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Research article

The finiteness of a certain class of geometric ergodic measures

Hangyue Zhang[*]

School of Mathematics, Nanjing University, Nanjing 210093, China

* **Correspondence:** Email: zhanghangyue@nju.edu.cn.

Abstract: From the perspective of the dominated splitting, under certain assumptions, there existed finitely many closed sets $A_1, A_2, \dots, A_m$ (with $A_i \neq A_j$ for $i \neq j$) such that the support of any ergodic geometric measure coincided with one of them.

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1. Introduction

Let $f : M \to M$ be a diffeomorphism of a compact smooth Riemannian manifold M. Suppose that f admits a dominated splitting $TM = E \oplus_{>} F$, that is, $E \oplus F$ is a Df-invariant continuous splitting and there exist constants $c > 0$ and $\sigma < 1$ such that for all unit vectors $v_E \in E$ and $v_F \in F$, and for all $n \geq 1$,

$$\frac{\|Df^n v_F\|}{\|Df^n v_E\|} \leq c\sigma^n.$$

We denote by $\mathcal{P}_f(M)$ the set of all f-invariant Borel probability measures on M, equipped with the weak-* topology, and by $\mathcal{P}_f^{\mathrm{erg}}(M)$ its subset consisting of ergodic measures.

Let μ be an f-invariant measure. We say that μ is a geometric measure for f if it has only positive Lyapunov exponents along E and only negative Lyapunov exponents along F, and if, for μ-almost every point $x \in M$, the support of μ contains the unstable manifold of x. We denote by $\mathrm{Geo}(M)$ the set of geometric measures on M. The basin of μ is defined by

$$\mathrm{Basin}(\mu) := \left\{ x \in M : \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \int_M \varphi \, d\mu, \ \forall \varphi \in C^0(M) \right\}.$$

We say that μ is a physical measure if $\mathrm{Basin}(\mu)$ has positive Lebesgue measure.

Before stating our main theorem, we first present the motivation behind it. We say that a diffeomorphism $g : M \to M$ is partially hyperbolic if there exists a continuous Dg-invariant splitting of the tangent bundle

$$TM = E^{uu} \oplus_{>} E^{cu} \oplus_{>} E^{cs},$$

such that E^{uu} is uniformly expanding and E^{cs} is nontrivial. It is a classical result (see [9]) that the strong unstable bundle E^{uu} uniquely integrates into a g-invariant strong unstable foliation $\mathcal{F}^{uu}(g)$. Moreover, the existence of a partially hyperbolic splitting is C^1-robust.

Assume that g is a C^{1+} partially hyperbolic diffeomorphism. We say that a probability measure $\mu \in \mathcal{P}_g(M)$ is a Gibbs u-state of g if its conditional measures along strong unstable leaves are absolutely continuous with respect to the Lebesgue measure on these leaves. The notion of Gibbs u-states was first introduced by Pesin and Sinai [15].

Dolgopyat, Bonatti, Viana, and Yang [5, 7] studied the case where every Gibbs u-state has only negative Lyapunov exponents along E^{cs} and E^{cu} is trivial. Mi, Cao, and Yang [13] considered the setting in which every Gibbs u-state has only positive Lyapunov exponents along E^{cu} and only negative exponents along E^{cs}. Andersson and Vásquez [2, 3] analyzed the case where every Gibbs u-state has only positive exponents along E^{cu} and E^{cs} is uniformly contracting. Alves, Bonatti, and Viana [1] considered the setting in which E^{cu} is mostly expanding in the sense of the Lebesgue measure, approaching the problem from a more general perspective, including the case in which E^{uu} is trivial. In all these works, the focus is on obtaining physical measures, which are always contained within the set of Gibbs u-states when E^{uu} is nontrivial. The set formed by Gibbs u-states is compact and depends upper semi-continuously on the diffeomorphism in the C^{1+} topology [6]. For a Gibbs u-state, almost every ergodic component is also a Gibbs u-state. In these cases, the physical measures are geometric measures in the sense of this paper. Most results in the study of physical measures show that the number of physical measures is finite and that the union of their basins has full Lebesgue measure. The study of physical measures may stem from the interest in understanding the asymptotic behavior of most points (in the sense of Lebesgue measure). Ures, Viana, F. Yang, and J. Yang [16] studied the setting in which measures of maximal u-entropy has only negative exponents along centers. Some of their results reveal the close connection between measures of maximal u-entropy and Gibbs u-states. At the same time, the set of measures of maximal u-entropy exhibits properties analogous to those of the set of Gibbs u-states. These measures of maximal u-entropy considered in [16] are also geometric measures in the sense of this paper. Motivated by the above, we consider the setting of the following main theorem, in which a certain form of finiteness is obtained. The novelty of this article may stem from Lemma 2.2.

Theorem 1.1 (Main theorem)**.** *Let $f : M \to M$ be a C^1 diffeomorphism admitting a dominated splitting*

$$TM = E \oplus_{>} F.$$

Assume that there exists a weak-$*$ compact subset

$$\mathcal{G} \subset \mathcal{P}_f(M),$$

such that, for every $\mu \in \mathcal{G}$:

(1) all Lyapunov exponents of μ along E are positive for f, while all Lyapunov exponents along F are negative for f;

(2) almost every ergodic component of μ belongs to $\mathcal{G}$.

If

$$\mathrm{Geo}(M) \cap \mathcal{G} \cap \mathcal{P}_f^{\mathrm{erg}}(M) \neq \emptyset,$$

then there exist finitely many measures $\mu_1, \ldots, \mu_k \in \mathrm{Geo}(M) \cap \mathcal{G} \cap \mathcal{P}_f^{\mathrm{erg}}(M)$ *such that for any* $\mu \in \mathrm{Geo}(M) \cap \mathcal{G} \cap \mathcal{P}_f^{\mathrm{erg}}(M)$*, there exists some* $1 \leq i \leq k$ *with*

$$\mathrm{supp}(\mu) = \mathrm{supp}(\mu_i).$$

We point out that the existence cannot be deduced without assuming non-emptiness in Theorem 1.1. Let A be a linear Anosov diffeomorphism on $\mathbb{T}^2$ and k a C^∞ Morse–Smale map on circle $\mathbb{S}^1$ with a sink p and a source q. Then,

$$\mathrm{Leb}_{\mathbb{T}^2} \times \delta_q$$

is an ergodic Gibbs u-state for $A \times k$. Set

$$\mathcal{G} := \{\mathrm{Leb}_{\mathbb{T}^2} \times \delta_q\}.$$

It is evident that $\mathcal{G}$ satisfies the assumptions of Theorem 1.1. However, there is no geometric measure in this case.

The finiteness of the geometric measures obtained here may be interpreted, in a mathematical sense, as a form of "physical" measures. It is worth noting that geometric measures with the same support can even be uncountably many in concrete examples. A relatively simple example is the direct product of Kan's diffeomorphism [10] and Anosov systems, whose verification can be found in [19, Lemma 5.8] (the author in [19] also admits that the argument indeed follows a rather circuitous route to arrive at the final result. The present paper aims to provide a more direct approach and, to some extent, remedy this issue. Alternatively, one may view the current work as starting from a more general setting than that considered in [19]). More intricate examples can be found in [18], where the author used a rather simple and elementary construction to establish the result.

Although related considerations have been discussed in the preceding paragraph, the results of this paper remain independent and are valid in arbitrary finite-dimensional settings.

2. New method

Throughout this article, let f be a C^1 diffeomorphism on M admitting a dominated splitting

$$TM = E \oplus_{>} F.$$

Lemma 2.1. *Assume that there exists a weak-* compact subset*

$$\mathcal{G} \subset \mathcal{P}_f(M),$$

such that, for every $\mu \in \mathcal{G}$*, all Lyapunov exponents of* μ *along E are positive for f, while all Lyapunov exponents along F are negative for f. There exist* $N \in \mathbb{N}$ *and* $\lambda > 0$ *such that for any* $\mu \in \mathcal{G}$*,*

$$\int \log \|Df^{-N}|_{E(x)}\| d\mu < -\lambda, \quad \int \log \left\|Df^{N}|_{F(x)}\right\| d\mu < -\lambda.$$

Proof. Since, for each $\mu \in \mathcal{G}$, the Lyapunov exponents along E are negative for f^{-1}, it is enough to prove the corresponding inequality for the exponents along F for f. By Oseledets' multiplicative ergodic theorem (see [14] and [17, Theorem 3.3.9]), for any $\mu \in \mathcal{G}$, we have

$$\lim_{n\to+\infty} \frac{1}{n} \int \log \|Df^n|_{F(x)}\| \, d\mu < 0.$$

It follows that for each $\mu \in \mathcal{G}$ there exist constants $N_\mu \in \mathbb{N}$ and $a_\mu > 0$, such that

$$\frac{1}{N_\mu} \int \log \|Df^{N_\mu}|_{F(x)}\| \, d\mu < -a_\mu < 0.$$

Equivalently,

$$\int \log \|Df^{N_\mu}|_{F(x)}\| \, d\mu < -N_\mu a_\mu < 0.$$

Now, for each $\mu \in \mathcal{G}$, choose a neighborhood $\mathcal{V}_\mu$ of μ in the space of probability measures on M such that the inequality

$$\int \log \|Df^{N_\mu}|_{F(x)}\| \, d\nu < -N_\mu a_\mu < 0$$

holds for all $\nu \in \mathcal{V}_\mu$. By the weak-* compactness of $\mathcal{G}$, there exists a finite subcover $\{\mathcal{V}_{\mu_i}\}_{i=1}^k$ of $\mathcal{G}$, so that for any $\tilde{\nu} \in \mathcal{V}_{\mu_i}$,

$$\int \log \|Df^{N_{\mu_i}}|_{F(x)}\| \, d\tilde{\nu} < -N_{\mu_i} a_{\mu_i} < 0. \tag{2.1}$$

Let

$$N := \prod_{i=1}^{k} N_{\mu_i} \quad \text{and} \quad a := \min\{a_{\mu_1}, \dots, a_{\mu_k}\}.$$

For any $\tilde{\mu} \in \mathcal{G}$, there exists some $1 \le i_0 \le k$ such that $\tilde{\mu} \in \mathcal{V}_{\mu_{i_0}}$.

Writing $j := N/N_{\mu_{i_0}} \in \mathbb{N}$, by the chain rule together with inequality (2.1) and the choice of $\mathcal{V}_{\mu_{i_0}}$, we obtain

$$\begin{aligned}
\int \log \|Df^N|_{F(x)}\| \, d\tilde{\mu} &\le \sum_{\ell=0}^{j-1} \int \log \|Df^{N_{\mu_{i_0}}}|_{F(f^{\ell N_{\mu_{i_0}}}(x))}\| \, d\tilde{\mu} \\
&= \sum_{\ell=0}^{j-1} \int \log \|Df^{N_{\mu_{i_0}}}|_{F(x)}\| \, d\tilde{\mu} \quad (\text{since } \tilde{\mu} \text{ is } f^{\ell N_{\mu_{i_0}}}\text{-invariant}) \\
&< \sum_{\ell=0}^{j-1} (-N_{\mu_{i_0}} a_{\mu_{i_0}}) \le -a.
\end{aligned}$$

Hence, the lemma follows by taking $\lambda = a$. □

Lemma 2.2. *Let $\mathcal{G}$ be as in Lemma 2.1. Then, there exists $\alpha_0 > 0$ such that for every $\mu \in \mathcal{G} \cap \mathcal{P}_f^{\mathrm{erg}}(M)$, μ-almost every point $x \in M$ satisfies*

$$\lim_{n\to+\infty} \frac{1}{n} \log \|Df^{-n}|_{E(x)}\| < -\alpha_0 \quad \textit{and} \quad \lim_{n\to+\infty} \frac{1}{n} \log \|Df^{n}|_{F(x)}\| < -\alpha_0.$$

Proof. For any $\mu \in \mathcal{G} \cap \mathcal{P}_f^{\mathrm{erg}}(M)$, by Lemma 2.1 we have

$$\int \log \|Df^{-N}|_{E(x)}\| \, d\mu < -\lambda, \qquad \int \log \|Df^{N}|_{F(x)}\| \, d\mu < -\lambda.$$

Since μ is also invariant under f^N, the Birkhoff ergodic theorem (see [17, Theorem 3.2.3]) implies that for any continuous function ϕ on M,

$$\int \lim_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^{-Ni}(x)) \, d\mu = \int \phi \, d\mu \quad \text{and} \quad \int \lim_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^{Ni}(x)) \, d\mu = \int \phi \, d\mu.$$

Let us define $\phi(x) := \log \|Df^N|_{F(x)}\|$, which is clearly a continuous function on M. It follows that there exists a set $\Gamma \subset M$ of positive μ-measure such that for every $x \in \Gamma$,

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^{iN}(x)) = \lim_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df^N|_{F(f^{iN}(x))}\| < -\lambda. \tag{2.2}$$

For any f-invariant measure ν, the maximal Lyapunov exponent along F is given by

$$\lambda_{\max}(F, x, f) = \lim_{n\to+\infty} \frac{1}{n} \log \left\| Df^n|_{F(x)} \right\|,$$

which exists for ν-almost every $x \in M$. Moreover, if ν is ergodic, then $\lambda_{\max}(F, x, f)$ is constant ν-almost everywhere (see [14] and [17, Theorem 3.3.9]). Similarly, for the Df-invariant subbundle E, the maximal Lyapunov exponent along E for the inverse map f^{-1} is

$$\lambda_{\max}(E, x, f^{-1}) = \lim_{n\to+\infty} \frac{1}{n} \log \left\| Df^{-n}|_{E(x)} \right\|,$$

which is constant ν-almost everywhere if ν is ergodic. It follows that there exist constants $\lambda_E(\mu)$ and $\lambda_F(\mu)$ such that, for μ-almost every point $x \in M$,

$$\lim_{n\to+\infty} \frac{1}{n} \log \|Df^{-n}|_{E(x)}\| = \lambda_E(\mu) \quad \text{and} \quad \lim_{n\to+\infty} \frac{1}{n} \log \|Df^{n}|_{F(x)}\| = \lambda_F(\mu).$$

However, for $x \in \Gamma$, we have

$$\lim_{n\to+\infty} \frac{1}{n} \log \|Df^n|_{F(x)}\| = \lim_{n\to+\infty} \frac{1}{nN} \log \|Df^{nN}|_{F(x)}\| \le \lim_{n\to+\infty} \frac{1}{nN} \sum_{i=0}^{n-1} \log \|Df^N|_{F(f^{iN}(x))}\| < -\frac{\lambda}{N} \text{ (recall (2.2))}.$$

Hence, we have

$$\lambda_F(\mu) < -\frac{\lambda}{N}.$$

Similarly, one can show that

$$\lambda_E(\mu) < -\frac{\lambda}{N}.$$

Since $\mu \in \mathcal{G} \cap \mathcal{P}_f^{\mathrm{erg}}(M)$ was arbitrary, it follows that we can take

$$\alpha_0 := \frac{\lambda}{N},$$

which completes the proof of the lemma. □

We point out that Lemma 2.2 may have applications beyond the setting considered in this paper, for instance, in random or multifractal frameworks such as those studied in [4].

3. Existing results

The stable manifold of a point $x \in M$ with respect to f is defined by

$$W^s(x, f) := \left\{ y \in M \,\middle|\, \lim_{n\to+\infty} d(f^n(y), f^n(x)) = 0 \right\}.$$

Similarly, the unstable manifold of x with respect to f is defined as

$$W^u(x, f) := W^s(x, f^{-1}).$$

For a periodic point p, the stable manifold of its orbit $\mathrm{Orb}(p, f)$ with respect to f is given by

$$W^s(\mathrm{Orb}(p, f), f) := \{x \in M \mid \lim_{n\to+\infty} d(f^n(x), \mathrm{Orb}(p, f)) = 0\}.$$

Similarly,

$$W^u(\mathrm{Orb}(p, f), f) := W^s(\mathrm{Orb}(p, f), f^{-1}).$$

For any $\alpha > 0$ and $l \in \mathbb{N}$, the Pesin blocks are defined by

$$\Lambda_f(\alpha, l, E, F) = \left\{ x \in M : \prod_{i=0}^{n-1} \|Df^l|_{F(f^{il}(x))}\| \le e^{-\alpha nl}, \ \prod_{i=0}^{n-1} \|Df^{-l}|_{E(f^{-il}(x))}\| \le e^{-\alpha nl}, \ \ \forall n \in \mathbb{N} \right\}.$$

From [12, Lemma 3.2], there exists a constant $R(\alpha, l) > 0$ such that every point $x \in \Lambda_f(\alpha, l, E, F)$ possesses local stable and unstable manifolds, $W^s_{\mathrm{loc}}(x, f)$ and $W^u_{\mathrm{loc}}(x, f)$, tangent to E and F, respectively, each of size at least $R(\alpha, l)$ and centered at x. It is clear that $W^\sigma_{\mathrm{loc}}(x, f) \subset W^\sigma(x, f)$ for each $\sigma = s, u$. For convenience, we denote by $R(W^u_{\mathrm{loc}}(x, f))$ and $R(W^s_{\mathrm{loc}}(x, f))$ the sizes of the Pesin local unstable and stable manifolds at x, respectively. Keep in mind that $R(W^\sigma_{\mathrm{loc}}(x, f))$ depends only on α and l for each $\sigma = s, u$. We can check that

$$\prod_{i=0}^{n-1} \left\| Df^{2l}|_{F(f^{2il}(x))} \right\| \le \prod_{i=0}^{n-1} \left(\left\| Df^l|_{F(f^{2il}(x))} \right\| \cdot \left\| Df^l|_{F(f^{(2i+1)l}(x))} \right\| \right) = \prod_{j=0}^{2n-1} \left\| Df^l|_{F(f^{jl}(x))} \right\|.$$

Consequently, if $x \in \Lambda_f(\alpha, l, E, F)$, we have

$$\prod_{i=0}^{n-1} \left\| Df^{2l}|_{F(f^{2il}(x))} \right\| \le e^{-2\alpha nl},$$

which implies

$$\Lambda_f(\alpha, l, E, F) \subset \Lambda_f(\alpha, 2l, E, F).$$

For the same reason, for any positive integer j, we have

$$\Lambda_f(\alpha, l, E, F) \subset \Lambda_f(\alpha, jl, E, F). \tag{3.1}$$

The existence of a dominated splitting is a C^1-robust property. Thus, we may assume that there exists a C^1-neighborhood $\mathcal{U}_f$ of f such that every diffeomorphism $g \in \mathcal{U}_f$ admits a dominated splitting

$$TM = E_g \oplus_> F_g.$$

Moreover, when $g = f$, this splitting coincides with the original one; that is,

$$E_g \oplus_> F_g = E \oplus_> F.$$

In the following, every C^1 neighborhood of f under consideration is assumed to be contained in $\mathcal{U}_f$.

Lemma 3.1. *[12, Lemma A.2] Fix a constant $\alpha \in (0, \alpha_0)$. Let $\mu \in \mathcal{P}_f(M)$ be a measure such that, for μ-almost every $x \in M$, the following hold:*

$$\lim_{n\to+\infty} \frac{1}{n} \log \|Df^{-n}|_{E(x)}\| < -\alpha_0 \quad \textit{and} \quad \lim_{n\to+\infty} \frac{1}{n} \log \|Df^{n}|_{F(x)}\| < -\alpha_0.$$

Then, there exist an integer $l_\mu \in \mathbb{N}$, a C^1 neighborhood $\mathcal{U}_\mu$ of f, and a neighborhood $\mathcal{V}_\mu$ of μ such that for every diffeomorphism $g \in \mathcal{U}_\mu$ and every g-invariant measure $\nu \in \mathcal{V}_\mu$, one has

$$\nu(\Lambda_g(\alpha, l_\mu, E_g, F_g)) > \tfrac{1}{2}.$$

In [12], where the perturbation of the skeleton is analyzed, this lemma plays a crucial role.

The combination of Liao-Gan's shadowing lemma [8, 11] and the Poincaré recurrence theorem [17] yields the following result.

Lemma 3.2. *Fix $\alpha > 0$ and $l \in \mathbb{N}$. There exist a C^1 neighborhood $\mathcal{U}$ of f and constants $\rho_{\alpha,l}, \delta_{\alpha,l}, L_{\alpha,l} > 0$ with the following property: for every $g \in \mathcal{U}$, if there exist $x \in M$, $\mu \in \mathcal{P}_g(M)$, and $0 < \rho \le \rho_{\alpha,l}$ such that*

$$\mu(B(x,\rho) \cap \Lambda_g(\alpha, l, E_g, F_g)) > 0,$$

then there exists a hyperbolic periodic point

$$p \in B(x, L_{\alpha,l} \cdot \rho)$$

of stable index $\dim F$ *satisfying*

$$R(W^s_{\text{loc}}(p, g)) \ge \delta_{\alpha,l}, \quad R(W^u_{\text{loc}}(p, g)) \ge \delta_{\alpha,l}.$$

4. Proof of Theorem 1.1

Before proving the main theorem, we first establish the following lemma.

Lemma 4.1. *Under the assumption of Theorem 1.1, there exist constants $\alpha > 0$ and $l \in \mathbb{N}$ such that, for every measure $\mu \in \mathcal{G}$,*

$$\mu(\Lambda_f(\alpha, l, E, F)) > \frac{1}{2},$$

where $\mathcal{G}$ is as in Theorem 1.1.

Proof. By combining the ergodic decomposition theorem [17, Theorem 5.1.3] and Lemma 2.2, we obtain that for every $\mu \in \mathcal{G}$,

$$\mu\left(\left\{x \in M : \limsup_{n\to+\infty} \frac{1}{n} \log \left\|Df^{-n}|_{E(x)}\right\| < -\alpha_0 \text{ and } \limsup_{n\to+\infty} \frac{1}{n} \log \left\|Df^{n}|_{F(x)}\right\| < -\alpha_0\right\}\right) = 1.$$

Moreover, by Lemma 3.1 and the compactness of $\mathcal{G}$, there exist finitely many triples

$$(l_{\mu_i}, \mathcal{U}_{\mu_i}, \mathcal{V}_{\mu_i}), \quad i = 1, \ldots, m,$$

such that $\bigcup_{i=1}^m \mathcal{V}_{\mu_i}$ covers $\mathcal{G}$. For each $\mu \in \bigcup_{i=1}^m \mathcal{V}_{\mu_i}$, there exists some i with $\mu \in \mathcal{V}_{\mu_i}$ and

$$\mu(\Lambda_g(\alpha, l_{\mu_i}, E_g, F_g)) > \frac{1}{2} \quad \text{for every } g \in \bigcap_{i=1}^m \mathcal{U}_{\mu_i}.$$

Define

$$l := \prod_{j=1}^m l_{\mu_j}.$$

By the chain rule (see relation (3.1)), we then have

$$\Lambda_g(\alpha, l_{\mu_i}, E_g, F_g) \subset \Lambda_g(\alpha, l, E_g, F_g) \quad \text{for each } i = 1, \ldots, m.$$

It follows that for every $g \in \bigcap_{i=1}^m \mathcal{U}_{\mu_i}$ and every $\mu \in \bigcup_{i=1}^m \mathcal{V}_{\mu_i}$, we have

$$\mu(\Lambda_g(\alpha, l, E_g, F_g)) > \frac{1}{2}.$$

The conclusion follows from the fact $f \in \bigcap_{i=1}^m \mathcal{U}_{\mu_i}$. □

For any set $\Lambda \subset M$, we denote by $\overline{\Lambda}$ its closure.

Proof of Theorem 1.1. We first prove the following claim.

Claim 4.1. *Under the assumptions of Theorem 1.1, there exists* $\delta > 0$ *such that for any*

$$\mu \in \mathrm{Geo}(M) \cap \mathcal{G} \cap \mathcal{P}_f^{\mathrm{erg}}(M),$$

there exists a hyperbolic periodic point p_μ *with stable index* $\dim(F)$ *such that*

$$\mathrm{supp}(\mu) = \overline{W^u(\mathrm{Orb}(p_\mu), f)}, \tag{4.1}$$

and

$$R(W^s_{\mathrm{loc}}(p_\mu, f)) \geq \delta, \qquad R(W^u_{\mathrm{loc}}(p_\mu, f)) \geq \delta. \tag{4.2}$$

Proof of Claim 4.1. By Lemma 4.1, we obtain that there exist $\alpha > 0$ and $l \in \mathbb{N}$ such that for any $\mu \in \mathrm{Geo}(M) \cap \mathcal{G} \cap \mathcal{P}_f^{\mathrm{erg}}(M)$,

$$\mu(\Lambda_f(\alpha, l, E, F)) > \frac{1}{2}.$$

Fix α and l satisfying the above condition. It is clear that

$$\mu(\Lambda_f(\alpha, l, E, F) \cap \mathrm{Basin}(\mu)) = \mu(\Lambda_f(\alpha, l, E, F)).$$

By the definition of $\mathrm{Geo}(M)$, we may assume that for μ-almost every point

$$x \in \Lambda_f(\alpha, l, E, F) \cap \mathrm{Basin}(\mu),$$

the unstable manifold $W^u(x, f)$ is contained in $\mathrm{supp}(\mu)$. Then, by Lemma 3.2, there exists a hyperbolic periodic point p_μ and some $x \in \Lambda_f(\alpha, l, E, F) \cap \mathrm{Basin}(\mu)$, such that

$$W^u_{\mathrm{loc}}(p_\mu, f) \pitchfork W^s_{\mathrm{loc}}(x, f) \neq \emptyset, \quad W^s_{\mathrm{loc}}(p_\mu, f) \pitchfork W^u_{\mathrm{loc}}(x, f) \neq \emptyset,$$

and

$$R(W^s_{\mathrm{loc}}(p_\mu, f)) \geq \delta_{\alpha,l}, \quad R(W^u_{\mathrm{loc}}(p_\mu, f)) \geq \delta_{\alpha,l}. \tag{4.3}$$

By the invariance of the support and the inclination Lemma, the condition

$$W^s_{\mathrm{loc}}(p_\mu, f) \pitchfork W^u(x, f) \neq \emptyset$$

implies that

$$W^u(\mathrm{Orb}(p_\mu), f) \subset \mathrm{supp}(\mu).$$

Since

$$W^s(x, f) \subset \mathrm{Basin}(\mu) \quad \text{and} \quad W^u_{\mathrm{loc}}(p_\mu, f) \pitchfork W^s(x, f) \neq \emptyset,$$

it follows that there exists a point

$$y \in W^u_{\mathrm{loc}}(p_\mu, f) \cap \mathrm{Basin}(\mu).$$

By the invariance of $W^u(\mathrm{Orb}(p_\mu), f)$, it follows that

$$\mathrm{supp}(\mu) \subset \overline{W^u(\mathrm{Orb}(p_\mu), f)}.$$

Combining the above inclusions, we conclude that

$$\mathrm{supp}(\mu) = \overline{W^u(\mathrm{Orb}(p_\mu), f)}.$$

This completes the proof of the claim by setting $\delta := \delta_{\alpha,l}$. □

By Claim 4.1, for any $\nu_1, \nu_2 \in \mathrm{Geo}(M) \cap \mathcal{G} \cap \mathcal{P}^{\mathrm{erg}}_f(M)$, there exist corresponding periodic points p_{ν_1} and p_{ν_2} that satisfy conditions (4.1) and (4.2), respectively. By (4.2), if p_{ν_1} and p_{ν_2} are sufficiently close, we have

$$W^u(p_{\nu_1}, f) \pitchfork W^s(p_{\nu_1}, f) \neq \emptyset \quad \text{and} \quad W^u(p_{\nu_1}, f) \pitchfork W^s(p_{\nu_2}, f) \neq \emptyset.$$

This implies

$$\mathrm{supp}(\nu_1) = \overline{W^u(\mathrm{Orb}(p_{\nu_1}), f)} = \overline{W^u(\mathrm{Orb}(p_{\nu_2}), f)} = \mathrm{supp}(\nu_2).$$

By the compactness of M, one can verify that there exist finitely many periodic points

$$p_1, p_2, \ldots, p_m,$$

such that for any

$$\nu \in \mathrm{Geo}(M) \cap \mathcal{G} \cap \mathcal{P}^{\mathrm{erg}}_f(M),$$

there exists some $1 \leq i \leq m$ satisfying

$$\mathrm{supp}(\nu) = \overline{W^u(\mathrm{Orb}(p_i), f)}.$$

□

In the mostly contracting setting of Bonatti and Viana, the supports of distinct ergodic Gibbs u-states are pairwise disjoint (see [5, 7]), which naturally motivates the following question.

Question 4.1. *For any* $\mu, \nu \in \mathrm{Geo}(M) \cap \mathcal{G} \cap \mathcal{P}_f^{\mathrm{erg}}(M)$*, if*

$$\mathrm{supp}(\nu) \cap \mathrm{supp}(\mu) \neq \emptyset,$$

must we have

$$\mathrm{supp}(\nu) = \mathrm{supp}(\mu)\,?$$

The answer to Question 4.1 is negative. Consider a smooth gluing of two C^∞ Kan-type examples, defined respectively on $\mathbb{T}^2 \times [0, 1]$ and $\mathbb{T}^2 \times [1, 2]$, where $\mathbb{T}^2 \times \{0\}$ is identified with $\mathbb{T}^2 \times \{2\}$. Let g denote the resulting diffeomorphism. Then, g^{-1} admits two ergodic Gibbs u-states, μ_1 and μ_2, corresponding to the ergodic physical measures in [20, Theorem 2.2] or [20, Lemma 3.4]. Each has only negative Lyapunov exponents along the center direction for g^{-1}, and their supports satisfy

$$\mathrm{supp}(\mu_1) \cap (\mathbb{T}^2 \times (0, 1)) \neq \emptyset, \quad \mathrm{supp}(\mu_2) \cap (\mathbb{T}^2 \times (1, 2)) \neq \emptyset,$$

and

$$\mathbb{T}^2 \times [0, 1] \supset \mathrm{supp}(\mu_1) \supset \mathbb{T}^2 \times \{0, 1\}, \quad \mathbb{T}^2 \times [1, 2] \supset \mathrm{supp}(\mu_2) \supset \mathbb{T}^2 \times \{1, 2\}.$$

Note that the map $t \mapsto t\mu_1 + (1 - t)\mu_2$ is continuous in the weak* topology. Consequently, if we set

$$\mathcal{G} = \{t\mu_1 + (1 - t)\mu_2 : t \in [0, 1]\},$$

then $\mathcal{G}$ is a compact set. Here, there are only two ergodic physical (geometric) measures μ_1 and μ_2, satisfying

$$\mathrm{supp}(\mu_1) \neq \mathrm{supp}(\mu_2) \quad \text{but} \quad \mathrm{supp}(\mu_1) \cap \mathrm{supp}(\mu_2) \neq \emptyset.$$

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflicts of interest in this paper.

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