.948)	0.051 (0.950)	0.050 (0.950)	0.042 (0.951)	0.048 (0.952)
	(1,1,1,1,1)	(5,5)	0.075 (0.942)	0.056 (0.943)	0.050 (0.948)	0.049 (0.949)	0.033 (0.953)	0.043 (0.951)
		(5,10)	0.070 (0.950)	0.056 (0.950)	0.052 (0.953)	0.051 (0.954)	0.039 (0.957)	0.048 (0.957)
		(10,10)	0.064 (0.948)	0.054 (0.946)	0.051 (0.949)	0.051 (0.949)	0.041 (0.951)	0.048 (0.951)
		(10,15)	0.060 (0.950)	0.051 (0.948)	0.049 (0.951)	0.048 (0.951)	0.040 (0.952)	0.046 (0.952)
	(1,3,5,7,10)	(5,5)	0.077 (0.943)	0.056 (0.945)	0.051 (0.950)	0.050 (0.950)	0.034 (0.956)	0.045 (0.954)
		(5,10)	0.069 (0.947)	0.055 (0.947)	0.052 (0.950)	0.051 (0.951)	0.039 (0.955)	0.049 (0.953)
		(10,10)	0.063 (0.948)	0.054 (0.947)	0.052 (0.950)	0.051 (0.950)	0.042 (0.952)	0.049 (0.952)
		(10,15)	0.059 (0.957)	0.051 (0.956)	0.049 (0.957)	0.049 (0.958)	0.041 (0.958)	0.047 (0.960)
0.5	(0.5,0.5,0.5,0.5,0.5)	(5,5)	0.095 (0.953)	0.060 (0.944)	0.054 (0.949)	0.053 (0.949)	0.030 (0.944)	0.047 (0.952)
		(5,10)	0.076 (0.953)	0.053 (0.946)	0.050 (0.949)	0.049 (0.949)	0.033 (0.944)	0.047 (0.953)
		(10,10)	0.066 (0.955)	0.048 (0.946)	0.046 (0.949)	0.046 (0.950)	0.030 (0.942)	0.041 (0.950)
		(10,15)	0.067 (0.956)	0.049 (0.948)	0.047 (0.950)	0.047 (0.951)	0.034 (0.945)	0.045 (0.952)
	(0.1,0.3,0.5,0.7,1.0)	(5,5)	0.089 (0.955)	0.052 (0.949)	0.046 (0.952)	0.045 (0.953)	0.026 (0.948)	0.040 (0.953)
		(5,10)	0.074 (0.954)	0.051 (0.945)	0.048 (0.948)	0.047 (0.949)	0.031 (0.943)	0.045 (0.954)
		(10,10)	0.075 (0.955)	0.054 (0.946)	0.051 (0.948)	0.051 (0.949)	0.035 (0.943)	0.047 (0.950)
		(10,15)	0.066 (0.958)	0.051 (0.951)	0.048 (0.952)	0.048 (0.953)	0.036 (0.946)	0.045 (0.953)
	(1,1,1,1,1)	(5,5)	0.091 (0.954)	0.057 (0.945)	0.052 (0.949)	0.051 (0.950)	0.026 (0.945)	0.043 (0.950)
		(5,10)	0.074 (0.956)	0.053 (0.947)	0.049 (0.950)	0.048 (0.950)	0.033 (0.944)	0.045 (0.953)
		(10,10)	0.078 (0.957)	0.056 (0.947)	0.053 (0.949)	0.053 (0.950)	0.038 (0.945)	0.051 (0.952)
		(10,15)	0.072 (0.956)	0.055 (0.949)	0.053 (0.950)	0.052 (0.951)	0.039 (0.945)	0.049 (0.951)
	(1,3,5,7,10)	(5,5)	0.089 (0.951)	0.057 (0.944)	0.049 (0.947)	0.048 (0.948)	0.029 (0.943)	0.045 (0.950)
		(5,10)	0.076 (0.956)	0.055 (0.947)	0.051 (0.950)	0.051 (0.951)	0.031 (0.945)	0.048 (0.955)
		(10,10)	0.070 (0.956)	0.051 (0.947)	0.047 (0.949)	0.047 (0.949)	0.033 (0.943)	0.042 (0.950)
		(10,15)	0.069 (0.956)	0.054 (0.947)	0.052 (0.950)	0.051 (0.950)	0.038 (0.943)	0.047 (0.952)
1.0	(0.5,0.5,0.5,0.5,0.5)	(5,5)	0.120 (0.971)	0.056 (0.951)	0.050 (0.954)	0.049 (0.954)	0.020 (0.925)	0.042 (0.950)
		(5,10)	0.093 (0.969)	0.048 (0.947)	0.045 (0.949)	0.044 (0.949)	0.021 (0.927)	0.041 (0.954)
		(10,10)	0.087 (0.964)	0.052 (0.950)	0.050 (0.951)	0.049 (0.951)	0.029 (0.930)	0.042 (0.951)
		(10,15)	0.081 (0.965)	0.050 (0.948)	0.049 (0.948)	0.048 (0.949)	0.030 (0.927)	0.045 (0.948)
	(0.1,0.3,0.5,0.7,1.0)	(5,5)	0.119 (0.969)	0.059 (0.946)	0.053 (0.949)	0.052 (0.950)	0.022 (0.921)	0.044 (0.948)
		(5,10)	0.097 (0.967)	0.057 (0.948)	0.052 (0.950)	0.051 (0.950)	0.026 (0.928)	0.050 (0.955)
		(10,10)	0.089 (0.967)	0.055 (0.949)	0.051 (0.950)	0.051 (0.951)	0.027 (0.932)	0.045 (0.950)
		(10,15)	0.080 (0.966)	0.053 (0.952)	0.052 (0.953)	0.051 (0.953)	0.032 (0.934)	0.046 (0.953)
	(1,1,1,1,1)	(5,5)	0.113 (0.968)	0.053 (0.950)	0.047 (0.953)	0.045 (0.954)	0.018 (0.927)	0.038 (0.952)
		(5,10)	0.097 (0.967)	0.052 (0.950)	0.049 (0.951)	0.048 (0.952)	0.025 (0.929)	0.045 (0.955)
		(10,10)	0.088 (0.965)	0.055 (0.951)	0.052 (0.952)	0.051 (0.952)	0.029 (0.932)	0.047 (0.953)
		(10,15)	0.082 (0.964)	0.051 (0.950)	0.050 (0.950)	0.049 (0.950)	0.031 (0.931)	0.046 (0.950)
	(1,3,5,7,10)	(5,5)	0.117 (0.968)	0.059 (0.948)	0.053 (0.950)	0.051 (0.951)	0.022 (0.923)	0.044 (0.948)
		(5,10)	0.102 (0.967)	0.058 (0.947)	0.053 (0.949)	0.052 (0.949)	0.028 (0.927)	0.050 (0.954)
		(10,10)	0.089 (0.964)	0.054 (0.947)	0.050 (0.948)	0.050 (0.948)	0.028 (0.929)	0.044 (0.949)
		(10,15)	0.083 (0.961)	0.054 (0.947)	0.052 (0.948)	0.051 (0.948)	0.031 (0.931)	0.047 (0.949)
2.0	(0.5,0.5,0.5,0.5,0.5)	(5,5)	0.179 (0.983)	0.058 (0.953)	0.053 (0.952)	0.051 (0.951)	0.014 (0.867)	0.040 (0.944)
		(5,10)	0.131 (0.976)	0.055 (0.947)	0.051 (0.947)	0.051 (0.947)	0.019 (0.892)	0.046 (0.951)
		(10,10)	0.115 (0.980)	0.050 (0.954)	0.048 (0.954)	0.048 (0.954)	0.020 (0.900)	0.040 (0.949)
		(10,15)	0.112 (0.976)	0.051 (0.950)	0.049 (0.951)	0.048 (0.950)	0.022 (0.905)	0.042 (0.947)
	(0.1,0.3,0.5,0.7,1.0)	(5,5)	0.171 (0.982)	0.056 (0.949)	0.049 (0.949)	0.048 (0.948)	0.013 (0.863)	0.037 (0.939)
		(5,10)	0.137 (0.977)	0.057 (0.947)	0.055 (0.948)	0.054 (0.947)	0.021 (0.892)	0.049 (0.952)
		(10,10)	0.119 (0.980)	0.051 (0.953)	0.050 (0.953)	0.050 (0.953)	0.021 (0.899)	0.041 (0.950)
		(10,15)	0.107 (0.976)	0.052 (0.955)	0.050 (0.954)	0.050 (0.954)	0.023 (0.913)	0.044 (0.952)
	(1,1,1,1,1)	(5,5)	0.172 (0.985)	0.055 (0.953)	0.051 (0.952)	0.050 (0.952)	0.016 (0.860)	0.040 (0.941)
		(5,10)	0.130 (0.977)	0.054 (0.950)	0.051 (0.950)	0.050 (0.950)	0.015 (0.892)	0.048 (0.951)
		(10,10)	0.117 (0.979)	0.052 (0.951)	0.050 (0.952)	0.050 (0.951)	0.020 (0.899)	0.043 (0.949)
		(10,15)	0.109 (0.974)	0.051 (0.949)	0.050 (0.949)	0.049 (0.949)	0.024 (0.906)	0.045 (0.948)
	(1,3,5,7,10)	(5,5)	0.176 (0.983)	0.052 (0.948)	0.046 (0.948)	0.045 (0.948)	0.012 (0.860)	0.035 (0.945)
		(5,10)	0.141 (0.980)	0.054 (0.951)	0.051 (0.951)	0.051 (0.950)	0.018 (0.890)	0.049 (0.952)
		(10,10)	0.121 (0.977)	0.053 (0.951)	0.051 (0.951)	0.050 (0.951)	0.021 (0.901)	0.045 (0.946)
		(10,15)	0.109 (0.976)	0.051 (0.954)	0.050 (0.955)	0.049 (0.954)	0.022 (0.912)	0.044 (0.953)

Table 3. Frequentist coverage probability of 0.05 (0.95) quantiles of θ_1 when $k = 10$.

θ_1	$(\sigma_1, \ldots, \sigma_k)$	$(n_i, m_i))$	π_1	π_2	π_3	π_m	Hedges	Bonett
0.0	(0.5,0.5,. . .,0.5)	(5,5)	0.067 (0.937)	0.058 (0.946)	0.053 (0.951)	0.052 (0.951)	0.038 (0.962)	0.046 (0.954)
		(5,10)	0.057 (0.943)	0.050 (0.949)	0.047 (0.953)	0.047 (0.954)	0.040 (0.960)	0.046 (0.955)
		(10,10)	0.060 (0.942)	0.055 (0.947)	0.052 (0.949)	0.052 (0.949)	0.046 (0.955)	0.051 (0.952)
		(10,15)	0.053 (0.942)	0.050 (0.945)	0.049 (0.948)	0.049 (0.947)	0.045 (0.952)	0.047 (0.950)
	(0.1,0.2,. . .,1.0)	(5,5)	0.063 (0.934)	0.056 (0.944)	0.052 (0.948)	0.051 (0.949)	0.039 (0.962)	0.047 (0.952)
		(5,10)	0.059 (0.938)	0.053 (0.944)	0.050 (0.947)	0.049 (0.948)	0.043 (0.956)	0.048 (0.950)
		(10,10)	0.055 (0.943)	0.051 (0.948)	0.049 (0.950)	0.049 (0.950)	0.044 (0.956)	0.047 (0.951)
		(10,15)	0.052 (0.946)	0.049 (0.952)	0.048 (0.953)	0.047 (0.953)	0.043 (0.958)	0.045 (0.954)
	(1,1,. . .,1)	(5,5)	0.065 (0.934)	0.055 (0.943)	0.049 (0.948)	0.049 (0.948)	0.036 (0.960)	0.046 (0.952)
		(5,10)	0.059 (0.944)	0.054 (0.949)	0.051 (0.952)	0.051 (0.952)	0.042 (0.959)	0.048 (0.956)
		(10,10)	0.056 (0.944)	0.051 (0.950)	0.049 (0.952)	0.049 (0.951)	0.042 (0.958)	0.045 (0.954)
		(10,15)	0.052 (0.944)	0.049 (0.948)	0.048 (0.950)	0.048 (0.950)	0.043 (0.955)	0.046 (0.953)
	(1,2,. . .,10)	(5,5)	0.065 (0.933)	0.056 (0.941)	0.051 (0.947)	0.051 (0.947)	0.039 (0.958)	0.048 (0.949)
		(5,10)	0.061 (0.944)	0.055 (0.949)	0.051 (0.952)	0.051 (0.953)	0.043 (0.959)	0.049 (0.956)
		(10,10)	0.049 (0.945)	0.045 (0.950)	0.042 (0.953)	0.042 (0.953)	0.037 (0.958)	0.039 (0.954)
		(10,15)	0.056 (0.948)	0.052 (0.951)	0.050 (0.952)	0.050 (0.952)	0.045 (0.957)	0.048 (0.955)
0.2	(0.5,0.5,. . .,0.5)	(5,5)	0.077 (0.945)	0.057 (0.943)	0.051 (0.947)	0.049 (0.947)	0.030 (0.950)	0.048 (0.951)
		(5,10)	0.074 (0.951)	0.058 (0.949)	0.055 (0.952)	0.055 (0.952)	0.040 (0.952)	0.051 (0.954)
		(10,10)	0.065 (0.950)	0.051 (0.945)	0.048 (0.949)	0.047 (0.949)	0.035 (0.948)	0.044 (0.950)
		(10,15)	0.066 (0.952)	0.053 (0.950)	0.051 (0.952)	0.051 (0.952)	0.042 (0.951)	0.049 (0.952)
	(0.1,0.2,. . .,1.0)	(5,5)	0.083 (0.947)	0.060 (0.944)	0.053 (0.949)	0.052 (0.950)	0.032 (0.952)	0.048 (0.952)
		(5,10)	0.068 (0.953)	0.054 (0.951)	0.051 (0.954)	0.050 (0.954)	0.038 (0.954)	0.050 (0.955)
		(10,10)	0.065 (0.947)	0.052 (0.944)	0.050 (0.946)	0.049 (0.946)	0.038 (0.945)	0.047 (0.948)
		(10,15)	0.064 (0.949)	0.053 (0.946)	0.051 (0.947)	0.051 (0.948)	0.040 (0.946)	0.049 (0.949)
	(1,1,. . .,1)	(5,5)	0.079 (0.946)	0.055 (0.945)	0.049 (0.949)	0.049 (0.950)	0.030 (0.952)	0.045 (0.952)
		(5,10)	0.068 (0.947)	0.052 (0.944)	0.049 (0.948)	0.049 (0.949)	0.035 (0.949)	0.045 (0.951)
		(10,10)	0.067 (0.949)	0.054 (0.947)	0.051 (0.949)	0.051 (0.949)	0.038 (0.949)	0.046 (0.951)
		(10,15)	0.058 (0.953)	0.049 (0.950)	0.047 (0.952)	0.047 (0.952)	0.036 (0.951)	0.044 (0.952)
	(1,2,. . .,10)	(5,5)	0.078 (0.949)	0.054 (0.947)	0.049 (0.953)	0.049 (0.953)	0.029 (0.956)	0.044 (0.954)
		(5,10)	0.067 (0.950)	0.052 (0.948)	0.049 (0.951)	0.048 (0.952)	0.035 (0.952)	0.047 (0.954)
		(10,10)	0.066 (0.950)	0.051 (0.947)	0.049 (0.949)	0.048 (0.950)	0.036 (0.948)	0.046 (0.953)
		(10,15)	0.061 (0.950)	0.050 (0.947)	0.047 (0.948)	0.047 (0.948)	0.037 (0.947)	0.047 (0.950)
0.5	(0.5,0.5,. . .,0.5)	(5,5)	0.101 (0.963)	0.058 (0.950)	0.051 (0.953)	0.050 (0.954)	0.023 (0.937)	0.043 (0.955)
		(5,10)	0.091 (0.962)	0.059 (0.951)	0.054 (0.953)	0.053 (0.953)	0.032 (0.940)	0.053 (0.956)
		(10,10)	0.081 (0.963)	0.052 (0.951)	0.050 (0.953)	0.050 (0.953)	0.031 (0.941)	0.047 (0.955)
		(10,15)	0.079 (0.960)	0.054 (0.948)	0.051 (0.950)	0.051 (0.950)	0.035 (0.938)	0.048 (0.951)
	(0.1,0.2,. . .,1.0)	(5,5)	0.097 (0.959)	0.056 (0.944)	0.051 (0.947)	0.050 (0.948)	0.023 (0.934)	0.043 (0.949)
		(5,10)	0.085 (0.962)	0.055 (0.949)	0.051 (0.952)	0.051 (0.952)	0.029 (0.937)	0.049 (0.955)
		(10,10)	0.076 (0.963)	0.050 (0.950)	0.048 (0.952)	0.048 (0.952)	0.031 (0.938)	0.044 (0.952)
		(10,15)	0.069 (0.957)	0.049 (0.947)	0.047 (0.948)	0.047 (0.948)	0.032 (0.937)	0.044 (0.949)
	(1,1,. . .,1)	(5,5)	0.104 (0.960)	0.058 (0.942)	0.052 (0.947)	0.051 (0.947)	0.023 (0.930)	0.045 (0.949)
		(5,10)	0.085 (0.959)	0.054 (0.946)	0.051 (0.948)	0.050 (0.948)	0.028 (0.936)	0.047 (0.953)
		(10,10)	0.078 (0.961)	0.054 (0.948)	0.052 (0.950)	0.052 (0.950)	0.031 (0.938)	0.048 (0.951)
		(10,15)	0.072 (0.961)	0.050 (0.949)	0.047 (0.951)	0.048 (0.951)	0.029 (0.940)	0.045 (0.953)
	(1,2,. . .,10)	(5,5)	0.099 (0.959)	0.055 (0.946)	0.049 (0.949)	0.049 (0.949)	0.022 (0.936)	0.046 (0.951)
		(5,10)	0.087 (0.961)	0.054 (0.948)	0.051 (0.950)	0.051 (0.950)	0.029 (0.939)	0.046 (0.955)
		(10,10)	0.080 (0.962)	0.053 (0.949)	0.051 (0.951)	0.051 (0.952)	0.031 (0.939)	0.047 (0.953)
		(10,15)	0.077 (0.957)	0.053 (0.947)	0.051 (0.948)	0.051 (0.948)	0.035 (0.938)	0.047 (0.950)
1.0	(0.5,0.5,. . .,0.5)	(5,5)	0.141 (0.975)	0.054 (0.944)	0.047 (0.946)	0.046 (0.946)	0.011 (0.891)	0.042 (0.944)
		(5,10)	0.115 (0.973)	0.054 (0.948)	0.050 (0.950)	0.049 (0.950)	0.019 (0.908)	0.050 (0.954)
		(10,10)	0.111 (0.974)	0.055 (0.949)	0.051 (0.950)	0.051 (0.950)	0.023 (0.914)	0.048 (0.951)
		(10,15)	0.100 (0.973)	0.055 (0.953)	0.054 (0.953)	0.053 (0.953)	0.028 (0.921)	0.051 (0.954)
	(0.1,0.2,. . .,1.0)	(5,5)	0.146 (0.975)	0.055 (0.946)	0.049 (0.947)	0.049 (0.948)	0.012 (0.893)	0.044 (0.945)
		(5,10)	0.120 (0.976)	0.058 (0.948)	0.055 (0.950)	0.055 (0.950)	0.019 (0.914)	0.052 (0.956)
		(10,10)	0.107 (0.971)	0.054 (0.952)	0.051 (0.953)	0.050 (0.953)	0.024 (0.918)	0.047 (0.953)
		(10,15)	0.101 (0.973)	0.052 (0.952)	0.051 (0.953)	0.051 (0.954)	0.026 (0.921)	0.051 (0.954)
	(1,1,. . .,1)	(5,5)	0.152 (0.978)	0.059 (0.948)	0.051 (0.950)	0.051 (0.950)	0.013 (0.894)	0.044 (0.948)
		(5,10)	0.116 (0.973)	0.056 (0.948)	0.051 (0.949)	0.051 (0.949)	0.020 (0.910)	0.050 (0.954)
		(10,10)	0.104 (0.973)	0.052 (0.948)	0.048 (0.949)	0.048 (0.950)	0.023 (0.913)	0.043 (0.950)
		(10,15)	0.097 (0.971)	0.052 (0.951)	0.050 (0.952)	0.049 (0.952)	0.024 (0.923)	0.048 (0.953)
	(1,2,. . .,10)	(5,5)	0.153 (0.980)	0.056 (0.952)	0.049 (0.953)	0.049 (0.954)	0.012 (0.901)	0.044 (0.950)
		(5,10)	0.115 (0.972)	0.052 (0.949)	0.048 (0.950)	0.048 (0.950)	0.019 (0.914)	0.048 (0.954)
		(10,10)	0.109 (0.973)	0.053 (0.951)	0.051 (0.953)	0.050 (0.953)	0.024 (0.916)	0.047 (0.953)
		(10,15)	0.094 (0.969)	0.049 (0.951)	0.047 (0.952)	0.047 (0.951)	0.025 (0.923)	0.047 (0.950)
2.0	(0.5,0.5,. . .,0.5)	(5,5)	0.248 (0.991)	0.062 (0.948)	0.054 (0.947)	0.053 (0.946)	0.007 (0.787)	0.047 (0.938)
		(5,10)	0.178 (0.988)	0.053 (0.952)	0.049 (0.951)	0.049 (0.952)	0.011 (0.846)	0.051 (0.955)
		(10,10)	0.152 (0.985)	0.051 (0.950)	0.049 (0.950)	0.048 (0.949)	0.015 (0.858)	0.044 (0.949)
		(10,15)	0.140 (0.983)	0.055 (0.950)	0.053 (0.950)	0.052 (0.950)	0.018 (0.878)	0.051 (0.951)
	(0.1,0.2,. . .,1.0)	(5,5)	0.240 (0.991)	0.058 (0.953)	0.050 (0.952)	0.049 (0.952)	0.007 (0.790)	0.047 (0.943)
		(5,10)	0.177 (0.985)	0.052 (0.949)	0.049 (0.949)	0.049 (0.949)	0.012 (0.847)	0.055 (0.954)
		(10,10)	0.159 (0.984)	0.053 (0.947)	0.051 (0.946)	0.050 (0.947)	0.014 (0.855)	0.049 (0.947)
		(10,15)	0.137 (0.981)	0.050 (0.947)	0.048 (0.947)	0.047 (0.947)	0.017 (0.875)	0.049 (0.949)
	(1,1,. . .,1)	(5,5)	0.242 (0.992)	0.055 (0.952)	0.050 (0.950)	0.049 (0.950)	0.006 (0.786)	0.042 (0.941)
		(5,10)	0.178 (0.987)	0.055 (0.949)	0.051 (0.949)	0.051 (0.948)	0.011 (0.849)	0.051 (0.957)
		(10,10)	0.158 (0.985)	0.050 (0.947)	0.048 (0.947)	0.047 (0.947)	0.013 (0.856)	0.044 (0.943)
		(10,15)	0.137 (0.985)	0.050 (0.953)	0.049 (0.953)	0.049 (0.953)	0.016 (0.877)	0.048 (0.953)
	(1,2,. . .,10)	(5,5)	0.238 (0.992)	0.055 (0.953)	0.047 (0.952)	0.046 (0.951)	0.005 (0.796)	0.042 (0.944)
		(5,10)	0.175 (0.988)	0.053 (0.953)	0.049 (0.952)	0.049 (0.952)	0.011 (0.852)	0.054 (0.956)
		(10,10)	0.157 (0.984)	0.053 (0.947)	0.050 (0.947)	0.049 (0.947)	0.014 (0.855)	0.045 (0.945)
		(10,15)	0.139 (0.983)	0.052 (0.947)	0.050 (0.947)	0.050 (0.947)	0.015 (0.876)	0.047 (0.949)

Table 4. Frequentist coverage probability of 90% (95%) credible intervals of θ_1 when $k = 3$.

θ_1	$(\sigma_1, \dots, \sigma_k)$	$(n_i, m_i))$	π_1	π_2	π_3	π_m	Hedges	Bonett
0.0	(0.5,0.5,0.5)	(5,5)	0.863 (0.925)	0.882 (0.939)	0.894 (0.947)	0.900 (0.951)	0.920 (0.964)	0.910 (0.961)
		(5,10)	0.878 (0.936)	0.891 (0.945)	0.898 (0.949)	0.900 (0.951)	0.917 (0.960)	0.908 (0.955)
		(10,10)	0.883 (0.939)	0.892 (0.946)	0.896 (0.950)	0.898 (0.951)	0.909 (0.957)	0.906 (0.954)
		(10,15)	0.887 (0.940)	0.893 (0.946)	0.898 (0.950)	0.899 (0.950)	0.908 (0.957)	0.904 (0.955)
	(0.1,0.5,1.0)	(5,5)	0.864 (0.925)	0.883 (0.940)	0.895 (0.949)	0.900 (0.951)	0.921 (0.965)	0.913 (0.958)
		(5,10)	0.874 (0.932)	0.886 (0.943)	0.891 (0.948)	0.894 (0.950)	0.909 (0.958)	0.902 (0.953)
		(10,10)	0.880 (0.938)	0.889 (0.943)	0.893 (0.946)	0.896 (0.947)	0.907 (0.955)	0.904 (0.951)
		(10,15)	0.888 (0.943)	0.895 (0.948)	0.899 (0.949)	0.899 (0.950)	0.907 (0.955)	0.904 (0.954)
	(1,1,1)	(5,5)	0.868 (0.928)	0.887 (0.943)	0.898 (0.950)	0.902 (0.953)	0.923 (0.965)	0.914 (0.960)
		(5,10)	0.875 (0.936)	0.889 (0.945)	0.895 (0.949)	0.897 (0.950)	0.913 (0.961)	0.906 (0.954)
		(10,10)	0.883 (0.939)	0.892 (0.945)	0.897 (0.948)	0.898 (0.949)	0.910 (0.956)	0.906 (0.954)
		(10,15)	0.888 (0.943)	0.894 (0.949)	0.898 (0.951)	0.900 (0.951)	0.908 (0.957)	0.907 (0.956)
	(1,5,10)	(5,5)	0.869 (0.927)	0.886 (0.940)	0.897 (0.948)	0.902 (0.951)	0.921 (0.965)	0.911 (0.958)
		(5,10)	0.878 (0.934)	0.891 (0.942)	0.899 (0.947)	0.901 (0.949)	0.914 (0.960)	0.907 (0.955)
		(10,10)	0.884 (0.938)	0.892 (0.945)	0.896 (0.948)	0.898 (0.950)	0.909 (0.957)	0.905 (0.954)
		(10,15)	0.887 (0.941)	0.895 (0.946)	0.899 (0.948)	0.901 (0.949)	0.909 (0.956)	0.905 (0.954)
0.2	(0.5,0.5,0.5)	(5,5)	0.862 (0.923)	0.883 (0.937)	0.894 (0.946)	0.898 (0.948)	0.917 (0.961)	0.907 (0.955)
		(5,10)	0.884 (0.938)	0.895 (0.947)	0.900 (0.951)	0.903 (0.953)	0.916 (0.963)	0.910 (0.956)
		(10,10)	0.885 (0.939)	0.895 (0.947)	0.900 (0.950)	0.901 (0.951)	0.911 (0.957)	0.907 (0.955)
		(10,15)	0.893 (0.945)	0.901 (0.949)	0.903 (0.951)	0.904 (0.952)	0.913 (0.956)	0.910 (0.954)
	(0.1,0.5,1.0)	(5,5)	0.872 (0.929)	0.890 (0.943)	0.901 (0.952)	0.904 (0.955)	0.924 (0.967)	0.917 (0.964)
		(5,10)	0.878 (0.934)	0.890 (0.942)	0.897 (0.946)	0.900 (0.948)	0.913 (0.955)	0.905 (0.952)
		(10,10)	0.886 (0.942)	0.896 (0.948)	0.900 (0.952)	0.902 (0.952)	0.912 (0.959)	0.907 (0.956)
		(10,15)	0.884 (0.941)	0.892 (0.946)	0.896 (0.949)	0.898 (0.950)	0.906 (0.955)	0.902 (0.952)
	(1,1,1)	(5,5)	0.867 (0.928)	0.884 (0.941)	0.896 (0.949)	0.899 (0.952)	0.921 (0.964)	0.912 (0.959)
		(5,10)	0.880 (0.934)	0.890 (0.943)	0.897 (0.948)	0.900 (0.949)	0.913 (0.958)	0.906 (0.952)
		(10,10)	0.889 (0.943)	0.898 (0.950)	0.903 (0.952)	0.904 (0.953)	0.914 (0.959)	0.911 (0.958)
		(10,15)	0.883 (0.938)	0.890 (0.945)	0.894 (0.947)	0.895 (0.948)	0.905 (0.955)	0.902 (0.952)
	(1,5,10)	(5,5)	0.859 (0.922)	0.880 (0.937)	0.892 (0.944)	0.897 (0.947)	0.917 (0.960)	0.907 (0.953)
		(5,10)	0.874 (0.933)	0.888 (0.942)	0.895 (0.946)	0.896 (0.947)	0.911 (0.958)	0.906 (0.954)
		(10,10)	0.883 (0.938)	0.891 (0.943)	0.897 (0.948)	0.899 (0.949)	0.910 (0.956)	0.906 (0.953)
		(10,15)	0.886 (0.941)	0.895 (0.946)	0.899 (0.949)	0.900 (0.949)	0.910 (0.955)	0.906 (0.952)
0.5	(0.5,0.5,0.5)	(5,5)	0.859 (0.921)	0.881 (0.936)	0.892 (0.945)	0.895 (0.949)	0.913 (0.960)	0.907 (0.956)
		(5,10)	0.882 (0.939)	0.894 (0.949)	0.901 (0.953)	0.904 (0.954)	0.915 (0.962)	0.911 (0.958)
		(10,10)	0.889 (0.942)	0.898 (0.947)	0.903 (0.950)	0.906 (0.951)	0.913 (0.957)	0.910 (0.955)
		(10,15)	0.891 (0.940)	0.900 (0.946)	0.903 (0.948)	0.904 (0.950)	0.911 (0.954)	0.908 (0.954)
	(0.1,0.5,1.0)	(5,5)	0.865 (0.921)	0.887 (0.938)	0.897 (0.945)	0.900 (0.948)	0.918 (0.959)	0.913 (0.956)
		(5,10)	0.883 (0.939)	0.895 (0.950)	0.902 (0.953)	0.904 (0.955)	0.916 (0.962)	0.912 (0.958)
		(10,10)	0.886 (0.943)	0.898 (0.950)	0.902 (0.952)	0.904 (0.952)	0.913 (0.958)	0.910 (0.957)
		(10,15)	0.887 (0.940)	0.895 (0.947)	0.897 (0.949)	0.899 (0.950)	0.907 (0.954)	0.905 (0.951)
	(1,1,1)	(5,5)	0.864 (0.922)	0.885 (0.940)	0.894 (0.947)	0.898 (0.950)	0.917 (0.961)	0.910 (0.958)
		(5,10)	0.879 (0.933)	0.893 (0.942)	0.898 (0.947)	0.900 (0.949)	0.913 (0.956)	0.907 (0.956)
		(10,10)	0.885 (0.938)	0.894 (0.944)	0.899 (0.947)	0.902 (0.948)	0.908 (0.954)	0.907 (0.952)
		(10,15)	0.890 (0.942)	0.899 (0.948)	0.902 (0.951)	0.902 (0.951)	0.909 (0.955)	0.906 (0.954)
	(1,5,10)	(5,5)	0.864 (0.923)	0.888 (0.939)	0.899 (0.946)	0.901 (0.949)	0.919 (0.960)	0.912 (0.957)
		(5,10)	0.875 (0.932)	0.889 (0.942)	0.895 (0.946)	0.897 (0.949)	0.910 (0.956)	0.902 (0.953)
		(10,10)	0.886 (0.939)	0.897 (0.944)	0.901 (0.948)	0.903 (0.949)	0.911 (0.956)	0.908 (0.954)
		(10,15)	0.885 (0.939)	0.891 (0.943)	0.896 (0.946)	0.898 (0.947)	0.904 (0.954)	0.902 (0.953)
1.0	(0.5,0.5,0.5)	(5,5)	0.857 (0.920)	0.886 (0.942)	0.896 (0.947)	0.900 (0.950)	0.907 (0.954)	0.909 (0.957)
		(5,10)	0.875 (0.933)	0.894 (0.946)	0.900 (0.950)	0.903 (0.951)	0.906 (0.955)	0.909 (0.957)
		(10,10)	0.881 (0.939)	0.897 (0.948)	0.901 (0.950)	0.901 (0.951)	0.904 (0.953)	0.907 (0.955)
		(10,15)	0.881 (0.937)	0.892 (0.943)	0.895 (0.946)	0.897 (0.946)	0.899 (0.949)	0.901 (0.950)
	(0.1,0.5,1.0)	(5,5)	0.853 (0.913)	0.882 (0.939)	0.891 (0.946)	0.896 (0.948)	0.903 (0.954)	0.906 (0.955)
		(5,10)	0.871 (0.928)	0.889 (0.941)	0.893 (0.945)	0.895 (0.947)	0.902 (0.951)	0.901 (0.951)
		(10,10)	0.881 (0.935)	0.893 (0.946)	0.896 (0.948)	0.897 (0.948)	0.902 (0.951)	0.903 (0.953)
		(10,15)	0.883 (0.940)	0.896 (0.949)	0.899 (0.951)	0.899 (0.952)	0.905 (0.953)	0.904 (0.955)
	(1,1,1)	(5,5)	0.856 (0.919)	0.887 (0.943)	0.896 (0.949)	0.900 (0.952)	0.908 (0.959)	0.911 (0.960)
		(5,10)	0.868 (0.925)	0.889 (0.941)	0.895 (0.944)	0.896 (0.946)	0.904 (0.950)	0.906 (0.952)
		(10,10)	0.882 (0.937)	0.898 (0.946)	0.902 (0.949)	0.903 (0.950)	0.905 (0.955)	0.907 (0.956)
		(10,15)	0.884 (0.937)	0.894 (0.945)	0.897 (0.947)	0.899 (0.947)	0.902 (0.951)	0.902 (0.954)
	(1,5,10)	(5,5)	0.856 (0.919)	0.886 (0.942)	0.895 (0.947)	0.899 (0.949)	0.907 (0.955)	0.908 (0.959)
		(5,10)	0.871 (0.930)	0.885 (0.944)	0.891 (0.948)	0.893 (0.948)	0.900 (0.952)	0.902 (0.953)
		(10,10)	0.879 (0.938)	0.896 (0.949)	0.901 (0.951)	0.902 (0.952)	0.906 (0.954)	0.907 (0.955)
		(10,15)	0.882 (0.938)	0.898 (0.947)	0.900 (0.950)	0.901 (0.950)	0.906 (0.952)	0.906 (0.954)
2.0	(0.5,0.5,0.5)	(5,5)	0.828 (0.905)	0.896 (0.946)	0.900 (0.949)	0.902 (0.951)	0.879 (0.936)	0.909 (0.955)
		(5,10)	0.857 (0.920)	0.894 (0.946)	0.898 (0.949)	0.898 (0.950)	0.886 (0.942)	0.908 (0.956)
		(10,10)	0.871 (0.932)	0.901 (0.950)	0.904 (0.951)	0.904 (0.951)	0.885 (0.941)	0.904 (0.955)
		(10,15)	0.871 (0.932)	0.898 (0.946)	0.900 (0.947)	0.900 (0.948)	0.890 (0.945)	0.902 (0.950)
	(0.1,0.5,1.0)	(5,5)	0.830 (0.899)	0.889 (0.942)	0.893 (0.946)	0.895 (0.947)	0.873 (0.930)	0.903 (0.952)
		(5,10)	0.861 (0.924)	0.896 (0.947)	0.900 (0.950)	0.899 (0.950)	0.887 (0.941)	0.908 (0.954)
		(10,10)	0.872 (0.932)	0.896 (0.947)	0.898 (0.947)	0.900 (0.948)	0.892 (0.940)	0.904 (0.952)
		(10,15)	0.867 (0.926)	0.892 (0.946)	0.893 (0.946)	0.894 (0.946)	0.890 (0.944)	0.903 (0.952)
	(1,1,1)	(5,5)	0.833 (0.904)	0.892 (0.943)	0.897 (0.947)	0.898 (0.948)	0.866 (0.928)	0.899 (0.951)
		(5,10)	0.858 (0.920)	0.890 (0.945)	0.895 (0.947)	0.895 (0.948)	0.881 (0.936)	0.904 (0.950)
		(10,10)	0.867 (0.929)	0.899 (0.949)	0.901 (0.951)	0.902 (0.952)	0.889 (0.944)	0.903 (0.953)
		(10,15)	0.876 (0.935)	0.898 (0.947)	0.900 (0.948)	0.901 (0.948)	0.888 (0.941)	0.905 (0.951)
	(1,5,10)	(5,5)	0.835 (0.902)	0.896 (0.947)	0.901 (0.950)	0.903 (0.951)	0.873 (0.934)	0.908 (0.956)
		(5,10)	0.859 (0.920)	0.892 (0.947)	0.896 (0.950)	0.897 (0.950)	0.885 (0.938)	0.904 (0.953)
		(10,10)	0.869 (0.928)	0.895 (0.947)	0.897 (0.948)	0.898 (0.949)	0.886 (0.942)	0.903 (0.952)
		(10,15)	0.870 (0.927)	0.892 (0.943)	0.893 (0.944)	0.894 (0.944)	0.884 (0.941)	0.899 (0.949)

Table 5. Frequentist coverage probability of 90% (95%) credible intervals of θ_1 when $k = 5$.

θ_1	$(\sigma_1,\ldots,\sigma_k)$	$(n_i, m_i))$	π_1	π_2	π_3	π_m	Hedges	Bonett
0.0	(0.5,0.5,0.5,0.5,0.5)	(5,5)	0.867 (0.927)	0.885 (0.940)	0.896 (0.948)	0.899 (0.948)	0.922 (0.964)	0.907 (0.956)
		(5,10)	0.882 (0.936)	0.894 (0.946)	0.901 (0.950)	0.903 (0.951)	0.918 (0.960)	0.906 (0.954)
		(10,10)	0.892 (0.942)	0.900 (0.948)	0.905 (0.952)	0.905 (0.953)	0.915 (0.961)	0.910 (0.956)
		(10,15)	0.885 (0.939)	0.894 (0.944)	0.897 (0.947)	0.899 (0.946)	0.908 (0.952)	0.903 (0.949)
	(0.1,0.3,0.5,0.7,1.0)	(5,5)	0.869 (0.929)	0.889 (0.940)	0.900 (0.948)	0.902 (0.948)	0.925 (0.965)	0.910 (0.957)
		(5,10)	0.878 (0.940)	0.890 (0.948)	0.897 (0.953)	0.897 (0.953)	0.916 (0.962)	0.909 (0.956)
		(10,10)	0.887 (0.942)	0.895 (0.949)	0.899 (0.953)	0.902 (0.953)	0.912 (0.961)	0.906 (0.957)
		(10,15)	0.891 (0.946)	0.898 (0.950)	0.901 (0.953)	0.902 (0.954)	0.911 (0.959)	0.906 (0.956)
	(1,1,1,1,1)	(5,5)	0.870 (0.927)	0.888 (0.939)	0.899 (0.946)	0.901 (0.948)	0.923 (0.963)	0.908 (0.955)
		(5,10)	0.881 (0.940)	0.895 (0.945)	0.903 (0.950)	0.904 (0.951)	0.920 (0.961)	0.909 (0.955)
		(10,10)	0.883 (0.940)	0.893 (0.946)	0.898 (0.949)	0.899 (0.950)	0.912 (0.959)	0.904 (0.955)
		(10,15)	0.885 (0.939)	0.892 (0.944)	0.894 (0.947)	0.895 (0.947)	0.904 (0.955)	0.899 (0.949)
	(1,3,5,7,10)	(5,5)	0.877 (0.934)	0.895 (0.947)	0.906 (0.953)	0.909 (0.955)	0.930 (0.968)	0.917 (0.961)
		(5,10)	0.882 (0.937)	0.894 (0.944)	0.900 (0.947)	0.902 (0.948)	0.916 (0.961)	0.906 (0.956)
		(10,10)	0.887 (0.940)	0.896 (0.948)	0.900 (0.950)	0.901 (0.951)	0.912 (0.960)	0.905 (0.955)
		(10,15)	0.891 (0.941)	0.896 (0.945)	0.899 (0.948)	0.899 (0.948)	0.908 (0.956)	0.903 (0.951)
0.2	(0.5,0.5,0.5,0.5,0.5)	(5,5)	0.871 (0.928)	0.889 (0.941)	0.901 (0.949)	0.903 (0.950)	0.922 (0.965)	0.911 (0.958)
		(5,10)	0.880 (0.938)	0.893 (0.948)	0.899 (0.952)	0.900 (0.952)	0.915 (0.961)	0.907 (0.955)
		(10,10)	0.883 (0.941)	0.891 (0.948)	0.897 (0.952)	0.898 (0.952)	0.911 (0.959)	0.904 (0.955)
		(10,15)	0.890 (0.943)	0.897 (0.948)	0.901 (0.950)	0.902 (0.951)	0.913 (0.956)	0.906 (0.953)
	(0.1,0.3,0.5,0.7,1.0)	(5,5)	0.865 (0.924)	0.887 (0.942)	0.897 (0.950)	0.898 (0.951)	0.920 (0.965)	0.905 (0.956)
		(5,10)	0.883 (0.936)	0.893 (0.945)	0.899 (0.949)	0.900 (0.950)	0.915 (0.960)	0.908 (0.954)
		(10,10)	0.889 (0.942)	0.899 (0.949)	0.904 (0.953)	0.905 (0.954)	0.915 (0.959)	0.911 (0.956)
		(10,15)	0.889 (0.941)	0.897 (0.947)	0.899 (0.949)	0.900 (0.949)	0.909 (0.956)	0.904 (0.953)
	(1,1,1,1,1)	(5,5)	0.866 (0.926)	0.887 (0.941)	0.897 (0.947)	0.900 (0.948)	0.919 (0.961)	0.908 (0.953)
		(5,10)	0.880 (0.937)	0.894 (0.947)	0.900 (0.953)	0.903 (0.954)	0.917 (0.964)	0.909 (0.956)
		(10,10)	0.884 (0.937)	0.893 (0.943)	0.898 (0.947)	0.898 (0.947)	0.910 (0.954)	0.903 (0.952)
		(10,15)	0.890 (0.943)	0.898 (0.949)	0.902 (0.951)	0.903 (0.952)	0.912 (0.957)	0.906 (0.955)
	(1,3,5,7,10)	(5,5)	0.866 (0.927)	0.888 (0.942)	0.898 (0.950)	0.900 (0.952)	0.923 (0.967)	0.909 (0.958)
		(5,10)	0.878 (0.936)	0.893 (0.945)	0.898 (0.948)	0.900 (0.950)	0.916 (0.958)	0.904 (0.954)
		(10,10)	0.884 (0.937)	0.893 (0.944)	0.897 (0.947)	0.898 (0.948)	0.910 (0.955)	0.903 (0.952)
		(10,15)	0.897 (0.947)	0.904 (0.953)	0.908 (0.956)	0.909 (0.957)	0.917 (0.963)	0.912 (0.958)
0.5	(0.5,0.5,0.5,0.5,0.5)	(5,5)	0.858 (0.921)	0.885 (0.941)	0.894 (0.947)	0.896 (0.949)	0.914 (0.961)	0.905 (0.954)
		(5,10)	0.877 (0.934)	0.893 (0.944)	0.898 (0.948)	0.900 (0.949)	0.911 (0.957)	0.906 (0.953)
		(10,10)	0.889 (0.942)	0.898 (0.950)	0.902 (0.953)	0.904 (0.954)	0.912 (0.957)	0.909 (0.955)
		(10,15)	0.889 (0.941)	0.900 (0.948)	0.903 (0.950)	0.904 (0.951)	0.911 (0.955)	0.907 (0.953)
	(0.1,0.3,0.5,0.7,1.0)	(5,5)	0.867 (0.930)	0.897 (0.945)	0.906 (0.951)	0.907 (0.953)	0.921 (0.964)	0.913 (0.960)
		(5,10)	0.879 (0.935)	0.894 (0.945)	0.900 (0.948)	0.902 (0.949)	0.911 (0.956)	0.908 (0.952)
		(10,10)	0.880 (0.937)	0.892 (0.946)	0.897 (0.949)	0.898 (0.950)	0.908 (0.954)	0.903 (0.952)
		(10,15)	0.892 (0.943)	0.900 (0.950)	0.904 (0.952)	0.904 (0.953)	0.910 (0.956)	0.908 (0.954)
	(1,1,1,1,1)	(5,5)	0.863 (0.923)	0.888 (0.943)	0.897 (0.951)	0.900 (0.952)	0.919 (0.963)	0.907 (0.957)
		(5,10)	0.882 (0.934)	0.894 (0.945)	0.901 (0.949)	0.902 (0.950)	0.911 (0.957)	0.909 (0.953)
		(10,10)	0.879 (0.934)	0.891 (0.944)	0.896 (0.946)	0.897 (0.947)	0.906 (0.954)	0.901 (0.949)
		(10,15)	0.884 (0.939)	0.894 (0.946)	0.897 (0.949)	0.899 (0.949)	0.905 (0.953)	0.902 (0.952)
	(1,3,5,7,10)	(5,5)	0.863 (0.924)	0.886 (0.938)	0.898 (0.946)	0.900 (0.947)	0.915 (0.957)	0.905 (0.951)
		(5,10)	0.879 (0.933)	0.893 (0.946)	0.899 (0.951)	0.900 (0.951)	0.914 (0.958)	0.907 (0.953)
		(10,10)	0.886 (0.942)	0.896 (0.948)	0.902 (0.951)	0.902 (0.952)	0.910 (0.956)	0.908 (0.955)
		(10,15)	0.887 (0.940)	0.893 (0.947)	0.898 (0.949)	0.899 (0.949)	0.905 (0.953)	0.905 (0.952)
1.0	(0.5,0.5,0.5,0.5,0.5)	(5,5)	0.851 (0.917)	0.895 (0.946)	0.904 (0.951)	0.905 (0.952)	0.905 (0.954)	0.909 (0.957)
		(5,10)	0.875 (0.937)	0.899 (0.953)	0.904 (0.956)	0.905 (0.956)	0.906 (0.957)	0.912 (0.959)
		(10,10)	0.877 (0.935)	0.897 (0.946)	0.902 (0.948)	0.902 (0.950)	0.901 (0.949)	0.909 (0.953)
		(10,15)	0.884 (0.937)	0.897 (0.946)	0.900 (0.948)	0.901 (0.948)	0.897 (0.951)	0.903 (0.951)
	(0.1,0.3,0.5,0.7,1.0)	(5,5)	0.851 (0.912)	0.887 (0.943)	0.895 (0.948)	0.898 (0.949)	0.898 (0.951)	0.903 (0.954)
		(5,10)	0.871 (0.926)	0.891 (0.949)	0.898 (0.952)	0.899 (0.953)	0.902 (0.955)	0.904 (0.955)
		(10,10)	0.878 (0.934)	0.894 (0.950)	0.899 (0.953)	0.899 (0.954)	0.905 (0.952)	0.906 (0.955)
		(10,15)	0.887 (0.938)	0.899 (0.946)	0.901 (0.948)	0.902 (0.948)	0.902 (0.950)	0.906 (0.951)
	(1,1,1,1,1)	(5,5)	0.855 (0.919)	0.897 (0.945)	0.906 (0.952)	0.909 (0.953)	0.909 (0.955)	0.914 (0.957)
		(5,10)	0.870 (0.931)	0.897 (0.948)	0.902 (0.953)	0.903 (0.954)	0.904 (0.954)	0.910 (0.957)
		(10,10)	0.877 (0.932)	0.896 (0.947)	0.900 (0.950)	0.901 (0.951)	0.902 (0.949)	0.905 (0.952)
		(10,15)	0.882 (0.938)	0.898 (0.950)	0.901 (0.951)	0.901 (0.952)	0.900 (0.951)	0.904 (0.954)
	(1,3,5,7,10)	(5,5)	0.851 (0.915)	0.888 (0.942)	0.897 (0.948)	0.900 (0.949)	0.901 (0.951)	0.904 (0.957)
		(5,10)	0.865 (0.925)	0.889 (0.944)	0.896 (0.948)	0.896 (0.949)	0.899 (0.953)	0.904 (0.954)
		(10,10)	0.874 (0.934)	0.893 (0.946)	0.898 (0.949)	0.899 (0.949)	0.901 (0.950)	0.904 (0.951)
		(10,15)	0.878 (0.936)	0.893 (0.946)	0.896 (0.948)	0.897 (0.949)	0.901 (0.948)	0.902 (0.951)
2.0	(0.5,0.5,0.5,0.5,0.5)	(5,5)	0.804 (0.880)	0.894 (0.946)	0.899 (0.950)	0.900 (0.950)	0.853 (0.920)	0.904 (0.953)
		(5,10)	0.845 (0.908)	0.892 (0.944)	0.896 (0.946)	0.896 (0.947)	0.872 (0.929)	0.905 (0.952)
		(10,10)	0.865 (0.924)	0.905 (0.952)	0.906 (0.953)	0.907 (0.954)	0.880 (0.938)	0.909 (0.956)
		(10,15)	0.865 (0.926)	0.899 (0.952)	0.901 (0.953)	0.902 (0.953)	0.883 (0.938)	0.905 (0.955)
	(0.1,0.3,0.5,0.7,1.0)	(5,5)	0.811 (0.885)	0.893 (0.945)	0.900 (0.948)	0.900 (0.948)	0.850 (0.915)	0.902 (0.953)
		(5,10)	0.841 (0.909)	0.890 (0.942)	0.893 (0.945)	0.893 (0.945)	0.871 (0.930)	0.903 (0.948)
		(10,10)	0.861 (0.924)	0.902 (0.951)	0.903 (0.952)	0.904 (0.952)	0.878 (0.939)	0.910 (0.956)
		(10,15)	0.870 (0.927)	0.903 (0.950)	0.904 (0.951)	0.904 (0.950)	0.890 (0.940)	0.909 (0.954)
	(1,1,1,1,1)	(5,5)	0.813 (0.887)	0.898 (0.948)	0.901 (0.951)	0.902 (0.952)	0.845 (0.913)	0.901 (0.951)
		(5,10)	0.847 (0.909)	0.896 (0.948)	0.900 (0.950)	0.900 (0.951)	0.877 (0.932)	0.903 (0.953)
		(10,10)	0.862 (0.922)	0.899 (0.950)	0.902 (0.952)	0.902 (0.952)	0.880 (0.936)	0.905 (0.957)
		(10,15)	0.865 (0.928)	0.897 (0.946)	0.899 (0.948)	0.900 (0.948)	0.882 (0.938)	0.902 (0.951)
	(1,3,5,7,10)	(5,5)	0.806 (0.884)	0.896 (0.946)	0.902 (0.949)	0.902 (0.950)	0.849 (0.917)	0.909 (0.955)
		(5,10)	0.839 (0.907)	0.896 (0.948)	0.899 (0.950)	0.899 (0.951)	0.872 (0.932)	0.903 (0.955)
		(10,10)	0.856 (0.920)	0.898 (0.946)	0.900 (0.947)	0.901 (0.948)	0.880 (0.935)	0.901 (0.950)
		(10,15)	0.866 (0.926)	0.903 (0.951)	0.905 (0.952)	0.905 (0.952)	0.890 (0.943)	0.909 (0.954)

Table 6. Frequentist coverage probability of 90% (95%) credible intervals of θ_1 when $k = 10$.

θ_1	$(\sigma_1,\ldots,\sigma_k)$	(n_i,m_i))	π_1	π_2	π_3	π_m	Hedges	Bonett
0.0	(0.5,0.5,. . .,0.5)	(5,5)	0.870 (0.929)	0.888 (0.944)	0.898 (0.950)	0.899 (0.951)	0.925 (0.965)	0.909 (0.954)
		(5,10)	0.886 (0.939)	0.899 (0.949)	0.906 (0.953)	0.907 (0.953)	0.920 (0.960)	0.909 (0.955)
		(10,10)	0.882 (0.936)	0.892 (0.943)	0.897 (0.947)	0.897 (0.947)	0.909 (0.955)	0.902 (0.950)
		(10,15)	0.889 (0.945)	0.895 (0.951)	0.899 (0.953)	0.899 (0.954)	0.907 (0.960)	0.903 (0.954)
	(0.1,0.2,. . .,1.0)	(5,5)	0.871 (0.927)	0.888 (0.942)	0.896 (0.949)	0.897 (0.950)	0.923 (0.963)	0.906 (0.952)
		(5,10)	0.879 (0.934)	0.891 (0.942)	0.897 (0.946)	0.899 (0.947)	0.913 (0.957)	0.902 (0.951)
		(10,10)	0.887 (0.941)	0.896 (0.948)	0.901 (0.951)	0.902 (0.951)	0.912 (0.960)	0.905 (0.954)
		(10,15)	0.894 (0.946)	0.903 (0.951)	0.905 (0.953)	0.905 (0.953)	0.915 (0.959)	0.909 (0.955)
	(1,1,. . .,1)	(5,5)	0.869 (0.928)	0.888 (0.942)	0.898 (0.950)	0.900 (0.949)	0.925 (0.965)	0.906 (0.953)
		(5,10)	0.885 (0.936)	0.894 (0.945)	0.901 (0.950)	0.901 (0.950)	0.917 (0.960)	0.907 (0.952)
		(10,10)	0.888 (0.942)	0.899 (0.948)	0.903 (0.951)	0.903 (0.952)	0.916 (0.960)	0.908 (0.954)
		(10,15)	0.892 (0.941)	0.899 (0.946)	0.902 (0.949)	0.902 (0.949)	0.911 (0.957)	0.907 (0.952)
	(1,2,. . .,10)	(5,5)	0.868 (0.924)	0.885 (0.940)	0.896 (0.948)	0.896 (0.949)	0.919 (0.963)	0.902 (0.951)
		(5,10)	0.883 (0.938)	0.894 (0.946)	0.901 (0.950)	0.901 (0.951)	0.916 (0.960)	0.907 (0.956)
		(10,10)	0.896 (0.948)	0.905 (0.953)	0.911 (0.956)	0.911 (0.957)	0.921 (0.964)	0.915 (0.959)
		(10,15)	0.892 (0.944)	0.898 (0.948)	0.902 (0.951)	0.902 (0.951)	0.912 (0.958)	0.907 (0.953)
0.2	(0.5,0.5,. . .,0.5)	(5,5)	0.868 (0.925)	0.886 (0.941)	0.896 (0.948)	0.898 (0.949)	0.920 (0.963)	0.903 (0.954)
		(5,10)	0.877 (0.933)	0.891 (0.944)	0.896 (0.949)	0.897 (0.950)	0.913 (0.961)	0.903 (0.953)
		(10,10)	0.884 (0.939)	0.894 (0.946)	0.901 (0.949)	0.901 (0.950)	0.912 (0.956)	0.906 (0.954)
		(10,15)	0.887 (0.940)	0.897 (0.946)	0.900 (0.949)	0.901 (0.949)	0.909 (0.956)	0.904 (0.953)
	(0.1,0.2,. . .,1.0)	(5,5)	0.863 (0.923)	0.884 (0.940)	0.896 (0.947)	0.898 (0.948)	0.920 (0.963)	0.903 (0.951)
		(5,10)	0.885 (0.936)	0.897 (0.947)	0.903 (0.952)	0.904 (0.952)	0.915 (0.961)	0.905 (0.953)
		(10,10)	0.882 (0.936)	0.892 (0.943)	0.896 (0.946)	0.897 (0.947)	0.907 (0.954)	0.901 (0.950)
		(10,15)	0.885 (0.941)	0.893 (0.948)	0.896 (0.950)	0.897 (0.950)	0.906 (0.956)	0.900 (0.951)
	(1,1,. . .,1)	(5,5)	0.867 (0.929)	0.890 (0.944)	0.900 (0.951)	0.901 (0.952)	0.922 (0.968)	0.907 (0.957)
		(5,10)	0.880 (0.937)	0.892 (0.946)	0.898 (0.951)	0.900 (0.952)	0.914 (0.961)	0.905 (0.953)
		(10,10)	0.882 (0.940)	0.893 (0.947)	0.897 (0.950)	0.898 (0.951)	0.910 (0.956)	0.904 (0.952)
		(10,15)	0.894 (0.948)	0.901 (0.953)	0.904 (0.956)	0.905 (0.956)	0.915 (0.961)	0.908 (0.958)
	(1,2,. . .,10)	(5,5)	0.871 (0.931)	0.893 (0.947)	0.903 (0.953)	0.904 (0.953)	0.927 (0.967)	0.910 (0.958)
		(5,10)	0.884 (0.938)	0.896 (0.947)	0.902 (0.952)	0.904 (0.952)	0.917 (0.962)	0.907 (0.954)
		(10,10)	0.885 (0.945)	0.896 (0.950)	0.901 (0.953)	0.901 (0.954)	0.912 (0.962)	0.907 (0.956)
		(10,15)	0.888 (0.943)	0.897 (0.947)	0.901 (0.949)	0.901 (0.950)	0.910 (0.955)	0.903 (0.951)
0.5	(0.5,0.5,. . .,0.5)	(5,5)	0.862 (0.920)	0.892 (0.942)	0.902 (0.948)	0.904 (0.949)	0.914 (0.960)	0.912 (0.954)
		(5,10)	0.872 (0.929)	0.892 (0.943)	0.899 (0.947)	0.900 (0.948)	0.908 (0.956)	0.903 (0.950)
		(10,10)	0.881 (0.939)	0.899 (0.947)	0.903 (0.951)	0.903 (0.951)	0.910 (0.954)	0.908 (0.954)
		(10,15)	0.881 (0.938)	0.894 (0.945)	0.899 (0.948)	0.899 (0.948)	0.903 (0.951)	0.904 (0.950)
	(0.1,0.2,. . .,1.0)	(5,5)	0.862 (0.921)	0.888 (0.943)	0.897 (0.949)	0.897 (0.949)	0.911 (0.957)	0.905 (0.955)
		(5,10)	0.876 (0.930)	0.894 (0.946)	0.900 (0.951)	0.900 (0.951)	0.907 (0.957)	0.906 (0.954)
		(10,10)	0.886 (0.939)	0.900 (0.948)	0.903 (0.951)	0.903 (0.952)	0.907 (0.956)	0.908 (0.953)
		(10,15)	0.888 (0.941)	0.898 (0.948)	0.900 (0.950)	0.901 (0.950)	0.905 (0.955)	0.904 (0.952)
	(1,1,. . .,1)	(5,5)	0.856 (0.917)	0.885 (0.940)	0.895 (0.947)	0.896 (0.948)	0.908 (0.958)	0.904 (0.950)
		(5,10)	0.874 (0.931)	0.892 (0.944)	0.898 (0.949)	0.898 (0.950)	0.908 (0.956)	0.906 (0.954)
		(10,10)	0.882 (0.935)	0.894 (0.947)	0.898 (0.952)	0.898 (0.952)	0.907 (0.957)	0.902 (0.952)
		(10,15)	0.889 (0.945)	0.899 (0.952)	0.903 (0.955)	0.903 (0.955)	0.910 (0.957)	0.907 (0.956)
	(1,2,. . .,10)	(5,5)	0.860 (0.922)	0.891 (0.941)	0.899 (0.947)	0.900 (0.948)	0.914 (0.956)	0.905 (0.951)
		(5,10)	0.873 (0.931)	0.893 (0.944)	0.899 (0.949)	0.900 (0.950)	0.910 (0.956)	0.908 (0.954)
		(10,10)	0.882 (0.936)	0.896 (0.947)	0.900 (0.950)	0.901 (0.950)	0.908 (0.954)	0.906 (0.953)
		(10,15)	0.880 (0.939)	0.894 (0.945)	0.897 (0.948)	0.897 (0.948)	0.904 (0.951)	0.903 (0.950)
1.0	(0.5,0.5,. . .,0.5)	(5,5)	0.834 (0.903)	0.890 (0.944)	0.899 (0.950)	0.900 (0.951)	0.880 (0.937)	0.902 (0.953)
		(5,10)	0.858 (0.920)	0.893 (0.947)	0.900 (0.950)	0.901 (0.951)	0.889 (0.945)	0.904 (0.953)
		(10,10)	0.863 (0.928)	0.894 (0.947)	0.898 (0.949)	0.899 (0.950)	0.890 (0.944)	0.903 (0.953)
		(10,15)	0.873 (0.929)	0.897 (0.946)	0.899 (0.949)	0.900 (0.949)	0.893 (0.946)	0.903 (0.950)
	(0.1,0.2,. . .,1.0)	(5,5)	0.828 (0.898)	0.891 (0.941)	0.898 (0.948)	0.899 (0.948)	0.881 (0.937)	0.901 (0.949)
		(5,10)	0.856 (0.917)	0.890 (0.947)	0.895 (0.953)	0.896 (0.953)	0.895 (0.945)	0.904 (0.952)
		(10,10)	0.865 (0.926)	0.897 (0.945)	0.902 (0.948)	0.903 (0.948)	0.894 (0.945)	0.906 (0.950)
		(10,15)	0.873 (0.933)	0.900 (0.950)	0.902 (0.951)	0.903 (0.952)	0.895 (0.948)	0.903 (0.956)
	(1,1,. . .,1)	(5,5)	0.826 (0.896)	0.888 (0.944)	0.898 (0.950)	0.900 (0.950)	0.881 (0.939)	0.904 (0.952)
		(5,10)	0.857 (0.917)	0.892 (0.944)	0.897 (0.948)	0.897 (0.948)	0.890 (0.945)	0.904 (0.952)
		(10,10)	0.869 (0.929)	0.896 (0.947)	0.901 (0.950)	0.901 (0.950)	0.890 (0.943)	0.907 (0.952)
		(10,15)	0.874 (0.932)	0.899 (0.950)	0.902 (0.951)	0.903 (0.952)	0.899 (0.949)	0.905 (0.954)
	(1,2,. . .,10)	(5,5)	0.827 (0.899)	0.896 (0.947)	0.904 (0.953)	0.905 (0.953)	0.889 (0.943)	0.905 (0.956)
		(5,10)	0.857 (0.923)	0.897 (0.944)	0.902 (0.949)	0.902 (0.950)	0.895 (0.944)	0.906 (0.956)
		(10,10)	0.864 (0.927)	0.898 (0.946)	0.902 (0.949)	0.902 (0.949)	0.892 (0.946)	0.906 (0.953)
		(10,15)	0.875 (0.936)	0.902 (0.948)	0.905 (0.949)	0.905 (0.950)	0.898 (0.947)	0.904 (0.952)
2.0	(0.5,0.5,. . .,0.5)	(5,5)	0.743 (0.832)	0.887 (0.947)	0.892 (0.951)	0.893 (0.951)	0.780 (0.862)	0.891 (0.949)
		(5,10)	0.810 (0.884)	0.899 (0.951)	0.902 (0.954)	0.903 (0.954)	0.836 (0.909)	0.903 (0.953)
		(10,10)	0.833 (0.902)	0.899 (0.946)	0.901 (0.948)	0.901 (0.948)	0.843 (0.912)	0.905 (0.951)
		(10,15)	0.842 (0.909)	0.896 (0.947)	0.897 (0.948)	0.897 (0.948)	0.860 (0.922)	0.900 (0.949)
	(0.1,0.2,. . .,1.0)	(5,5)	0.751 (0.837)	0.895 (0.948)	0.901 (0.951)	0.902 (0.951)	0.783 (0.870)	0.896 (0.947)
		(5,10)	0.808 (0.885)	0.897 (0.946)	0.899 (0.948)	0.900 (0.948)	0.835 (0.904)	0.900 (0.949)
		(10,10)	0.826 (0.898)	0.894 (0.945)	0.895 (0.948)	0.897 (0.948)	0.841 (0.911)	0.898 (0.953)
		(10,15)	0.845 (0.915)	0.897 (0.948)	0.899 (0.949)	0.900 (0.950)	0.857 (0.922)	0.900 (0.949)
	(1,1,. . .,1)	(5,5)	0.749 (0.840)	0.896 (0.948)	0.900 (0.950)	0.901 (0.950)	0.780 (0.866)	0.899 (0.952)
		(5,10)	0.809 (0.886)	0.894 (0.946)	0.898 (0.949)	0.898 (0.950)	0.838 (0.906)	0.906 (0.954)
		(10,10)	0.827 (0.898)	0.897 (0.948)	0.899 (0.949)	0.899 (0.950)	0.843 (0.908)	0.900 (0.951)
		(10,15)	0.848 (0.911)	0.903 (0.951)	0.904 (0.952)	0.904 (0.952)	0.861 (0.926)	0.905 (0.953)
	(1,2,. . .,10)	(5,5)	0.754 (0.835)	0.898 (0.951)	0.904 (0.955)	0.905 (0.955)	0.790 (0.872)	0.901 (0.952)
		(5,10)	0.812 (0.884)	0.899 (0.948)	0.904 (0.951)	0.904 (0.951)	0.841 (0.908)	0.902 (0.952)
		(10,10)	0.827 (0.898)	0.894 (0.946)	0.897 (0.947)	0.898 (0.947)	0.841 (0.909)	0.900 (0.948)
		(10,15)	0.843 (0.911)	0.896 (0.950)	0.897 (0.951)	0.897 (0.951)	0.861 (0.919)	0.902 (0.951)

Table 7. Data on placebo interventions for pain.

Study	n_T	$\bar{X}_T$	S_T	n_C	$\bar{X}_C$	S_C
Reading (1982) [34]	18	1.60	1.30	20	2.30	2.00
Conn (1986) [35]	13	28.20	18.40	14	44.40	15.70
Hashish (1986) [36]	25	16.00	11.70	50	30.00	18.90
Hashish (1988) [37]	25	42.00	25.00	25	60.00	23.00
Hargreaves (1989) [38]	25	4.50	2.50	25	4.90	2.40
Blanchard (1990b) [39]	18	11.90	23.90	24	20.70	34.80
Blanchard (1990a) [40]	13	8.30	13.60	11	22.50	25.10
Sprott (1993) [41]	10	7.90	3.00	10	7.40	3.00
Forster (1994) [42]	15	3.20	2.80	15	4.60	2.20
Parker (1995) [43]	49	4.00	1.90	45	3.80	2.20
Rowbotham (1996) [44]	35	-4.40	8.70	35	1.90	8.70
Wang (1997) [45]	25	10.70	7.30	26	13.40	5.80
Robinson (2001) [46]	13	3.85	3.48	10	4.25	3.74
Cupal (2001) [47]	10	2.70	0.95	10	2.70	1.34
Rawling (2001) [48]	89	5.30	4.72	96	5.60	4.90
Kotani (2001) [49]	23	15.00	4.50	24	18.00	6.00
Lin (2002) [50]	25	30.20	14.40	25	38.10	16.00

Table 8. Estimates and 90% and 95% confidence intervals for θ_1.

Method	Estimate	90% confidence interval	95% confidence interval
Hedges (1982) [2]	−0.339	(−0.450, −0.228)	(−0.472, −0.206)
Bonett (2009) [10]	−0.395	(−0.528, −0.261)	(−0.554, −0.236)
JP	−0.357	(−0.469, −0.244)	(−0.491, −0.223)
TR	−0.350	(−0.461, −0.239)	(−0.483, −0.217)
OR	−0.350	(−0.462, −0.238)	(−0.483, −0.217)
MP	−0.350	(−0.462, −0.238)	(−0.483, −0.216)

Table 9. Data for four eyewitness identification studies.

Study	n_1	$\bar{X}_1$	S_1	n_2	$\bar{X}_2$	S_2
1	40	7.4	1.7	40	6.3	2.3
2	20	6.9	1.5	20	5.7	2.0
3	25	6.8	1.6	25	5.8	1.8
4	30	6.6	1.8	30	5.5	2.1

Table 10. Estimates and 90% and 95% confidence intervals for θ_1.

Method	Estimate	90% confidence interval	95% confidence interval
Hedges (1982) [2]	0.573	(0.352, 0.795)	(0.310, 0.837)
Bonett (2009) [10]	0.584	(0.354, 0.815)	(0.310, 0.859)
JP	0.593	(0.371, 0.815)	(0.328, 0.858)
TR	0.583	(0.362, 0.803)	(0.320, 0.846)
OR	0.583	(0.361, 0.804)	(0.319, 0.846)
MP	0.583	(0.361, 0.804)	(0.319, 0.847)

5. Conclusions

This paper developed an objective Bayesian framework for inference on the common standardized mean difference (SMD) in normal models and systematically compared probability matching priors with reference priors. We established that, in general, a second-order probability matching prior does not exist for the common SMD. Nevertheless, when the two arms of each study have equal sample sizes, we derived a valid second-order matching prior. Among the reference priors considered, the two-group and one-at-a-time reference priors satisfy a first-order matching criterion, whereas Jeffreys' prior does not.

Our numerical studies spanning a range of sample size allocations, variances, and numbers of studies show that the matching prior and the one-at-a-time reference prior deliver frequentist coverage probabilities close to nominal levels and generally outperform Jeffreys' prior. These findings are consistent across scenarios and are further supported by two meta-analytic applications, indicating that probability matching and reference-based constructions can provide reliable objective Bayesian inference for SMD models.

When an objective prior is desired for a common SMD under homogeneity, we recommend using the matching prior or, alternatively, the one-at-a-time reference prior. In practice, the two yield very similar interval estimates, with coverage advantages over Jeffreys' prior. In the special case of equal arm sizes across studies, the second-order matching prior offers an additional theoretical guarantee.

Our development assumes normal outcomes and focuses on the common-effect (homogeneous) setting; moreover, a general second-order matching solution is unavailable when arm sizes are unequal. Extending the framework to random-effects models (between-study heterogeneity), unequal-variance settings, and small-k regimes are important possible directions. It would also be useful to study robustness to deviations from normality, to compare alternative standardized effect definitions, and to investigate computational strategies (e.g., accurate Laplace approximations or Markov chain Monte Carlo (MCMC) diagnostics) for complex designs. Finally, exploring predictive and empirical-Bayes priors that incorporate external evidence may offer improved performance in sparse-data meta-analyses.

Author contributions

S.-G.K.: Conceptualization, Data curation, Methodology, Writing original draft; Y.K.: Conceptualization, Formal Analysis, Writing original draft, Writing review & editing. All authors read and approved the final version of the article.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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