Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(6): 16613–16634.
DOI: 10.3934/math.2026682
Received: 20 February 2026
Revised: 27 May 2026
Accepted: 04 June 2026
Published: 10 June 2026

Research article

Hilfer–Taylor expansions and fractional Appell-type sequences

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Abstract: We develop a Hilfer-adapted Taylor-type framework that is compatible with the natural initial trace of the Hilfer fractional derivative. For $0 < \alpha < 1$ and $\beta \in [0, 1]$, we introduce a shifted (α,β)-fractional power series (FPS) with $\delta = \alpha(1 - \beta) + \beta - 1$ and define Hilfer–Taylor coefficients via the regularized trace $\mathcal{T}_n(f) = (I^{(1-\beta)(1-\alpha)}(D^{\alpha,\beta})^n f)(0+)$. This yields an explicit coefficient formula and a Taylor-type expansion in the normalized basis $t^{n\alpha+\delta}/\Gamma(n\alpha + \delta + 1)$. Using the associated Mittag–Leffler eigenfunction kernel $G_{\alpha,\delta}(t, x) = x^\delta E_{\alpha,\delta+1}(t^\alpha x^\alpha)$, we define fractional Appell-type sequences through a Hilfer-adapted generating identity and establish their main operational properties, including a lowering relation under $D_x^{\alpha,\beta}$. As an application, we introduce Bernoulli-type objects and derive a convolution recurrence for the corresponding fractional Bernoulli numbers, recovering the classical case when $(\alpha,\beta) = (1, 1)$.

Keywords: Hilfer fractional derivative; fractional Taylor formula; Mittag–Leffler function; fractional Appell-type sequences; Bernoulli sequences
Mathematics Subject Classification: 26A33, 33E12, 34A08, 33E12, 11B68, 11B83

1. Introduction

Taylor formulas are a central tool in analysis and approximation theory, and their fractional counterparts have been investigated from multiple viewpoints; see, for instance, [1, 2, 5] and the references therein. Riemann's early work already contains fractional Taylor-type representations in the Riemann–Liouville setting. In particular, Cheng [7] proposed the formal generalized Taylor expansion

$$f(x + h) = \sum_{m=-\infty}^{\infty} \frac{h^{m+r}}{\Gamma(m + r + 1)} (I_x^{m+r} f)(x), \tag{1.1}$$

where I_x^{m+r} denotes the Riemann–Liouville fractional integral of order $m + r$. Later, Watanabe [25] obtained representations of the form

$$f(x) = \sum_{k=-m}^{n-1} \frac{(x - x_0)^{\alpha+k}}{\Gamma(\alpha + k + 1)} {}_R D_x^{m+r} f(x_0) + R_{n,m}, \qquad x_0 > 0, \tag{1.2}$$

where ${}_R D_x^{\alpha+n}$ in the fractional derivative of Riemann–Liouville type of order $\alpha + n$ and

$$R_{n,m} = (I_x^{\alpha+n} {}_R D_x^{\alpha+n} f)(x) + \frac{1}{\Gamma(\alpha - m)} \int_a^{x_0} (x - t)^{-\alpha-m-1} \left({}_R D_t^{\alpha-m-1} f \right)(t) dt.$$

More recently, Trujillo et al. [24] introduced a generalized Taylor formula (under appropriate assumptions on f and $\alpha \in [0, 1]$) expressed in terms of Riemann–Liouville derivatives and weighted traces at the initial point:

$$f(x) = \sum_{j=0}^{n} \frac{c_j \, x^{(j+1)\alpha-1}}{\Gamma((j + 1)\alpha)} + R_n(x), \qquad x > 0, \tag{1.3}$$

where the coefficients are given by

$$c_j = \Gamma(\alpha) \Big[x^{1-\alpha} \, {}_R D_x^{j\alpha} f(x) \Big](0+) = (I_x^{1-\alpha} \, {}_R D_x^{j\alpha} f)(0+), \qquad j = 0, 1, \ldots, n \tag{1.4}$$

and the remainder admits a Lagrange-type representation. In a complementary direction, El-Ajou et al. [11] derived a generalized Taylor-type expansion using the Caputo fractional derivative, leading to fractional power series (FPS) representations of the form

$$f(x) = \sum_{n=0}^{\infty} \frac{{}_C D_x^{n\alpha} f(0)}{\Gamma(n\alpha + 1)} x^{n\alpha} \tag{1.5}$$

under suitable regularity assumptions.

While formulas such as (1.1)–(1.3) and (1.5) provide powerful representations within the Riemann–Liouville and Caputo frameworks, several modern applications require Taylor-type expansions compatible with alternative fractional operators and their natural initial data. A prominent example is the Hilfer fractional derivative, which interpolates between the Riemann–Liouville and Caputo operators and whose natural initial datum is expressed through a regularized fractional integral trace. In applied settings, such regularized trace-type data may be interpreted as generalized initial data for memory-dependent models, a perspective consistent with the use of fractional operators in anomalous diffusion and viscoelasticity [16, 19]. For $\beta < 1$, this typically entails singular behavior at the origin, so standard α-FPS are not fully adapted to the operator-theoretic structure of Hilfer-type problems.

The first goal of this article is to develop a Hilfer-adapted Taylor calculus that is explicitly compatible with the Hilfer initial trace. For $0 < \alpha < 1$ and $\beta \in [0, 1]$, we introduce the shifted exponent

$$\delta = \alpha(1 - \beta) + \beta - 1 = -(1 - \alpha)(1 - \beta) \le 0,$$

and the consider generalized (α,β)-FPS in the normalized basis $t^{n\alpha+\delta}/\Gamma(n\alpha + \delta + 1)$. We define Hilfer–Taylor coefficients through the regularized trace

$$\mathcal{T}_n(f) := \left(I^{(1-\beta)(1-\alpha)} (D^{\alpha,\beta})^n f \right)(0+)$$

and prove an explicit coefficient extraction formula yielding a Hilfer–Taylor-type expansion. This construction is closely related to weighted-trace formulations such as (1.4): In the Riemann–Liouville limit $\beta = 0$ (and under standard trace/regularity assumptions), our coefficients reduce to equivalent weighted traces at the initial point in the spirit of Trujillo et al. [24]. In the Caputo limit $\beta = 1$ (so that $\delta = 0$ and the regularized trace reduces to point evaluation whenever it exists), our coefficient extraction yields the usual α-fractional Taylor-type expansion, in line with Caputo-based representations such as (1.5). For general $\beta \in (0, 1)$, the extraction mechanism remains intrinsically tailored to the Hilfer trace and is particularly effective in the singular regime $\beta < 1$.

A second motivation comes from the interplay between FPS and special polynomial families. Classical Appell polynomials $\{\mathscr{A}_n(x)\}_{n\in\mathbb{N}_0}$ are characterized by the lowering relation

$$\frac{d}{dx}\mathscr{A}_n(x) = n\,\mathscr{A}_{n-1}(x), \qquad n \in \mathbb{N}, \tag{1.6}$$

together with an exponential generating function $A(t)e^{xt}$. Within this class, the Bernoulli and Euler polynomials play a distinguished role due to their rich algebraic structure and broad applicability; see, e.g., [9, 21]. In recent years, Appell-type constructions have been extended in various directions using fractional calculus, typically by replacing the classical derivative in (1.6) with a fractional operator and by replacing the exponential kernel with a Mittag–Leffler-type kernel; see [6, 13, 23]. For instance, Caratelli et al. [4] introduced the fractional Bernoulli family $\{\mathscr{B}_n^\alpha(x)\}_{n\in\mathbb{N}_0}$ and the fractional Euler family $\{\mathscr{E}_n^\alpha(x)\}_{n\in\mathbb{N}_0}$ of order $\alpha > 0$ via generating functions involving the Mittag–Leffler function $E_{\alpha,1}(t)$:

$$\frac{t^\alpha E_{\alpha,1}\left(x^\alpha t^\alpha\right)}{E_{\alpha,1}\left(t^\alpha\right) - 1} = \sum_{n=0}^{\infty} \mathscr{B}_n^\alpha(x)\frac{t^{\alpha n}}{\Gamma(n\alpha + 1)}, \quad \frac{2\,E_{\alpha,1}\left(x^\alpha t^\alpha\right)}{E_{\alpha,1}\left(t^\alpha\right) + 1} = \sum_{n=0}^{\infty} \mathscr{E}_n^\alpha(x)\frac{t^{\alpha n}}{\Gamma(n\alpha + 1)},$$

where the classical Mittag–Leffler function is defined by

$$E_{\alpha,\beta}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n + \beta)}.$$

It is worth noting that these constructions often depart from the classical notion of a polynomial, since they lead to expansions in fractional powers, e.g., $a_0 + a_1 x^\alpha + a_2 x^{2\alpha} + \cdots$, rather than polynomials in the integer powers of x. Continuing the study of these Appell-type constructions, Díaz [8] proposed a more abstract and unified framework that simultaneously covers the Caputo and the Riemann–Liouville settings. Within this approach, two families were introduced: the Appell–Caputo sequence $\{\mathscr{C}_n^\alpha(x)\}_{n\in\mathbb{N}_0}$ and the Appell–Riemann sequence $\{\mathscr{R}_n^\alpha(x)\}_{n\in\mathbb{N}_0}$, defined through the identities

$${}_{\mathrm{C}}D_x^\alpha \mathscr{C}_n^\alpha(x) = n\mathscr{C}_{n-1}^\alpha(x), \quad {}_{\mathrm{R}}D_x^\alpha \mathscr{R}_n^\alpha(x) = n\mathscr{R}_{n-1}^\alpha(x), \quad n \in \mathbb{N}.$$

These generalized Appell-type sequences admit generating functions of the form

$$a(t^\alpha)E_\alpha\left(x^\alpha t^\alpha\right) = \sum_{n=0}^{\infty} \mathscr{C}_n^\alpha(x)\frac{t^n}{n!}, \quad b(t^\alpha)\,x^{\alpha-1}E_{\alpha,\alpha}\left(x^\alpha t^\alpha\right) = \sum_{n=0}^{\infty} \mathscr{R}_n^\alpha(x)\frac{t^n}{n!},$$

where $a(t)$ and $b(t)$ are formal power series. This reveals a key difference from the generating functions in [4], which rely explicitly on α-fractional powers in the expansion parameter.

Despite this progress, a Hilfer-based Appell-type theory that is intrinsically compatible with the natural Hilfer initial trace and with a shifted FPS calculus in the singular regime $\beta < 1$ has not been systematically developed. Motivated by this gap, the second goal of this article is to introduce and investigate new families of fractional Appell-type functions constructed via the Hilfer fractional derivative. We define fractional Appell-type sequences through a Hilfer-adapted generating identity based on the Mittag–Leffler eigenfunction kernel

$$G_{\alpha,\delta}(t,x) = x^{\delta} E_{\alpha,\delta+1}(t^{\alpha} x^{\alpha}),\ 0 < \alpha \leq 1,\ \delta \in \mathbb{R},$$

which satisfies $D_x^{\alpha,\beta} G_{\alpha,\delta}(t,x) = t^{\alpha} G_{\alpha,\delta}(t,x)$. We establish their fundamental operational properties (lowering relations under $D_x^{\alpha,\beta}$, finite expansions in the normalized Hilfer basis, and coefficient recovery via Hilfer–Taylor traces) and introduce Bernoulli-type objects as a distinguished example, deriving a clean convolution-type recurrence for the associated fractional Bernoulli numbers. The classical Appell/Bernoulli theory is recovered in the limiting case $(\alpha,\beta) = (1,1)$.

The paper is organized as follows. Section 2 recalls basic material from fractional calculus and the Mittag–Leffler function. In Section 3, we develop the shifted (α,β)-FPS framework and prove the Hilfer–Taylor coefficient extraction formula. Section 4 introduces fractional Appell-type sequences through a Hilfer-adapted generating identity and establishes their basic differential property, while Section 5 collects operational properties and coefficient-recovery identities. Finally, we present Bernoulli-type constructions and derive recurrence relations for the corresponding fractional Bernoulli numbers.

2. Fractional calculus and the Mittag–Leffler function

We recall standard definitions from fractional calculus together with basic properties of the Mittag–Leffler function. These preliminaries will be used repeatedly in the subsequent development.

Throughout the paper, $*$ denotes the Volterra convolution on the positive half-line

$$(f * g)(x) := \int_0^x f(x-s) g(s)\, ds, \qquad x > 0.$$

Equivalently, this coincides with the usual convolution on $\mathbb{R}$ when the functions are extended by zero to $(-\infty, 0)$.

For $\beta > 0$, define

$$g_{\beta}(t) := \begin{cases} \frac{1}{\Gamma(\beta)} t^{\beta-1}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

where Γ denotes Euler's gamma function; see, for instance, [17] for standard background on fractional kernels and fractional integral operators. When $\alpha = 0$, g_0 is understood as the identity element for the Volterra convolution in the standard operational sense. Then the family $\{g_{\alpha}\}_{\alpha \geq 0}$ satisfies the semigroup property

$$g_{\alpha+\beta} = g_{\alpha} * g_{\beta}, \qquad (g_{\alpha} * g_{\beta})(x) := \int_0^x g_{\alpha}(x-s)\, g_{\beta}(s)\, ds, \quad \alpha,\beta \geq 0,$$

with the understanding that δ_0 acts as the identity element under convolution. This identity follows from the classical beta integral

$$\int_0^1 (1-r)^{\alpha-1} r^{\beta-1}\, dr = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \qquad \alpha,\beta > 0.$$

See, for example, [14, Formula 3.197].

Let X be a Banach space and let $u : [0,\infty) \to X$ be locally integrable (in the Bochner sense). The *Riemann–Liouville fractional integral* of order $\alpha > 0$ is defined by

$$I_x^\alpha u(x) := (g_\alpha * u)(x).$$

A basic identity for the Riemann–Liouville fractional integral of power functions is [17, Section 2.1]: If $\gamma > -1$, then

$$I_x^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}\, x^{\gamma+\alpha}. \tag{2.1}$$

Let $\alpha > 0$ and set $m := \lceil \alpha \rceil$ (so that $m-1 < \alpha \le m$). For sufficiently smooth u, the *Caputo fractional derivative* is defined by

$${}_C D_x^\alpha u(x) := I_x^{m-\alpha} u^{(m)}(x), \quad x \ge 0.$$

In particular, for $0 < \alpha < 1$, one has ${}_C D_x^\alpha u = I_x^{1-\alpha} u'$, and if $\alpha = n \in \mathbb{N}$, then ${}_C D_x^n u = \frac{d^n}{dx^n} u$.

The *Riemann–Liouville fractional derivative* of order $\alpha > 0$ is given by

$${}_R D_x^\alpha u(x) := \frac{d^m}{dx^m}\Big(I_x^{m-\alpha} u(x)\Big), \qquad x \ge 0.$$

For the power function $u(x) = x^\gamma$ with $\gamma > 0$ and $0 < \alpha < 1$, both derivatives coincide and admit the closed form

$${}_C D_x^\alpha x^\gamma = {}_R D_x^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}\, x^{\gamma-\alpha}. \tag{2.2}$$

Moreover, for $0 < \alpha < 1$, one has

$${}_C D_x^\alpha k = 0 \quad \text{for any constant } k \in \mathbb{R}, \qquad {}_R D_x^\alpha x^{\alpha-1} = 0. \tag{2.3}$$

The first identity follows from $u' \equiv 0$ for constants. For the second, note that $I_x^{1-\alpha} x^{\alpha-1} = \Gamma(\alpha)$ is constant, hence its derivative is zero. Unlike the Caputo derivative, the Riemann–Liouville derivative does not cancel constants; for $0 < \alpha < 1$, one has

$${}_R D_x^\alpha k = k\, \frac{x^{-\alpha}}{\Gamma(1-\alpha)}, \quad x > 0.$$

Let $0 < \alpha < 1$ and $\beta \in [0,1]$. The *Hilfer fractional derivative* of order α and type β is defined by

$$D_x^{\alpha,\beta} u(x) := I_x^{\beta(1-\alpha)} \left(\frac{d}{dx}\, I_x^{(1-\beta)(1-\alpha)} u(x)\right).$$

Two limiting cases are

$$D_x^{\alpha,0} u(x) = {}_R D_x^\alpha u(x), \qquad D_x^{\alpha,1} u(x) = {}_C D_x^\alpha u(x).$$

For monomials $u(x) = x^\gamma$ with $\gamma > 0$, the action of $D_x^{\alpha,\beta}$ is independent of β and equals

$$D_x^{\alpha,\beta} u(x) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}\, x^{\gamma-\alpha}. \tag{2.4}$$

This invariance does not extend to general functions. For instance, if

$$u(x) = x^{(1-\beta)(\alpha-1)} = x^{-(1-\beta)(1-\alpha)},$$

then $I_x^{(1-\beta)(1-\alpha)}u(x)$ is constant and therefore

$$D_x^{\alpha,\beta}u(x) = 0,$$

showing that the Hilfer derivative is not injective on typical function classes. For further background, we refer to [16, 20, 22].

The two-parameter Mittag–Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}, \qquad \alpha > 0,\ \beta \in \mathbb{C},\ z \in \mathbb{C}. \tag{2.5}$$

We write $E_\alpha(z) := E_{\alpha,1}(z)$. The function $E_{\alpha,\beta}$ is entire and generalizes the exponential since $E_1(z) = e^z$; for example, $E_2(-z^2) = \cos(z)$.

A classical Laplace transform identity (see [3, 15]) is

$$\int_0^\infty e^{-\lambda x}\, x^{\beta-1} E_{\alpha,\beta}(\pm z x^\alpha)\, dx = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha \mp z}, \qquad \Re(\lambda) > 0,\ \ |z| < |\lambda|^\alpha.$$

In particular, one obtains (for $x > 0$ and the corresponding admissible parameters) the eigenfunction relations

$$_C D_x^\alpha E_\alpha(z x^\alpha) = z\, E_\alpha(z x^\alpha), \qquad {}_R D_x^\alpha\Big(x^{\alpha-1} E_{\alpha,\alpha}(z x^\alpha)\Big) = z\, x^{\alpha-1} E_{\alpha,\alpha}(z x^\alpha).$$

Similarly, the Mittag–Leffler function provides the fundamental solution of the Hilfer-type equation

$$D_x^{\alpha,\beta}u(x) = z\, u(x), \qquad 0 < \alpha < 1,\ \beta \in [0, 1].$$

More precisely, for a prescribed value $u_0 = (I_x^{(1-\beta)(1-\alpha)}u)(0+)$, the solution is

$$u(x) = u_0\, x^{(1-\beta)(\alpha-1)}\, E_{\alpha,(1-\beta)\alpha+\beta}(z x^\alpha),$$

which reduces to the classical Riemann–Liouville and Caputo cases when $\beta = 0$ and $\beta = 1$, respectively.

3. Fractional power series and generalized Taylor-type expansions

Building on the fractional operators and Mittag–Leffler functions recalled in Section 2, we now introduce FPS representations adapted to fractional differentiation. While the classical Taylor series provides a local integer-power expansion for sufficiently smooth functions, fractional calculus naturally leads to FPS, where integer exponents are replaced by fractional ones. These expansions are useful both in the analysis of fractional differential equations and in the analytic representation of special functions.

3.1. α-fractional power series

We begin by formalizing FPS and their convergence. Let $\alpha > 0$ and $I = [0, b)$. A function $f : I \to \mathbb{R}$ is said to be representable by an α-FPS on I if it can be written as

$$f(x) = \sum_{n=0}^{\infty} a_n x^{n\alpha}, \qquad x \in I,$$

for some coefficients $a_n \in \mathbb{R}$; see [10, 12]. More generally, an α-FPS centered at $x_0 \in \mathbb{R}$ takes the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^{n\alpha}, \qquad x \geq x_0. \tag{3.1}$$

Its (real) radius of convergence is conveniently described by viewing (3.1) as an ordinary power series in $y = (x - x_0)^\alpha$. If

$$R := \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}},$$

then the corresponding radius in the variable x is

$$r = \begin{cases} R^{1/\alpha}, & R < \infty, \\ \infty, & R = \infty. \end{cases}$$

A basic example is provided by the two-parameter Mittag–Leffler function. Indeed, for $\nu > 0$ and $\lambda \in \mathbb{R}$,

$$E_{\alpha,\nu}(\lambda(x - x_0)^\alpha) = \sum_{n=0}^{\infty} \frac{\lambda^n}{\Gamma(\alpha n + \nu)} (x - x_0)^{n\alpha}, \qquad x \geq x_0$$

is an α-FPS centered at x_0, with coefficients

$$a_n = \frac{\lambda^n}{\Gamma(\alpha n + \nu)}.$$

Thus, Mittag–Leffler functions provide classical special functions admitting FPS representations. In particular, $E_{\alpha,1}(\lambda x^\alpha)$ appears as the fundamental solution of fractional relaxation-type equations, and Mittag–Leffler-type functions play a central role in fractional relaxation, anomalous diffusion, and viscoelasticity [17, 19].

3.2. α-FPS and the Hilfer fractional derivative

We next study how the Hilfer derivative acts an α-FPS. Throughout this subsection, we fix $0 < \alpha < 1$ and $\beta \in [0, 1]$. We denote by $D_t^{\alpha,\beta}$ the Hilfer fractional derivative defined in Section 2, and for $n \in \mathbb{N}$, we set

$$(D_t^{\alpha,\beta})^n := \underbrace{D_t^{\alpha,\beta} \circ \cdots \circ D_t^{\alpha,\beta}}_{n \text{ times}}, \qquad (D_t^{\alpha,\beta})^0 := \mathrm{Id}.$$

In the following result, we justify termwise Hilfer differentiation of an α-FPS on its interval of convergence.

Theorem 3.1. *Fix* $0 < \alpha < 1$ *and* $\beta \in [0, 1]$. *Let*

$$f(t) = \sum_{n=0}^{\infty} a_n t^{n\alpha}, \qquad t \in (0, r)$$

be an α*-FPS with radius of convergence* $r > 0$. *Then for every* $t \in (0, r)$, *the derivative* $D_t^{\alpha,\beta} f(t)$ *exists and is given by the termwise formula*

$$D_t^{\alpha,\beta} f(t) = \sum_{n=0}^{\infty} a_n \frac{\Gamma(n\alpha + 1)}{\Gamma(n\alpha + 1 - \alpha)} t^{n\alpha-\alpha}, \qquad t \in (0, r)$$

whenever $0 \le \beta < 1$, *whereas in the Caputo case* $\beta = 1$, *one has*

$$D_t^{\alpha,1} f(t) = \sum_{n=1}^{\infty} a_n \frac{\Gamma(n\alpha + 1)}{\Gamma(n\alpha + 1 - \alpha)} t^{n\alpha-\alpha}, \qquad t \in (0, r).$$

In both cases, the resulting series converges for all $t \in (0, r)$.

Proof. Let

$$S_N(t) = \sum_{n=0}^{N} a_n t^{n\alpha}.$$

By linearity of the Hilfer fractional derivative and (2.4), for each $n \ge 1$,

$$D_t^{\alpha,\beta} t^{n\alpha} = \frac{\Gamma(n\alpha + 1)}{\Gamma(n\alpha + 1 - \alpha)} t^{n\alpha-\alpha}.$$

Moreover, for $0 \le \beta < 1$, the constant term satisfies

$$D_t^{\alpha,\beta} 1 = \frac{1}{\Gamma(1 - \alpha)} t^{-\alpha},$$

whereas in the Caputo case $\beta = 1$ one has

$$D_t^{\alpha,1} 1 = 0.$$

Therefore,

$$D_t^{\alpha,\beta} S_N(t) = \sum_{n=0}^{N} a_n \frac{\Gamma(n\alpha + 1)}{\Gamma(n\alpha + 1 - \alpha)} t^{n\alpha-\alpha}, \qquad 0 \le \beta < 1,$$

and

$$D_t^{\alpha,1} S_N(t) = \sum_{n=1}^{N} a_n \frac{\Gamma(n\alpha + 1)}{\Gamma(n\alpha + 1 - \alpha)} t^{n\alpha-\alpha}.$$

It remains to justify the passage to the limit. Since the original FPS has radius of convergence r, the ordinary power series

$$\sum_{n=0}^{\infty} a_n y^n, \qquad y = t^{\alpha}$$

has radius of convergence $R = r^{\alpha}$. Moreover,

$$\frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha+1-\alpha)} \sim (n\alpha)^{\alpha} \qquad \text{as } n \to \infty.$$

Hence,

$$\lim_{n\to\infty}\left(\frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha+1-\alpha)}\right)^{1/n} = 1.$$

Consequently, the series

$$\sum_{n=0}^{\infty} a_n \frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha+1-\alpha)} t^{n\alpha}$$

has the same radius of convergence as the original series. Therefore, after multiplying by $t^{-\alpha}$, the resulting series converges for every $t \in (0, r)$.

Thus, passing to the limit as $N \to \infty$ gives

$$D_t^{\alpha,\beta} f(t) = \sum_{n=0}^{\infty} a_n \frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha+1-\alpha)} t^{n\alpha-\alpha}, \qquad 0 \le \beta < 1,$$

whereas in the Caputo case,

$$D_t^{\alpha,1} f(t) = \sum_{n=1}^{\infty} a_n \frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha+1-\alpha)} t^{n\alpha-\alpha}.$$

This completes the proof. □

The distinction in Theorem 3.1 reflects the behavior on constants: The Caputo case $\beta = 1$ annihilates constants, while for $\beta < 1$, the Hilfer derivative of a nonzero constant produces a $t^{-\alpha}$-type singularity. Consequently, for $\beta < 1$, an α-FPS generally yields a singular term at $t = 0$ whenever $a_0 \neq 0$. This motivates the shifted (α,β)-FPS introduced next, which is compatible with the natural Hilfer initial trace.

3.3. A generalized (α,β)-fractional power series

The standard α-FPS does not naturally reproduce the structure of fundamental solutions associated with the Hilfer derivative when $\beta < 1$, notably those involving the two-parameter Mittag–Leffler function (2.5). Motivated by the Hilfer eigenfunction (cf. Section 2)

$$t^{(1-\beta)(\alpha-1)} E_{\alpha,(1-\beta)\alpha+\beta}(zt^{\alpha}),$$

we introduce a shifted exponent

$$\delta := \alpha(1-\beta) + \beta - 1 = -(1-\alpha)(1-\beta) \in [-(1-\alpha), 0]. \tag{3.2}$$

We say that f admits an (α,β)-*FPS* about $t = 0$ if there exist $a_n \in \mathbb{R}$ and $r > 0$ such that

$$f(t) = \sum_{n=0}^{\infty} a_n t^{n\alpha+\delta}, \qquad t \in (0, r). \tag{3.3}$$

Another example is provided by the Wright function (see [3]). For $\rho > -1$ and $\eta \in \mathbb{R}$ such that the Gamma function is well defined, let

$$W_{\rho,\eta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(\rho n + \eta)}.$$

Then

$$t^{\delta} W_{\rho,\eta}(zt^{\alpha}) = \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(\rho n + \eta)} t^{n\alpha+\delta},$$

which is an (α,β)-FPS. Wright-type functions are especially relevant in fractional diffusion and related evolution equations.

A key feature of the choice (3.2) is that the Hilfer derivative acts as a shift on the normalized basis

$$\phi_n(t) := \frac{t^{n\alpha+\delta}}{\Gamma(n\alpha + \delta + 1)}, \qquad n \in \mathbb{N}_0.$$

Lemma 3.2 (Shift property)**.** *Let* $0 < \alpha < 1$*,* $\beta \in [0, 1]$*, and* δ *be given by* (3.2)*. Then*

$$D_t^{\alpha,\beta}\phi_0(t) = 0, \qquad D_t^{\alpha,\beta}\phi_n(t) = \phi_{n-1}(t), \quad n \geq 1, \qquad t > 0.$$

Proof. Set $\mu := (1 - \beta)(1 - \alpha) = -\delta \geq 0$. Since $0 < \alpha < 1$ and $\beta \in [0, 1]$, we have $0 \leq \mu < 1$. For $u(t) := t^{\delta} = t^{-\mu}$, by (2.1) with $\gamma = -\mu$, one has

$$I_t^{\mu} u(t) = I_t^{\mu} t^{-\mu} = \frac{\Gamma(1 - \mu)}{\Gamma(1)} t^0 = \Gamma(1 - \mu),$$

where the case $\mu = 0$ is understood through the convention $I_t^0 = \mathrm{Id}$. Hence, $\frac{d}{dt} I_t^{\mu} u(t) = 0$ and therefore $D_t^{\alpha,\beta} u(t) = 0$, and consequently $D_t^{\alpha,\beta}\phi_0(t) = 0$, proving the first claim.

For $n \geq 1$, consider $u(t) = t^{n\alpha+\delta}$. Then $I_t^{\mu} u(t)$ is proportional to $t^{n\alpha}$, so differentiation produces a multiple of $t^{n\alpha-1}$, and applying $I_t^{\beta(1-\alpha)}$ yields a multiple of $t^{(n-1)\alpha+\delta}$. More precisely, using again (2.1),

$$I_t^{\mu} t^{n\alpha+\delta} = I_t^{\mu} t^{n\alpha-\mu} = \frac{\Gamma(n\alpha + \delta + 1)}{\Gamma(n\alpha + 1)} t^{n\alpha}.$$

Hence,

$$\frac{d}{dt} I_t^{\mu} t^{n\alpha+\delta} = \frac{\Gamma(n\alpha + \delta + 1)}{\Gamma(n\alpha)} t^{n\alpha-1}.$$

Applying $I_t^{\beta(1-\alpha)}$ and using $n\alpha + \beta(1 - \alpha) = (n - 1)\alpha + \delta + 1$, we obtain

$$D_t^{\alpha,\beta} t^{n\alpha+\delta} = I_t^{\beta(1-\alpha)} \left(\frac{\Gamma(n\alpha + \delta + 1)}{\Gamma(n\alpha)} t^{n\alpha-1} \right) = \frac{\Gamma(n\alpha + \delta + 1)}{\Gamma((n - 1)\alpha + \delta + 1)} t^{(n-1)\alpha+\delta}.$$

Thus,

$$D_t^{\alpha,\beta} t^{n\alpha+\delta} = \frac{\Gamma(n\alpha + \delta + 1)}{\Gamma((n - 1)\alpha + \delta + 1)}\, t^{(n-1)\alpha+\delta},$$

and dividing by $\Gamma(n\alpha + \delta + 1)$ yields $D_t^{\alpha,\beta}\phi_n = \phi_{n-1}$. □

3.4. Hilfer–Taylor-type coefficient extraction

We now introduce a coefficient-extraction mechanism compatible with the Hilfer initial trace. For $\beta < 1$, the value $f(0)$ may fail to exist (typically $f(t) \sim t^{\delta}$ as $t \to 0+$ with $\delta < 0$). In the Hilfer setting, the natural initial datum is instead given by the regularized trace

$$(I_t^{(1-\beta)(1-\alpha)} f)(0+).$$

This motivates defining the following generalized Taylor coefficients.

Definition 3.3. *Let* $0 < \alpha < 1$, $\beta \in [0, 1]$, *and* $\mu := (1-\beta)(1-\alpha)$. *For* $n \in \mathbb{N}_0$, *whenever the limit exists, define*

$$\mathcal{T}_n(f) := \Big(I_t^{\mu} (D_t^{\alpha,\beta})^n f\Big)(0+).$$

Theorem 3.4. *Let* $0 < \alpha < 1$, $\beta \in [0, 1]$, *and* δ *be given by* (3.2). *Assume that* f *admits an* (α,β)*-FPS representation* (3.3) *on* $(0, r)$ *for some* $r > 0$. *Then for each* $n \in \mathbb{N}_0$,

$$a_n = \frac{\mathcal{T}_n(f)}{\Gamma(n\alpha + \delta + 1)} = \frac{1}{\Gamma(n\alpha + \delta + 1)}\Big(I_t^{(1-\beta)(1-\alpha)} (D_t^{\alpha,\beta})^n f\Big)(0+). \tag{3.4}$$

Equivalently,

$$f(t) = \sum_{n=0}^{\infty} \frac{\mathcal{T}_n(f)}{\Gamma(n\alpha + \delta + 1)}\, t^{n\alpha+\delta}, \qquad t \in (0, r). \tag{3.5}$$

In the Caputo case $\beta = 1$ *(hence* $\delta = 0$ *and* $\mu = 0$*), one has* $\mathcal{T}_n(f) = (D_t^{\alpha,1})^n f(0)$ *whenever* $f(0)$ *exists and these values are finite, and* (3.5) *reduces to the usual* α*-fractional Taylor-type expansion.*

Proof. Write (3.3) in the normalized basis from Lemma 3.2 as

$$f(t) = \sum_{n=0}^{\infty} c_n\, \phi_n(t), \qquad c_n := a_n\, \Gamma(n\alpha + \delta + 1).$$

By Lemma 3.2, $(D_t^{\alpha,\beta})^n \phi_k = \phi_{k-n}$ for $k \geq n$ and equals 0 for $k < n$. Hence,

$$(D_t^{\alpha,\beta})^n f(t) = \sum_{k=n}^{\infty} c_k\, \phi_{k-n}(t).$$

Applying I_t^{μ} (with $\mu = -\delta$) yields

$$I_t^{\mu} \phi_m(t) = \frac{t^{m\alpha}}{\Gamma(m\alpha + 1)}, \qquad m \in \mathbb{N}_0,$$

so

$$I_t^{\mu} (D_t^{\alpha,\beta})^n f(t) = c_n + \sum_{k=n+1}^{\infty} c_k\, \frac{t^{(k-n)\alpha}}{\Gamma((k-n)\alpha + 1)}.$$

Letting $t \to 0+$ gives $\mathcal{T}_n(f) = c_n$, and therefore $a_n = c_n/\Gamma(n\alpha + \delta + 1)$, which is (3.4). □

Remark 3.5. *Set $\mu := (1-\beta)(1-\alpha) \in [0,1)$ so that $\delta = -\mu$ and $\delta + 1 = (1-\beta)\alpha + \beta$. In the Riemann–Liouville setting, it is common to express initial data either through a fractional integral trace or through a weighted limit; see, e.g., the use of weighted Cauchy-type conditions in the work of Trujillo and coauthors.**

In our Hilfer framework, the functional $\mathcal{T}_n(f)$ from Definition 3.3 admits an equivalent weighted form. Namely, whenever the limit exists,

$$\mathcal{T}_n(f) = \Gamma(1-\mu) \lim_{t\to 0+} t^{\mu} (D_t^{\alpha,\beta})^n f(t) = \Gamma(\delta+1) \lim_{t\to 0+} t^{-\delta} (D_t^{\alpha,\beta})^n f(t). \tag{3.6}$$

In particular, if f admits an (α,β)-FPS expansion (3.3) *on $(0,r)$, then the limit in* (3.6) *exists for every $n \in \mathbb{N}_0$ and coincides with $\mathcal{T}_n(f)$.*

Consequently, the coefficients in (3.3) *can be recovered as*

$$a_n = \frac{\mathcal{T}_n(f)}{\Gamma(n\alpha+\delta+1)} = \frac{\Gamma(\delta+1)}{\Gamma(n\alpha+\delta+1)} t^{-\delta} (D_t^{\alpha,\beta})^n f(0+), \qquad n \in \mathbb{N}_0. \tag{3.7}$$

Remark 3.6. *If u solves the Hilfer eigenvalue problem $D_t^{\alpha,\beta} u = \lambda u$ with initial condition*

$$(I_t^{(1-\beta)(1-\alpha)} u)(0+) = u_0,$$

then (3.5) *yields*

$$u(t) = u_0\, t^{\delta} E_{\alpha,\delta+1}(\lambda t^{\alpha}), \qquad \delta + 1 = (1-\beta)\alpha + \beta$$

in agreement with the standard Mittag–Leffler representation (cf. Section 2).

4. Fractional Appell-type sequences

The power of the generalized (α,β)-fractional Taylor series FPS lies in its ability to serve as a generating function for sequences of polynomials that generalize the classical Appell sequences. Classical Appell polynomials are characterized by the exponential generating function $A(t)e^{xt}$. We define the fractional Appell type sequences (FAPS), denoted by $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$, by replacing the standard exponential function e^t with the fundamental solution of the Hilfer fractional differential equation when the eigenvalue is 1, which is $x^{\delta} E_{\alpha,\delta+1}(t^{\alpha})$.

Definition 4.7. *Let $\alpha \in (0,1)$, $\beta \in [0,1]$, and $A^{(\alpha,\beta)}(t)$ be a coefficient generating function defined by a convergent generalized FPS:*

$$A^{(\alpha,\beta)}(t) = \sum_{n=0}^{\infty} a_n t^{n\alpha+\delta}, \quad a_n \in \mathbb{R}.$$

We define the (α,β)-generating function for the sequence $\{P_n^{(\alpha,\beta)}(x)\}$ as the product:

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x) \frac{t^{n\alpha+\delta}}{\Gamma(n\alpha+\delta+1)} = A^{(\alpha,\beta)}(t) G_{\alpha,\delta}(t,x), \tag{4.1}$$

where $\delta = \alpha(1-\beta) + \beta - 1$ and $G_{\alpha,\delta}(t,x) = x^{\delta} E_{\alpha,\delta+1}(t^{\alpha} x^{\alpha})$.

*A prototype identity reported in this context is $\Gamma(\rho)[(x-a)^{1-\rho} D_a^{m\rho} f(x)](a+) = (I_a^{1-\rho} D_a^{m\rho} f)(a+)$, which links a fractional integral trace at a with a weighted limit as $x \to a+$.

Proposition 4.8. *The FAPS* $\{P_n^{(\alpha,\beta)}(x)\}$*, defined by* (4.1)*, satisfies the following fractional differential relation for* $n \geq 1$*:*

$$D_x^{\alpha,\beta} P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(n\alpha + \delta + 1)}{\Gamma((n-1)\alpha + \delta + 1)} P_{n-1}^{(\alpha,\beta)}(x). \tag{4.2}$$

Proof. We apply the Hilfer fractional derivative $D_x^{\alpha,\beta}$ with respect to x to both sides of the generating function definition.

$$\begin{aligned} D_x^{\alpha,\beta}\left\{A^{(\alpha,\beta)}(t)G_{\alpha,\delta}(t,x)\right\} &= t^\alpha A^{(\alpha,\beta)}(t)G_{\alpha,\delta}(t,x) \\ &= t^\alpha \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\frac{t^{n\alpha+\delta}}{\Gamma(n\alpha+\delta+1)} \\ &= \sum_{n=1}^{\infty} P_{n-1}^{(\alpha,\beta)}(x)\frac{t^{n\alpha+\delta}}{\Gamma((n-1)\alpha+\delta+1)}. \end{aligned}$$

We substitute the original series definition of the generating function:

$$\sum_{n=0}^{\infty} D_x^{\alpha,\beta} P_n^{(\alpha,\beta)}(x)\frac{t^{n\alpha+\delta}}{\Gamma(n\alpha+\delta+1)} = \sum_{n=1}^{\infty} P_{n-1}^{(\alpha,\beta)}(x)\frac{t^{n\alpha+\delta}}{\Gamma((n-1)\alpha+\delta+1)}.$$

We then compare the coefficients of $t^{n\alpha+\delta}$ for $n \geq 1$ on both sides:

$$D_x^{\alpha,\beta} P_n^{(\alpha,\beta)}(x)\frac{1}{\Gamma(n\alpha+\delta+1)} = P_{n-1}^{(\alpha,\beta)}(x)\frac{1}{\Gamma((n-1)\alpha+\delta+1)}.$$

This completes the proof. □

Remark 4.9. *The fractional differential property* (4.2) *serves as a robust generalization, recovering canonical relations through the choice of* β*: In the Caputo case, setting* $\beta = 1$ *forces* $\delta = \alpha(1-1)+1-1 = 0$*. The operator* $D_x^{\alpha,1}$ *is the Caputo derivative* ${}^{C}D_x^{\alpha}$*, and the relation simplifies to*

$${}_{C}D_x^{\alpha} P_n^{(\alpha,1)}(x) = \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} P_{n-1}^{(\alpha,1)}(x).$$

Furthermore, if $\alpha = 1$ *(the classical derivative* $\frac{d}{dx}$*), the relation fully recovers the defining property of classical Appell sequences:*

$$\frac{d}{dx} P_n^{(1,1)}(x) = \frac{\Gamma(n+1)}{\Gamma(n)} P_{n-1}^{(1,1)}(x) = n P_{n-1}^{(1,1)}(x).$$

5. Operational properties of fractional Appell-type sequences

Throughout this section, we fix $\alpha \in (0,1), \beta \in [0,1]$, and $\delta = \alpha(1-\beta)+\beta-1$.

Proposition 5.10. *Let* $\{P_n^{(\alpha,\beta)}\}_{n\geq 0}$ *be an FAPS defined by* (4.1)*. Then for every* $n \in \mathbb{N}_0$ *and* $m \in \mathbb{N}_0$*,*

$$(D_x^{\alpha,\beta})^m P_n^{(\alpha,\beta)}(x) = \begin{cases} \dfrac{\Gamma(n\alpha+\delta+1)}{\Gamma((n-m)\alpha+\delta+1)} P_{n-m}^{(\alpha,\beta)}(x), & 0 \leq m \leq n, \\ 0, & m > n, \end{cases} \qquad x > 0.$$

Proof. The case $m = 0$ is trivial. For $m = 1$, the identity is exactly (4.2). Assume the claim holds for some $m \le n-1$. Applying $D_x^{\alpha,\beta}$ and using (4.2) gives

$$(D_x^{\alpha,\beta})^{m+1} P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(n\alpha+\delta+1)}{\Gamma((n-m)\alpha+\delta+1)} D_x^{\alpha,\beta} P_{n-m}^{(\alpha,\beta)}(x) = \frac{\Gamma(n\alpha+\delta+1)}{\Gamma((n-m-1)\alpha+\delta+1)} P_{n-m-1}^{(\alpha,\beta)}(x),$$

which is the desired formula for $m+1$. If $m > n$, repeated application eventually reaches $D_x^{\alpha,\beta} P_0^{(\alpha,\beta)} = 0$, so $(D_x^{\alpha,\beta})^m P_n^{(\alpha,\beta)} \equiv 0$. □

Now, recall the normalized Hilfer basis functions (cf. Section 3)

$$\phi_k(x) := \frac{x^{k\alpha+\delta}}{\Gamma(k\alpha+\delta+1)}, \qquad k \in \mathbb{N}_0,\ x > 0.$$

Proposition 5.11. *For each $n \in \mathbb{N}_0$ and $x > 0$, the following identity holds,*

$$P_n^{(\alpha,\beta)}(x) = \Gamma(n\alpha+\delta+1) \sum_{k=0}^{n} a_{n-k}\, \phi_k(x). \tag{5.1}$$

Equivalently,

$$P_n^{(\alpha,\beta)}(x) = \Gamma(n\alpha+\delta+1) \sum_{k=0}^{n} a_{n-k} \frac{x^{k\alpha+\delta}}{\Gamma(k\alpha+\delta+1)}. \tag{5.2}$$

In particular, $P_n^{(\alpha,\beta)}(x) = x^\delta Q_n(x^\alpha)$, where Q_n is a polynomial of degree at most n.

Proof. Expand the kernel

$$G_{\alpha,\delta}(t,x) = x^\delta E_{\alpha,\delta+1}(t^\alpha x^\alpha) = x^\delta \sum_{k=0}^{\infty} \frac{(t^\alpha x^\alpha)^k}{\Gamma(k\alpha+\delta+1)} = \sum_{k=0}^{\infty} \phi_k(x)\, t^{k\alpha}.$$

Multiplying by $A^{(\alpha,\beta)}(t) = \sum_{m\ge0} a_m t^{m\alpha+\delta}$ and collecting powers of $t^{n\alpha+\delta}$ in (4.1) yields (5.1), which is equivalent to (5.2). □

Let $\mu := (1-\beta)(1-\alpha) = -\delta \in [0,1)$ and define, for functions of x, the Hilfer–Taylor trace functional

$$\mathcal{T}_m^{(x)}(f) := (I_x^\mu (D_x^{\alpha,\beta})^m f)(0+), \qquad m \in \mathbb{N}_0$$

whenever the limit exists (this is the x-analogue of Definition 3.3).

Proposition 5.12 (Kronecker property and coefficient identification)**.** *For the basis functions $\{\phi_k\}_{k\ge0}$, one has*

$$\mathcal{T}_m^{(x)}(\phi_k) = \delta_{mk}, \qquad (m,k \in \mathbb{N}_0),$$

where δ_{mk} is the Kronecker symbol. Consequently, for an FAPS, one has, for all $n, m \in \mathbb{N}_0$,

$$\mathcal{T}_m^{(x)}(P_n^{(\alpha,\beta)}) = \begin{cases} \Gamma(n\alpha+\delta+1)\, a_{n-m}, & 0 \le m \le n, \\ 0, & m > n. \end{cases}$$

In particular,

$$a_n = \frac{\mathcal{T}_0^{(x)}(P_n^{(\alpha,\beta)})}{\Gamma(n\alpha+\delta+1)}.$$

Proof. Using the shift property from Lemma 3.2 (with t replaced by x), $(D_x^{\alpha,\beta})^m\phi_k = \phi_{k-m}$ if $k \geq m$ and 0 if $k < m$. Applying I_x^μ gives $I_x^\mu\phi_{k-m}(x) = \frac{x^{(k-m)\alpha}}{\Gamma((k-m)\alpha+1)}$, whose limit at $x \to 0+$ equals 1 if $k = m$ and 0 if $k > m$. Hence, $\mathcal{T}_m^{(x)}(\phi_k) = \delta_{mk}$. The formula for $\mathcal{T}_m^{(x)}(P_n^{(\alpha,\beta)})$ follows by applying $\mathcal{T}_m^{(x)}$ to (5.1) and using linearity. □

Corollary 5.13. *Define the normalized sequence*

$$\widetilde{P}_n^{(\alpha,\beta)}(x) := \frac{P_n^{(\alpha,\beta)}(x)}{\Gamma(n\alpha+\delta+1)}.$$

Then $\widetilde{P}_0^{(\alpha,\beta)}(x) = a_0\,\phi_0(x)$ *and*

$$D_x^{\alpha,\beta}\widetilde{P}_n^{(\alpha,\beta)}(x) = \widetilde{P}_{n-1}^{(\alpha,\beta)}(x), \qquad n \geq 1,\ x > 0.$$

Moreover, $\widetilde{P}_n^{(\alpha,\beta)}$ *admits the finite expansion*

$$\widetilde{P}_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} a_{n-k}\phi_k(x).$$

Proof. Divide (4.2) and (5.1) by $\Gamma(n\alpha+\delta+1)$. □

6. Fractional Bernoulli sequences and numbers $\mathcal{B}_n^{(\alpha,\beta)}(x)$

In this section, we introduce a Bernoulli-type example within the class of FAPS constructed in Section 4. Throughout, $0 < \alpha < 1, \beta \in [0, 1]$, and

$$\delta = \alpha(1-\beta) + \beta - 1 = -(1-\alpha)(1-\beta) \in [-(1-\alpha), 0].$$

We recall the Hilfer eigenfunction kernel

$$G_{\alpha,\delta}(t,x) = x^\delta E_{\alpha,\delta+1}(t^\alpha x^\alpha), \qquad t > 0,\ x > 0, \tag{6.1}$$

which satisfies $D_x^{\alpha,\beta}G_{\alpha,\delta}(t,x) = t^\alpha G_{\alpha,\delta}(t,x)$.

6.1. Definition via a Hilfer-adapted generating function

Motivated by the classical Bernoulli factor $t/(e^t-1)$, we define a fractional coefficient generator by subtracting the constant term of $E_{\alpha,\delta+1}(t^\alpha)$.

Definition 6.14. *Define the coefficient generator*

$$\mathcal{A}_{\mathcal{B}}^{(\alpha,\beta)}(t) := t^\delta\,\frac{t^\alpha}{E_{\alpha,\delta+1}(t^\alpha) - \frac{1}{\Gamma(\delta+1)}}, \qquad t > 0. \tag{6.2}$$

The (α,β)*-fractional Bernoulli sequences* $\{\mathcal{B}_n^{(\alpha,\beta)}(x)\}_{n\geq 0}$ *are defined by the generating identity*

$$\sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha,\beta)}(x)\,\frac{t^{n\alpha+\delta}}{\Gamma(n\alpha+\delta+1)} = \mathcal{A}_{\mathcal{B}}^{(\alpha,\beta)}(t)\,G_{\alpha,\delta}(t,x), \qquad t > 0,\ x > 0. \tag{6.3}$$

The fractional Bernoulli numbers $\{\mathcal{B}_n^{(\alpha,\beta)}\}_{n\geq 0}$ *are the coefficients of the* (α,β)*-FPS expansion of the generator:*

$$\mathcal{A}_{\mathcal{B}}^{(\alpha,\beta)}(t) = \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha,\beta)} \frac{t^{n\alpha+\delta}}{\Gamma(n\alpha+\delta+1)}. \tag{6.4}$$

Remark 6.15. *The fractional generating identity* (6.3) *recovers the classical Bernoulli generating function as a limiting case. Indeed, for* $\alpha = 1$ *and* $\beta = 1$*, one has* $\delta = 0$*,* $E_{1,1}(z) = e^z$*, and* $1/\Gamma(\delta+1) = 1$*. Hence,*

$$\mathcal{A}_{\mathcal{B}}^{(1,1)}(t) = \frac{t}{e^t - 1}, \qquad G_{1,0}(t,x) = E_{1,1}(tx) = e^{tx}.$$

Substituting into (6.3) *gives*

$$\sum_{n=0}^{\infty} \mathcal{B}_n^{(1,1)}(x) \frac{t^n}{\Gamma(n+1)} = \frac{t}{e^t-1} e^{tx} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!},$$

so $\mathcal{B}_n^{(1,1)}(x) = B_n(x)$ *and* $\mathcal{B}_n^{(1,1)} = B_n$*.*

Remark 6.16 (On the value at $x = 0$ and the role of β)*. Since* $\delta \leq 0$*, the factor* x^δ *in* (6.1) *is either harmless (*$\delta = 0$*) or singular at the origin (*$\delta < 0$*). In the Caputo limit* $\beta = 1$*, one has* $\delta = 0$ *and therefore* $G_{\alpha,0}(t,0) = 1$*, so it is consistent to set* $\mathcal{B}_n^{(\alpha,1)} = \mathcal{B}_n^{(\alpha,1)}(0)$ *(when* $x = 0$ *is included in the domain). For* $\beta < 1$ *(hence* $\delta < 0$*),* $\mathcal{B}_n^{(\alpha,\beta)}(x)$ *is naturally defined for* $x > 0$ *and typically diverges as* $x \to 0+$*; in this regime, the numbers are understood intrinsically through the coefficient definition* (6.4) *(equivalently, through Hilfer-trace extraction).*

6.2. Appell-type property and finite expansion

As a particular instance of the general FAPS theory, the fractional Bernoulli sequences inherit an Appell-type lowering relation under the Hilfer derivative.

Proposition 6.17. *For* $n \geq 1$ *and* $x > 0$*,*

$$D_x^{\alpha,\beta} \mathcal{B}_n^{(\alpha,\beta)}(x) = \frac{\Gamma(n\alpha+\delta+1)}{\Gamma((n-1)\alpha+\delta+1)} \mathcal{B}_{n-1}^{(\alpha,\beta)}(x). \tag{6.5}$$

Proof. Differentiate (6.3) termwise with $D_x^{\alpha,\beta}$ and use

$$D_x^{\alpha,\beta} G_{\alpha,\delta}(t,x) = t^\alpha G_{\alpha,\delta}(t,x).$$

Comparing coefficients of $t^{n\alpha+\delta}$ yields (6.5). □

Corollary 6.18. *Define*

$$\widetilde{\mathcal{B}}_n^{(\alpha,\beta)}(x) := \mathcal{B}_n^{(\alpha,\beta)}(x)/\Gamma(n\alpha+\delta+1).$$

Then

$$D_x^{\alpha,\beta} \widetilde{\mathcal{B}}_n^{(\alpha,\beta)}(x) = \widetilde{\mathcal{B}}_{n-1}^{(\alpha,\beta)}(x), \qquad n \geq 1,\ x > 0.$$

Moreover, writing $\phi_k(x) := x^{k\alpha+\delta}/\Gamma(k\alpha+\delta+1)$*, one has the finite expansion*

$$\widetilde{\mathcal{B}}_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \frac{\mathcal{B}_{n-k}^{(\alpha,\beta)}}{\Gamma((n-k)\alpha+\delta+1)} \phi_k(x), \qquad n \in \mathbb{N}_0,\ x > 0. \tag{6.6}$$

Proof. The normalized lowering relation follows by dividing (6.5) by $\Gamma(n\alpha+\delta+1)$. For (6.6), expand both $\mathcal{A}_{\mathcal{B}}^{(\alpha,\beta)}(t)$ and $G_{\alpha,\delta}(t,x)$ using (6.4) and the Mittag–Leffler series, multiply the resulting series, and identify coefficients of $t^{n\alpha+\delta}$ in (6.3). □

6.3. *A convolution-type recurrence for* $\mathcal{B}_n^{(\alpha,\beta)}$

From (6.2), we obtain the identity

$$\left(E_{\alpha,\delta+1}(t^{\alpha})-\tfrac{1}{\Gamma(\delta+1)}\right)\mathcal{A}_{\mathcal{B}}^{(\alpha,\beta)}(t)=t^{\alpha+\delta}. \tag{6.7}$$

Since

$$E_{\alpha,\delta+1}(t^{\alpha})-\frac{1}{\Gamma(\delta+1)}=\sum_{k=1}^{\infty}\frac{t^{k\alpha}}{\Gamma(k\alpha+\delta+1)},$$

coefficient comparison in (6.7) yields the following recurrence.

Proposition 6.19. *The numbers* $\{\mathcal{B}_n^{(\alpha,\beta)}\}_{n\geq 0}$ *satisfy*

$$\sum_{k=1}^{n}\frac{\mathcal{B}_{n-k}^{(\alpha,\beta)}}{\Gamma(k\alpha+\delta+1)\,\Gamma((n-k)\alpha+\delta+1)}=\begin{cases}1, & n=1,\\ 0, & n\geq 2.\end{cases} \tag{6.8}$$

In particular,

$$\mathcal{B}_0^{(\alpha,\beta)}=\Gamma(\delta+1)\,\Gamma(\alpha+\delta+1). \tag{6.9}$$

Proof. Insert the series expansions of $E_{\alpha,\delta+1}(t^{\alpha})-1/\Gamma(\delta+1)$ and $\mathcal{A}_{\mathcal{B}}^{(\alpha,\beta)}(t)$ into (6.7) and identify the coefficients of $t^{n\alpha+\delta}$. For $n=1$, one obtains $\mathcal{B}_0^{(\alpha,\beta)}/(\Gamma(\alpha+\delta+1)\Gamma(\delta+1))=1$, which gives (6.9); for $n\geq 2$, one obtains the homogeneous relation in (6.8). □

6.4. *Example: the Caputo case* $\beta=1$

When $\beta=1$, one has $\delta=0$, and the generator reduces to the standard fractional Bernoulli factor

$$\mathcal{A}_{\mathcal{B}}^{(\alpha,1)}(t)=\frac{t^{\alpha}}{E_{\alpha,1}(t^{\alpha})-1},$$

so that $\mathcal{B}_0^{(\alpha,1)}=\Gamma(\alpha+1)$ by (6.9). The recurrence (6.8) becomes, for $n\geq 2$,

$$\sum_{k=1}^{n}\frac{\mathcal{B}_{n-k}^{(\alpha,1)}}{\Gamma(k\alpha+1)\,\Gamma((n-k)\alpha+1)}=0,$$

which determines the sequence recursively.

For $\alpha=0.2$, the first values of $\mathcal{B}_n^{(0.2,1)}$ (computed from (6.8)) are reported in Table 1.

Table 1. Fractional Bernoulli numbers $\mathcal{B}_n^{(0.2,1)}$ (Caputo case), rounded.

n	$\mathcal{B}_n^{(0.2,1)}$
0	0.9181687424
1	−0.8723981582
2	0.03526265367
3	0.02689256575
4	0.02016394995
5	0.01474683379
6	0.01040593949
7	0.006963538469
8	0.004278731168
9	0.002235223923

The corresponding first five fractional Bernoulli sequences (more precisely, functions expanded in fractional powers of x) are

$$\mathcal{B}_0^{(0.2,1)}(x) = 0.9181687424,$$

$$\mathcal{B}_1^{(0.2,1)}(x) = -0.8723981582 + 0.9181687424\,x^{0.2},$$

$$\mathcal{B}_2^{(0.2,1)}(x) = 0.03526265367 - 0.9181687424\,x^{0.2} + 0.9181687424\,x^{0.4},$$

$$\mathcal{B}_3^{(0.2,1)}(x) = 0.02689256575 + 0.03867601625\,x^{0.2} - 0.9568447586\,x^{0.4} + 0.9181687424\,x^{0.6},$$

$$\mathcal{B}_4^{(0.2,1)}(x) = 0.020163949 + 0.03053067345\,x^{0.2} + 0.04171940589\,x^{0.4} - 0.9904188217\,x^{0.6} + 0.9181687424\,x^{0.8}.$$

6.5. *Example: the Riemann–Liouville limit $\beta = 0$*

We consider the Riemann–Liouville limit $\beta = 0$. In this case,

$$\delta = \alpha(1-\beta) + \beta - 1 = \alpha - 1 < 0, \qquad \delta + 1 = \alpha,$$

and the Hilfer eigenfunction kernel becomes

$$G_{\alpha,\alpha-1}(t,x) = x^{\alpha-1}E_{\alpha,\alpha}(t^\alpha x^\alpha),$$

which is consistent with the classical Riemann–Liouville eigenfunction relation for the Mittag–Leffler kernel.

Moreover, the Bernoulli-type generator (6.2) simplifies to

$$\mathcal{A}_{\mathcal{B}}^{(\alpha,0)}(t) = t^{\alpha-1}\,\frac{t^\alpha}{E_{\alpha,\alpha}(t^\alpha) - \frac{1}{\Gamma(\alpha)}} = \frac{t^{2\alpha-1}}{E_{\alpha,\alpha}(t^\alpha) - \frac{1}{\Gamma(\alpha)}}.$$

The fractional Bernoulli numbers $\{\mathcal{B}_n^{(\alpha,0)}\}_{n\geq 0}$ are defined by

$$\mathcal{A}_{\mathcal{B}}^{(\alpha,0)}(t) = \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha,0)} \frac{t^{n\alpha+\alpha-1}}{\Gamma(n\alpha+\alpha)}.$$

In particular, Proposition 6.19 yields

$$\mathcal{B}_0^{(\alpha,0)} = \Gamma(\alpha)\Gamma(2\alpha).$$

For $\alpha = 0.2$ (hence $\delta = -0.8$), the first values of $\mathcal{B}_n^{(0.2,0)}$ are displayed in Table 2.

Table 2. Fractional Bernoulli numbers $\mathcal{B}_n^{(0.2,0)}$ (Riemann–Liouville limit), rounded.

n	$\mathcal{B}_n^{(0.2,0)}$
0	10.1832237937
1	−7.3287106135
2	1.0351321940
3	0.3950457138
4	0.1402091223
5	0.0360936267
6	−0.0038171836
7	−0.0157108314
8	−0.0158533456
9	−0.0117911205

The corresponding first five fractional Bernoulli sequences (more precisely, Appell-type functions expanded in fractional powers of x) are naturally defined for $x > 0$ and take the form

$$\mathcal{B}_n^{(0.2,0)}(x) = x^{-0.8} \sum_{k=0}^{n} c_{n,k}\, x^{0.2k}.$$

Using the finite expansion identity induced by (6.3), one obtains:

$$\mathcal{B}_0^{(0.2,0)}(x) = 2.2181595438\, x^{-0.8},$$

$$\mathcal{B}_1^{(0.2,0)}(x) = x^{-0.8}\Big(-1.5963755408 + 2.2181595438\, x^{0.2}\Big),$$

$$\mathcal{B}_2^{(0.2,0)}(x) = x^{-0.8}\Big(0.2254775503 - 2.2181595438\, x^{0.2} + 2.2181595438\, x^{0.4}\Big),$$

$$\mathcal{B}_3^{(0.2,0)}(x) = x^{-0.8}\Big(0.0860507869 + 0.3648303401\, x^{0.2} - 2.5829898838\, x^{0.4} + 2.2181595438\, x^{0.6}\Big).$$

7. Conclusions

This work develops a Hilfer-adapted analytic framework that unifies FPS expansions, coefficient extraction, and Appell-type structures under a single operational viewpoint. Starting from the Hilfer fractional derivative and the associated Mittag–Leffler eigenfunction, we introduced a shifted (α,β)-FPS with exponent $\delta = \alpha(1-\beta)+\beta-1$, which removes the incompatibility of standard α-FPS with the natural Hilfer initial trace when $\beta < 1$. On this basis, we defined generalized Hilfer–Taylor coefficients via the regularized trace $\mathcal{T}_n(f) = (I^{(1-\beta)(1-\alpha)}(D^{\alpha,\beta})^n f)(0+)$ and proved an explicit coefficient formula yielding a Taylor-type expansion consistent with both the Caputo limit $\beta = 1$ and the Riemann–Liouville-type regime $\beta < 1$.

Building on this Hilfer–Taylor calculus, we constructed fractional Appell-type sequences through a generating identity driven by the Hilfer eigenfunction kernel $G_{\alpha,\delta}(t, x)$. The resulting families satisfy a fractional Appell-type lowering relation under $D_x^{\alpha,\beta}$, admit finite expansions in the normalized Hilfer basis, and allow recovery of generator coefficients via Hilfer–Taylor traces. As an illustrative class, we introduced Bernoulli-type objects through an appropriate fractional coefficient generator and derived a clean convolution recurrence for the associated fractional Bernoulli numbers, recovering the classical Bernoulli structure in the limit $(\alpha,\beta) = (1, 1)$.

Several directions merit further investigation. First, the analytic scope of the (α,β)-FPS expansion and the associated trace functionals can be extended to broader function spaces and to vector-valued settings relevant for fractional evolution problems. Second, the Appell-type calculus developed here suggests natural analogues of classical identities (addition theorems, operational rules, and orthogonality-type properties) in the Hilfer context, which may yield new families of special functions. Finally, the proposed Bernoulli-type constructions invite a systematic study of asymptotic behavior, zero distributions, and numerical schemes for computing fractional Bernoulli numbers and sequences, as well as applications to Hilfer-type fractional differential equations. Another promising direction is suggested by recent developments in the asymptotic analysis of branching processes. In particular, Kutsenko [18] obtained complete left-tail asymptotic expansions for the density of branching processes in the Schröder case, where coefficients involving factors of the form

$$\frac{1}{\Gamma(n\alpha + \beta)}$$

naturally appear. This structure is closely related to Mittag–Leffler-type expansions and suggests that fractional differentiation methods, including Hilfer-adapted techniques, may provide a useful perspective for future investigations on martingale limits of branching processes.

Author contributions

J. Hernandez: Funding acquisition. All authors contributed to conceptualization, formal analysis, investigation, methodology, validation, writing – original draft preparation, and writing – review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The research of Juan Hernández has been partially supported by the Fondo Nacional de Innovación y Desarrollo Científico y Tecnológico (FONDOCYT), Dominican Republic, under grant 2023-1-1D1-0490.

Conflict of interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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