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*Research article*

# Well-posedness and stability of a coupled suspension bridge system with distributed internal feedback

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**Abstract:** This paper studies a coupled suspension bridge system featuring a unilateral nonlinear coupling of positive-part type and distributed delay feedback acting on the velocity components. Under appropriate assumptions linking the instantaneous damping coefficients and the delay kernels, we prove well-posedness and exponential stability via a carefully designed Lyapunov functional and multiplier technique. This work extends the existing theory by treating, for the first time in this setting, the combined effect of nonlinear unilateral coupling and distributed delay feedback.

**Keywords:** well-posedness; energy decay; suspension bridge; distributed delay; Lyapunov functional
**Mathematics Subject Classification:** 35B35, 35L51, 74D10, 93D15

## 1. Introduction

We consider a coupled suspension bridge model describing the transverse motion of the bridge deck and the vertical displacement of the sustaining cables. Let $\phi = \phi(x,t)$ denote the transverse displacement of the bridge deck and let $\varphi = \varphi(x,t)$ represent the displacement of the cable where $x \in (0,\ell)$ and $t > 0$. The evolution of the system is governed by the following coupled equations:

$$m_1\phi_{tt} + b_1\phi_{xxxx} + k[\phi - \varphi]^+ + \delta_1\phi_t + a(x)\int_{t-\tau_2}^{t-\tau_1} \delta_2(t-\tau)\,\phi_t(\tau)\,d\tau = 0, \tag{1.1}$$

$$m_2\varphi_{tt} - b_2\varphi_{xx} - k[\phi - \varphi]^+ + \beta_1\varphi_t + b(x)\int_{t-\tau_2}^{t-\tau_1} \beta_2(t-\tau)\,\varphi_t(\tau)\,d\tau = 0. \tag{1.2}$$

Here, $\ell > 0$ denotes the span length of the bridge. The functions $\phi = \phi(x,t)$ and $\varphi = \varphi(x,t)$ describe, respectively, the transverse motion of the bridge deck and the displacement of the cable. The positive parameters $m_1, m_2 > 0$ denote the mass densities of the deck and cable, while $b_1$ and $b_2$ characterize the stiffness coefficients of the deck and the cable, respectively. The constant $k > 0$ measures the strength of the coupling between the deck and the cable through the nonlinear term $[\phi-\varphi]^+ := \max\{\phi-\varphi, 0\}$, which models a unilateral restoring force acting only when $\phi > \varphi$. The instantaneous damping coefficients are given by $\delta_1 > 0, \beta_1 > 0$, and the distributed internal feedback is introduced through the delay integrals acting on the velocities $\varphi_t$ and $\phi_t$. The functions $a(x)$, $b(x) \in L^\infty(0, \ell)$ are nonnegative spatial weights describing the distribution of the delayed feedback. The delay interval is bounded and determined by two constants $0 < \tau_1 < \tau_2$, and the kernel functions $\delta_2(\cdot), \beta_2(\cdot) \in L^1(\tau_1, \tau_2)$ describe the distributed delay influence.

The systems (1.1) and (1.2) are complemented with the hinged boundary conditions for the deck

$$\phi(0, t) = \phi(\ell, t) = \phi_{xx}(0, t) = \phi_{xx}(\ell, t) = 0, \quad t \geq 0, \tag{1.3}$$

and homogeneous Dirichlet boundary conditions for the cable

$$\varphi(0, t) = \varphi(\ell, t) = 0, \quad t \geq 0. \tag{1.4}$$

The initial conditions are prescribed as

$$\varphi(x, 0) = \varphi_0(x), \quad \varphi_t(x, 0) = \varphi_1(x), \quad \phi(x, 0) = \phi_0(x), \quad \phi_t(x, 0) = \phi_1(x), \quad x \in (0, \ell). \tag{1.5}$$

Since our model contains distributed delay terms involving past velocities, we further prescribe the history data

$$\varphi_t(x, t) = f_0(x, t), \quad \phi_t(x, t) = g_0(x, t), \quad (x, t) \in (0, \ell) \times (-\tau_2, 0), \tag{1.6}$$

where $f_0$ and $g_0$ describe the past states of the velocity components.

The study of suspension bridge models has remained a central topic in nonlinear structural dynamics, primarily due to the intricate coupling mechanisms between the deck and the supporting cables. The foundational model $(1.2)_1$, introduced by Lazer and McKenna [1] (without the last term), was proposed to describe nonlinear oscillatory phenomena in suspension bridges. Since its introduction, a substantial body of work has focused on the mathematical analysis of this class of systems, including results on well-posedness, qualitative behavior of solutions, and numerical simulations. For representative contributions, we refer to [2–5] and the references cited therein. Furthermore, the analytical properties of coupled suspension bridge systems have been explored in several directions. In [6, 7], the authors proved the existence of strong solutions and established the presence of global attractors. More recently, Mukiawa et al. [8] examined a nonlinear coupled suspension bridge model closely related to (1.1) and (1.2), but without distributed delay effects. They demonstrated that the dissipation generated by the bridge's infinite memory, together with thermal effects described by the Gurtin–Pipkin non-Fourier heat law in the cables, yields sufficient damping to ensure stability of the system. In another contribution, Mukiawa and Messaoudi [9] analyzed a different nonlinear coupled suspension bridge model incorporating both infinite memory and external forcing, proving well-posedness and the existence of a global attractor. Additionally, the influence of distributed delay terms has been investigated in related structures such as Timoshenko and laminated beams; see [10–12] and the references therein.

We note that two main Lyapunov-based frameworks have been developed for the stability analysis of delay systems: the Lyapunov–Krasovskii approach, which operates on the full history of the state, and the Lyapunov–Razumikhin approach, which works with functions of the current state only; see, e.g., [13, 14] and the references therein. In the present infinite-dimensional partial differential equation setting, the Lyapunov–Krasovskii framework is the natural choice, as it accommodates the distributed delay structure through the transport-type reformulation. For related stability and controllability results in infinite-dimensional systems, we also refer the reader to [15, 16].

In this work, under the assumption

$$n_0 := \delta_1 - \|a\|_\infty \int_{\tau_1}^{\tau_2} |\delta_2(\tau)| d\tau > 0, \; n_1 := \beta_1 - \|b\|_\infty \int_{\tau_1}^{\tau_2} |\beta_2(\tau)| d\tau > 0, \tag{1.7}$$

we prove the well-posedness and exponential stability of systems (1.1)–(1.6). This condition requires the instantaneous damping coefficients $\delta_1$ and $\beta_1$ to be strong enough to dominate the cumulative effect of the distributed delay feedback over the interval $[\tau_1, \tau_2]$. Simply put, the direct damping must outweigh the total delayed influence. This balance ensures that the delay does not destabilize the system, and that guarantees that $n_0$ and $n_1$ are positive, which are the key quantities driving the dissipation relation (3.2). We note that condition (1.7) is sufficient but not known to be necessary in general. Whether it can be relaxed, for instance by exploiting weighted norms in the history space or by considering sign-varying kernels $\delta_2$ and $\beta_2$, remains an interesting open question. The remainder of this paper is organized as follows. Section 2 introduces suitable history variables that allow the system to be reformulated as an abstract Cauchy problem in an appropriately defined extended Hilbert space. The global existence and uniqueness of solutions are then established via nonlinear semigroup arguments. In Section 3, we construct a Lyapunov functional equivalent to the natural energy and establish an exponential decay estimate under condition (1.7). Section 4 provides a numerical illustration that confirms the exponential energy decay predicted by Theorem 3.1.

## 2. Problem transformation and well-posedness

To handle the distributed delay terms appearing in the model, we introduce auxiliary variables that encode the past history of the velocity components. Following a standard transport-type reformulation (see, e.g., [17]), we define, for $(x, \rho, \tau, t) \in (0, \ell) \times (0, 1) \times (\tau_1, \tau_2) \times \mathbb{R}_+$,

$$\begin{aligned} w(x, \rho, \tau, t) &= \phi_t(x, t - \tau\rho), \\ z(x, \rho, \tau, t) &= \varphi_t(x, t - \tau\rho). \end{aligned} \tag{2.1}$$

A direct differentiation with respect to $t$ and $\rho$ shows that the newly introduced variables satisfy the first-order transport equations

$$\begin{cases} w_t(x, \rho, \tau, t) + \dfrac{1}{\tau} w_\rho(x, \rho, \tau, t) = 0, \\ z_t(x, \rho, \tau, t) + \dfrac{1}{\tau} z_\rho(x, \rho, \tau, t) = 0, \end{cases} \quad (x, \rho, \tau, t) \in (0, \ell) \times (0, 1) \times (\tau_1, \tau_2) \times \mathbb{R}_+. \tag{2.2}$$

Using the change of variable $y = t - \tau$ and on account of (2.1) and (2.2), the original systems (1.1) and (1.2) become

$$\begin{cases} m_1\phi_{tt} + b_1\phi_{xxxx} + k\,[\phi-\varphi]^+ + \delta_1\phi_t + a(x)\displaystyle\int_{\tau_1}^{\tau_2} \delta_2(\tau)w\,(x,1,\tau)\ d\tau = 0,\ (x,t)\in(0,\ell)\times\mathbb{R}_+, \\ \tau\, w_t(x,\rho,\tau,t) + w_\rho(x,\rho,\tau,t) = 0, \quad (x,\rho,\tau,t)\in(0,\ell)\times(0,1)\times(\tau_1,\tau_2)\times\mathbb{R}_+, \\ m_2\varphi_{tt} - b_2\varphi_{xx} - k\,[\phi-\varphi]^+ + \beta_1\,\varphi_t + b(x)\displaystyle\int_{\tau_1}^{\tau_2} \beta_2(\tau)\,z(x,1,\tau)\ d\tau = 0,\ (x,t)\in(0,\ell)\times\mathbb{R}_+, \\ \tau\, z_t(x,\rho,\tau,t) + z_\rho(x,\rho,\tau,t) = 0, \quad (x,\rho,\tau,t)\in(0,\ell)\times(0,1)\times(\tau_1,\tau_2)\times\mathbb{R}_+, \end{cases} \tag{2.3}$$

with the transformed boundary conditions

$$\phi(0,t) = \phi_{xx}(0,t) = \phi(\ell,t) = \phi_{xx}(\ell,t) = \varphi(0,t) = \varphi(\ell,t) = 0, \forall t\geq 0, \tag{2.4}$$

and the initial conditions

$$\begin{cases} w(x,0,\tau,t) = \phi_t(x,t),\ \ z(x,0,\tau,t) = \varphi_t(x,t), \quad (x,t)\in(0,\ell)\times\mathbb{R}_+, \\ \quad \phi(x,0) = \phi_0(x),\ \ \phi_t(x,0) = \phi_1(x),\ \ \varphi(x,0) = \varphi_0(x),\ \ \varphi_t(x,0) = \varphi_1(x),\ \ x\in(0,\ell), \\ w(x,\rho,\tau,0) = \phi_t(x,-\tau\rho) = g_0(x,\tau\rho), (x,\rho,\tau)\in(0,\ell)\times(0,1)\times(\tau_1,\tau_2), \\ z(x,\rho,\tau,0) = \varphi_t(x,-\tau\rho) = f_0(x,\tau\rho), (x,\rho,\tau)\in(0,\ell)\times(0,1)\times(\tau_1,\tau_2). \end{cases} \tag{2.5}$$

Define

$$\mathbb{V} = (\phi,\Phi,w,\varphi,\Psi,z)^T,$$

with $\Phi = \phi_t$ and $\Psi = \varphi_t$. The semigroup representation of systems (2.3)–(2.5) is expressed as

$$\begin{cases} \mathbb{V}_t(t) + \mathcal{G}\mathbb{V}(t) = Q(\mathbb{V}), \quad t>0, \\ \mathbb{V}(0) = \mathbb{V}_0 = (\phi_0,\phi_1,g_0,\varphi_0,\varphi_1,f_0)^T, \end{cases} \tag{2.6}$$

$\mathcal{G}$: $\mathcal{D}(\mathcal{G})\subset\mathcal{H}\to\mathcal{H}$ is defined by

$$\mathcal{G}\begin{pmatrix}\phi\\ \Phi\\ w\\ \Psi\\ \varphi\\ z\end{pmatrix} = \begin{pmatrix} -\Phi \\ \dfrac{b_1}{m_1}\phi_{xxxx} + \dfrac{\delta_1}{m_1}\Phi + \dfrac{a(x)}{m_1}\displaystyle\int_{\tau_1}^{\tau_2}\delta_2(\tau)\,w(x,1,\tau)\,d\tau \\ \dfrac{1}{\tau}w_\rho \\ -\Psi \\ -\dfrac{b_2}{m_2}\varphi_{xx} + \dfrac{\beta_1}{m_2}\Psi + \dfrac{b(x)}{m_2}\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(\tau)\,z(x,1,\tau)\,d\tau \\ \dfrac{1}{\tau}z_\rho \end{pmatrix}, \quad Q(\mathbb{V}) = \begin{pmatrix} 0 \\ \dfrac{k}{m_1}[\phi-\varphi]^+ \\ 0 \\ 0 \\ -\dfrac{k}{m_2}[\phi-\varphi]^+ \\ 0 \end{pmatrix}. \tag{2.7}$$

Let

$$H_*^2(0,\ell) := H^2(0,\ell)\cap H_0^1(0,\ell) \tag{2.8}$$

be the space with the inner product

$$\langle u, v\rangle_{H_*^2(0,\ell)} := \int_0^\ell u_{xx}(x)\, v_{xx}(x)\, dx. \tag{2.9}$$

The induced norm is therefore

$$\|u\|^2_{H_*^2(0,\ell)} = \langle u, u\rangle_{H_*^2(0,\ell)} = \int_0^\ell |u_{xx}(x)|^2\, dx.$$

It is well known that this norm is equivalent to the standard $H^2(0,\ell)$-norm on $H_*^2(0,\ell)$; consequently, $(H_*^2(0,\ell), \|\cdot\|_{H_*^2(0,\ell)})$ is a Hilbert space.

For the semigroup formulation of (2.6), we define the underlying Hilbert space as

$$\mathcal{H} := H_*^2(0,\ell)\times L^2(0,\ell)\times L^2((0,\ell)\times(0,1)\times(\tau_1,\tau_2))\times H_0^1(0,\ell)\times L^2(0,\ell)\times L^2((0,\ell)\times(0,1)\times(\tau_1,\tau_2)),$$

and domain of $\mathcal{G}$ is defined by

$$\mathcal{D}(\mathcal{G}) := \left\{ \mathbb{V} = (\phi, \Phi, w, \varphi, \Psi, z) \in \mathcal{H} \;\middle|\; \begin{array}{l} \phi \in H_*^2(0,\ell)\cap H^4(0,\ell), \Phi \in H_*^2(0,\ell) \\ \phi_{xx}(0) = \phi_{xx}(\ell) = 0, \\ \varphi \in H_0^1(0,\ell)\cap H^2(0,\ell),\ \Psi \in H_0^1(0,\ell), \\ w(x,0,\tau) = \Phi, z(x,0,\tau) = \Psi, \\ w, w_y, z, z_y \in L^2((0,\ell)\times(0,1)\times(\tau_1,\tau_2)) \end{array} \right\}.$$

For any

$$\mathbb{V} = (\phi, \Phi, w, \varphi, \Psi, z), \qquad \widehat{\mathbb{V}} = (\widehat{\phi}, \widehat{\Phi}, \widehat{w}, \widehat{\varphi}, \widehat{\Psi}, \widehat{z}) \in \mathcal{H},$$

we endow $\mathcal{H}$ with the inner product defined by

$$\begin{aligned} \langle \mathbb{V}, \widehat{\mathbb{V}}\rangle_{\mathcal{H}} :=& m_1\int_0^\ell \Phi\widehat{\Phi}dx + m_2\int_0^\ell \Psi\widehat{\Psi}dx + b_1\int_0^\ell \phi_{xx}\widehat{\phi}_{xx}dx + b_2\int_0^\ell \varphi_x\widehat{\varphi}_x dx \\ &+ k\int_0^\ell [\phi-\varphi]^+[\widehat{\phi}-\widehat{\varphi}]^+dx + \int_0^\ell a(x)\int_0^1\int_{\tau_1}^{\tau_2} \tau|\delta_2(\tau)|\, w(x,\rho,\tau)\widehat{w}(x,\rho,\tau)d\tau d\rho dx \\ &+ \int_0^\ell b(x)\int_0^1\int_{\tau_1}^{\tau_2} \tau|\beta_2(\tau)|\, z(x,\rho,\tau)\widehat{z}(x,\rho,\tau)d\tau d\rho dx, \end{aligned} \tag{2.10}$$

and norm

$$\begin{aligned} \|\mathbb{V}\|^2_{\mathcal{H}} :=& \Big[m_1\|\Phi\|^2 + m_2\|\Psi\|^2 + b_1\|\phi_{xx}\|^2 + b_2\|\varphi_x\|^2 + k\|[\phi-\varphi]^+\|^2\Big] \\ &+ \int_0^\ell a(x)\int_0^1\int_{\tau_1}^{\tau_2} \tau|\delta_2(\tau)|\, w^2(x,\rho,\tau)d\tau d\rho dx \\ &+ \int_0^\ell b(x)\int_0^1\int_{\tau_1}^{\tau_2} \tau|\beta_2(\tau)|\, z^2(x,\rho,\tau)d\tau d\rho dx. \end{aligned} \tag{2.11}$$

We now state the well-posedness result for system (2.6).

**Theorem 2.1.** *Let*
$$\mathbb{V}_0 = (\phi_0, \phi_1, f_0, \varphi_0, \varphi_1, g_0) \in \mathcal{H}.$$
*Then, the system* (2.6) *possesses a unique global weak solution*
$$\mathbb{V} \in C([0,\infty); \mathcal{H}).$$
*Furthermore, if*
$$\mathbb{V}_0 \in D(\mathcal{G}),$$
*then the corresponding solution satisfies*
$$\mathbb{V} \in C([0,\infty); D(\mathcal{G})) \cap C^1([0,\infty); \mathcal{H}).$$

*Proof.* **Step1: the monotonicity of $\mathcal{G}$.** First, we show that the operator $\mathcal{G}$: $\mathcal{D}(\mathcal{G}) \subset \mathcal{H} \to \mathcal{H}$ defined by (2.7) is monotone. Setting $\mathbb{V} = (\phi, \Phi, w, \varphi, \Psi, z)^T \in \mathcal{D}(\mathcal{G})$, in view of (2.7), (2.10), and applying integration by parts on the $b_1$, $b_2$ terms, we obtain

$$\begin{aligned}
\langle \mathcal{G}\mathbb{V}, \mathbb{V}\rangle_{\mathcal{H}} = {} & \delta_1 \int_0^\ell \Phi^2 \, dx + \beta_1 \int_0^\ell \Psi^2 \, dx \\
& + \int_0^\ell a(x)\Phi(x) \left( \int_{\tau_1}^{\tau_2} \delta_2(\tau) \, w(x,1,\tau) \, d\tau \right) dx \\
& + \int_0^\ell b(x)\Psi(x) \left( \int_{\tau_1}^{\tau_2} \beta_2(\tau) \, z(x,1,\tau) \, d\tau \right) dx \\
& + \int_0^\ell a(x) \int_0^1 \int_{\tau_1}^{\tau_2} |\delta_2(\tau)| \, w_\rho \, w \, d\tau \, d\rho \, dx \\
& + \int_0^\ell b(x) \int_0^1 \int_{\tau_1}^{\tau_2} |\beta_2(\tau)| \, z_\rho \, z \, d\tau \, d\rho \, dx.
\end{aligned} \tag{2.12}$$

Further integration by parts on the last two terms in (2.12), and using the conditions $w(x,0,\tau) = \Phi(x)$, $z(x,0,\tau) = \Psi(x)$, we arrive at

$$\begin{aligned}
\langle \mathcal{G}\mathbb{V}, \mathbb{V}\rangle_{\mathcal{H}} = {} & \delta_1 \int_0^\ell \Phi^2 \, dx + \beta_1 \int_0^\ell \Psi^2 \, dx \\
& + \int_0^\ell a(x)\Phi(x) \left( \int_{\tau_1}^{\tau_2} \delta_2(\tau) \, w(x,1,\tau) \, d\tau \right) dx \\
& + \int_0^\ell b(x)\Psi(x) \left( \int_{\tau_1}^{\tau_2} \beta_2(\tau) \, z(x,1,\tau) \, d\tau \right) dx \\
& + \frac{1}{2} \int_0^\ell a(x) \int_{\tau_1}^{\tau_2} |\delta_2(\tau)| \, w^2(x,1,\tau) \, d\tau \, dx - \frac{1}{2} \left( \int_{\tau_1}^{\tau_2} |\delta_2(\tau)| \, d\tau \right) \int_0^\ell a(x)\Phi^2(x) \, dx \\
& + \frac{1}{2} \int_0^\ell b(x) \int_{\tau_1}^{\tau_2} |\beta_2(\tau)| \, z^2(x,1,\tau) \, d\tau \, dx - \frac{1}{2} \left( \int_{\tau_1}^{\tau_2} |\beta_2(\tau)| \, d\tau \right) \int_0^\ell b(x)\Psi^2(x) \, dx.
\end{aligned} \tag{2.13}$$

Now, applying Cauchy–Schwartz and Young's inequalities to the third and fourth terms of (2.13), we obtain

$$\langle \mathcal{G}\mathbb{V}, \mathbb{V}\rangle_{\mathcal{H}} \geq \left(\delta_1 - \|a\|_\infty \int_{\tau_1}^{\tau_2} |\delta_2(\tau)|\, d\tau\right) \|\Phi\|^2 + \left(\beta_1 - \|b\|_\infty \int_{\tau_1}^{\tau_2} |\beta_2(\tau)|\, d\tau\right) \|\Psi\|^2. \tag{2.14}$$

In view of the conditions in (1.7), we deduce from the last line above, that $\langle \mathcal{G}\mathbb{V}, \mathbb{V}\rangle_{\mathcal{H}} \geq 0, \forall \mathbb{V} \in \mathcal{D}(\mathcal{G})$. Hence, the monotonicity of $\mathcal{G}$ is established.

**Step 2: maximality of $\mathcal{G}$.**

Let

$$Q = (q_1, q_2, q_3, q_4, q_5, q_6)^T \in \mathcal{H}.$$

To establish that $\mathcal{G}$ is maximal, we solve the resolvent equation

$$(I + \mathcal{G})\mathbb{V} = Q \quad \text{in } \mathcal{H}, \tag{2.15}$$

for an unknown

$$\mathbb{V} = (\phi, \Phi, w, \varphi, \Psi, z)^T \in D(\mathcal{G}).$$

Using the definition of $\mathcal{G}$ in (2.7), the problem in (2.15) is equivalent to

$$\begin{cases} \phi - \Phi = q_1, \text{ in } H_*^2(0,\ell), \\ \Phi + \dfrac{b_1}{m_1}\phi_{xxxx} + \dfrac{\delta_1}{m_1}\Phi + \dfrac{a(x)}{m_1}\displaystyle\int_{\tau_1}^{\tau_2} \delta_2(\tau)\, w(x,1,\tau)\, d\tau = q_2, \text{ in } L^2(0,\ell), \\ w + \dfrac{1}{\tau} w_\rho = q_3, \text{ in } L^2((0,\ell)\times(0,1)\times(\tau_1,\tau_2)), \\ \varphi - \Psi = q_4, \text{ in } H_0^1(0,1), \\ \Psi - \dfrac{b_2}{m_2}\varphi_{xx} + \dfrac{\beta_1}{m_2}\Psi + \dfrac{b(x)}{m_2}\displaystyle\int_{\tau_1}^{\tau_2} \beta_2(\tau)\, z(x,1,\tau)\, d\tau = q_5, \text{ in } L^2(0,\ell), \\ z + \dfrac{1}{\tau} z_\rho = q_6 \text{ in } L^2((0,\ell)\times(0,1)\times(\tau_1,\tau_2)). \end{cases} \tag{2.16}$$

From Eq $(2.16)_1$ and $(2.16)_4$, we have

$$\Phi = \phi - q_1, \quad \Psi = \varphi - q_4. \tag{2.17}$$

Solving the first-order equations appearing in $(2.16)_3$ and $(2.16)_6$, and using the boundary conditions,

$$w(x,0,\tau) = \Phi(x), \quad z(x,0,\tau) = \Psi(x),$$

we obtain explicit expressions for $w$ and $z$ in terms of $\Phi$ and $\Psi$ as follows:

$$\begin{cases} w(x,\rho,\tau) = e^{-\tau\rho}\Phi + \tau e^{-\tau\rho}\displaystyle\int_0^\rho e^{\tau s}\, q_3(x,s,\tau) ds, \\ z(x,\rho,\tau) = e^{-\tau\rho}\Psi + \tau e^{-\tau\rho}\displaystyle\int_0^\rho e^{\tau s}\, q_6(x,s,\tau) ds. \end{cases} \tag{2.18}$$

In view of (2.17), we obtain

$$\begin{cases} w(x,\rho,\tau) = e^{-\tau\rho}\phi - e^{-\tau\rho}q_1 + \tau e^{-\tau\rho}\displaystyle\int_0^\rho e^{\tau s}\, q_3(x,s,\tau)ds, \\ z(x,\rho,\tau) = e^{-\tau\rho}\varphi - e^{-\tau\rho}q_4 + \tau e^{-\tau\rho}\displaystyle\int_0^\rho e^{\tau s}\, q_6(x,s,\tau)ds. \end{cases} \tag{2.19}$$

In particular,

$$\begin{cases} w(x,1,\tau) = e^{-\tau}\phi - e^{-\tau}q_1 + \tau e^{-\tau}\displaystyle\int_0^1 e^{\tau s}\, q_3(x,s,\tau)ds, \\ z(x,1,\tau) = e^{-\tau}\varphi - e^{-\tau}q_4 + \tau e^{-\tau}\displaystyle\int_0^1 e^{\tau s}\, q_6(x,s,\tau)ds. \end{cases} \tag{2.20}$$

Substituting (2.17) into $(2.16)_2$, $(2.16)_5$, and multiplying the results by $m_1$ and $m_2$ respectively, we obtain

$$\begin{cases} m_1(\phi - q_1) + b_1\phi_{xxxx} + \delta_1(\phi - q_1) + a(x)\displaystyle\int_{\tau_1}^{\tau_2} \delta_2(\tau)\, w(x,1,\tau)\, d\tau = m_1 q_2, \\ m_2(\varphi - q_4) - b_2\psi_{xx} + \beta_1(\varphi - q_4) + a(x)\displaystyle\int_{\tau_1}^{\tau_2} \beta_2(\tau)\, z(x,1,\tau)\, d\tau = m_2 q_5. \end{cases} \tag{2.21}$$

Furthermore, substituting (2.20) into (2.21), we obtain

$$\begin{cases} b_1\phi_{xxxx} + \alpha_1\,\phi = g_1, \\ -\, b_2\,\varphi_{xx} + \alpha_2\,\varphi = g_2, \end{cases} \tag{2.22}$$

where the terms $\alpha_1, \alpha_2, g_1, g_2$ are given by

$$\begin{cases} \alpha_1 := (m_1 + \delta_1) + a(x)\displaystyle\int_{\tau_1}^{\tau_2} e^{-\tau}\delta_2(\tau)d\tau, \\ g_1 := m_1 q_2 + (m_1 + \delta_1)q_1 - a(x)\left(\displaystyle\int_{\tau_1}^{\tau_2} e^{-\tau}\delta_2(\tau)d\tau\right)q_1 \\ \qquad + a(x)\displaystyle\int_{\tau_1}^{\tau_2} \tau e^{-\tau}\,\delta_2(\tau)\left(\int_0^1 e^{\tau s} q_3(x,s,\tau)ds\right)d\tau, \\ \alpha_2 := (m_2 + \beta_1) + b(x)\displaystyle\int_{\tau_1}^{\tau_2} e^{-\tau}\beta_2(\tau)d\tau, \\ g_2 := m_2 q_5 + (m_2 + \beta_1)q_4 - b(x)\left(\displaystyle\int_{\tau_1}^{\tau_2} e^{-\tau}\beta_2(\tau)d\tau\right)q_4 \\ \qquad + b(x)\displaystyle\int_{\tau_1}^{\tau_2} \tau e^{-\tau}\beta_2(\tau)\left(\int_0^1 e^{\tau s} q_6(x,s,\tau)ds\right)d\tau. \end{cases} \tag{2.23}$$

We now introduce the product space $\mathcal{V} := H_*^2(0,\ell) \times H_0^1(0,\ell)$. We then define the bilinear mapping $\mathcal{B}$: $\mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{R}$ by

$$\mathcal{B}((\phi,\varphi),(\eta,\xi)) = b_1\int_0^\ell \phi_{xx}\eta_{xx}\,dx + b_2\int_0^\ell \varphi_x\xi_x\,dx + \int_0^\ell \alpha_1\,\phi\,\eta\,dx + \int_0^\ell \alpha_2\,\varphi\,\xi\,dx. \tag{2.24}$$

In addition, we introduce the linear functional $\mathcal{L}$: $\mathcal{V} \longrightarrow \mathbb{R}$ defined by

$$\mathcal{L}(\eta, \xi) = \int_0^\ell g_1\, \eta\, dx + \int_0^\ell g_2\, \xi\, dx, \quad (\eta, \xi) \in \mathcal{V}. \tag{2.25}$$

The weak formulation associated with (2.22) is the following: Find $(\phi, \varphi) \in \mathcal{V}$ such that

$$\mathcal{B}((\phi, \varphi), (\eta, \xi)) = \mathcal{L}(\eta, \xi), \quad \forall\, (\eta, \xi) \in \mathcal{V}. \tag{2.26}$$

Applying the Cauchy–Schwarz inequality together with the Poincaré inequality on $H^2_*(0, \ell)$ and $H^1_0(0, \ell)$, one verifies that both $\mathcal{B}$ and $\mathcal{L}$ are continuous on $\mathcal{V}$. More precisely, there exist positive constants $C_1$ and $C_2$ such that

$$\begin{aligned} \left|\mathcal{B}((\phi, \varphi), (\eta, \xi))\right| &\le C_1 \|(\phi, \varphi)\|_{\mathcal{V}} \|(\eta, \xi)\|_{\mathcal{V}}, \\ \left|\mathcal{L}(\eta, \xi)\right| &\le C_2 \|(g_1, g_2)\|_{L^2(0,\ell)\times L^2(0,\ell)} \|(\eta, \xi)\|_{\mathcal{V}}. \end{aligned} \tag{2.27}$$

In addition, $\mathcal{B}$ is coercive, since there exists $C > 0$, such that

$$\mathcal{B}((\phi, \varphi), (\phi, \varphi)) \ge C \|(\phi, \varphi)\|^2_{\mathcal{V}}. \tag{2.28}$$

Since $\mathcal{B}$ is continuous and coercive on $\mathcal{V}$, and $\mathcal{L}$ is continuous, the Lax–Milgram theorem guarantees the existence of a unique element $(\phi, \varphi) \in \mathcal{V}$ satisfying (2.26). Furthermore, relation (2.17) implies that $\Phi \in H^2_*(0, \ell), \Psi \in H^1_0(0, \ell)$. Substituting $\phi$ and $\varphi$ into $(2.20)_1$ and $(2.20)_2$, respectively, we get

$$w, w_p, z, z_p \in L^2((0, \ell) \times (0, 1) \times (\tau_1, \tau_2)) \text{ and } w(x, 0, \tau) = \Psi, \;\; z(x, 0, \tau) = \Phi.$$

Taking $(\phi, \varphi) = (\phi, 0) \in \mathcal{V}$ in (2.26), we obtain

$$b_1 \int_0^\ell \phi_{xx}\eta_{xx} dx + \int_0^\ell [\alpha_1 \phi - g_1]\eta dx = 0, \;\; \forall \eta \in H^2_*(0, \ell). \tag{2.29}$$

It follows that

$$b_1 \phi_{xxxx} = -\alpha_1 \phi + g_1 \in L^2(0, \ell). \tag{2.30}$$

This implies

$$\phi \in H^4(0, \ell) \cap H^2_*(0, \ell).$$

Let $u \in C^2[0, \ell]$ be an arbitrary test function with $u_{xx}(0) = u_{xx}(\ell) = 0$, Eq (2.26) remains valid and we have

$$b_1 \int_0^\ell \phi_{xx} u_{xx} dx + \int_0^\ell [\alpha_1 \phi - g_1] u dx = 0, \;\; \forall u \in C^2[0, \ell], \;\; u_{xx}(0) = u_{xx}(\ell) = 0. \tag{2.31}$$

Integrating by parts in view of (2.33), yields

$$\phi_{xx}(\ell) u_{xx}(\ell) - \phi_{xx}(0) u_{xx}(0) = 0, \;\; \forall u \in C^2[0, \ell], u_{xx}(0) = u_{xx}(\ell) = 0. \tag{2.32}$$

By the arbitrariness of $u$, it follows that

$$\phi_{xx}(0) = \phi_{xx}(\ell) = 0.$$

Similarly, taking $(\phi,\varphi)=(0,\varphi)\in\mathcal{V}$ in (2.26), we obtain

$$b_2\int_0^\ell \varphi_x\xi_x dx+\int_0^\ell[\alpha_2\varphi-g_2]\xi dx=0,\ \forall\xi\in H_0^1(0,\ell). \tag{2.33}$$

It follows that

$$-b_2\varphi_{xx}=-\alpha_2\varphi+g_2\in L^2(0,\ell). \tag{2.34}$$

This implies

$$\varphi\in H^2(0,\ell)\cap H_0^1(0,\ell).$$

Therefore, $\mathbb{V}=(\phi,\Phi,w,\varphi,\Psi,z)\in\mathcal{D}(\mathcal{G})$ and solves (2.15). Hence, the operator $\mathcal{G}$ is maximal.

**Step 3: Lipschitz continuity of $Q$.**

Let $\mathbb{V}_1=(\phi_1,\Phi_1,w_1,\varphi_1,\Psi_1,z_1),\ \mathbb{V}_2=(\phi_2,\Phi_2,w_2,\varphi_2,\Psi_2,z_2)\in\mathcal{H}$. Then, in view of (2.7), we obtain

$$Q(\mathbb{V}_1)-Q(\mathbb{V}_2)=\begin{pmatrix}0\\ \dfrac{k}{m_1}([\phi_1-\varphi_1]^+-[\phi_2-\varphi_2]^+)\\ 0\\ 0\\ -\dfrac{k}{m_2}([\phi_1-\varphi_1]^+-[\phi_2-\varphi_2]^+)\\ 0\end{pmatrix}.$$

Thus, we have

$$\begin{aligned}\|Q(\mathbb{V}_1)-Q(\mathbb{V}_2)\|_{\mathcal{H}}^2&=m_1\left\|\frac{k}{m_1}\Big([\phi_1-\varphi_1]^+-[\phi_2-\varphi_2]^+\Big)\right\|_{L^2(0,\ell)}^2+m_2\left\|\frac{k}{m_2}\Big([\phi_1-\varphi_1]^+-[\phi_2-\varphi_2]^+\Big)\right\|_{L^2(0,\ell)}^2\\&=\left(\frac{k^2}{m_1}+\frac{k^2}{m_2}\right)\left\|[\phi_1-\varphi_1]^+-[\phi_2-\varphi_2]^+\right\|_{L^2(0,\ell)}^2.\end{aligned}$$

It is evident that,

$$\left\|[\phi_1-\varphi_1]^+-[\phi_2-\varphi_2]^+\right\|_{L^2(0,\ell)}\le\|\phi_1-\phi_2\|_{L^2(0,\ell)}+\|\varphi_1-\varphi_2\|_{L^2(0,\ell)}.$$

It follows that, there exists $C\ge 0$, depending on $k$, $m_1,m_2$ such that

$$\|Q(\mathbb{V}_1)-Q(\mathbb{V}_2)\|_{\mathcal{H}}\le C\|\mathbb{V}_1-\mathbb{V}_2\|_{\mathcal{H}},\ \ \forall\mathbb{V}_1,\mathbb{V}_2\in\mathcal{H}.$$

Hence, the nonlinear mapping $Q$: $\mathcal{H}\longrightarrow\mathcal{H}$ is globally Lipschitz continuous. Finally, applying the semi-group theory, see [18, 19], the well-posedness is deduced from **Steps** 1–3.

□

## 3. Stability of solution

**Lemma 3.1.** *Let* $(\phi, w, \varphi, z)$ *denote the solution of* (2.3)–(2.5) *under assumption* (1.7). *The continuous energy functional associated with the systems* (2.3)–(2.5) *is defined as*

$$\begin{aligned} E(t) =& \frac{1}{2}\left[m_1\|\phi_t\|^2 + m_2\|\varphi_t\|^2 + b_1\|\phi_{xx}\|^2 + b_2\|\varphi_x\|^2 + k\|[\phi-\varphi]^+\|^2\right] \\ &+ \frac{1}{2}\int_0^\ell a(x)\int_0^1\int_{\tau_1}^{\tau_2} \tau|\delta_2(\tau)|\, w^2(x,\rho,\tau,t)d\tau d\rho dx \\ &+ \frac{1}{2}\int_0^\ell b(x)\int_0^1\int_{\tau_1}^{\tau_2} \tau|\beta_2(\tau)|\, z^2(x,\rho,\tau,t)d\tau d\rho dx, \end{aligned} \tag{3.1}$$

*and satisfies the dissipative inequality*

$$\frac{d}{dt}\left[E(t)\right] = -n_0\|\phi_t\|^2 - n_1\|\varphi_t\|^2 \le 0, \ \forall t \ge 0, \tag{3.2}$$

*where* $n_0$ *and* $n_1$ *are positive parameters defined in* (1.7).

*Proof.* Testing $(2.3)_1$ and $(2.3)_3$ with $\phi_t$ and $\varphi_t$, respectively, integrating over $(0,\ell)$, and performing integration by parts yield

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(m_1\|\phi_t\|^2 + b_1\|\phi_{xx}\|^2\right) + k\int_0^\ell \phi_t[\phi-\varphi]^+dx + \delta_1\|\phi_t\|^2 + \int_0^\ell \phi_t a(x)\int_{\tau_1}^{\tau_2}\delta_2(\tau)w\left(x,1,\tau\right)\,d\tau dx = 0, \\ &\frac{1}{2}\frac{d}{dt}\left(m_2\|\varphi_t\|^2 + b_2\|\varphi_x\|^2\right) - k\int_0^\ell \varphi_t[\phi-\varphi]^+dx + \beta_1\|\varphi_t\|^2 + \int_0^\ell \varphi_t b(x)\int_{\tau_1}^{\tau_2}\beta_2(\tau)z\left(x,1,\tau\right)\,d\tau dx = 0, \end{aligned} \tag{3.3}$$

Summing the equations in (3.3) yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(m_1\|\phi_t\|^2 + m_2\|\varphi_t\|^2 + b_1\|\phi_{xx}\|^2 + b_2\|\varphi_x\|^2 + k\|[\phi-\varphi]^+\|^2\right) \\ &= -\delta_1\|\phi_t\|^2 - \int_0^\ell \phi_t a(x)\int_{\tau_1}^{\tau_2}\delta_2(\tau)w\left(x,1,\tau\right)\,d\tau dx - \beta_1\|\varphi_t\|^2 \\ &\quad - \int_0^\ell \varphi_t b(x)\int_{\tau_1}^{\tau_2}\beta_2(\tau)z\left(x,1,\tau\right)\,d\tau dx. \end{aligned} \tag{3.4}$$

Next, multiplying $(2.3)_2$ and $(2.3)_4$ by $a(x)|\delta_2(\tau)|w$ and $b(x)|\beta_2(\tau)|z$, respectively, and integrating over $(0,\ell)\times(0,1)\times(\tau_1,\tau_2)$, we arrive at

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\int_0^\ell a(x)\int_0^1\int_{\tau_1}^{\tau_2}\tau|\delta_2(\tau)|\, w^2(x,\rho,\tau,t)d\tau d\rho dx\right] \\ &= -\frac{1}{2}\int_0^\ell a(x)\int_{\tau_1}^{\tau_2}|\delta_2(\tau)|\int_0^1 \frac{\partial}{\partial\rho}\left[w^2(x,\rho,\tau,t)\right]d\rho d\tau dx \\ &= -\frac{1}{2}\int_0^\ell a(x)\int_{\tau_1}^{\tau_2}|\delta_2(\tau)|w^2(x,1,\tau,t)d\tau dx + \frac{1}{2}\int_0^\ell a(x)\int_{\tau_1}^{\tau_2}|\delta_2(\tau)|w^2(x,0,\tau,t)d\tau dx \\ &= -\frac{1}{2}\int_0^\ell a(x)\int_{\tau_1}^{\tau_2}|\delta_2(\tau)|w^2(x,1,\tau,t)d\tau dx + \frac{1}{2}\int_0^\ell a(x)\phi_t^2dx\int_{\tau_1}^{\tau_2}|\delta_2(\tau)|d\tau \end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\int_0^\ell b(x)\int_0^1\int_{\tau_1}^{\tau_2}\tau|\beta_2(\tau)|\, z^2(x,\rho,\tau,t)d\tau d\rho dx\right]\\
&=-\frac{1}{2}\int_0^\ell b(x)\int_{\tau_1}^{\tau_2}|\beta_2(\tau)|\,\int_0^1\frac{\partial}{\partial\rho}\left[z^2(x,\rho,\tau,t)\right]d\rho d\tau dx\\
&=-\frac{1}{2}\int_0^\ell b(x)\int_{\tau_1}^{\tau_2}|\beta_2(\tau)|z^2(x,1,\tau,t)d\tau dx+\frac{1}{2}\int_0^\ell b(x)\int_{\tau_1}^{\tau_2}|\beta_2(\tau)|z^2(x,0,\tau,t)d\tau dx\\
&=-\frac{1}{2}\int_0^\ell b(x)\int_{\tau_1}^{\tau_2}|\beta_2(\tau)|z^2(x,1,\tau,t)d\tau dx+\frac{1}{2}\int_0^\ell b(x)\varphi_t^2dx\,\int_{\tau_1}^{\tau_2}|\beta_2(\tau)|d\tau.
\end{aligned}\tag{3.6}$$

Using (3.4)–(3.6), it follows that

$$\begin{aligned}
\frac{d}{dt}\left[E(t)\right]=&-\delta_1\|\phi_t\|^2-\int_0^\ell\phi_t a(x)\int_{\tau_1}^{\tau_2}\delta_2(\tau)w\left(x,1,\tau\right)\,d\tau dx\\
&-\beta_1\|\varphi_t\|^2-\int_0^\ell\varphi_t b(x)\int_{\tau_1}^{\tau_2}\beta_2(\tau)z\left(x,1,\tau\right)\,d\tau dx\\
&-\frac{1}{2}\int_0^\ell a(x)\int_{\tau_1}^{\tau_2}|\delta_2(\tau)|w^2(x,1,\tau,t)d\tau dx+\frac{1}{2}\int_0^\ell a(x)\phi_t^2dx\,\int_{\tau_1}^{\tau_2}|\delta_2(\tau)|d\tau\\
&-\frac{1}{2}\int_0^\ell b(x)\int_{\tau_1}^{\tau_2}|\beta_2(\tau)|z^2(x,1,\tau,t)d\tau dx+\frac{1}{2}\int_0^\ell b(x)\varphi_t^2dx\,\int_{\tau_1}^{\tau_2}|\beta_2(\tau)|d\tau\\
\le&-\left(\delta_1-\frac{1}{2}\|a\|_\infty\int_{\tau_1}^{\tau_2}|\delta_2(\tau)|d\tau\right)\int_0^\ell\phi_t^2dx-\left(\beta_1-\frac{1}{2}\|b\|_\infty\int_{\tau_1}^{\tau_2}|\beta_2(\tau)|d\tau\right)\int_0^\ell\varphi_t^2dx\\
&-\int_0^\ell\phi_t a(x)\int_{\tau_1}^{\tau_2}\delta_2(\tau)w\left(x,1,\tau\right)\,d\tau dx-\int_0^\ell\varphi_t b(x)\int_{\tau_1}^{\tau_2}\beta_2(\tau)z\left(x,1,\tau\right)\,d\tau dx\\
&-\frac{1}{2}\int_0^\ell a(x)\int_{\tau_1}^{\tau_2}|\delta_2(\tau)|w^2(x,1,\tau,t)d\tau dx-\frac{1}{2}\int_0^\ell b(x)\int_{\tau_1}^{\tau_2}|\beta_2(\tau)|z^2(x,1,\tau,t)d\tau dx.
\end{aligned}\tag{3.7}$$

Applying the Cauchy–Schwarz and Young inequalities, we deduce

$$\begin{cases}
-\int_0^\ell\phi_t a(x)\int_{\tau_1}^{\tau_2}\delta_2(\tau)w\left(x,1,\tau\right)\,d\tau dx\\
\le\frac{1}{2}\int_0^\ell a(x)\phi_t^2dx\,\int_{\tau_1}^{\tau_2}|\delta_2(\tau)|d\tau+\frac{1}{2}\int_0^\ell a(x)\int_{\tau_1}^{\tau_2}|\delta_2(\tau)|w^2(x,1,\tau,t)d\tau dx\\
\le\frac{1}{2}\|a\|_\infty\,\int_{\tau_1}^{\tau_2}|\delta_2(\tau)|d\tau\int_0^\ell\phi_t^2dx+\frac{1}{2}\int_0^\ell a(x)\int_{\tau_1}^{\tau_2}|\delta_2(\tau)|w^2(x,1,\tau,t)d\tau dx,\\
-\int_0^\ell\varphi_t b(x)\int_{\tau_1}^{\tau_2}\beta_2(\tau)z\left(x,1,\tau\right)\,d\tau dx\\
\le\frac{1}{2}\int_0^\ell b(x)\varphi_t^2dx\,\int_{\tau_1}^{\tau_2}|\beta_2(\tau)|d\tau+\frac{1}{2}\int_0^\ell b(x)\int_{\tau_1}^{\tau_2}|\beta_2(\tau)|z^2(x,1,\tau,t)d\tau dx\\
\le\frac{1}{2}\|b\|_\infty\int_0^\ell\varphi_t^2dx\,\int_{\tau_1}^{\tau_2}|\beta_2(\tau)|d\tau+\frac{1}{2}\int_0^\ell b(x)\int_{\tau_1}^{\tau_2}|\beta_2(\tau)|z^2(x,1,\tau,t)d\tau dx.
\end{cases}\tag{3.8}$$

On account of (3.7), (3.8) and condition (1.7), we obtain

$$\frac{d}{dt}\left[E(t)\right] \le -n_0 \int_0^\ell \phi_t^2 dx - n_1 \int_0^\ell \varphi_t^2 dx \le 0, \ \forall t \ge 0. \tag{3.9}$$

This completes the proof. □

**Lemma 3.2.** *Let* $(\phi, w, \varphi, z)$ *be the sufficiently regular solution of* (2.3)–(2.5). *Define the functional*

$$I_1(t) = \int_0^\ell (m_1 \phi\phi_t + m_2 \varphi\varphi_t)\, dx. \tag{3.10}$$

*The functional* $I_1(t)$ *is differentiable for* $t > 0$*, and its time derivative satisfies*

$$\begin{aligned} I_1'(t) \le& -\frac{b_1}{2}\|\phi_{xx}\|^2 - \frac{b_2}{2}\|\varphi_x\|^2 - k\|[\phi-\varphi]^+\|^2 + c\|\phi_t\|^2 + c\|\varphi_t\|^2 \\ &+ c\int_0^\ell a(x)\int_{\tau_1}^{\tau_2} |\delta_2(\tau)|\, w^2(1,\tau) d\tau dx + c\int_0^\ell b(x)\int_{\tau_1}^{\tau_2} |\beta_2(\tau)|\, z^2(1,\tau)\, d\tau dx, \ \forall t \ge 0. \end{aligned} \tag{3.11}$$

*Proof.* Differentiating (3.10) and using $(2.3)_1$ and $(2.3)_3$, then integration by parts leads to

$$\begin{aligned} I_1'(t) =& \int_0^\ell \left(m_1\phi_t^2 + m_1\phi\,\phi_{tt} + m_2\varphi_t^2 + m_2\varphi\,\varphi_{tt}\right) dx \\ =& -b_1\|\phi_{xx}\|^2 - b_2\|\varphi_x\|^2 - k\|[\phi-\varphi]^+\|^2 + m_1\|\phi_t\|^2 + m_2\|\varphi_t\|^2 - \delta_1\int_0^\ell \phi\,\phi_t\, dx - \beta_1\int_0^\ell \varphi\,\varphi_t\, dx \\ &- \int_0^\ell a(x)\phi\left(\int_{\tau_1}^{\tau_2} \delta_2(\tau) w(x,1,\tau)\, d\tau\right) dx - \int_0^\ell b(x)\varphi\left(\int_{\tau_1}^{\tau_2} \beta_2(\tau)\, z(x,1,\tau)\, d\tau\right) dx. \end{aligned} \tag{3.12}$$

By applying the Cauchy–Schwarz, Young, and Poincaré inequalities together with the embedding result stated in [4, Lemma 2.1], we derive the following estimates,

$$-\delta_1\int_0^\ell \phi\,\phi_t\, dx \le \frac{3(\delta_1 c_e)^2}{4b_1}\|\phi_t\|^2 + \frac{b_1}{4}\|\phi_{xx}\|^2, \quad -\beta_1\int_0^\ell \varphi\,\varphi_t\, dx \le \frac{3(\beta_1 c_p)}{4b_2}\|\varphi_t\|^2 + \frac{b_2}{4}\|\varphi_x\|^2. \tag{3.13}$$

Furthermore,

$$\begin{aligned} &- \int_0^\ell a(x)\,\phi\left(\int_{\tau_1}^{\tau_2} \delta_2(\tau)\, w(x,1,\tau,t)\, d\tau\right) dx \\ &\le \left(\int_0^\ell \phi^2 dx\right)^{\frac{1}{2}} \left(\int_0^\ell \left(a(x)\int_{\tau_1}^{\tau_2} |\delta_2(\tau)|\, |w(x,1,\tau,t)|\, d\tau\right)^2 dx\right)^{\frac{1}{2}} \\ &\le c_e \left(\int_0^\ell \phi_{xx}^2 dx\right)^{\frac{1}{2}} \left(\|a\|_{L^\infty}\int_{\tau_1}^{\tau_2} |\delta_2(\tau)|\, d\tau\right)^{1/2} \left(\int_0^\ell a(x)\int_{\tau_1}^{\tau_2} |\delta_2(\tau)|\, w^2(x,1,\tau,t)\, d\tau\, dx\right)^{\frac{1}{2}} \\ &\le \frac{b_1}{4}\|\phi_{xx}\|^2 + \frac{3c_e^2\delta_1}{4b_1}\int_0^\ell a(x)\int_{\tau_1}^{\tau_2} |\delta_2(\tau)|\, w^2(x,1,\tau,t)\, d\tau\, dx. \end{aligned} \tag{3.14}$$

Similarly,

$$- \int_0^\ell b(x)\,\varphi\left(\int_{\tau_1}^{\tau_2} \beta_2(\tau)\, z(x,1,\tau,t)\, d\tau\right) dx \le \frac{b_2}{4}\|\varphi_x\|^2 + \frac{3c_p^2\beta_1}{4b_2}\int_0^\ell b(x)\int_{\tau_1}^{\tau_2} |\beta_2(\tau)|\, z^2(x,1,\tau)\, d\tau dx. \tag{3.15}$$

Substituting (3.13)–(3.15) into (3.12), we obtain (3.11). □

**Lemma 3.3.** *Let $(\phi, w, \varphi, z)$ be a solution of systems (2.3)–(2.5). Then, the functional*

$$I_2(t) = \int_0^\ell a(x) \int_0^1 \int_{\tau_1}^{\tau_2} \tau e^{-\tau\rho} |\delta_2(\tau)| w^2(x, \rho, \tau, t) \, d\tau d\rho \, dx \tag{3.16}$$

*satisfies*

$$\begin{aligned} I_2'(t) \leq & - m_0 \int_0^\ell a(x) \int_0^1 \int_{\tau_1}^{\tau_2} |\delta_2(\tau)| w^2(x, \rho, \tau, t) \; d\tau \, d\rho \, dx \\ & - m_0 \int_0^\ell a(x) \int_{\tau_1}^{\tau_2} |\delta_2(\tau)| w^2(x, 1, \tau, t) d\tau dx + \delta_1 \int_0^\ell \phi_t^2(x, t) dx, \end{aligned} \tag{3.17}$$

*where $m_0 = e^{-\tau_2}$.*

*Proof.* We differentiate $I_2(t)$ with respect to $t$. Using the transport equation satisfied by $w$, namely

$$w_t + \frac{1}{\tau} w_\rho = 0,$$

we obtain,

$$\begin{aligned} I_2'(t) = & 2 \int_0^\ell a(x) \int_0^1 \int_{\tau_1}^{\tau_2} \tau e^{-\tau\rho} |\delta_2(\tau)| w(x, \rho, \tau, t) \, w_t(x, \rho, \tau, t) \, d\tau \, d\rho \, dx \\ = & - 2 \int_0^\ell a(x) \int_0^1 \int_{\tau_1}^{\tau_2} e^{-\tau\rho} |\delta_2(\tau)| w(x, \rho, \tau, t) \, w_\rho(x, \rho, \tau, t) \, d\tau \, d\rho \, dx \\ = & - \int_0^\ell a(x) \int_0^1 \int_{\tau_1}^{\tau_2} |\delta_2(\tau)| \frac{\partial}{\partial_\rho} [e^{-\tau\rho} w^2(x, \rho, \tau, t)] \, d\tau \, d\rho \, dx \\ & - \int_0^\ell a(x) \int_0^1 \int_{\tau_1}^{\tau_2} e^{-\tau\rho} |\delta_2(\tau)| w^2(x, \rho, \tau, t) \; d\tau \, d\rho \, dx \\ = & - \int_0^\ell a(x) \int_{\tau_1}^{\tau_2} |\delta_2(\tau)| \Big[ e^{-\tau} w^2(x, 1, \tau, t) - w^2(x, 0, \tau, t) \Big] \, d\tau \, dx \\ & - \int_0^\ell a(x) \int_0^1 \int_{\tau_1}^{\tau_2} e^{-\tau\rho} |\delta_2(\tau)| w^2(x, \rho, \tau, t) \; d\tau \, d\rho \, dx. \end{aligned} \tag{3.18}$$

Recalling that $w(x, 0, \tau, t) = \phi_t(x, t)$ and using (1.7), we arrive at

$$\begin{aligned} I_2'(t) \leq & - \int_0^\ell a(x) \int_{\tau_1}^{\tau_2} e^{-\tau} |\delta_2(\tau)| w^2(x, 1, \tau, t) d\tau dx + \|a\|_\infty \int_{\tau_1}^{\tau_2} |\delta_2(\tau)| d\tau \int_0^\ell \phi_t^2(x, t) dx \\ & - \int_0^\ell a(x) \int_0^1 \int_{\tau_1}^{\tau_2} \tau e^{-\tau\rho} |\delta_2(\tau)| w^2(x, \rho, \tau, t) \; d\tau \, d\rho \, dx. \end{aligned} \tag{3.19}$$

Observe that $e^{-\tau} \leq e^{-\tau\rho}$ for all $\rho \in [0, 1]$. Moreover, $e^{-\tau}$ is decreasing on the interval $[\tau_1, \tau_2]$, it follows that

$$e^{-\tau_2} \leq e^{-\tau}, \quad \forall \tau \in [\tau_1, \tau_2].$$

Consequently, (3.19) leads to (3.17). □

**Lemma 3.4.** *Assume* $(\phi, w, \varphi, z)$ *is the sufficiently regular solution of systems* (2.3)–(2.5). *We define the functional*

$$I_3(t) = \int_0^{\ell} b(x) \int_0^1 \int_{\tau_1}^{\tau_2} \tau e^{-\tau\rho} |\beta_2(\tau)| \, z^2(x,\rho,\tau,t) \, d\tau d\rho dx. \tag{3.20}$$

*Then,* $I_3(t)$ *is differentiable for* $t > 0$*, and its time derivative satisfies*

$$\begin{aligned} I_3'(t) \leq & - m_0 \int_0^{\ell} b(x) \int_0^1 \int_{\tau_1}^{\tau_2} |\beta_2(\tau)| \, z^2(x,\rho,\tau,t) \; d\tau \, d\rho \, dx \\ & - m_0 \int_0^{\ell} b(x) \int_{\tau_1}^{\tau_2} |\beta_2(\tau)| z^2(x,1,\tau,t) d\tau dx + \beta_1 \int_0^{\ell} \varphi_t^2(x,t) dx, \end{aligned} \tag{3.21}$$

*where*

$$m_0 = e^{-\tau_2}.$$

*Proof.* Similarly, differentiating $I_3(t)$ with respect to $t$, using $(2.3)_4$ and following the same steps as in (3.18) and (3.19), we obtain (3.21). □

To establish exponential decay, we combine the natural energy $E(t)$ with three auxiliary functionals into a single Lyapunov functional $\mathcal{F}(t)$. Each component plays a specific role. The functional $I_1(t)$ is built from a cross term between displacement and velocity, and it produces the negative potential energy terms that drive the decay. Once $I_1'(t)$ is estimated, delay terms involving $w$ and $z$ appear on the right-hand side, and these are precisely what $I_2(t)$ and $I_3(t)$ are designed to absorb. The constants $\varepsilon$, $\varepsilon_2$, $\varepsilon_3$ are then chosen to make $\mathcal{F}(t)$ equivalent to $E(t)$.

**Lemma 3.5.** *Let* $(\phi, w, \varphi, z)$ *be a solution of* (2.3)–(2.5) *and assume that condition* (1.7) *is satisfied. Define the Lyapunov functional*

$$\mathcal{F}(t) = \varepsilon E(t) + I_1(t) + \varepsilon_2 I_2(t) + \varepsilon_3 I_3(t), \tag{3.22}$$

*where* $\varepsilon$*,* $\varepsilon_2$*, and* $\varepsilon_3$ *are suitably chosen positive constants. Then, there exist positive constants* $\lambda_1$*,* $\lambda_2$*, and* $\lambda_3$ *such that*

$$\lambda_1 E(t) \leq \mathcal{F}(t) \leq \lambda_2 E(t), \quad \forall\, t \geq 0 \tag{3.23}$$

*and*

$$\mathcal{F}'(t) \leq -\lambda_3 E(t), \quad \forall\, t \geq 0. \tag{3.24}$$

*Proof.* Using Cauchy–Schwarz, Young's, Poincaré's inequalities, and embedding, we have

$$|\mathcal{F}(t) - \epsilon E(t)| \leq |I_1(t)| + \epsilon_2 |I_2(t)| + \epsilon_3 |I_3(t)| \leq c E(t) \tag{3.25}$$

This implies,

$$(\epsilon - c) E(t) \leq \mathcal{F}(t) \leq (\epsilon + c) E(t).$$

Taking $\epsilon$ large such that $\epsilon - c > 0$, we obtain (3.23). Next, differentiating the Lyapunov functional $\mathcal{F}$ in (3.22) and using Lemmas 3.1–3.4, we have

$$\begin{aligned}\mathcal{F}'(t) \leq & -\Big[n_0\varepsilon - \delta_1\varepsilon_2 - C\Big]\|\phi_t\|^2 - \Big[n_1\varepsilon - \beta_1\varepsilon_3 - C\Big]\|\varphi_t\|^2 \\ & - \frac{b_1}{2}\|\phi_{xx}\|^2 - \frac{b_2}{2}\|\varphi_x\|^2 - k\|[\phi-\varphi]^+\|^2 \\ & - \varepsilon_2 m_0 \int_0^\ell a(x)\int_0^1\int_{\tau_1}^{\tau_2} \tau|\delta_2(\tau)|\, w^2(x,\rho,\tau,t)\, d\tau\, d\rho dx \\ & - \varepsilon_3 m_0 \int_0^\ell b(x)\int_0^1\int_{\tau_1}^{\tau_2} \tau|\beta_2(\tau)|\, z^2(x,\rho,\tau,t)\, d\tau\, d\rho dx \\ & - \Big[\varepsilon_2 m_0 - c\Big]\int_0^\ell a(x)\int_{\tau_1}^{\tau_2} |\delta_2(\tau)|\, w^2(x,1,\tau,t)\, d\tau dx \\ & - \Big[\varepsilon_3 m_0 - c\Big]\int_0^\ell b(x)\int_{\tau_1}^{\tau_2} |\beta_2(\tau)|\, z^2(x,1,\tau,t)\, d\tau dx.\end{aligned} \tag{3.26}$$

Choose $\varepsilon_2$ and $\varepsilon_3$ large such that

$$\varepsilon_2 m_0 - c > 0, \quad \varepsilon_3 m_0 - c > 0,$$

followed by selecting $\varepsilon$ even larger such that (3.23) stays valid and

$$n_0\varepsilon - \delta_1\varepsilon_2 - c > 0, \;\; n_1\varepsilon - \beta_1\varepsilon_3 - c > 0.$$

In view of (3.1), we obtain (3.24). □

**Theorem 3.1.** *Let* $(\phi, w, \varphi, z)$ *be a solution of systems* (2.3)–(2.5) *and suppose* (1.7) *holds. Then, the energy functional* (3.1) *decays exponentially. That is, we can find positive constants* $A_0$ *and* $\lambda$ *such that*

$$E(t) \leq A_0 \exp(-\lambda\, t), \;\; \forall t \geq 0. \tag{3.27}$$

*Proof.* From Lemma 3.5, on account of (3.23) and (3.24), we have for some $\lambda_0 > 0$,

$$\mathcal{F}'(t) \leq -\lambda_0 \mathcal{F}(t), \;\; \forall t \geq 0. \tag{3.28}$$

Integrating (3.28) over $(0, t)$, we obtain

$$\mathcal{F}(t) \leq \mathcal{F}(0)\exp(-\bar{\lambda}_0\, t), \;\; \forall t \geq 0, \tag{3.29}$$

for some positive constant $\bar{\lambda}_0$. Finally, using (3.23) again and (3.29), we obtain the result in (3.27). This completes the proof. □

## 4. Numerical illustration

To support the theoretical findings, we present a numerical simulation of systems (1.1)–(1.6). The spatial domain $(0, \pi)$ is discretized using $N = 20$ interior finite difference points, and the time integration is carried out via a Crank–Nicolson scheme with time step $\Delta t = 0.01$, in which the fourth-order and second-order spatial operators are treated implicitly while the damping, delay, and nonlinear coupling terms are treated explicitly. The parameters are chosen as $\ell = \pi$, $m_1 = m_2 = 1$, $b_1 = b_2 = 1$, $k = 0.5$, $\delta_1 = \beta_1 = 3$, $\tau_1 = 0.1$, $\tau_2 = 0.5$, with constant kernel functions $\delta_2(\tau) = \beta_2(\tau) = 1$ and uniform spatial weights $a(x) = b(x) = 1$. One can verify that condition (1.7) is satisfied, since

$$n_0 = n_1 = \delta_1 - \int_{\tau_1}^{\tau_2} |\delta_2(\tau)|\, d\tau = 3 - 0.4 = 2.6 > 0.$$

The initial conditions are

$$\phi_0(x) = \sin(x), \quad \phi_1(x) = 0, \quad \varphi_0(x) = \tfrac{1}{2}\sin(x), \quad \varphi_1(x) = 0,$$

with zero past history $g_0 \equiv 0$ and $f_0 \equiv 0$ on $(0, \ell) \times (-\tau_2, 0)$, consistent with the zero initial velocity.

The left panel of Figure 1 displays the energy $E(t)$ on a logarithmic scale, showing an initial transient followed by a clear linear trend, which confirms the exponential decay predicted by Theorem 3.1. The right panel shows the displacement profiles of the deck $\phi$ (solid lines) and the cable $\varphi$ (dashed lines) at times $t = 0, 1, 3, 6$, illustrating how both components are progressively damped to rest by the combined action of the instantaneous damping and the distributed delay feedback.

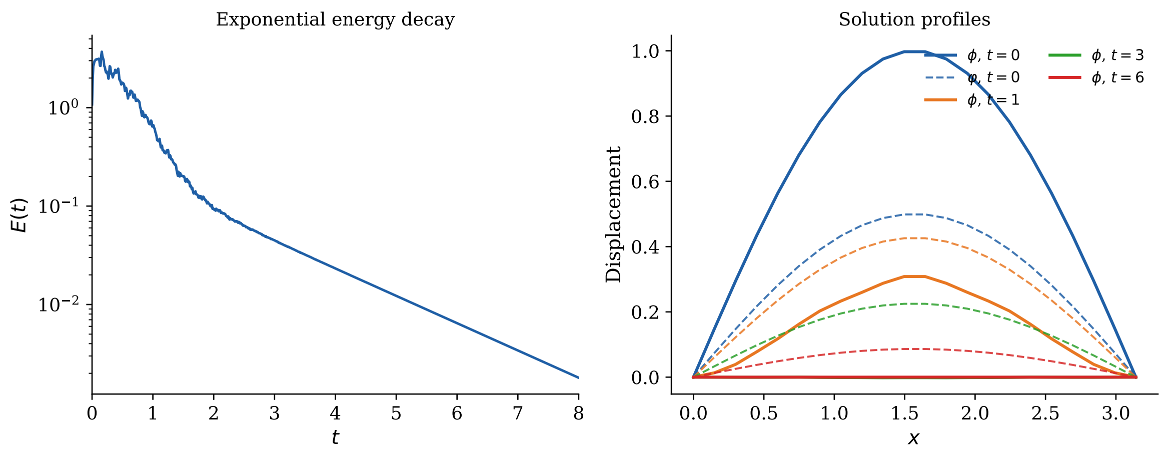


**Figure 1.** Left: energy $E(t)$ on a logarithmic scale, confirming exponential decay; Right: displacement profiles of the deck $\phi$ (solid) and cable $\varphi$ (dashed) at times $t = 0, 1, 3, 6$.

## 5. Conclusions

In this paper, we have studied a coupled suspension bridge system in which the deck and cable are linked through a nonlinear unilateral coupling term of positive-part type, and both components are subject to distributed delay feedback acting on their velocity fields. By introducing suitable auxiliary variables, the original integro-differential system was reformulated as an abstract Cauchy problem in an appropriate product Hilbert space, and well-posedness was established via semigroup theory. Under the natural balance condition (1.7) between the instantaneous damping coefficients and the delay kernels, we constructed a Lyapunov functional equivalent to the natural energy and derived an exponential decay estimate for the energy.

## Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The author declares no potential conflict of interest.

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