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Research article

Hessian operator on weighted m-subharmonic classes

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Abstract: This paper studied the characterization and existence of solutions to complex Hessian equations associated with a given weight. We provided a complete characterization of the Radon measures that could be represented as the complex Hessian measure of an $\mathcal{F}_{m,\chi}(\Omega)$-function, where χ was a decreasing weight function. Our main results provided both global and local characterizations of the range of the complex Hessian operator acting on these classes. Specifically, we demonstrated that the solvability of the equation $H_m(\varphi) = \mu$ in the class $\mathcal{F}_{m,\chi}(\Omega)$ is equivalent to a certain functional inequality involving the measure μ and the weighted energy $\delta_{m,\chi}$. Furthermore, we demonstrate that this global condition could be localized: a solution existed globally if, and only if, for every point in the closure $\overline{\Omega}$ of the domain, a solution existed in some neighborhood of that point.

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1. Introduction

The theory of m-subharmonic functions in a domain Ω of $\mathbb{C}^n$ was introduced by Błocki [5] to provide a natural generalization of plurisubharmonic functions that arises in the study of complex Hessian equations. These functions play a crucial role in several complex geometric problems, including the construction of metrics with prescribed curvature properties and the study of fully nonlinear elliptic equations in complex geometry. The associated complex Hessian operator $H_m(u) = (dd^c u)^m \wedge \omega^{n-m}$ defines a nonlinear elliptic operator when restricted to the cone of m-subharmonic functions, interpolating between the Laplacian ($m = 1$) and the complex Monge-Ampère operator ($m = n$).

The problem of characterizing which Radon measures can be represented as complex Hessian measures of functions with prescribed singularities has attracted considerable attention in recent years.

In the plurisubharmonic case (corresponding to $m = n$), this problem was extensively studied by Cegrell [6, 7], who introduced energy classes to capture the asymptotic behavior of plurisubharmonic functions near their singular sets. These ideas were further developed by Benelkourchi, Guedj, and Zeriahi [4] and Hiep [10], who considered weighted energy classes to obtain more precise control over singular behavior. Based on this work, Benelkourchi [3] and Quy [20] gave a characterization of the Radon measures that can be represented as the complex Monge-Ampère measure of functions in the weighted plurisubharmonic class.

For general $m \in \{1, ..., n\}$, Hung [16] and Hbil and Zaway [22] introduced and studied weighted energy classes $\mathcal{F}_{m,\chi}(\Omega)$ for m-subharmonic functions, where $\chi : \mathbb{R}^- \to \mathbb{R}^+$ is a decreasing weight function. These classes generalize the $\mathcal{F}_m^p$ classes of Cegrell and provide a natural framework for studying Hessian equations with measure data. When $\chi(-\infty) = +\infty$, the class consists of functions whose Hessian measures do not charge m-polar sets; when $\chi(-\infty) < +\infty$, the class includes functions with nontrivial singular mass.

An important aspect of our investigation concerns the interaction between m-polar sets and the weighted energy class $\mathcal{F}_{m,\chi}(\Omega)$. We show that, regardless of the asymptotic behavior of χ at $-\infty$, every m-polar set can be contained in the singular locus of some function in $\mathcal{F}_{m,\chi}(\Omega)$. This result extends known constructions in the plurisubharmonic case to the Hessian setting and provides a powerful tool for analyzing the singular structure of solutions to Hessian equations.

The main objective of this paper is to provide a complete characterization of measures that can be represented as $H_m(\varphi)$ for some $\varphi \in \mathcal{F}_{m,\chi}(\Omega)$. Our approach yields two types of characterizations: a global functional inequality condition and a local existence condition. Under the hypothesis that μ vanishes on m-polar sets, we establish that the existence of a solution $\varphi \in \mathcal{F}_{m,\chi}(\Omega)$ to $H_m(\varphi) = \mu$ is equivalent to the condition that for all $u \in \mathcal{E}_m^0(\Omega)$,

$$\int_\Omega \chi(u)d\mu \leq A_\mu \delta_{m,\chi}(u) + B_\mu,$$

with $A_\mu \in [0, 1)$ and $B_\mu > 0$. In the general case of measures, we also prove that when μ is a nonnegative Radon measure, then an additional local condition is required: for each point $z \in \overline{\Omega}$, one can find a neighborhood U_z of z and a function $v \in \mathcal{F}_{m,\chi}(\Omega \cap U_z)$ such that $\mu|_{\Omega\cap U_z} = H_m(v)$.

Our results unify and extend several lines of research. When $m = n$, we recover the characterization of measures classes obtained in [2, 3, 8, 11, 20]. It also improves results of the general case of weighted $m-$subharmonic classes studied by various authors (see [14, 21]). Moreover, our local characterization theorem provides a powerful tool for constructing global solutions from local data, which is particularly useful in applications to complex geometry.

The structure of this work is as follows: Section 2 provides the necessary background and preliminary results on m-subharmonic functions, complex Hessian measures, and the weighted energy classes $\mathcal{F}_{m,\chi}(\Omega)$. We also recall some basic properties of these classes that will be needed in the sequel. Section 3 studies the relationship between m-polar sets and the class $\mathcal{F}_{m,\chi}(\Omega)$, showing that every m-polar set can be contained in the singular locus of some function in this class. Section 4 presents our main global characterization theorems, in which we first prove that a solution $\varphi \in \mathcal{F}_{m,\chi}(\Omega)$ to $H_m(\varphi) = \mu$ exists precisely when μ satisfies the above growth condition. Then, we extend this result to more general measures and provide a bridge between global and local conditions. Finally, Section 5 synthesizes our characterization by establishing the equivalence between global solvability and local solvability in neighborhoods of every point in $\overline{\Omega}$.

2. Preliminary results

We begin by recalling the definitions of the well-known operators: $d = \bar{\partial} + \partial$, $d^c = -i(\partial - \bar{\partial})$, and define $\omega = dd^c|z|^2$.

We first review the notion of $(1,1)$-forms satisfying the m-positivity condition, following [5]:

Definition 1. *We call a real* $(1,1)$*-form* η *on* Ω m-positive *provided that*

$$\eta^k \wedge \omega^{n-k} \geq 0 \quad \textit{for all } k = 1, \ldots, m$$

pointwise on Ω.

Building on this, Blocki developed in [5] the theory of m-subharmonic functions:

Definition 2. *An m-subharmonic function* $v : \Omega \to \mathbb{R} \cup \{-\infty\}$ *is characterized by being subharmonic and by the condition that*

$$dd^c v \wedge \eta_1 \wedge \cdots \wedge \eta_{m-1} \wedge \omega^{n-m} \geq 0$$

holds pointwise on Ω *for every collection of m-positive forms* $\eta_1, \ldots, \eta_{m-1}$. *The family of these functions on* Ω *is written as* $SH_m(\Omega)$.

Example 1. *Take* $v(z_1, z_2, z_3) = 2|z_1|^2 + 2|z_2|^2 - |z_3|^2$. *One can verify that* $v \in SH_2(\mathbb{C}^3)$*, but* $v \notin SH_3(\mathbb{C}^3)$ *since its restriction to* $(0, 0, z_3)$ *fails to be subharmonic.*

Definition 3. *(1) Let* $\Omega \subset \mathbb{C}^n$ *be a bounded domain. The set* Ω *is called* m-hyperconvex *if one can find a continuous m-subharmonic function* $\rho : \Omega \to \mathbb{R}^-$ *for which* $\{\rho < t\} \Subset \Omega$ *holds for every* $t < 0$.

(2) Define a subset $E \subset \Omega$ *to be* m-polar *whenever there exists* $v \in SH_m(\Omega)$ *with* $E \subset \{v = -\infty\}$.

Additional results concerning m-subharmonic functions appear in [5, 18].

Consider $v_1, \ldots, v_m \in \mathrm{SH}_m(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$. By the same idea of the framework established in [1], one can define the following m-positive current

$$dd^c v_1 \wedge \cdots \wedge dd^c v_m \wedge \omega^{n-m}.$$

It is referred to as the *complex Hessian measure* for the m-tuple $(v_1, \ldots, v_m)$.

When $v_1 = \cdots = v_m = v$, we denote

$$\mathrm{H}_m(v) := (dd^c v)^m \wedge \omega^{n-m}.$$

From now on, we assume that $\Omega \subset \mathbb{C}^n$ is a bounded m-hyperconvex domain.

Definition 4. *Define the classes:*

$$\mathcal{E}^0_m(\Omega) := \left\{ v \in L^\infty(\Omega) \cap SH^-_m(\Omega) : \lim_{t \to \partial\Omega} v(t) = 0, \ \int_\Omega H_m(v) < +\infty \right\},$$

$$\mathcal{F}_m(\Omega) := \left\{ v \in SH^-_m(\Omega) : \exists\, (v_k) \subset \mathcal{E}^0_m, \ v_k \searrow v, \ \sup_k \int_\Omega H_m(v_k) < +\infty \right\}.$$

If in addition the sequence v_k *in the definition of* $\mathcal{F}_m(\Omega)$ *satisfies* $\sup_{k \geq 1} \int_\Omega (-v_k)^p\, H_m(v_k) < +\infty$ *for* $p > 0$*, we say that* $v \in \mathcal{F}^p_m(\Omega)$.

We set also

$$\mathcal{F}_m^{\infty}(\Omega) = \bigcap_{p>0} \mathcal{F}_m^{p}(\Omega).$$

Finally the class $\mathcal{E}_m(\Omega)$ consists of all functions that are locally in $\mathcal{E}_m(\Omega)$ and $v \in \mathcal{E}_m^a(\Omega)$ if, and only if, $v \in \mathcal{E}_m(\Omega)$ and $H_m(v)$ does not charge any $m-$polar set.

In this paper, we fix $\chi : \mathbb{R}^- \to \mathbb{R}^+$ a decreasing function. Building upon the work of Benelkourchi, Guedj, and Zeriahi [4], Hung [16] introduced the following class, named $\mathcal{F}_{m,\chi}(\Omega)$.

Definition 5. *Let $v \in SH_m^-(\Omega)$. We say that v belongs to the class $\mathcal{F}_{m,\chi}(\Omega)$ if there exists a sequence $(v_j) \subset \mathcal{E}_m^0(\Omega)$ such that v_j decreases to v and*

$$\sup_j \int_\Omega \chi(v_j) H_m(v_j) < +\infty.$$

The weighted energy of v will be denoted as $\delta_{m,\chi}(v) := \displaystyle\int_\Omega \chi(v) H_m(v)$.

Properties of the Class $\mathcal{F}_{m,\chi}(\Omega)$

The set $\mathcal{F}_{m,\chi}(\Omega)$ enjoys several important properties that we will use throughout the paper:

Remark 1. *The class $\mathcal{F}_{m,\chi}(\Omega)$ contains, as special cases, all analogous Cegrell-type classes introduced by Lu [18]. More precisely,*

(1) If the weight function χ is bounded and satisfies $\chi(0) \neq 0$, then the class $\mathcal{F}_{m,\chi}(\Omega)$ coincides with $\mathcal{F}_m(\Omega)$.
(2) For the power weight $\chi(t) = 1 + (-t)^p$, we recover the class $\mathcal{F}_{m,\chi}(\Omega) = \mathcal{F}_m^p(\Omega)$.
(3) Under the condition $\chi(0) \neq 0$, every function in $\mathcal{F}_{m,\chi}(\Omega)$ also belongs to $\mathcal{F}_m(\Omega)$.
(4) If $\chi(-\infty) < +\infty$, then $\mathcal{F}_m(\Omega) \subset \mathcal{F}_{m,\chi}(\Omega)$.

Theorem 1. *[22] Suppose χ is not identically zero. Then $\mathcal{F}_{m,\chi}(\Omega) \subset \mathcal{E}_m(\Omega)$. Therefore, for every $v \in \mathcal{F}_{m,\chi}(\Omega)$, the measure $H_m(v)$ is well-defined. As a consequence, we have $\chi(v) \in L^1(H_m(v))$.*

Proposition 1. *[22] The condition $\chi(-\infty) = +\infty$ holds if, and only if, the inclusion $\mathcal{F}_{m,\chi}(\Omega) \subset \mathcal{E}_m^a(\Omega)$ is satisfied.*

Proposition 2. *[22] Assume that $\chi \not\equiv 0$ and that there exists a sequence $(v_k) \subset \mathcal{E}_m^0(\Omega)$ with uniformly bounded weighted energy. Then, $v := \lim v_k$ is not identically $-\infty$ and $v \in \mathcal{F}_{m,\chi}(\Omega)$.*

3. m-polar sets in the class $\mathcal{F}_{m,\chi}(\Omega)$

This section is devoted to study the relationship between m-polar sets and the weighted energy classes. We will establish that every m-polar set can be contained in the singular locus of some function belonging to $\mathcal{F}_{m,\chi}(\Omega)$.

We begin with a fundamental result on the representation of m-polar sets. This result was proved by Hiep [15] in the context of plurisubharmonic functions. As the techniques used in our setting are essentially the same, we provide a sketch of the proof for completeness.

Theorem 2. *If $E \subset \Omega$ is an m-polar set, then*

$$\exists\, \varphi \in \mathcal{F}_m^{\infty}(\Omega) \text{ such that } E \subset \{\varphi = -\infty\}.$$

Sketch of the proof: Since E is a Borelian m-polar set, there exists a sequence of open subests $U_j \subset \Omega$ such that $E \subset U_j$ and $\lim_{j\to\infty} Cap_m(U_j) = 0$. Take an exhaustion $\Omega_j \Subset \Omega$ of Ω and the m-relative extremal functions $h_j := h^*_{m,U_j\cap\Omega_j,\Omega}$ relative to the subsets $U_j \cap \Omega_j$. Using [18], we deduce that for every j, one has: $h_j \in \mathcal{E}_m^0(\Omega)$, $-1 \le h_j \le 0$, and $h_j = -1$ on $U_j \cap \Omega_j$. Hence, $\lim_{j\to\infty} \int_\Omega H_m(h_j) = 0$. It follows that a subsequence $\{h_{j_k}\}$ can be inductively selected such that for partial sums $S_N = \sum_{k=1}^N h_{j_k}$, the quantities $\int_\Omega (-S_N)^p H_m(S_N)$ are uniformly bounded for every $p > 0$. Defining $u := \sum_{k=1}^\infty h_{j_k}$, we obtain $u \in \mathcal{F}_m^p(\Omega)$ for all $p > 0$ and, hence, $u \in \mathcal{F}_m^\infty(\Omega)$. The result follows since we have, by the construction of u, that $E \subset \{u = -\infty\}$.

We now establish the main result of this section, which shows that m-polar sets can be captured by functions in the weighted energy class. This result was proved in the plurisubharmonic case by Quy in [19].

Theorem 3. *If $E \subset \Omega$ is an m-polar set, then*

$$\exists\, \varphi \in \mathcal{F}_{m,\chi}(\Omega) \quad \text{satisfying} \quad E \subset \{\varphi = -\infty\}.$$

Proof. Consider $\tau : \mathbb{R}^- \to \mathbb{R}^+$ to be a convex, smooth, and strictly decreasing function satisfying

$$\chi(t) \le \tau(t) + t, \qquad \forall\, t \le 0. \quad (*)$$

Since τ is strictly decreasing, the function $-\tau$ admits an inverse in its range. Theorem 2 implies the existence of a function

$$u \in \mathcal{F}_m^\infty(\Omega) \quad \text{satisfying} \quad E \subset \{u = -\infty\}.$$

We define

$$\varphi := \max\Big((-\tau)^{-1}(u - \tau(0)),\, u\Big).$$

By the stability of $\mathcal{F}_m^\infty(\Omega)$ under taking the maximum with nonpositive m-subharmonic functions, we have

$$\varphi \in \mathcal{F}_m^\infty(\Omega) \quad \text{and} \quad E \subset \{\varphi = -\infty\}.$$

By the approximation theorem for Hessian energy classes, there exists a decreasing sequence

$$u_j \in \mathcal{E}_m^0(\Omega), \qquad u_j \searrow u.$$

We set

$$\varphi_j := \max\Big((-\tau)^{-1}(u_j - \tau(0)),\, u_j\Big).$$

Then, $\varphi_j \in \mathcal{E}_m^0(\Omega)$ and $\varphi_j \searrow \varphi$. Using $(*)$, the fact that χ is decreasing, and $\varphi_j \ge u_j$, we have

$$\chi(\varphi_j) \le \tau(\varphi_j) + \varphi_j \le \tau(\varphi_j) \le -u_j + \tau(0).$$

Using the m-Hessian comparison principle and the fact that $\varphi_j \geq u_j$, it follows that

$$\int_\Omega \chi(\varphi_j)\, H_m(\varphi_j) \leq \int_\Omega [-u_j + \tau(0)]\, H_m(\varphi_j) \leq \int_\Omega [-u_j + \tau(0)]\, H_m(u_j).$$

Since $u \in \mathcal{F}_m^\infty(\Omega)$, by definition,

$$\sup_{j\geq 1} \int_\Omega (-u_j)\, H_m(u_j) < +\infty.$$

Therefore,

$$\sup_{j\geq 1} \int_\Omega \chi(\varphi_j)\, H_m(\varphi_j) \leq \sup_{j\geq 1} \int_\Omega [-u_j + \tau(0)]\, H_m(u_j) < +\infty.$$

By definition of the weighted Hessian energy class $\mathcal{F}_{m,\chi}(\Omega)$, this implies that $\varphi \in \mathcal{F}_{m,\chi}(\Omega)$, and this completes the proof. □

4. Hessian equations in weighted energy classes

This section develops characterizations for solutions to Hessian equations within the classes $\mathcal{F}_{m,\chi}(\Omega)$. In the following, unless mentioned, we assume that μ is a measure defined in Ω such that $\mu(E) = 0$ for any m-polar set $E \subset \Omega$. We now address the central problem of this section; which is to solve, on Ω, the following equation:

$$H_m(.) = \mu \qquad (E).$$

For the remainder of this work, we operate under the assumption that χ satisfies:

$$\exists\, M > 0,\ \chi(2t) \leq M\chi(t) \qquad \forall t \in \mathbb{R}^-.$$

The following result gives a sufficient and necessary condition so that (E) has a solution in $\mathcal{F}_{m,\chi}(\Omega)$. This result improves the finding theorem in [14]. We prove essentially the following result:

Theorem 4. *The following assertions are equivalent:*

(a) (E) *has a solution* $\varphi \in \mathcal{F}_{m,\chi}(\Omega)$*.*
(b) There exist constants $0 \leq A_\mu < 1$ *and* $B_\mu \in]0, +\infty[$ *satisfying for all* $u \in \mathcal{E}_m^0(\Omega)$*,*

$$\int_\Omega \chi(u) d\mu \leq A_\mu \delta_{m,\chi}(u) + B_\mu.$$

Proof. We first prove $(b) \Rightarrow (a)$, then $(a) \Rightarrow (b)$.
(b) ⇒ (a). By Theorem 5.3 in [18], there exist $\xi \in \mathcal{E}_m^0(\Omega)$ and $f \in L^1_{\text{loc}}(H_m(\xi))$ such that $\mu = f H_m(\xi)$. Lemma 5.1 in [18] provides a sequence $(\varphi_j) \subset \mathcal{E}_m^0(\Omega)$ that satisfies

$$H_m(\varphi_j) = \min(j, f) H_m(\xi).$$

As $H_m(\varphi_j)$ converges increasingly to μ, the sequence φ_j converges decreasingly to a function φ in $\mathcal{SH}_m^-(\Omega)$.

From assumption (b),

$$\int_\Omega \chi(\varphi_j)\, d\mu \le A_\mu \delta_{m,\chi}(\varphi_j) + B_\mu,$$

and because $H_m(\varphi_j) \le \mu$,

$$\delta_{m,\chi}(\varphi_j) \le \int_\Omega \chi(\varphi_j)\, d\mu \le A_\mu \delta_{m,\chi}(\varphi_j) + B_\mu.$$

Hence,

$$(1 - A_\mu) \int_\Omega \chi(\varphi_j)\, H_m(\varphi_j) \le B_\mu.$$

So,

$$\sup_{j\ge 1} \int_\Omega \chi(\varphi_j)\, H_m(\varphi_j) \le \frac{B_\mu}{1 - A_\mu} < +\infty.$$

Thus, $\varphi \in \mathcal{F}_{m,\chi}(\Omega) \subset \mathcal{E}_m(\Omega)$ and $H_m(\varphi) = \lim_{j\to\infty} H_m(\varphi_j) = \mu$.

(a) ⇒ (b). Assume that there is a function $\varphi \in \mathcal{F}_{m,\chi}(\Omega)$ whose complex Hessian measure satisfies $H_m(\varphi) = \mu$. First, suppose $\chi \in C^\infty(\mathbb{R})$. Then, by Fubini's theorem and the fact that $\{u < -t\} \subset \{u < \frac{\varphi}{\varepsilon} - \frac{t}{2}\} \cup \{\frac{\varphi}{\varepsilon} < -\frac{t}{2}\}$, we get for any $\varepsilon > 0$,

$$\begin{aligned}
\int_\Omega \chi(u)\, H_m(\varphi) &= -\int_\Omega \int_0^{-u} \chi'(-t) dt H_m(\varphi) \ + \ \chi(0) \int_\Omega H_m(\varphi) \\
&= -\int_0^{+\infty} \chi'(-t) \int_{\{u<-t\}} H_m(\varphi)\, dt \ + \ \chi(0) \int_\Omega H_m(\varphi) \\
&\le \left[-\int_0^{+\infty} \chi'(-t) \int_{\{\frac{\varphi}{\varepsilon} - \frac{t}{2} > u\}} H_m(\varphi)\, dt \right] \\
&\quad + \left[-\int_0^{+\infty} \chi'(-t) \int_{\{-\frac{t}{2} > \frac{\varphi}{\varepsilon}\}} H_m(\varphi)\, dt \right] \ + \ \chi(0) \int_\Omega H_m(\varphi).
\end{aligned}$$

For the first term, the comparison principle gives

$$\begin{aligned}
-\int_0^{+\infty} \chi'(-t) \int_{\{u<\frac{\varphi}{\varepsilon}-\frac{t}{2}\}} H_m(\varphi)\, dt &= \varepsilon^m \int_0^{+\infty} -\chi'(-t) \int_{\{u<\frac{\varphi}{\varepsilon}-\frac{t}{2}\}} H_m\Big(\frac{\varphi}{\varepsilon} - \frac{t}{2}\Big) dt \\
&\le \varepsilon^m \int_0^{+\infty} -\chi'(-t) \int_{\{u<\frac{\varphi}{\varepsilon}-\frac{t}{2}\}} H_m(u)\, dt \\
&\le \frac{\varepsilon^m}{2^m} \int_0^{+\infty} -\chi'(-t) \int_{\{2u<-t\}} H_m(2u)\, dt \ + \ \chi(0) \int_\Omega H_m(2u) \\
&= \frac{\varepsilon^m}{2^m} \int_\Omega \chi(2u)\, H_m(2u) = \varepsilon^m \int_\Omega \chi(2u)\, H_m(u) \\
&\le \varepsilon^m M \delta_{m,\chi}(u) = A\delta_{m,\chi}(u),
\end{aligned}$$

where $A = \varepsilon^m M$.

For the second term,

$$\begin{aligned}
-\int_0^{+\infty}\chi'(-t)\int_{\{\frac{\varphi}{\varepsilon}<-\frac{t}{2}\}}H_m(\varphi)\,dt &= \left(\frac{\varepsilon}{2}\right)^m\int_0^{+\infty}-\chi'(-t)\int_{\{\frac{2\varphi}{\varepsilon}<-t\}}H_m\left(\frac{2\varphi}{\varepsilon}\right)dt\\
&= \left(\frac{\varepsilon}{2}\right)^m\left[\int_\Omega\chi\left(\frac{2\varphi}{\varepsilon}\right)H_m\left(\frac{2\varphi}{\varepsilon}\right)-\chi(0)\int_\Omega H_m\left(\frac{2\varphi}{\varepsilon}\right)\right]\\
&= \int_\Omega\chi\left(\frac{2\varphi}{\varepsilon}\right)H_m(\varphi)-\chi(0)\int_\Omega H_m(\varphi)\\
&\leq M^{\log_2(\frac{2}{\varepsilon})+1}\int_\Omega\chi(\varphi)\,H_m(\varphi)-\chi(0)\int_\Omega H_m(\varphi)\\
&= M^2\left(\frac{1}{\varepsilon}\right)^{\log_2 M}\delta_{m,\chi}(\varphi)-\chi(0)\int_\Omega H_m(\varphi)\\
&= M^2\left(\frac{M}{A}\right)^{\frac{\log_2 M}{m}}\delta_{m,\chi}(\varphi)-\chi(0)\int_\Omega H_m(\varphi).
\end{aligned}$$

Putting these estimates together yields

$$\int_\Omega\chi(u)\,H_m(\varphi)\leq A\delta_{m,\chi}(u)+B\delta_{m,\chi}(\varphi),$$

with $B=M^2(\frac{M}{A})^{\frac{\log_2 M}{m}}$.

For the general case, we proceed by regularization. Choose $\rho\in C^\infty(\mathbb{R})$ with $\rho\geq 0$, $\operatorname{supp}\rho\subset(0,1)$, $\int\rho=1$, and consider the regularized functions

$$\pi_j(t):=j\int_0^{\frac{1}{j}}\chi(t-x)\rho(jx)\,dx.$$

These enjoy the following properties: $\chi\leq\pi_j$, $\pi_j\in C^\infty(\mathbb{R})\cap\mathcal{A}_M$, and $\pi_j(t)$ tends to $\chi(t)$ for all $t\in\mathbb{R}^-$ outside a countable set. We modify π_j at discontinuities to obtain $\tilde{\pi}_j\in C^\infty(\mathbb{R})\cap\mathcal{A}_M$ with $\tilde{\pi}_j\geq\chi$ and $\tilde{\pi}_j\to\chi$ everywhere on $\mathbb{R}^-$.

Applying the smooth case to $\tilde{\pi}_j$ gives

$$\int_\Omega\chi(u)\,H_m(\varphi)\leq\liminf_{j\to+\infty}\int_\Omega\tilde{\pi}_j(u)\,H_m(\varphi)\leq\liminf_{j\to+\infty}(A\delta_{m,\tilde{\pi}_j}(u)+B\delta_{m,\tilde{\pi}_j}(\varphi))\leq A\delta_{m,\chi}(u)+B\delta_{m,\chi}(\varphi).$$

Thus, condition (b) holds with the same constants A and B. □

To illustrate the practical utility of Theorem 4, we present below an explicit example demonstrating how the functional inequality condition can be applied to solve complex Hessian equations within the weighted class $\mathcal{F}_{m,\chi}(\Omega)$.

Example 2. *Let $\Omega\subset\mathbb{C}^n$ be a bounded m-hyperconvex domain, and $\chi(t)=(-t)^p$, $t\leq 0$, with $p\geq 1$. Fix $\tau\in\mathcal{E}_m^0(\Omega)$ and let*

$$\mu=f\,H_m(\tau),\qquad f\in L^\infty(\Omega),\ f\geq 0.$$

For any $u\in\mathcal{E}_m^0(\Omega)$,

$$\int_\Omega\chi(u)\,d\mu\leq\|f\|_\infty\int_\Omega(-u)^pH_m(\tau).$$

By Lemma 3.5 of Lu [18], there exists $D_p > 0$

$$\int_\Omega (-u)^p H_m(\tau) \le D_p(\delta_{m,\chi}(u))^{\frac{p}{m+p}}(\delta_{m,\chi}(\tau))^{\frac{m}{m+p}}.$$

Since $\delta_{m,\chi}(\tau) < \infty$, there exists $C_\tau > 0$ such that

$$\int_\Omega (-u)^p H_m(\tau) \le C_\tau(\delta_{m,\chi}(u))^\alpha, \qquad \alpha = \frac{p}{m+p} \in (0,1).$$

By Young's inequality, for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$(\delta_{m,\chi}(u))^\alpha \le \varepsilon\delta_{m,\chi}(u) + C_\varepsilon.$$

Consequently,

$$\int_\Omega \chi(u)\, d\mu \le \|f\|_\infty C_\tau \varepsilon\, \delta_{m,\chi}(u) + \|f\|_\infty C_\tau C_\varepsilon.$$

Choosing $\varepsilon > 0$ small enough so that $\|f\|_\infty C_\tau \varepsilon < 1$, condition (b) *of Theorem 4 holds. Hence, there exists $\varphi \in \mathcal{F}_{m,\chi}(\Omega)$ such that $H_m(\varphi) = \mu$.*

The following example shows that the hypothesis "μ puts no mass on m-polar sets" used in Theorem 4 is essential; without it, the equivalence between (a) and (b) breaks down.

Example 3. *Suppose that $m \ge 2$. Take $\chi(t) = 1$ for all $t \le 0$, so that $\mathcal{F}_{m,\chi}(\Omega) = \mathcal{F}_m(\Omega)$.*

Let $E \subset \Omega$ be a compact smooth complex curve, which is m-polar. Define μ as the area measure in E normalized to $\mu(E) = 1$; thus, μ charges an m-polar set.

For any $u \in \mathcal{E}_m^0(\Omega)$, the function u is bounded on E, and $\chi(u) = 1$ on E. Hence,

$$\int_\Omega \chi(u)\, d\mu = 1 \le 0 \cdot \delta_{m,\chi}(u) + 1.$$

So condition (b) of Theorem 4 holds with $A_\mu = 0$, $B_\mu = 1$.

Suppose that there exists $\varphi \in \mathcal{F}_{m,\chi}(\Omega)$ with $H_m(\varphi) = \mu$. Approximate φ by a decreasing sequence $(\varphi_j) \subset \mathcal{E}_m^0(\Omega)$ with $\sup_j \int_\Omega H_m(\varphi_j) < \infty$. Since each φ_j is bounded, $H_m(\varphi_j)(E) = 0$. The weak convergence $H_m(\varphi_j) \to H_m(\varphi)$ then gives $H_m(\varphi)(E) = 0$, contradicting $\mu(E) = 1$.

As this example illustrates, further sufficient conditions are required to guarantee a solution to equation (E) when μ is a nonnegative measure on Ω in general (which means μ may have mass on some $m-$polar set). This is the objective of the following theorem.

Theorem 5. *We have the equivalence of the following:*

(a) $\mu = H_m(\varphi)$ for some $\varphi \in \mathcal{F}_{m,\chi}(\Omega)$.

(b) There exist constants $0 \le A_\mu < 1$ and $B_\mu \in]0, +\infty[$ satisfying for all $u \in \mathcal{E}_m^0(\Omega)$,

$$\int_\Omega \chi(u)\, d\mu \le A_\mu \delta_{m,\chi}(u) + B_\mu,$$

and for each point $z \in \Omega$ there is an $m-$hyperconvex neighborhood U of z and $\varphi_z \in \mathcal{E}_m(U)$ satisfying $\mu \le H_m(\varphi_z)$ on U.

Proof. **Proof of (a) ⇒ (b).** Assume $\varphi \in \mathcal{F}_{m,\chi}(\Omega)$ solves $H_m(\varphi) = \mu$. The inclusion $\mathcal{F}_{m,\chi}(\Omega) \subset \mathcal{E}_m(\Omega)$ implies $\varphi \in \mathcal{E}_m(\Omega)$. The estimate in (b) follows directly from Theorem 1 (with the same constants A_μ, B_μ), while the local majorization condition is satisfied by taking $U = \Omega$ and $\varphi_z = \varphi$.
Proof of (b) ⇒ (a). Assume that (b) holds. According to Theorem 5.3 in [18], we can decompose μ as

$$\mu = fH_m(\xi) + \nu,$$

where $\xi \in \mathcal{E}^0_m(\Omega)$, $f \in L^1_{\text{loc}}(H_m(\xi))$, and ν is a measure that vanishes outside of a certain Borel $m-$polar set $E \subset \Omega$.

We treat two cases depending on the behavior of χ at $-\infty$.

Case 1: $\chi(-\infty) = +\infty$**.** In this situation, Proposition 4.4 of [22] tells us that $\mathcal{F}_{m,\chi}(\Omega) \subset \mathcal{E}^a_m(\Omega)$. We first verify that μ does not have mass on any $m-$polar set. Let $E \subset \Omega$ be any $m-$polar set. By Theorem 3 we can pick $\varphi_0 \in \mathcal{F}_{m,\chi}(\Omega)$ with $E \subset \{\varphi_0 = -\infty\}$. Applying the inequality in (b) to the sequence in $\mathcal{E}^0_m(\Omega)$ associated to φ_0 and using the continuity of $\delta_{m,\chi}$ with the monotone convergence theorem, we deduce that

$$\int_\Omega \chi(\varphi_0)\, d\mu \le A_\mu \delta_{m,\chi}(\varphi_0) + B_\mu < \infty.$$

Consequently, for every $j \in \mathbb{N}$,

$$\mu(E) \le \mu(\{\varphi_0 < -j\}) \le \frac{1}{\chi(-j)} \int_{\{\varphi_0 < -j\}} \chi(\varphi_0)\, d\mu \le \frac{1}{\chi(-j)}(A_\mu \delta_{m,\chi}(\varphi_0) + B_\mu) \xrightarrow[j\to\infty]{} 0.$$

Hence, $\mu(E) = 0$, so μ puts no mass on $m-$polar sets. Now, Theorem 4 guarantees the existence of $\varphi \in \mathcal{F}_{m,\chi}(\Omega)$ with $H_m(\varphi) = \mu$.

Case 2: $\chi(-\infty) < +\infty$**.** Since

$$\mu = fH_m(\xi) + \nu.$$

We will prove that each of the measure on the righthand side of the above equality can be written as the Hessian of a function in $\mathcal{F}_{m,\chi}(\Omega)$. We have

$$\int_\Omega \chi(u) f H_m(\xi) \le \int_\Omega \chi(u)\, d\mu \le A_\mu \delta_{m,\chi}(u) + B_\mu \qquad (u \in \mathcal{E}^0_m(\Omega)).$$

Since the measure $fH_m(\xi)$ puts no mass on $m-$polar sets, then Theorem 4 yields the existence of a function $g \in \mathcal{F}_{m,\chi}(\Omega)$ such that $H_m(g) = fH_m(\xi)$.

Let us now focus on the measure ν. Using the hypothesis (b), for each $z \in \Omega$ we choose an $m-$hyperconvex neighborhood U_z and a function $\varphi_z \in \mathcal{E}_m(U_z)$ with $\nu|_{U_z} \le \mu|_{U_z} \le H_m(\varphi_z)$. Take an $m-$hyperconvex open set $V_z \Subset U_z$ and define

$$\tau_z = \sup\{\tau \in SH^-_m(U_z) : \tau \le \varphi_z \text{ on } V_z\} \in \mathcal{F}_m(U_z).$$

As $\tau_z = \varphi_z$ on V_z, we get $\nu|_{V_z} \le H_m(\tau_z)$. Lemma 3.2 in [12] provides a subextension $\tilde{\tau}_z \in \mathcal{F}_m(\Omega)$ such that $H_m(\tilde{\tau}_z) \le \mathbf{1}_{U_z} H_m(\tau_z)$.

Select $\{z_p\}_{p\in\mathbb{N}} \subset \Omega$ and $\varepsilon_p := 2^{-p}\Big(\int_{U_{z_p}} H_m(\varepsilon_p \tau_{z_p})\Big)^{-1/m}$.

Define $\tilde{\tau} := \sum_{p=1}^{\infty} \varepsilon_p \tilde{\tau}_{z_p}$. Using the subadditivity of the Hessian mass, we obtain

$$\Big[\int_\Omega H_m\Big(\sum_{p=1}^{k} \varepsilon_p \tilde{\tau}_{z_p}\Big)\Big]^{1/m} \leq \sum_{p=1}^{k}\Big(\int_\Omega H_m(\varepsilon_p \tilde{\tau}_{z_p})\Big)^{1/m} \leq \sum_{p=1}^{k} \varepsilon_p\Big(\int_{U_{z_p}} H_m(\tau_{z_p})\Big)^{1/m} = \sum_{p=1}^{k} \frac{1}{2^p} \leq 1,$$

hence, $\tilde{\tau} \in \mathcal{F}_m(\Omega)$. Moreover, on V_{z_p} we have $\nu \leq \varepsilon_p^{-m} H_m(\tilde{\tau})$ for every $p \in \mathbb{N}$.

By Radon-Nykodim theorem and Lemma 1 in [13], we can write $\nu = \theta H_m(\tilde{\tau})$ for some $\theta \in L^\infty_{\text{loc}}(\Omega)$. Take $g_j := \frac{1}{j} \min(\theta, j) 1_{\{\tilde{\tau}=-\infty\}}$ then $0 \leq g_j \leq 1$. Applying Theorem 4.3 in [9] on g_j gives a sequence $h_j \in \mathcal{E}_m(\Omega)$ with

$$H_m(h_j) = \min(\theta, j)\, H_m(\tilde{\tau}).$$

Since $\tilde{\tau} \in \mathcal{F}_m(\Omega)$, then $h_j \in \mathcal{F}_m(\Omega)$, and it decreases, by the Comparison principle, to $h \in SH_m^-(\Omega)$. Applying hypothesis (b) to h_j yields

$$\delta_{m,\chi}(h_j) \leq \int_\Omega \chi(h_j)\theta H_m(\tilde{\tau}) = \int_\Omega \chi(h_j)\, d\nu \leq \int_\Omega \chi(h_j)\, d\mu \leq A_\mu \delta_{m,\chi}(h_j) + B_\mu.$$

Thus, $\delta_{m,\chi}(h_j) \leq \frac{B_\mu}{1-A_\mu}$ for all $j \in \mathbb{N}$. Because ν vanishes outside E, the same holds for $H_m(h_j)$; consequently,

$$\delta_{m,\chi}(h_j) = \int_{\{h_j=-\infty\}} \chi(h_j) H_m(h_j) = \chi(-\infty) \int_\Omega H_m(h_j).$$

Hence,

$$\sup_{j\geq 1} \int_\Omega H_m(h_j) \leq \frac{B_\mu}{(1-A_\mu)\chi(-\infty)} < \infty.$$

This shows that $h_j \searrow h$ in $\mathcal{F}_m(\Omega)$ and $H_m(h) = \nu$.

Finally, from the decomposition $\mu = f H_m(\xi) + \nu$ we obtain

$$\mu = H_m(g) + H_m(h) \leq H_m(g + h).$$

Theorem 4.7 of [9] provides $\varphi \in \mathcal{E}_m(\Omega)$ with $h + g \leq \varphi$ and $H_m(\varphi) = \mu$. Since $\chi(-\infty) < +\infty$, the class $\mathcal{F}_m(\Omega)$ is contained in $\mathcal{F}_{m,\chi}(\Omega)$; therefore, $h \in \mathcal{F}_{m,\chi}(\Omega)$ and, consequently, $\varphi \in \mathcal{F}_{m,\chi}(\Omega)$. This completes the proof. □

Remark 2. *(1) For the plurisubharmonic case $m = n$, the previous result recovers the characterizations of Benelkourchi [3] and Quy [20] for weighted Monge–Ampère equations. However, Theorem 5 applies to the full range $1 \leq m \leq n$.*

(2) Compared to the recent work of Hbil [14] and Zaway and Hbil [21] on weighted m-subharmonic classes, Theorem 5 provides a necessary and sufficient local condition (the existence of local dominants φ_z) that was missing in those studies. In particular, [14] only gave a global inequality without addressing measures charging m-polar sets. Theorem 5 fills this gap by handling the general case $\chi(-\infty) < +\infty$ via a decomposition $\mu = f H_m(\xi) + \nu$ and a delicate subextension argument.

5. Local characterizations

We now provide local descriptions of the range of the Hessian operator acting on the classes $\mathcal{F}_{m,\chi}(\Omega)$. These results strengthen Theorem 5 by working with points z in the closure $\overline{\Omega}$ and dispensing with the global estimate condition.

Theorem 6. *The statements below are pairwise equivalent:*

(a) $\mu = H_m(\varphi)$ *for some* $\varphi \in \mathcal{F}_{m,\chi}(\Omega)$.
(b) Locally, for any $z \in \overline{\Omega}$ *and any m-hyperconvex neighborhood U of z, the restriction* $\mu|_{\Omega\cap U}$ *coincides with* $H_m(\tau)$ *for a suitable* $\tau \in \mathcal{F}_{m,\chi}(\Omega \cap U)$.
(c) For every $z \in \overline{\Omega}$*, there exists an m-hyperconvex neighborhood U of z and a function* $\tau \in \mathcal{F}_{m,\chi}(\Omega \cap U)$ *satisfying* $\mu|_{\Omega\cap U} = H_m(\tau)$.

Proof. **(a) ⇒ (b).** Suppose there is $\varphi \in \mathcal{F}_{m,\chi}(\Omega)$ that solves $H_m(\varphi) = \mu$. Fix $z \in \overline{\Omega}$ and an $m-$hyperconvex neighborhood U of z. We distinguish two possibilities for the weight χ.

If $\chi(-\infty) = +\infty$. By Proposition 4.4 [22], the measure μ does not charge Borel $m-$polar sets. Theorem 5.3 in [18] gives a representation $\mu = fH_m(\xi)$ on $\Omega \cap U$, where $\xi \in \mathcal{E}^0_m(\Omega \cap U)$ and $f \in L^1_{\mathrm{loc}}(H_m(\xi))$. Applying Lemma 5.1 of [18], we obtain a decreasing sequence $(\tau_j) \subset \mathcal{E}^0_m(\Omega \cap U)$ with

$$H_m(\tau_j) = \min(f, j)\, H_m(\xi).$$

Since $H_m(\tau_j) \le H_m(\varphi)$, the the comparison principle yields $\tau_j \searrow \tau \ge \varphi|_{\Omega\cap U}$ and $H_m(\tau) = \mu|_{\Omega\cap U}$. Moreover,

$$\sup_{j\ge1} \int_{\Omega\cap U} \chi(\tau_j)\, H_m(\tau_j) \le \int_{\Omega\cap U} \chi(\varphi)\, H_m(\varphi) < \infty,$$

hence, $\tau \in \mathcal{F}_{m,\chi}(\Omega \cap U)$.

If $\chi(-\infty) < +\infty$. Then,

$$\int_{\{\varphi=-\infty\}} H_m(\varphi) \le \frac{1}{\chi(-\infty)} \int_{\Omega} \chi(\varphi)\, H_m(\varphi) < \infty.$$

Following the same technique as in the proof of Theorem 5.8 in [17] (namely, approximating $\mathbf{1}_{\{\varphi=-\infty\}\cap U}$ by compactly supported functions and using Proposition 4.7 of [17]), we conclude the existence of a function $v \in \mathcal{F}_m(\Omega \cap U)$ such that

$$H_m(v) = \mathbf{1}_{\{\varphi=-\infty\}\cap U}\, H_m(\varphi).$$

As in the previous case, we can construct a decreasing sequence $u_j \in \mathcal{E}^0_m(\Omega \cap U)$ with $u_j \ge \varphi|_{U\cap\Omega}$ and

$$H_m(u_j) \nearrow \mathbf{1}_{\{\varphi>-\infty\}\cap U}\, H_m(\varphi).$$

Define

$$\tau_j = \sup\{\, u \in \mathcal{E}_m(U \cap \Omega) : H_m(u_j) \le H_m(u),\ u \le v\,\}.$$

Since $H_m(v) \le H_m(\varphi)$ and $H_m(u_j) \le H_m(v)$, then $\varphi \le \tau_j$. Proposition 4.6 of [9] tells us that $H_m(\tau_j) = H_m(u_j) + H_m(v)$ and, hence, $u_j + v \le \tau_j \le v$. Because $H_m(u_j)$ increases, $\tau_j \searrow \tau \ge \varphi + v$ and $H_m(\tau_j)$ converges to $\mathbf{1}_{U\cap\Omega} H_m(\varphi)$, so $H_m(\tau) = \mathbf{1}_{U\cap\Omega} H_m(\varphi)$. Furthermore,

$$\sup_{j\ge1} \int_{U\cap\Omega} \chi(u_j) H_m(u_j) \le \int_{U\cap\Omega} \chi(\varphi) \mathbf{1}_{\{\varphi>-\infty\}} H_m(\varphi) < \infty,$$

hence $u_j \searrow h \in \mathcal{F}_{m,\chi}(U \cap \Omega)$. Since $\chi(-\infty) < \infty$, we have $v \in \mathcal{F}_m(U \cap \Omega) \subset \mathcal{F}_{m,\chi}(U \cap \Omega)$ and $h + v \leq \tau \leq v$; consequently, $\tau \in \mathcal{F}_{m,\chi}(U \cap \Omega)$.

(b) ⇒ (c). Trivial, because (b) is stronger than (c).

(c) ⇒ (a). Assume (c) holds. We shall verify the hypotheses of Theorem 5; it is enough to produce constants $A_\mu \in [0, 1)$, $B_\mu > 0$ satisfying

$$\int_\Omega \chi(u)\, d\mu \leq A_\mu \delta_{m,\chi}(u) + B_\mu \qquad (u \in \mathcal{E}^0_m(\Omega)).$$

Because $\overline{\Omega}$ is compact, we may cover it by finitely many $m-$hyperconvex sets $\Omega_1, \ldots, \Omega_p$. For each $j \in \{1, ..., p\}$ there exist an $m-$hyperconvex subset $U_j \Subset \Omega_j$ and a function $\tau_j \in \mathcal{F}_{m,\chi}(\Omega_j \cap \Omega)$ with $\overline{\Omega} \subset \bigcup_{j=1}^p U_j$ and $\mu|_{\Omega_j \cap \Omega} = H_m(\tau_j)$. Set

$$\varphi_j = \sup\{\, \tau \in SH^-_m(\Omega_j \cap \Omega) : \tau \leq \tau_j \text{ on } U_j \cap \Omega \,\} \in \mathcal{F}_{m,\chi}(\Omega_j \cap \Omega).$$

Then, $\mu|_{U_j \cap \Omega} \leq H_m(\varphi_j)$ and $\operatorname{supp} H_m(\varphi_j) \subset \overline{U_j} \cap \Omega$.

Given $u \in \mathcal{E}^0_m(\Omega)$, define

$$u_j = \sup\{\, \tau \in SH^-_m(\Omega_j \cap \Omega) : \tau \leq u \text{ on } \overline{U_j} \cap \Omega \,\} \in \mathcal{E}^0_m(\Omega_j \cap \Omega).$$

Using Lemma 4 and the reasoning of Theorem 2 in [16], we obtain the existence of a constant $C_m > 0$

$$\int_{\Omega_j \cap \Omega} \chi(u_j) H_m(u_j) \leq C_m \int_\Omega \chi(u) H_m(u).$$

Theorem 5, applied to the domain $\Omega_j \cap \Omega$ provides a pair of constants (A'_j, B'_j) with $A'_j \in [0, 1)$ and $B'_j > 0$, such that for every $w \in \mathcal{E}^0_m(\Omega_j \cap \Omega)$,

$$\int_{\Omega_j \cap \Omega} \chi(w) d\mu \leq A'_j \delta_{m,\chi}(w) + B'_j.$$

To obtain an estimate with a sufficiently small constant multiplying $\delta_{m,\chi}$, we examine the origin of A'_j in the proof of Theorem 4. There it is shown that one can actually achieve any prescribed constant $A \in (0, 1)$ by choosing the free parameter $\varepsilon > 0$ small enough in the estimate

$$A = \varepsilon^m M,$$

at the possible expense of increasing the additive constant. Therefore, there exists

$$A_j \leq \min\Big(\frac{1}{2pC_m}, \frac{1}{2p}\Big),$$

such that

$$\begin{aligned}
\int_\Omega \chi(u)\, d\mu \;&\leq\; \sum_{j=1}^p \int_{U_j \cap \Omega} \chi(u)\, d\mu \leq \sum_{j=1}^p \int_{U_j \cap \Omega} \chi(u_j)\, H_m(\varphi_j).\\
&\leq\; \sum_{j=1}^p \int_{\Omega_j \cap \Omega} \chi(u_j)\, H_m(\varphi_j) \leq \sum_{j=1}^p A_j \delta_{m,\chi}(u_j) + B_j\\
&\leq\; \sum_{j=1}^p A_j C_m \delta_{m,\chi}(u) + B_j \leq \sum_{j=1}^p \frac{1}{2p} \delta_{m,\chi}(u) + B_j = \frac{1}{2} \delta_{m,\chi}(u) + D
\end{aligned}$$

where $D = \sum_{j=1}^p B_j$. Thus, Theorem 5 yields a function $\varphi \in \mathcal{F}_{m,\chi}(\Omega)$ with $H_m(\varphi) = \mu$, which is exactly statement (a). □

6. Conclusions

This paper provided a complete characterization of the range of the complex Hessian operator on the weighted energy class $\mathcal{F}_{m,\chi}(\Omega)$, establishing that solvability of $H_m(\varphi) = \mu$ is equivalent to a global functional inequality together with a local condition. We also proved that every m-polar set can be captured within the singular locus of such functions. These results unify and extend previous work on complex Monge–Ampère and Hessian equations. As perspectives, extending these characterizations to non-compact manifolds and studying the stability of solutions under convergence of measures.

Author contributions

Jawhar Hbil: Conceptualization of the weighted energy class $\mathcal{F}_{m,\chi}(\Omega)$ and the complex Hessian equation problem, development of the global characterization theorem (Theorem 4) relating solvability to the functional inequality, and writing original draft.

Hadi Obaid Alshammari: Development of the local characterization theorem (Theorems 5 and 6), analysis of m-polar sets and their inclusion in $\mathcal{F}_{m,\chi}(\Omega)$ (Theorem 3), and Hessian measure estimates.

Mohamed Zaway: Development of the approximation and regularization techniques for the weighted Hessian operator, analysis of the comparison principle and stability under convergence, and writing, review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Authors declare no conflicts of interest.

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