Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(6): 16334–16365.
DOI: 10.3934/math.2026671
Received: 03 April 2026
Revised: 23 May 2026
Accepted: 03 June 2026
Published: 08 June 2026

Research article

Matrix-induced quantum calculus in higher dimensions

Mohamed Jleli and Bessem Samet*

Department of Mathematics, College of Science, King Saud University, Riyadh 11451, Saudi Arabia

* **Correspondence:** Email: bsamet@ksu.edu.sa.

Abstract: In this work, we introduced a matrix-induced framework for quantum calculus on discrete forward orbits generated by affine contractions of the form $T(x) = a + Q(x - a)$ in $\mathbb{R}^N$, where the matrix Q has a spectral radius less than one. The proposed approach extends the classical one-dimensional q-calculus to higher dimensions by replacing scalar dilations with matrix-driven dynamics. We defined matrix-induced directional derivatives and integrals on the associated forward orbit and established their fundamental properties, including integration by parts, and first and second fundamental theorems of calculus. As applications, we proved a Hermite–Hadamard-type inequality for convex functions under a segment-preserving condition on Q, derived a Poincaré-type inequality on the orbit, and obtained an existence–uniqueness result for a nonlinear first-order problem driven by the new derivative. In dimension one, the framework reduces to the standard q-calculus.

Keywords: quantum calculus; higher dimensions; forward orbit; matrix-induced directional derivative; matrix-induced integral
Mathematics Subject Classification: 39A13, 47B39, 39A27, 26D15

1. Introduction

Quantum calculus, or q-calculus, is a limit-free framework in which differentiation and integration are formulated through q-difference operators and q-integrals. Its origins go back to the pioneering works of F. H. Jackson [1, 2]. Since then, it has become a useful analytical tool with connections to special functions [3–5], orthogonal polynomials [6–8], combinatorics [9–11], and representation theory [12]. Recent developments include (p, q)-deformations [13–15] and various q-fractional operators [16–18], often motivated by applications to difference equations, q-Schrödinger-type equations [19], and related nonlocal models. For further background, we refer the reader to [20, 21].

Another relevant direction is the study of Jackson-type constructions on shifted grids over finite intervals. This leads to anchored operators, such as the left q_a-derivative and the associated q_a-integral on $[a, b]$, where the contraction is taken with respect to the base point a rather than the origin. This viewpoint was initiated, to the best of our knowledge, in [22] and is particularly useful in boundary-value settings, where the discrete structure must be adapted to a prescribed endpoint.

Some contributions in the literature have addressed extensions of quantum calculus to higher dimensions; see, for example, [20, Chap. 10] and [23–25]. A common feature of these works is that differentiation is introduced coordinatewise. More precisely, given a function $f(x_1, \ldots, x_N)$ and a fixed parameter $q \in (0, 1)$, the q-partial derivative with respect to x_i is defined by applying the classical one-dimensional Jackson q-derivative in the x_i-variable, while keeping the remaining variables fixed:

$$\partial_{q,x_i} f(x_1, \ldots, x_N) = \frac{f(x_1, \ldots, x_i, \ldots, x_N) - f(x_1, \ldots, qx_i, \ldots, x_N)}{(1-q)x_i}, \qquad x_i \neq 0.$$

The value at $x_i = 0$ is understood in the appropriate limiting sense whenever this limit exists. Thus, if all variables except x_i are fixed, then ∂_{q,x_i} coincides with the usual Jackson derivative applied to the one-variable function

$$s \mapsto f(x_1, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_N);$$

see, for example, [16, 21]. This approach has the advantage of being conceptually simple and directly connected with the well-established one-dimensional theory. It also provides a practical multivariable toolkit, since q-analogues of higher-order derivatives, Taylor-type expansions, and Jacobian-type matrices can be constructed by iterating these q-partial operators. On the other hand, the coordinatewise character of the classical multivariate Jackson construction is inherently separable: each q-partial derivative is generated by an independent one-dimensional dilation $x_i \mapsto qx_i$. As a consequence, this framework does not naturally encode genuinely coupled transformations between variables. In particular, it is less naturally adapted to anisotropic settings in which the underlying contraction mixes coordinates (for instance, through non-diagonal linear maps), since such couplings cannot be represented by independent axis dilations. More broadly, the resulting calculus is organized around N decoupled q-directions and does not, by itself, provide an intrinsic directional framework tailored to a prescribed linear transformation. This can be a drawback when one seeks operator-driven identities and integral structures, such as fundamental theorems of calculus or integration-by-parts formulas, that are canonically aligned with a coupled discrete evolution.

In this paper, we introduce a framework for quantum calculus in higher dimensions that is intrinsically adapted to affine contractions and their associated discrete orbits. More precisely, we consider the affine map $T : \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$T(x) = a + Q(x - a),$$

where $a \in \mathbb{R}^N$ is the unique fixed point of T and $Q \in \mathbb{R}^{N\times N}$ satisfies $\rho(Q) < 1$, so that T is a strict contraction. The associated discrete evolution is described by the forward orbit $\Gamma_Q(b) = \{T^k(b) : k \in \mathbb{N}_0\}$ for a given $b \in \mathbb{R}^N$ with $b \neq a$. Rather than relying on coordinatewise q-dilations, this viewpoint permits genuinely coupled dynamics through non-diagonal matrices Q. Such matrices allow different coordinates to interact along the same discrete evolution, which is natural when modeling coupled systems or anisotropic behavior. This leads to directional difference and integral operators that are naturally aligned with the contraction generated by T. Within this setting, we establish fundamental theorems of calculus, integration-by-parts formulas, and basic convergence results, while recovering the classical one-dimensional q-calculus, including the anchored q_a setting, as special cases.

The paper is structured as follows. In Section 2, we introduce the matrix-induced affine map and its forward orbit, and we formulate the notion of Q-admissibility, which plays a central role throughout the paper. Section 3 is devoted to the definition of the matrix-induced directional derivative ${}_aD_{Q,e}$ and to the study of its basic properties, including algebraic rules and its connection with the classical directional derivative in the isotropic limit. In Section 4, we introduce the associated (a, Q, e)-integral and establish the fundamental theorems of calculus, integration-by-parts formulas, and a dominated convergence theorem within this framework. Finally, in Section 5, we present several applications of the matrix-induced quantum calculus, including a Hermite–Hadamard-type inequality, a Poincaré-type inequality on the forward orbit $\Gamma_Q(b)$, and existence and uniqueness results for a nonlinear first-order problem driven by the operator ${}_aD_{Q,e}$.

2. Affine maps, orbits, and Q-admissibility

Notation. We denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{N}_0$ the set of nonnegative integers. Throughout the paper, $N \in \mathbb{N}$ is fixed.

We write $\mathbb{R}^{N\times N}$ for the space of real $N \times N$ matrices, and denote by I the identity matrix.

For $A \in \mathbb{R}^{N\times N}$, we denote by $\rho(A)$ its spectral radius, that is, the maximum of the moduli of its eigenvalues. We also denote by A^{T} the transpose of A.

The Euclidean inner product on $\mathbb{R}^N$ is denoted by $\langle \cdot, \cdot \rangle$, and the associated norm by $\|\cdot\|$.

We assume that $a, b \in \mathbb{R}^N$, with $a \neq b$, and that $Q \in \mathbb{R}^{N\times N}$ satisfies $\rho(Q) < 1$.

Definition 2.1. Define $T : \mathbb{R}^N \to \mathbb{R}^N$ by

$$T(x) = a + Q(x - a), \qquad x \in \mathbb{R}^N.$$

We call T the matrix-induced affine map associated with Q. For each $k \in \mathbb{N}_0$, the k-th iterate of T is defined recursively by

$$T^0(x) = x, \qquad T^{k+1}(x) = T(T^k(x)), \qquad x \in \mathbb{R}^N.$$

Definition 2.2. The forward orbit of b under T is the countable set

$$\Gamma_Q(b) = \{T^k(b) : k \in \mathbb{N}_0\}.$$

We collect below a few elementary properties of the iterates of T and of the forward orbit $\Gamma_Q(b)$.

Lemma 2.3. *The affine map T and the forward orbit $\Gamma_Q(b)$ satisfy the following properties.*

(i) *The map T has a unique fixed point, namely, a.*

(ii) *For every $x \in \mathbb{R}^N$ and every $k \in \mathbb{N}_0$, one has*

$$T^k(x) = a + Q^k(x - a).$$

(iii) *The forward orbit of b under T admits the representation*

$$\Gamma_Q(b) = \{a + Q^k(b - a) : k \in \mathbb{N}_0\}.$$

(iv) *For every* $x \in \mathbb{R}^N$*, one has*

$$T^k(x) \to a \quad \text{as } k \to \infty.$$

(v) *One has*

$$T(\Gamma_Q(b)) = \Gamma_Q(b) \setminus \{b\}.$$

Moreover, if Q *is invertible, then the restriction*

$$T : \Gamma_Q(b) \to \Gamma_Q(b) \setminus \{b\}$$

is a bijection.

(vi) *The set* $\{a\} \cup \Gamma_Q(b)$ *is compact in* $\mathbb{R}^N$*.*

Proof. (i) Let $x \in \mathbb{R}^N$ be a fixed point of T. Then

$$x = T(x) = a + Q(x - a),$$

hence $(I - Q)(x - a) = 0$. Since $\rho(Q) < 1$, the number 1 is not an eigenvalue of Q, and therefore $I - Q$ is invertible. It follows that $x = a$, and a is the unique fixed point of T.

(ii) Fix $x \in \mathbb{R}^N$ and argue by induction on $k \in \mathbb{N}_0$. For $k = 0$, the identity is immediate since $T^0(x) = x$. Assume that $T^k(x) = a + Q^k(x - a)$ for some $k \in \mathbb{N}_0$. Then

$$T^{k+1}(x) = T(T^k(x)) = a + Q(T^k(x) - a) = a + Q^{k+1}(x - a),$$

which completes the induction.

(iii) This follows from (ii) by taking $x = b$.

(iv) Let $x \in \mathbb{R}^N$. Since $\rho(Q) < 1$, one has $Q^k \to 0$ as $k \to \infty$. By (ii),

$$T^k(x) - a = Q^k(x - a) \to 0 \quad \text{as } k \to \infty,$$

and hence $T^k(x) \to a$.

(v) For every $k \in \mathbb{N}_0$,

$$T(T^k(b)) = T^{k+1}(b) \in \Gamma_Q(b),$$

so $T(\Gamma_Q(b)) \subset \Gamma_Q(b)$. Moreover, $T^{k+1}(b) \neq b$ for all $k \in \mathbb{N}_0$. Indeed, if $T^{k+1}(b) = b$ for some k, then by (ii),

$$b = a + Q^{k+1}(b - a),$$

so $(I - Q^{k+1})(b - a) = 0$. Since $\rho(Q^{k+1}) = \rho(Q)^{k+1} < 1$, the matrix $I - Q^{k+1}$ is invertible, which forces $b = a$, which is a contradiction. Therefore,

$$T(\Gamma_Q(b)) \subset \Gamma_Q(b) \setminus \{b\}.$$

Conversely, if $y \in \Gamma_Q(b) \setminus \{b\}$, then $y = T^k(b)$ for some $k \geq 1$, and hence

$$y = T(T^{k-1}(b)) \in T(\Gamma_Q(b)).$$

This proves that $T(\Gamma_Q(b)) = \Gamma_Q(b) \setminus \{b\}$.

Assume finally that Q is invertible. If $T(x) = T(y)$ for some $x, y \in \Gamma_Q(b)$, then

$$a + Q(x - a) = a + Q(y - a),$$

and the invertibility of Q implies $x = y$. Thus T is injective on $\Gamma_Q(b)$. Combined with surjectivity onto $\Gamma_Q(b) \setminus \{b\}$, this shows that the restriction of T is a bijection.

(vi) Since $\rho(Q) < 1$, one has $Q^k \to 0$ as $k \to \infty$. By (iii), the set $\{a\} \cup \Gamma_Q(b) = \{a\} \cup \{a + Q^k(b - a) : k \in \mathbb{N}_0\}$ is bounded and has a as its only accumulation point. Hence it is closed in $\mathbb{R}^N$ and therefore compact. □

Definition 2.4. Let $e \in \mathbb{R}^N$ satisfy $\|e\| = 1$. We say that e is *Q-admissible* if it satisfies the following properties:

$$\begin{cases} \text{there exists } q > 0 \text{ such that } Q^{\mathsf{T}}(I - Q)^{\mathsf{T}} e = q\,(I - Q)^{\mathsf{T}} e, \\ \langle b - a, (I - Q)^{\mathsf{T}} e \rangle > 0. \end{cases} \tag{2.1}$$

In this case, the constant q is called the Q-admissibility eigenvalue associated with e.

Condition (2.1) has a simple geometric meaning. It requires the direction selected by the linear functional

$$x \mapsto \langle x, (I - Q)^{\mathsf{T}} e \rangle$$

to be invariant under Q, up to multiplication by the positive factor q. Thus, along the orbit generated by T, the scalar quantity measured in the direction e is contracted by the same factor q at each step. The positivity condition

$$\langle b - a, (I - Q)^{\mathsf{T}} e \rangle > 0$$

fixes the orientation of this direction and will be used below to obtain positive normalizing factors along the forward orbit.

Remark 2.5. If e is Q-admissible, then the constant q in (2.1) is unique. Indeed, the second condition in (2.1) gives

$$\langle b - a, (I - Q)^{\mathsf{T}} e \rangle > 0,$$

so $(I - Q)^{\mathsf{T}} e \neq 0$. If $q_1, q_2 > 0$ both satisfy

$$Q^{\mathsf{T}}(I - Q)^{\mathsf{T}} e = q_1 (I - Q)^{\mathsf{T}} e \qquad \text{and} \qquad Q^{\mathsf{T}}(I - Q)^{\mathsf{T}} e = q_2 (I - Q)^{\mathsf{T}} e,$$

then $(q_1 - q_2)(I - Q)^{\mathsf{T}} e = 0$, hence $q_1 = q_2$.

Remark 2.6. Let us clarify how Q-admissible directions may be constructed. Such directions do not necessarily exist for an arbitrary real matrix Q. The admissibility condition is a compatibility condition between the matrix Q, the endpoint b, and the direction e.

Since $\rho(Q) < 1$, the matrix $I - Q$ is invertible. Hence $(I - Q)^{\mathsf{T}}$ is also invertible. Let $w \neq 0$ be a real eigenvector of Q^{T} associated with a positive real eigenvalue q. Assume that

$$\langle b - a, w \rangle \neq 0.$$

Replacing w by $-w$, if necessary, we may suppose that

$$\langle b - a, w \rangle > 0.$$

Define

$$e = \frac{(I - Q)^{-\mathsf{T}} w}{\left\|(I - Q)^{-\mathsf{T}} w\right\|}.$$

Then $\|e\| = 1$ and

$$(I - Q)^{\mathsf{T}} e = \frac{w}{\left\|(I - Q)^{-\mathsf{T}} w\right\|}.$$

Consequently,

$$Q^{\mathsf{T}}(I - Q)^{\mathsf{T}} e = q(I - Q)^{\mathsf{T}} e.$$

Moreover,

$$\langle b - a, (I - Q)^{\mathsf{T}} e\rangle = \frac{\langle b - a, w\rangle}{\left\|(I - Q)^{-\mathsf{T}} w\right\|} > 0.$$

Thus e is Q-admissible.

Therefore, Q-admissible directions can be constructed from positive real eigenvalues of Q^{T}, provided that the corresponding eigendirection is not orthogonal to $b - a$. In particular, the existence of such directions is not automatic for a general matrix Q.

Lemma 2.7. *Assume that e is Q-admissible with constant q. Then* $0 < q < 1$.

Proof. By Definition 2.4, the admissibility constant q is assumed to be a positive real number. Moreover, since e is Q-admissible, the vector $(I - Q)^{\mathsf{T}} e$ is nonzero and satisfies

$$Q^{\mathsf{T}}(I - Q)^{\mathsf{T}} e = q(I - Q)^{\mathsf{T}} e.$$

Hence q is a real eigenvalue of Q^{T}. Therefore,

$$q = |q| \leq \rho(Q^{\mathsf{T}}) = \rho(Q).$$

Since $\rho(Q) < 1$, we conclude that $0 < q < 1$. □

Lemma 2.8. *Let $e \in \mathbb{R}^N$ be Q-admissible with constant q. Then, for every $x \in \Gamma_Q(b)$, one has*

$$\langle x - T(x), e\rangle = \langle (I - Q)(x - a), e\rangle > 0.$$

Proof. Let $x \in \Gamma_Q(b)$. Then there exists $k \in \mathbb{N}_0$ such that

$$x = T^k(b) = a + Q^k(b - a).$$

Since $x - T(x) = (I - Q)(x - a)$, we compute

$$\begin{aligned} \langle x - T(x), e\rangle &= \langle (I - Q)(x - a), e\rangle \\ &= \langle x - a, (I - Q)^{\mathsf{T}} e\rangle \\ &= \langle Q^k(b - a), (I - Q)^{\mathsf{T}} e\rangle. \end{aligned}$$

Using $\langle Q^k u, v\rangle = \langle u, (Q^{\mathsf{T}})^k v\rangle$, we obtain

$$\langle x - T(x), e\rangle = \langle b - a, (Q^{\mathsf{T}})^k (I - Q)^{\mathsf{T}} e\rangle.$$

By (2.1), one has $(Q^{\mathsf{T}})^k(I-Q)^{\mathsf{T}}e = q^k(I-Q)^{\mathsf{T}}e$, and hence

$$\langle x - T(x), e\rangle = q^k\,\langle b-a, (I-Q)^{\mathsf{T}}e\rangle.$$

Since $q > 0$ and $\langle b-a, (I-Q)^{\mathsf{T}}e\rangle > 0$ by (2.1), it follows that

$$\langle x - T(x), e\rangle > 0,$$

as claimed. □

In the remainder of the paper, we always assume that $e \in \mathbb{R}^N$ is Q-admissible with the associated constant $q \in (0,1)$ and satisfies $\|e\| = 1$.

3. The ${}_aD_{Q,e}$-directional derivative

In this section, we introduce the ${}_aD_{Q,e}$-directional derivative and establish its basic properties.

Definition 3.1. Let $u : \Gamma_Q(b) \to \mathbb{R}$. Since e is Q-admissible, Lemma 2.8 gives

$$\big\langle (I-Q)(x-a), e\big\rangle > 0, \qquad x \in \Gamma_Q(b).$$

The ${}_aD_{Q,e}$-directional derivative of u at a point $x \in \Gamma_Q(b)$, in the direction e, is defined by

$${}_aD_{Q,e}u(x) = \frac{u(x) - u(a + Q(x-a))}{\big\langle (I-Q)(x-a), e\big\rangle}.$$

Remark 3.2. Since $T(x) = a + Q(x-a)$ and $x - T(x) = (I-Q)(x-a)$, we may rewrite

$${}_aD_{Q,e}u(x) = \frac{u(x) - u(T(x))}{\langle x - T(x), e\rangle}, \qquad x \in \Gamma_Q(b).$$

Remark 3.3. Let $N = 1$ and let $a, b \in \mathbb{R}$ satisfy $a < b$. Fix $q \in (0,1)$, and set $Q = [q]$ and $e = 1$. Since $Q^{\mathsf{T}} = Q = [q]$ and $(I-Q)^{\mathsf{T}} = I - Q = [1-q]$, we have

$$Q^{\mathsf{T}}(I-Q)^{\mathsf{T}}e = q(1-q)e = q\,(I-Q)^{\mathsf{T}}e.$$

Thus the first condition in (2.1) holds with the constant $q > 0$. Moreover,

$$\langle b-a, (I-Q)^{\mathsf{T}}e\rangle = (b-a)(1-q) > 0,$$

so the second condition in (2.1) is also satisfied. Therefore, e is Q-admissible with constant $q \in (0,1)$. In this case, the forward orbit is

$$\Gamma_Q(b) = \{a + q^k(b-a) : k \in \mathbb{N}_0\}.$$

For $x \in \Gamma_Q(b)$, Definition 3.1 yields

$${}_aD_{Q,e}u(x) = \frac{u(x) - u(a + Q(x-a))}{\langle (I-Q)(x-a), e\rangle} = \frac{u(x) - u(a + q(x-a))}{(1-q)(x-a)}.$$

In particular, ${}_aD_{Q,e}u(x)$ coincides with the q_a-derivative ${}_aD_qu(x)$ in the sense of Tariboon and Ntouyas [22], restricted to the q-grid $\Gamma_Q(b)$.

We now present a few concrete examples in which ${}_aD_{Q,e}u$ can be computed explicitly for selected classes of functions u on $\Gamma_Q(b)$.

Example 3.4. Let $N = 2$ and

$$Q = \begin{pmatrix} s & 0 \\ 0 & r \end{pmatrix}, \qquad s, r \in (0,1).$$

Set

$$a = (0,0)^{\mathsf{T}}, \qquad b = (b_1, b_2)^{\mathsf{T}}, \qquad b_1 > 0,\ b_2 > 0, \qquad e_1 = (1,0)^{\mathsf{T}}, \qquad e_2 = (0,1)^{\mathsf{T}}.$$

Since $\rho(Q) = \max\{s, r\} < 1$, the standing assumption $\rho(Q) < 1$ is satisfied.

We check that e_1 and e_2 are Q-admissible. Since $Q^{\mathsf{T}} = Q$ and

$$(I - Q)^{\mathsf{T}} = I - Q = \begin{pmatrix} 1-s & 0 \\ 0 & 1-r \end{pmatrix},$$

we have

$$(I - Q)^{\mathsf{T}} e_1 = (1 - s)e_1, \qquad (I - Q)^{\mathsf{T}} e_2 = (1 - r)e_2,$$

and

$$Q^{\mathsf{T}}(I - Q)^{\mathsf{T}} e_1 = s(1 - s)e_1 = s\,(I - Q)^{\mathsf{T}} e_1, \qquad Q^{\mathsf{T}}(I - Q)^{\mathsf{T}} e_2 = r(1 - r)e_2 = r\,(I - Q)^{\mathsf{T}} e_2.$$

Thus the first condition in (2.1) holds for e_1 with constant s and for e_2 with constant r. Moreover,

$$\langle b - a, (I - Q)^{\mathsf{T}} e_1\rangle = \langle b, (1 - s)e_1\rangle = (1 - s)b_1 > 0$$

and

$$\langle b - a, (I - Q)^{\mathsf{T}} e_2\rangle = \langle b, (1 - r)e_2\rangle = (1 - r)b_2 > 0,$$

so the second condition in (2.1) is also satisfied in both cases. Therefore, e_1 is Q-admissible with constant s, and e_2 is Q-admissible with constant r.

Let $n, m \in \mathbb{N}_0$ and define $u : \Gamma_Q(b) \to \mathbb{R}$ by

$$u(x) = x_1^n x_2^m, \qquad x = (x_1, x_2)^{\mathsf{T}} \in \Gamma_Q(b).$$

Since $a = (0,0)^{\mathsf{T}}$, we have $a + Q(x - a) = Qx$ with

$$Qx = (sx_1, rx_2)^{\mathsf{T}}, \qquad u(Qx) = (sx_1)^n (rx_2)^m = s^n r^m\, x_1^n x_2^m.$$

Hence

$$u(x) - u(Qx) = (1 - s^n r^m)\, x_1^n x_2^m.$$

For $e_1 = (1,0)^{\mathsf{T}}$, one has $\langle (I - Q)(x - a), e_1\rangle = (1 - s)x_1$, and therefore

$${}_aD_{Q,e_1}u(x) = \frac{(1 - s^n r^m)\, x_1^n x_2^m}{(1 - s)x_1} = \frac{1 - s^n r^m}{1 - s}\, x_1^{n-1} x_2^m.$$

For $e_2 = (0,1)^{\mathsf{T}}$, one has $\langle (I - Q)(x - a), e_2\rangle = (1 - r)x_2$, and therefore

$${}_aD_{Q,e_2}u(x) = \frac{(1 - s^n r^m)\, x_1^n x_2^m}{(1 - r)x_2} = \frac{1 - s^n r^m}{1 - r}\, x_1^n x_2^{m-1}.$$

By Lemma 2.8, the denominators above are positive on $\Gamma_Q(b)$, so both expressions are well-defined.

This example corresponds to a separable anisotropic orbit: The two coordinates are contracted independently, with rates s and r, and the two admissible directions e_1 and e_2 recover the corresponding coordinatewise q-derivatives along the orbit.

Example 3.5. Let

$$Q = \begin{pmatrix} \delta & s & 0 \\ 0 & r & 0 \\ 0 & 0 & p \end{pmatrix}, \qquad \delta, r, p \in (0,1), \quad s > 0,$$

and set

$$a = (0,0,0)^{\mathsf{T}}, \qquad b = (0,1,0)^{\mathsf{T}}, \qquad e = \frac{1}{\sqrt{(\delta - r)^2 + s^2}}\,(\delta - r, s, 0)^{\mathsf{T}}.$$

Since Q is upper triangular with diagonal entries δ, r, and p, one has

$$\rho(Q) = \max\{\delta, r, p\} < 1.$$

We verify that e is Q-admissible. Writing

$$Q^{\mathsf{T}} = \begin{pmatrix} \delta & 0 & 0 \\ s & r & 0 \\ 0 & 0 & p \end{pmatrix}, \qquad (I - Q)^{\mathsf{T}} = I - Q^{\mathsf{T}} = \begin{pmatrix} 1-\delta & 0 & 0 \\ -s & 1-r & 0 \\ 0 & 0 & 1-p \end{pmatrix},$$

a direct computation yields

$$(I - Q)^{\mathsf{T}} e = \frac{1}{\sqrt{(\delta - r)^2 + s^2}}\big((1-\delta)(\delta - r),\ -s(\delta - r) + (1-r)s,\ 0\big)^{\mathsf{T}} = (1-\delta)e,$$

and

$$Q^{\mathsf{T}} e = \frac{1}{\sqrt{(\delta - r)^2 + s^2}}\big(\delta(\delta - r),\ s(\delta - r) + rs,\ 0\big)^{\mathsf{T}} = \delta e.$$

Consequently,

$$Q^{\mathsf{T}}(I - Q)^{\mathsf{T}} e = Q^{\mathsf{T}}((1-\delta)e) = \delta(1-\delta)e = \delta\,(I - Q)^{\mathsf{T}} e,$$

so the first condition in (2.1) holds with constant $\delta \in (0,1)$. Moreover, since $a = (0,0,0)^{\mathsf{T}}$ and $b = (0,1,0)^{\mathsf{T}}$,

$$\langle b - a, (I - Q)^{\mathsf{T}} e\rangle = \langle b, (1-\delta)e\rangle = (1-\delta)\,\frac{s}{\sqrt{(\delta - r)^2 + s^2}} > 0.$$

Thus e is Q-admissible with constant δ.

Let $u : \Gamma_Q(b) \to \mathbb{R}$ be defined by

$$u(x) = x_1^{\ell} x_2^{m} x_3^{n}, \qquad x = (x_1, x_2, x_3)^{\mathsf{T}} \in \Gamma_Q(b), \qquad \ell, m, n \in \mathbb{N}_0.$$

Since $a = (0,0,0)^{\mathsf{T}}$, Definition 3.1 gives

$$_{a}D_{Q,e}u(x) = \frac{u(x) - u(Qx)}{\langle (I - Q)x, e\rangle}, \qquad x \in \Gamma_Q(b).$$

A direct computation shows that

$$Qx = \begin{pmatrix} \delta x_1 + s x_2 \\ r x_2 \\ p x_3 \end{pmatrix}, \qquad (I-Q)x = \begin{pmatrix} (1-\delta)x_1 - s x_2 \\ (1-r)x_2 \\ (1-p)x_3 \end{pmatrix},$$

and hence

$$u(Qx) = (\delta x_1 + s x_2)^{\ell} (r x_2)^m (p x_3)^n.$$

Moreover,

$$\langle (I-Q)x, e\rangle = \frac{1-\delta}{\sqrt{(\delta-r)^2+s^2}}((\delta-r)x_1 + s x_2).$$

Therefore,

$$_aD_{Q,e}u(x) = \frac{x_1^{\ell} x_2^m x_3^n - (\delta x_1 + s x_2)^{\ell}(r x_2)^m (p x_3)^n}{\frac{1-\delta}{\sqrt{(\delta-r)^2+s^2}}((\delta-r)x_1 + s x_2)}, \qquad x \in \Gamma_Q(b).$$

After simplification, this can be written as

$$_aD_{Q,e}u(x) = \frac{\sqrt{(\delta-r)^2+s^2}}{(1-\delta)((\delta-r)x_1 + s x_2)}\, x_2^m x_3^n \Big(x_1^{\ell} - r^m p^n (\delta x_1 + s x_2)^{\ell} \Big), \qquad x \in \Gamma_Q(b).$$

This example illustrates the genuinely matrix-induced nature of the construction: Because of the nonzero off-diagonal entry s, the first coordinate of Qx depends on both x_1 and x_2. Thus the orbit is no longer generated by independent coordinate dilations, and the admissible direction e captures this coupled behavior.

We now establish several basic properties of the $_aD_{Q,e}$-directional derivative.

The following result connects $_aD_{Q,e}$ with the classical directional derivative in the isotropic case.

Proposition 3.6. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $u \in C^1(\Omega)$. Fix $a \in \mathbb{R}^N$ and a unit vector $e \in \mathbb{R}^N$. Let $t > 0$ and set $b = a + te$. For $h \in (0,1)$, define*

$$Q_h = (1-h)I \qquad \text{and} \qquad T_h(x) = a + Q_h(x-a) = a + (1-h)(x-a).$$

Assume that $\Gamma_{Q_h}(b) \subset \Omega$. Then:

- (i) *For every $h \in (0,1)$, the vector e is Q_h-admissible with constant $q = 1-h$.*
- (ii) *Fix $k \in \mathbb{N}_0$ and set $x_h = T_h^k(b) \in \Gamma_{Q_h}(b)$. Then*

$$\lim_{h \downarrow 0} {}_aD_{Q_h,e}u(x_h) = \partial_e u(b),$$

where $\partial_e u(b) = \langle \nabla u(b), e\rangle$.

Proof. (i) Fix $h \in (0,1)$. Since $Q_h = (1-h)I$, one has $\rho(Q_h) = 1-h < 1$. Moreover,

$$(I-Q_h)^{\mathsf{T}} e = he \quad \text{and} \quad Q_h^{\mathsf{T}}(I-Q_h)^{\mathsf{T}} e = (1-h)he = (1-h)(I-Q_h)^{\mathsf{T}} e,$$

so (2.1) holds with constant $q = 1-h \in (0,1)$. In addition,

$$\langle b-a, (I-Q_h)^{\mathsf{T}} e\rangle = \langle te, he\rangle = th > 0.$$

Thus e is Q_h-admissible.

(ii) Fix $k \in \mathbb{N}_0$ and set

$$x_h = T_h^k(b) = a + t(1-h)^k e.$$

Then

$$T_h(x_h) = x_h - th(1-h)^k e, \qquad \langle x_h - T_h(x_h), e\rangle = th(1-h)^k.$$

Therefore,

$$_aD_{Q_h,e}u(x_h) = \frac{u(x_h) - u(T_h(x_h))}{\langle x_h - T_h(x_h), e\rangle} = \frac{u(x_h) - u(x_h - \delta_h e)}{\delta_h}, \qquad \delta_h = th(1-h)^k.$$

Define $\phi_h : [0, \delta_h] \to \mathbb{R}$ by $\phi_h(s) = u(x_h - se)$. Since $u \in C^1(\Omega)$, the function ϕ_h is differentiable on $[0, \delta_h]$ and

$$\phi_h'(s) = -\langle \nabla u(x_h - se), e\rangle.$$

By the mean value theorem, there exists $\xi_h \in (0, \delta_h)$ such that

$$u(x_h) - u(x_h - \delta_h e) = \phi_h(0) - \phi_h(\delta_h) = -\phi_h'(\xi_h)\,\delta_h = \langle \nabla u(x_h - \xi_h e), e\rangle\,\delta_h.$$

Hence

$$_aD_{Q_h,e}u(x_h) = \langle \nabla u(x_h - \xi_h e), e\rangle.$$

Since $\delta_h \to 0$ as $h \downarrow 0$, one has $\xi_h \to 0$. Moreover, $x_h \to b$ as $h \downarrow 0$, so $x_h - \xi_h e \to b$. By the continuity of ∇u at b, it follows that

$$\lim_{h\downarrow 0} {}_aD_{Q_h,e}u(x_h) = \langle \nabla u(b), e\rangle = \partial_e u(b).$$

□

Proposition 3.7. *Let $u, v : \Gamma_Q(b) \to \mathbb{R}$ and let $\alpha, \beta \in \mathbb{R}$. Then, for every $x \in \Gamma_Q(b)$, the following identities hold:*

(i) *Linearity:*

$$_aD_{Q,e}(\alpha u + \beta v)(x) = \alpha\, {}_aD_{Q,e}u(x) + \beta\, {}_aD_{Q,e}v(x).$$

(ii) *Product rule:*

$$_aD_{Q,e}(uv)(x) = u(x)\, {}_aD_{Q,e}v(x) + v(a + Q(x-a))\, {}_aD_{Q,e}u(x).$$

(iii) *Quotient rule: If $v(x) \neq 0$ and $v(a + Q(x-a)) \neq 0$, then*

$$_aD_{Q,e}\left(\frac{u}{v}\right)(x) = \frac{v(a + Q(x-a))\, {}_aD_{Q,e}u(x) - u(a + Q(x-a))\, {}_aD_{Q,e}v(x)}{v(x)\, v(a + Q(x-a))}.$$

Proof. Fix $x \in \Gamma_Q(b)$ and set

$$y = a + Q(x-a) \in \Gamma_Q(b).$$

(i) This follows directly from Definition 3.1.

(ii) We write

$$u(x)v(x) - u(y)v(y) = u(x)(v(x) - v(y)) + v(y)(u(x) - u(y)),$$

and divide by $\langle (I-Q)(x-a), e\rangle$.

(iii) Assume that $v(x)\neq 0$ and $v(y)\neq 0$. Then

$$\frac{u(x)}{v(x)}-\frac{u(y)}{v(y)}=\frac{u(x)v(y)-u(y)v(x)}{v(x)v(y)}.$$

Moreover,

$$u(x)v(y)-u(y)v(x)=v(y)(u(x)-u(y))-u(y)(v(x)-v(y)).$$

Dividing by $\langle (I-Q)(x-a), e\rangle$ yields the stated formula. □

Proposition 3.8 (Chain rule in integral form)**.** *Let $u:\Gamma_Q(b)\to\mathbb{R}$ and let $\phi\in C^1(\mathbb{R})$. Then, for every $x\in\Gamma_Q(b)$, one has*

$${}_aD_{Q,e}(\phi\circ u)(x)=\left(\int_0^1 \phi'(\tau u(x)+(1-\tau)u(a+Q(x-a)))\,d\tau\right){}_aD_{Q,e}u(x).$$

Proof. Fix $x\in\Gamma_Q(b)$ and set $y=a+Q(x-a)$. Using the identity

$$\phi(u(x))-\phi(u(y))=\left(\int_0^1 \phi'(\tau u(x)+(1-\tau)u(y))\,d\tau\right)(u(x)-u(y)),$$

and dividing by $\langle (I-Q)(x-a), e\rangle$, we obtain the claim. □

We may iterate the operator ${}_aD_{Q,e}$ along the forward orbit $\Gamma_Q(b)$ and thus define higher-order, matrix-induced directional derivatives.

Definition 3.9. Let $u:\Gamma_Q(b)\to\mathbb{R}$ and let $m\in\mathbb{N}_0$. The m-th-order matrix-induced directional derivative ${}_aD^m_{Q,e}u:\Gamma_Q(b)\to\mathbb{R}$ is defined recursively by

$${}_aD^0_{Q,e}u=u,\qquad {}_aD^{m+1}_{Q,e}u(x)={}_aD_{Q,e}\left({}_aD^m_{Q,e}u\right)(x),\qquad x\in\Gamma_Q(b).$$

In particular, for $x\in\Gamma_Q(b)$, set

$$x^{(1)}=a+Q(x-a),\qquad x^{(2)}=a+Q^2(x-a).$$

Then

$${}_aD^2_{Q,e}u(x)=\frac{{}_aD_{Q,e}u(x)-{}_aD_{Q,e}u(x^{(1)})}{\langle (I-Q)(x-a), e\rangle}.$$

Moreover,

$${}_aD_{Q,e}u(x^{(1)})=\frac{u(x^{(1)})-u(x^{(2)})}{\langle (I-Q)(x^{(1)}-a), e\rangle}=\frac{u(x^{(1)})-u(x^{(2)})}{\langle (I-Q)Q(x-a), e\rangle},$$

and therefore

$${}_aD^2_{Q,e}u(x)=\frac{1}{\langle (I-Q)(x-a), e\rangle}\left(\frac{u(x)-u(x^{(1)})}{\langle (I-Q)(x-a), e\rangle}-\frac{u(x^{(1)})-u(x^{(2)})}{\langle (I-Q)Q(x-a), e\rangle}\right).$$

4. (a, Q, e)-Integrals

In this section, we introduce an integral operator naturally associated with the ${}_aD_{Q,e}$-directional derivative. Throughout this section, e is assumed to be Q-admissible with the associated constant $q \in (0,1)$, as in (2.1).

Definition 4.1. Let $u : \{a\} \cup \Gamma_Q(b) \to \mathbb{R}$. For $x \in \Gamma_Q(b)$, the (a, Q, e)-integral of u from a to x is defined by

$$\int_a^x u(t)\, d_{Q,e}t = \langle (I-Q)(x-a), e\rangle \sum_{j=0}^{\infty} q^j\, u(a + Q^j(x-a)),$$

whenever the series converges. We also set $\int_a^a u(t)\, d_{Q,e}t = 0$.

For $x, y \in \Gamma_Q(b)$, the (a, Q, e)-integral of u from x to y is defined by

$$\int_x^y u(t)\, d_{Q,e}t = \int_a^y u(t)\, d_{Q,e}t - \int_a^x u(t)\, d_{Q,e}t.$$

Remark 4.2. Assume that u is bounded on $\Gamma_Q(b)$, that is, there exists $M > 0$ such that $|u(y)| \le M$ for all $y \in \Gamma_Q(b)$. Since e is Q-admissible, Lemma 2.7 yields $0 < q < 1$. Therefore, for every $x \in \Gamma_Q(b)$,

$$\sum_{j=0}^{\infty} \left| q^j\, u(a + Q^j(x-a)) \right| \le M \sum_{j=0}^{\infty} q^j < \infty,$$

so the series in Definition 4.1 converges absolutely and the integral $\int_a^x u(t)\, d_{Q,e}t$ is well-defined.

Since $\Gamma_Q(b)$ is a discrete subset of $\mathbb{R}^N$, every function $u : \Gamma_Q(b) \to \mathbb{R}$ is automatically continuous on $\Gamma_Q(b)$ with respect to the relative topology. Consequently, if $u : \{a\} \cup \Gamma_Q(b) \to \mathbb{R}$ is continuous at a, then u is continuous on $\{a\} \cup \Gamma_Q(b)$.

In addition, the set $\{a\} \cup \Gamma_Q(b)$ is compact by Lemma 2.3(vi). Therefore, if u is continuous at a, then u is bounded on $\Gamma_Q(b)$, and the integral in Definition 4.1 is well-defined.

Remark 4.3. The boundedness assumption is only a sufficient condition for the convergence of the (a, Q, e)-integral. For a fixed $x \in \Gamma_Q(b)$, the integral

$$\int_a^x u(t)\, d_{Q,e}t$$

is well-defined whenever the series

$$\sum_{j=0}^{\infty} q^j u(a + Q^j(x-a))$$

converges. In particular, absolute convergence holds if

$$\sum_{j=0}^{\infty} q^j \left| u(a + Q^j(x-a)) \right| < \infty.$$

Thus, unbounded functions may also be integrated, provided that their growth along the forward orbit is controlled by the geometric weight q^j. For example, it is enough to assume that

$$\left| u(a + Q^j(x-a)) \right| \le C q^{-\alpha j}$$

for some $C > 0$ and some $\alpha < 1$.

Remark 4.4. Consider the one-dimensional setting $N = 1$ and fix $q \in (0, 1)$. Let $Q = [q]$, take $e = 1$, and keep the same points $a, b \in \mathbb{R}$ with $a \neq b$. Then

$$a + Q(x - a) = a + q(x - a), \quad a + Q^j(x - a) = a + q^j(x - a), \quad \langle (I - Q)(x - a), e\rangle = (1 - q)(x - a).$$

Moreover, the Q-admissibility constant is q. Hence, for $x \in \Gamma_Q(b)$, Definition 4.1 yields

$$\int_a^x u(t)\, d_{Q,e}t = (1 - q)(x - a) \sum_{j=0}^{\infty} q^j\, u(a + q^j(x - a)).$$

Therefore, the (a, Q, e)-integral coincides with the q_a-integral introduced by Tariboon and Ntouyas [22].

We now provide an example in which we compute the (a, Q, e)-integral explicitly for a specific function.

Example 4.5. Consider

$$Q = \frac{1}{10}\begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}, \qquad a = (0, 0, 0)^{\mathsf{T}}, \qquad b = (1, 1, 1)^{\mathsf{T}}, \qquad e = \frac{1}{\sqrt{3}}(1, 1, 1)^{\mathsf{T}}.$$

An elementary computation gives

$$\rho(Q) = \frac{1}{2} < 1.$$

We verify that e is Q-admissible. Since $Q^{\mathsf{T}} = Q$ and $(I - Q)^{\mathsf{T}} = I - Q$, we have

$$Qe = \frac{1}{2}\, e \qquad \text{and} \qquad (I - Q)^{\mathsf{T}} e = (I - Q)e = \frac{1}{2}\, e.$$

Hence

$$Q^{\mathsf{T}}(I - Q)^{\mathsf{T}} e = Q(I - Q)e = \frac{1}{4}\, e = \frac{1}{2}\, (I - Q)^{\mathsf{T}} e,$$

so the first condition in (2.1) holds with constant $q = \frac{1}{2} \in (0, 1)$. Moreover,

$$\langle b - a, (I - Q)^{\mathsf{T}} e\rangle = \left\langle (1, 1, 1)^{\mathsf{T}}, \frac{1}{2}\, e \right\rangle = \frac{\sqrt{3}}{2} > 0.$$

Therefore, e is Q-admissible with constant $q = \frac{1}{2}$.

Let $u : \{a\} \cup \Gamma_Q(b) \to \mathbb{R}$ be defined by

$$u(x) = x_2 x_3, \qquad x = (x_1, x_2, x_3)^{\mathsf{T}} \in \Gamma_Q(b).$$

Fix $x \in \Gamma_Q(b)$. Then there exists $k \in \mathbb{N}_0$ such that

$$x = a + Q^k(b - a) = Q^k b.$$

Since $b = (1, 1, 1)^{\mathsf{T}}$ is an eigenvector of Q associated with the eigenvalue $1/2$, we have

$$Q^k b = \left(\frac{1}{2}\right)^k b, \qquad \text{hence} \qquad x = \left(2^{-k}, 2^{-k}, 2^{-k}\right)^{\mathsf{T}}.$$

In particular,

$$u(x) = x_2 x_3 = 2^{-2k}.$$

Recall that e is Q-admissible with constant $q = \frac{1}{2}$. Moreover, since x is collinear with b, one has

$$(I - Q)x = \left(1 - \frac{1}{2}\right)x = \frac{1}{2}\,x,$$

and therefore

$$\big\langle (I - Q)x, e\big\rangle = \frac{1}{2}\langle x, e\rangle = \frac{1}{2}\left\langle 2^{-k}(1, 1, 1)^{\mathsf{T}}, \frac{1}{\sqrt{3}}(1, 1, 1)^{\mathsf{T}}\right\rangle = \frac{\sqrt{3}}{2^{k+1}}.$$

Furthermore, for every $j \in \mathbb{N}_0$,

$$Q^j x = Q^{j+k} b = \left(\frac{1}{2}\right)^{j+k} b = \left(\frac{1}{2}\right)^j x,$$

so

$$u(Q^j x) = \big((Q^j x)_2\big)\big((Q^j x)_3\big) = \left(\frac{1}{2}\right)^{2j} u(x).$$

Using Definition 4.1, we obtain

$$\begin{aligned}
\int_a^x u(t)\, d_{Q,e}t &= \big\langle (I - Q)x, e\big\rangle \sum_{j=0}^{\infty} q^j\, u(Q^j x) \\
&= \big\langle (I - Q)x, e\big\rangle \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j \left(\frac{1}{2}\right)^{2j} u(x) \\
&= \big\langle (I - Q)x, e\big\rangle\, u(x) \sum_{j=0}^{\infty} \left(\frac{1}{8}\right)^j \\
&= \frac{8}{7}\, \big\langle (I - Q)x, e\big\rangle\, u(x).
\end{aligned}$$

Substituting $\big\langle (I - Q)x, e\big\rangle = \frac{\sqrt{3}}{2^{k+1}}$ and $u(x) = 2^{-2k}$ yields

$$\int_a^x u(t)\, d_{Q,e}t = \frac{8}{7} \cdot \frac{\sqrt{3}}{2^{k+1}} \cdot 2^{-2k} = \frac{\sqrt{3}}{7}\, 2^{2-3k}.$$

Equivalently, since $x_1 = x_2 = x_3 = 2^{-k}$, one may write

$$\int_a^x u(t)\, d_{Q,e}t = \frac{4\sqrt{3}}{7}\, x_1^3.$$

We next record several basic properties of the (a, Q, e)-integral.

Proposition 4.6. *Let* $f, g : \{a\} \cup \Gamma_Q(b) \to \mathbb{R}$ *be bounded and fix* $x \in \Gamma_Q(b)$.

(i) *If* $f \geq 0$ *on* $\Gamma_Q(b)$ *and*

$$\int_a^x f(t)\, d_{Q,e}t = 0,$$

then

$$f(a + Q^j(x - a)) = 0 \quad \textit{for all } j \in \mathbb{N}_0.$$

In particular, $f(x) = 0$.

(ii) *If* $f \leq g$ *on* $\Gamma_Q(b)$, *then*

$$\int_a^x f(t)\, d_{Q,e}t \leq \int_a^x g(t)\, d_{Q,e}t.$$

Proof. Fix $x \in \Gamma_Q(b)$. Since e is Q-admissible, Lemma 2.8 yields

$$\langle (I - Q)(x - a), e\rangle > 0.$$

(i) By Definition 4.1, we have

$$\int_a^x f(t)\, d_{Q,e}t = \langle (I - Q)(x - a), e\rangle \sum_{j=0}^{\infty} q^j f(a + Q^j(x - a)).$$

Assume that $f \geq 0$ on $\Gamma_Q(b)$ and $\int_a^x f(t)\, d_{Q,e}t = 0$. Dividing by $\langle (I - Q)(x - a), e\rangle$ gives

$$\sum_{j=0}^{\infty} q^j f(a + Q^j(x - a)) = 0.$$

Each term in the above series is nonnegative since $q^j > 0$ and $f \geq 0$. Hence every term must vanish, that is,

$$f(a + Q^j(x - a)) = 0 \quad \text{for all } j \in \mathbb{N}_0.$$

In particular, taking $j = 0$ yields $f(x) = 0$.

(ii) Assume that $f \leq g$ on $\Gamma_Q(b)$. Then, for every $j \in \mathbb{N}_0$,

$$f(a + Q^j(x - a)) \leq g(a + Q^j(x - a)).$$

Multiplying by $q^j \geq 0$ and summing over $j \in \mathbb{N}_0$, we obtain

$$\sum_{j=0}^{\infty} q^j f(a + Q^j(x - a)) \leq \sum_{j=0}^{\infty} q^j g(a + Q^j(x - a)).$$

Finally, multiplying by $\langle (I - Q)(x - a), e\rangle$ and using Definition 4.1, we conclude that

$$\int_a^x f(t)\, d_{Q,e}t \leq \int_a^x g(t)\, d_{Q,e}t.$$

□

Proposition 4.7. *Let* $f, g : \{a\} \cup \Gamma_Q(b) \to \mathbb{R}$ *be bounded and let* $\alpha, \beta \in \mathbb{R}$. *Then, for every* $x \in \Gamma_Q(b)$, *one has*

$$\int_a^x (\alpha f(t) + \beta g(t))\, d_{Q,e}t = \alpha \int_a^x f(t)\, d_{Q,e}t + \beta \int_a^x g(t)\, d_{Q,e}t.$$

Proof. Fix $x \in \Gamma_Q(b)$. The identity follows directly from Definition 4.1. □

We next establish the first fundamental theorem of the matrix-induced quantum calculus, showing that the ${}_aD_{Q,e}$-directional derivative inverts the (a, Q, e)-integral along the forward orbit $\Gamma_Q(b)$.

Theorem 4.8 (First fundamental theorem)**.** *Let* $u : \{a\} \cup \Gamma_Q(b) \to \mathbb{R}$ *be bounded on* $\Gamma_Q(b)$, *and define* $F : \{a\} \cup \Gamma_Q(b) \to \mathbb{R}$ *by*

$$F(a) = 0, \qquad F(x) = \int_a^x u(t)\, d_{Q,e}t, \qquad x \in \Gamma_Q(b).$$

Then, for every $x \in \Gamma_Q(b)$, *one has*

$${}_aD_{Q,e}F(x) = u(x).$$

Proof. Fix $x \in \Gamma_Q(b)$ and set

$$S(x) = \sum_{j=0}^{\infty} q^j\, u(a + Q^j(x - a)).$$

Since $x \in \Gamma_Q(b)$, one has $a + Q(x - a) \in \Gamma_Q(b)$ and, for every $j \in \mathbb{N}_0$,

$$a + Q^j((a + Q(x - a)) - a) = a + Q^{j+1}(x - a).$$

Hence,

$$S(a + Q(x - a)) = \sum_{j=0}^{\infty} q^j\, u(a + Q^{j+1}(x - a)) = \frac{1}{q} \sum_{j=1}^{\infty} q^j\, u(a + Q^j(x - a)).$$

Moreover, since e is Q-admissible with constant q, one has

$$\begin{aligned} \langle (I - Q)((a + Q(x - a)) - a), e \rangle &= \langle (I - Q)Q(x - a), e \rangle \\ &= \langle x - a, Q^{\mathsf{T}}(I - Q)^{\mathsf{T}} e \rangle \\ &= q\, \langle (I - Q)(x - a), e \rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} F(a + Q(x - a)) &= \langle (I - Q)((a + Q(x - a)) - a), e \rangle\, S(a + Q(x - a)) \\ &= q\, \langle (I - Q)(x - a), e \rangle \cdot \frac{1}{q} \sum_{j=1}^{\infty} q^j\, u(a + Q^j(x - a)) \\ &= \langle (I - Q)(x - a), e \rangle \sum_{j=1}^{\infty} q^j\, u(a + Q^j(x - a)). \end{aligned}$$

The key point is the telescoping cancellation produced by subtracting the shifted series from the original one: All terms with $j \geq 1$ cancel, and only the first term remains. Subtracting this identity from

$$F(x) = \langle (I - Q)(x - a), e \rangle \sum_{j=0}^{\infty} q^j \, u(a + Q^j(x - a))$$

yields

$$F(x) - F(a + Q(x - a)) = \langle (I - Q)(x - a), e \rangle \, u(x).$$

Dividing by

$$\langle x - (a + Q(x - a)), e \rangle = \langle (I - Q)(x - a), e \rangle,$$

we obtain

$${}_aD_{Q,e}F(x) = u(x).$$

□

We next establish the second fundamental theorem.

Theorem 4.9 (Second fundamental theorem)**.** *Let* $u : \{a\} \cup \Gamma_Q(b) \to \mathbb{R}$*. Assume that* u *is continuous at* a *and that* ${}_aD_{Q,e}u$ *is bounded on* $\Gamma_Q(b)$*. Then, for every* $x \in \Gamma_Q(b)$*, one has*

$$\int_a^x {}_aD_{Q,e}u(t) \, d_{Q,e}t = u(x) - u(a).$$

Proof. Fix $x \in \Gamma_Q(b)$ and set $x_j = a + Q^j(x - a)$ for $j \in \mathbb{N}_0$. Then $x_j \in \Gamma_Q(b)$ and

$$x_{j+1} = a + Q(x_j - a).$$

By Definition 4.1 and Definition 3.1, we obtain

$$\begin{aligned}\int_a^x {}_aD_{Q,e}u(t) \, d_{Q,e}t &= \langle (I - Q)(x - a), e \rangle \sum_{j=0}^{\infty} q^j \, {}_aD_{Q,e}u(x_j) \\ &= \langle (I - Q)(x - a), e \rangle \sum_{j=0}^{\infty} q^j \, \frac{u(x_j) - u(x_{j+1})}{\langle (I - Q)(x_j - a), e \rangle}.\end{aligned}$$

Since $x_j - a = Q^j(x - a)$, we have

$$\begin{aligned}\langle (I - Q)(x_j - a), e \rangle &= \langle (I - Q)Q^j(x - a), e \rangle \\ &= \langle x - a, (Q^{\mathsf{T}})^j (I - Q)^{\mathsf{T}} e \rangle.\end{aligned}$$

Using the Q-admissibility condition $Q^{\mathsf{T}}(I - Q)^{\mathsf{T}}e = q\,(I - Q)^{\mathsf{T}}e$, we obtain

$$(Q^{\mathsf{T}})^j (I - Q)^{\mathsf{T}} e = q^j (I - Q)^{\mathsf{T}} e,$$

and therefore

$$\langle (I - Q)(x_j - a), e \rangle = q^j \langle x - a, (I - Q)^{\mathsf{T}} e \rangle = q^j \langle (I - Q)(x - a), e \rangle.$$

Hence, for every $j \in \mathbb{N}_0$,

$$\langle (I-Q)(x-a), e\rangle \frac{q^j}{\langle (I-Q)(x_j-a), e\rangle} = 1,$$

and consequently

$$\int_a^x {}_aD_{Q,e}u(t)\, d_{Q,e}t = \sum_{j=0}^{\infty}(u(x_j) - u(x_{j+1})) = u(x_0) - \lim_{k\to\infty} u(x_{k+1}).$$

Finally, since $x_{k+1} = a + Q^{k+1}(x-a) \to a$ as $k \to \infty$ and u is continuous at a, we have $u(x_{k+1}) \to u(a)$ as $k \to \infty$. Hence

$$\int_a^x {}_aD_{Q,e}u(t)\, d_{Q,e}t = u(x_0) - u(a) = u(x) - u(a),$$

which completes the proof. □

Proposition 4.10 (Integration by parts). *Let $u, v : \{a\}\cup\Gamma_Q(b) \to \mathbb{R}$. Assume that u and v are continuous at a, and that ${}_aD_{Q,e}u$ and ${}_aD_{Q,e}v$ are bounded on $\Gamma_Q(b)$. Then, for every $x \in \Gamma_Q(b)$,*

$$\int_a^x u(t)\, {}_aD_{Q,e}v(t)\, d_{Q,e}t = u(x)v(x) - u(a)v(a) - \int_a^x v(a + Q(t-a))\, {}_aD_{Q,e}u(t)\, d_{Q,e}t.$$

Proof. Fix $x \in \Gamma_Q(b)$. By the product rule in Proposition 3.7(ii), for every $t \in \Gamma_Q(b)$, one has

$${}_aD_{Q,e}(uv)(t) = u(t)\, {}_aD_{Q,e}v(t) + v(a + Q(t-a))\, {}_aD_{Q,e}u(t).$$

Rearranging, we obtain

$$u(t)\, {}_aD_{Q,e}v(t) = {}_aD_{Q,e}(uv)(t) - v(a + Q(t-a))\, {}_aD_{Q,e}u(t).$$

Integrating from a to x and using the linearity of the (a, Q, e)-integral, we get

$$\int_a^x u(t)\, {}_aD_{Q,e}v(t)\, d_{Q,e}t = \int_a^x {}_aD_{Q,e}(uv)(t)\, d_{Q,e}t - \int_a^x v(a + Q(t-a))\, {}_aD_{Q,e}u(t)\, d_{Q,e}t.$$

Since uv is continuous at a, Theorem 4.9 applied to uv yields

$$\int_a^x {}_aD_{Q,e}(uv)(t)\, d_{Q,e}t = (uv)(x) - (uv)(a) = u(x)v(x) - u(a)v(a).$$

Substituting into the previous identity gives the desired formula. □

We conclude this section by establishing a dominated convergence theorem for the (a, Q, e)-integral.

Theorem 4.11 (Dominated convergence). *Let $(u_n)_{n\in\mathbb{N}}$ and u be functions*

$$u_n, u : \{a\} \cup \Gamma_Q(b) \to \mathbb{R}.$$

Assume that:

(i) *for every* $y \in \Gamma_Q(b)$*, one has* $u_n(y) \to u(y)$ *as* $n \to \infty$*;*
(ii) *there exists a function* $g : \Gamma_Q(b) \to [0, \infty)$ *such that*

$$|u_n(y)| \le g(y) \quad \textit{for all } n \ge 1 \textit{ and all } y \in \Gamma_Q(b);$$

(iii) *the function* g *is bounded on* $\Gamma_Q(b)$*.*

Then u *is bounded on* $\Gamma_Q(b)$ *and, for every* $x \in \Gamma_Q(b)$*,*

$$\lim_{n\to\infty} \int_a^x u_n(t)\, d_{Q,e}t = \int_a^x u(t)\, d_{Q,e}t.$$

Proof. Fix $x \in \Gamma_Q(b)$. Since e is Q-admissible, Lemma 2.7 yields $0 < q < 1$. Set

$$A_x = \langle (I - Q)(x - a), e \rangle.$$

By Definition 4.1, one has

$$\int_a^x u_n(t)\, d_{Q,e}t = A_x \sum_{j=0}^{\infty} q^j\, u_n(a + Q^j(x - a)), \qquad \int_a^x u(t)\, d_{Q,e}t = A_x \sum_{j=0}^{\infty} q^j\, u(a + Q^j(x - a)).$$

For each fixed $j \in \mathbb{N}_0$, assumption (i) implies

$$u_n(a + Q^j(x - a)) \to u(a + Q^j(x - a)) \quad \text{as } n \to \infty.$$

Moreover, by (ii) and (iii), there exists $M > 0$ such that $g(y) \le M$ for all $y \in \Gamma_Q(b)$. Hence

$$\left| q^j\, u_n(a + Q^j(x - a)) \right| \le q^j\, g(a + Q^j(x - a)) \le M q^j, \qquad n \ge 1,\ j \in \mathbb{N}_0.$$

Since $\sum_{j=0}^{\infty} M q^j < \infty$, the dominated convergence theorem for the series yields

$$\lim_{n\to\infty} \sum_{j=0}^{\infty} q^j\, u_n(a + Q^j(x - a)) = \sum_{j=0}^{\infty} q^j\, u(a + Q^j(x - a)).$$

Multiplying by A_x gives

$$\lim_{n\to\infty} \int_a^x u_n(t)\, d_{Q,e}t = \int_a^x u(t)\, d_{Q,e}t.$$

Finally, by (ii) and (iii), one has

$$|u(y)| = \lim_{n\to\infty} |u_n(y)| \le g(y) \le M \quad \text{for all } y \in \Gamma_Q(b),$$

and thus u is bounded on $\Gamma_Q(b)$. □

5. Applications

In this section, we present several applications of matrix-induced quantum calculus. These applications show how the proposed framework can be used to derive functional inequalities and solvability results on discrete orbits generated by affine contractions. They are particularly relevant for discrete models in which the variables are coupled through prescribed matrix dynamics, including anisotropic contractions and nonlinear difference equations along matrix-generated orbits.

We first establish a Hermite–Hadamard-type inequality for convex functions. We then derive a Poincaré-type inequality on $\Gamma_Q(b)$. Finally, we address the existence and uniqueness of solutions to a nonlinear first-order problem driven by the directional derivative ${}_aD_{Q,e}$.

5.1. A Hermite–Hadamard-type inequality

We first introduce the following notion.

Definition 5.1 (Segment-preserving matrix)**.** The matrix Q is said to be segment-preserving along $[a, b]$ if there exists $\lambda \in [0, 1)$ such that

$$Q(b - a) = \lambda(b - a).$$

The scalar $\lambda \in [0, 1)$ satisfying this identity is uniquely determined and is called the segment eigenvalue of Q along $[a, b]$.

Remark 5.2. The segment-preserving condition is a restriction on the pair $(Q, b - a)$, not a diagonality condition on Q. It requires the direction $b - a$ to span a one-dimensional invariant subspace of Q. This assumption is natural in the Hermite–Hadamard inequality, since it ensures that the forward orbit remains on the segment $[a, b]$. However, Q may be non-diagonal.

For example, let $N = 2$,

$$a = (0, 0)^\mathsf{T}, \qquad b = (1, 0)^\mathsf{T},$$

and

$$Q = \begin{pmatrix} \lambda & c \\ 0 & \mu \end{pmatrix}, \qquad 0 < \lambda, \mu < 1, \quad c \neq 0.$$

Then Q is non-diagonal and $\rho(Q) < 1$. Moreover,

$$Q(b - a) = Q(1, 0)^\mathsf{T} = \lambda(1, 0)^\mathsf{T} = \lambda(b - a).$$

Thus Q is segment-preserving along $[a, b]$ with the segment eigenvalue λ, although Q is not diagonal.

Lemma 5.3. *Assume that Q is segment-preserving along $[a, b]$ with the segment eigenvalue $\lambda \in [0, 1)$. Then the following assertions hold:*

(i) $\{a\} \cup \Gamma_Q(b) \subset [a, b]$*;*

(ii) $a + (1 - q)(I - qQ)^{-1}(b - a) \in [a, b]$*, where $q \in (0, 1)$ is the Q-admissibility eigenvalue associated with e.*

Proof. Set $v = b - a \neq 0$.

(i) Since $Qv = \lambda v$, one has $Q^j v = \lambda^j v$ for every $j \in \mathbb{N}_0$. Hence

$$a + Q^j(b - a) = a + \lambda^j(b - a) = (1 - \lambda^j)a + \lambda^j b \in [a, b],$$

and therefore $\Gamma_Q(b) \subset [a, b]$. Since $a \in [a, b]$, it follows that $\{a\} \cup \Gamma_Q(b) \subset [a, b]$.

(ii) Since $\rho(Q) < 1$ and $q \in (0, 1)$, one has $\rho(qQ) = q\rho(Q) < 1$, so $I - qQ$ is invertible. Moreover,

$$(I - qQ)v = v - qQv = (1 - q\lambda)v,$$

and thus

$$(I - qQ)^{-1}v = \frac{1}{1 - q\lambda}\, v.$$

Consequently,

$$a + (1-q)(I-qQ)^{-1}(b-a) = a + t(b-a), \qquad t = \frac{1-q}{1-q\lambda}.$$

Since $\lambda \in [0,1)$ and $q \in (0,1)$, we have $1 - q\lambda > 0$ and

$$0 < \frac{1-q}{1-q\lambda} < 1,$$

so $t \in (0,1)$. Hence $a + t(b-a) \in [a,b]$. □

Theorem 5.4 (Hermite–Hadamard-type inequality)**.** *Assume that Q is segment-preserving along $[a,b]$ with segment eigenvalue $\lambda \in [0,1)$. Let $u : [a,b] \to \mathbb{R}$ be continuous and convex on the segment $[a,b]$. Then*

$$\begin{aligned} u\Big(a + (1-q)(I-qQ)^{-1}(b-a)\Big) &\le \frac{1-q}{\langle (I-Q)(b-a), e\rangle} \int_a^b u(t)\, d_{Q,e}t \\ &\le \frac{q(1-\lambda)}{1-q\lambda} u(a) + \frac{1-q}{1-q\lambda} u(b), \end{aligned} \tag{5.1}$$

where $q \in (0,1)$ is the Q-admissibility eigenvalue associated with e.

Proof. By Lemma 5.3(i), we have $\{a\} \cup \Gamma_Q(b) \subset [a,b]$. Since u is continuous on the compact segment $[a,b]$, it is bounded on $[a,b]$, and in particular on $\Gamma_Q(b)$. Hence the integral $\int_a^b u(t)\, d_{Q,e}t$ is well-defined. Moreover, by Lemma 5.3(ii), one has

$$a + (1-q)(I-qQ)^{-1}(b-a) \in [a,b],$$

so the value $u\Big(a + (1-q)(I-qQ)^{-1}(b-a)\Big)$ is well-defined.

We next prove the right-hand inequality in (5.1). Set $v = b - a$. For $t \in [a,b]$, there exists a unique $\theta(t) \in [0,1]$ such that

$$t = (1-\theta(t))a + \theta(t)b.$$

Moreover,

$$\theta(t) = \frac{\langle t-a,\, v\rangle}{\|v\|^2}.$$

Indeed, from $t = (1-\theta(t))a + \theta(t)b$, we get $t - a = \theta(t)(b-a) = \theta(t)v$, and taking the inner product with v yields

$$\langle t-a,\, v\rangle = \theta(t)\|v\|^2.$$

Since u is convex on the segment $[a,b]$, we obtain

$$u(t) \le (1-\theta(t))u(a) + \theta(t)u(b), \qquad t \in [a,b].$$

In particular, since $\Gamma_Q(b) \subset [a,b]$ by Lemma 5.3(i), it follows that

$$u(t) \le (1-\theta(t))u(a) + \theta(t)u(b), \qquad t \in \Gamma_Q(b).$$

Then, by Proposition 4.6(ii) and Proposition 4.7, we obtain

$$\int_a^b u(t)\, d_{Q,e}t \le u(a) \int_a^b (1-\theta(t))\, d_{Q,e}t + u(b) \int_a^b \theta(t)\, d_{Q,e}t. \tag{5.2}$$

By Definition 4.1 (with $x = b$), we have

$$\int_a^b 1\, d_{Q,e}t = \langle (I - Q)v,\, e\rangle \sum_{j=0}^{\infty} q^j = \frac{\langle (I - Q)v,\, e\rangle}{1 - q}.$$

Moreover, since Q is segment-preserving along $[a, b]$ with the segment eigenvalue $\lambda \in [0, 1)$, one has

$$\theta(a + Q^j v) = \frac{\langle Q^j v,\, v\rangle}{\|v\|^2} = \frac{\langle \lambda^j v,\, v\rangle}{\|v\|^2} = \lambda^j,$$

and therefore

$$\begin{aligned}\int_a^b \theta(t)\, d_{Q,e}t &= \langle (I - Q)v,\, e\rangle \sum_{j=0}^{\infty} q^j\, \theta(a + Q^j v)\\ &= \langle (I - Q)v,\, e\rangle \sum_{j=0}^{\infty} q^j \lambda^j = \frac{\langle (I - Q)v,\, e\rangle}{1 - q\lambda}.\end{aligned}$$

Consequently,

$$\int_a^b (1 - \theta(t))\, d_{Q,e}t = \frac{\langle (I - Q)v,\, e\rangle}{1 - q} - \frac{\langle (I - Q)v,\, e\rangle}{1 - q\lambda}.$$

Multiplying (5.2) by $(1 - q)/\langle (I - Q)v,\, e\rangle$ and substituting the above identities, we get

$$\frac{1 - q}{\langle (I - Q)v,\, e\rangle} \int_a^b u(t)\, d_{Q,e}t \le \frac{q(1 - \lambda)}{1 - q\lambda}\, u(a) + \frac{1 - q}{1 - q\lambda}\, u(b).$$

This completes the proof of the right-hand inequality in (5.1).

We now prove the left-hand inequality in (5.1). Set again $v = b - a$ and define

$$t_j = a + Q^j v \in \Gamma_Q(b), \qquad j \in \mathbb{N}_0.$$

By Lemma 5.3(i), we have $t_j \in [a, b]$ for all $j \in \mathbb{N}_0$, and hence $u(t_j)$ is well-defined.

For $j \ge 0$, set

$$w_j = (1 - q)q^j.$$

Then $w_j \ge 0$ and $\sum_{j=0}^{\infty} w_j = 1$. Moreover, by Definition 4.1 (with $x = b$),

$$\frac{1 - q}{\langle (I - Q)v,\, e\rangle} \int_a^b u(t)\, d_{Q,e}t = (1 - q) \sum_{j=0}^{\infty} q^j\, u(t_j) = \sum_{j=0}^{\infty} w_j\, u(t_j). \tag{5.3}$$

Next, we compute the corresponding barycenter. Using $\sum_{j=0}^{\infty} w_j = 1$, we have

$$\sum_{j=0}^{\infty} w_j\, t_j = (1 - q) \sum_{j=0}^{\infty} q^j (a + Q^j v) = a + (1 - q) \sum_{j=0}^{\infty} q^j Q^j v.$$

Since $\rho(qQ) = q\,\rho(Q) < 1$, the Neumann series gives

$$\sum_{j=0}^{\infty} q^j Q^j = (I - qQ)^{-1},$$

and therefore

$$\sum_{j=0}^{\infty} w_j t_j = a + (1-q)(I-qQ)^{-1}v = a + (1-q)(I-qQ)^{-1}(b-a). \tag{5.4}$$

We now apply Jensen's inequality to the convex function u. For $n \in \mathbb{N}_0$, set

$$W_n = \sum_{j=0}^{n} w_j = 1 - q^{n+1}, \qquad \widetilde{w}_j^{(n)} = \frac{w_j}{W_n} \quad (0 \le j \le n),$$

so that $\widetilde{w}_j^{(n)} \ge 0$ and $\sum_{j=0}^{n} \widetilde{w}_j^{(n)} = 1$. By the convexity of u,

$$u\left(\sum_{j=0}^{n} \widetilde{w}_j^{(n)} t_j\right) \le \sum_{j=0}^{n} \widetilde{w}_j^{(n)} u(t_j).$$

Multiplying by W_n yields

$$W_n\, u\left(\sum_{j=0}^{n} \widetilde{w}_j^{(n)} t_j\right) \le \sum_{j=0}^{n} w_j\, u(t_j). \tag{5.5}$$

Moreover, $W_n \to 1$. Since $t_j \in [a,b]$ for all $j \in \mathbb{N}_0$ and $\sum_{j=0}^{\infty} w_j = 1$, the series

$$\sum_{j=0}^{\infty} w_j t_j$$

converges in $\mathbb{R}^N$. Hence

$$\sum_{j=0}^{n} \widetilde{w}_j^{(n)} t_j = \frac{1}{W_n} \sum_{j=0}^{n} w_j t_j \longrightarrow \sum_{j=0}^{\infty} w_j t_j, \qquad \text{as } n \to \infty.$$

Since u is continuous on $[a,b]$, it follows that

$$u\left(\sum_{j=0}^{n} \widetilde{w}_j^{(n)} t_j\right) \longrightarrow u\left(\sum_{j=0}^{\infty} w_j t_j\right).$$

Also, u is bounded on $[a,b]$, and therefore the series

$$\sum_{j=0}^{\infty} w_j u(t_j)$$

converges absolutely. Thus

$$\sum_{j=0}^{n} w_j u(t_j) \longrightarrow \sum_{j=0}^{\infty} w_j u(t_j).$$

Passing to the limit in (5.5), we obtain

$$u\left(\sum_{j=0}^{\infty} w_j t_j\right) \le \sum_{j=0}^{\infty} w_j\, u(t_j).$$

Combining this with (5.3) and (5.4) gives

$$u\Big(a + (1-q)(I-qQ)^{-1}(b-a)\Big) \le \frac{1-q}{\langle (I-Q)(b-a),\, e\rangle} \int_a^b u(t)\, d_{Q,e}t,$$

which is the left-hand inequality in (5.1). □

Remark 5.5. Consider the one-dimensional setting $N = 1$. Fix $q \in (0,1)$, take $Q = [q]$, and set $e = 1$. Then Q is segment-preserving along $[a, b]$ with the segment eigenvalue $\lambda = q$. Moreover,

$$(I-qQ)^{-1} = \frac{1}{1-q^2}, \qquad a + (1-q)(I-qQ)^{-1}(b-a) = a + \frac{1-q}{1-q^2}(b-a) = \frac{qa+b}{1+q}.$$

In addition, since $(I-Q)(b-a) = (1-q)(b-a)$, Definition 4.1 yields

$$\begin{aligned} \frac{1-q}{\langle (I-Q)(b-a), e\rangle} \int_a^b u(t)\, d_{Q,e}t &= \frac{1-q}{(1-q)(b-a)} \int_a^b u(t)\, d_{Q,e}t \\ &= \frac{1}{b-a} \sum_{j=0}^{\infty} (1-q)q^j\, u\Big(a + q^j(b-a)\Big) \\ &= \frac{1}{b-a} \int_a^b u(t)\, {}_a d_q t, \end{aligned}$$

where ${}_a d_q t$ denotes the left q-integral on the q-grid.

Therefore, Theorem 5.4 reduces to the one-dimensional q-Hermite–Hadamard-type inequality (see [26–28]):

$$u\left(\frac{qa+b}{1+q}\right) \le \frac{1}{b-a} \int_a^b u(t)\, {}_a d_q t \le \frac{q}{1+q}\, u(a) + \frac{1}{1+q}\, u(b),$$

for every convex and continuous function $u : [a, b] \to \mathbb{R}$.

For further contributions to q-Hermite–Hadamard-type inequalities in the one-dimensional setting, we refer to [29–32] and the references therein.

5.2. A Poincaré-type inequality

In this subsection, we establish a Poincaré-type inequality on $\Gamma_Q(b)$ linking the size of a function to the size of its ${}_a D_{Q,e}$-directional derivative.

Theorem 5.6 (Poincaré-type inequality)**.** *Let* $u : \{a\} \cup \Gamma_Q(b) \to \mathbb{R}$ *be continuous at* a *and assume that* $u(a) = 0$*. Assume in addition that* ${}_a D_{Q,e} u$ *is bounded on* $\Gamma_Q(b)$*. Then, for every* $x \in \Gamma_Q(b)$*,*

$$\int_a^x |u(t)|\, d_{Q,e}t \le \frac{\langle (I-Q)(x-a), e\rangle}{1-q} \int_a^x \left| {}_a D_{Q,e} u(t) \right| d_{Q,e}t,$$

where $q \in (0,1)$ *is the* Q*-admissibility eigenvalue associated with* e*.*

Proof. Fix $x \in \Gamma_Q(b)$ and set

$$x_j = a + Q^j(x-a) \in \Gamma_Q(b), \qquad j \in \mathbb{N}_0.$$

Write

$$A_x = \langle (I - Q)(x - a), e \rangle > 0.$$

For each $j \in \mathbb{N}_0$, using the Q-admissibility condition as in the proof of Theorem 4.9, we have

$$\langle (I - Q)(x_j - a), e \rangle = q^j A_x. \tag{5.6}$$

Moreover,

$$_aD_{Q,e}u(x_j) = \frac{u(x_j) - u(x_{j+1})}{\langle (I - Q)(x_j - a), e \rangle},$$

so

$$u(x_j) - u(x_{j+1}) = \langle (I - Q)(x_j - a), e \rangle \, {}_aD_{Q,e}u(x_j) = q^j A_x \, {}_aD_{Q,e}u(x_j). \tag{5.7}$$

Since $x_j \to a$ as $j \to \infty$ and u is continuous at a with $u(a) = 0$, we have $u(x_j) \to 0$. Hence, for every $j \in \mathbb{N}_0$,

$$u(x_j) = \sum_{m=j}^{\infty} (u(x_m) - u(x_{m+1})).$$

Taking absolute values and using (5.7), we obtain

$$|u(x_j)| \leq \sum_{m=j}^{\infty} q^m A_x \left| {}_aD_{Q,e}u(x_m) \right| = A_x \sum_{m=j}^{\infty} q^m \left| {}_aD_{Q,e}u(x_m) \right|.$$

Multiplying by q^j and summing over $j \geq 0$ gives

$$\begin{aligned}
\sum_{j=0}^{\infty} q^j |u(x_j)| &\leq A_x \sum_{j=0}^{\infty} \sum_{m=j}^{\infty} q^{j+m} \left| {}_aD_{Q,e}u(x_m) \right| \\
&= A_x \sum_{m=0}^{\infty} q^m \left| {}_aD_{Q,e}u(x_m) \right| \sum_{j=0}^{m} q^j \\
&\leq \frac{A_x}{1 - q} \sum_{m=0}^{\infty} q^m \left| {}_aD_{Q,e}u(x_m) \right|.
\end{aligned}$$

Finally, by Definition 4.1,

$$\int_a^x |u(t)| \, d_{Q,e}t = A_x \sum_{j=0}^{\infty} q^j |u(x_j)|, \qquad \int_a^x \left| {}_aD_{Q,e}u(t) \right| d_{Q,e}t = A_x \sum_{m=0}^{\infty} q^m \left| {}_aD_{Q,e}u(x_m) \right|.$$

Combining the last two displays yields

$$\int_a^x |u(t)| \, d_{Q,e}t \leq \frac{A_x}{1 - q} \int_a^x \left| {}_aD_{Q,e}u(t) \right| d_{Q,e}t,$$

which is the desired inequality. □

Remark 5.7. The constant in Theorem 5.6 depends on the endpoint x. However, a uniform version follows directly from the structure of the orbit. Indeed, if $x = T^k(b) = a + Q^k(b - a)$, then

$$\langle (I - Q)(x - a), e\rangle = q^k \langle (I - Q)(b - a), e\rangle \le \langle (I - Q)(b - a), e\rangle.$$

Therefore, for every $x \in \Gamma_Q(b)$,

$$\int_a^x |u(t)|\, d_{Q,e}t \le \frac{\langle (I - Q)(b - a), e\rangle}{1 - q} \int_a^x \left|{}_aD_{Q,e}u(t)\right| d_{Q,e}t.$$

Thus the Poincaré-type inequality admits a global constant on the whole forward orbit $\Gamma_Q(b)$.

5.3. Existence and uniqueness for a nonlinear first-order problem

In this subsection, we investigate the existence and uniqueness of solutions to a first-order problem governed by the directional derivative ${}_aD_{Q,e}$ on $\Gamma_Q(b)$.

We consider the nonlinear first-order problem

$$\begin{cases} {}_aD_{Q,e}u(x) = f(x, u(x)), & x \in \Gamma_Q(b), \\ u(a) = u_0, \end{cases} \tag{5.8}$$

where $f : \Gamma_Q(b) \times \mathbb{R} \to \mathbb{R}$ is given and $u_0 \in \mathbb{R}$.

Definition 5.8. A function $u : \{a\} \cup \Gamma_Q(b) \to \mathbb{R}$ is a solution of (5.8) if:

(i) u is continuous at a;
(ii) $u(a) = u_0$;
(iii) ${}_aD_{Q,e}u(x) = f(x, u(x))$ for all $x \in \Gamma_Q(b)$.

Remark 5.9. The continuity assumption at a in this definition is precisely the one needed in Theorem 4.9.

Lemma 5.10. *Assume that $f : \Gamma_Q(b) \times \mathbb{R} \to \mathbb{R}$ is bounded. Let $u_0 \in \mathbb{R}$ and let $u : \{a\} \cup \Gamma_Q(b) \to \mathbb{R}$. Then the following assertions are equivalent:*

(i) *u is a solution of* (5.8) *in the sense of Definition 5.8;*
(ii) *u is continuous at a and, for every $x \in \Gamma_Q(b)$, one has*

$$u(x) = u_0 + \int_a^x f(t, u(t))\, d_{Q,e}t.$$

Proof. (i)⇒(ii) If u is a solution, then u is continuous at a and $u(a) = u_0$. Moreover, ${}_aD_{Q,e}u(t) = f(t, u(t))$ for all $t \in \Gamma_Q(b)$. Since f is bounded on $\Gamma_Q(b) \times \mathbb{R}$, the function ${}_aD_{Q,e}u$ is bounded on $\Gamma_Q(b)$. Theorem 4.9 yields

$$\int_a^x {}_aD_{Q,e}u(t)\, d_{Q,e}t = u(x) - u(a) = u(x) - u_0.$$

Substituting ${}_aD_{Q,e}u(t) = f(t, u(t))$ gives (ii).

(ii)⇒(i) Assume that (ii) holds. Evaluating at $x = a$ gives $u(a) = u_0$. Define $F : \{a\} \cup \Gamma_Q(b) \to \mathbb{R}$ by

$$F(a) = 0, \qquad F(x) = \int_a^x f(t, u(t))\, d_{Q,e}t, \qquad x \in \Gamma_Q(b).$$

Since f is bounded on $\Gamma_Q(b) \times \mathbb{R}$, the integral defining F is well defined. By Theorem 4.8, for every $x \in \Gamma_Q(b)$, one has

$$_aD_{Q,e}F(x) = f(x, u(x)).$$

From (ii), $u(x) = u_0 + F(x)$ for all $x \in \Gamma_Q(b)$, and therefore, by linearity,

$$_aD_{Q,e}u(x) = f(x, u(x)), \qquad x \in \Gamma_Q(b).$$

Thus u is a solution of (5.8) in the sense of Definition 5.8. □

Theorem 5.11 (Existence and uniqueness)**.** *Assume that $f : \Gamma_Q(b) \times \mathbb{R} \to \mathbb{R}$ is bounded. Assume in addition that there exists a bounded function $\varphi : \Gamma_Q(b) \to [0, \infty)$ such that*

$$|f(t, y) - f(t, z)| \le \varphi(t)\, |y - z| \quad \textit{for all } t \in \Gamma_Q(b) \textit{ and all } y, z \in \mathbb{R}, \tag{5.9}$$

and

$$\sup_{x \in \Gamma_Q(b)} \int_a^x \varphi(t)\, d_{Q,e}t < 1. \tag{5.10}$$

Then the problem (5.8) *admits a unique solution $u : \{a\} \cup \Gamma_Q(b) \to \mathbb{R}$ in the sense of Definition 5.8.*

Proof. Set $X = C(\{a\} \cup \Gamma_Q(b))$ endowed with the supremum norm

$$\|u\|_\infty = \sup_{x \in \{a\} \cup \Gamma_Q(b)} |u(x)|.$$

Since $\{a\} \cup \Gamma_Q(b)$ is compact, $(X, \|\cdot\|_\infty)$ is a Banach space.

For $u \in X$, define $\mathcal{T}u : \{a\} \cup \Gamma_Q(b) \to \mathbb{R}$ by

$$(\mathcal{T}u)(a) = u_0, \qquad (\mathcal{T}u)(x) = u_0 + \int_a^x f(t, u(t))\, d_{Q,e}t, \qquad x \in \Gamma_Q(b).$$

Fix $u \in X$. We claim that $\mathcal{T}u$ is continuous at a. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\Gamma_Q(b)$ such that $x_n \to a$ as $n \to \infty$. Since f is bounded on $\Gamma_Q(b) \times \mathbb{R}$, there exists $M > 0$ such that

$$|f(t, y)| \le M \quad \text{for all } t \in \Gamma_Q(b) \text{ and all } y \in \mathbb{R}.$$

Hence, for every $n \in \mathbb{N}$,

$$\left|(\mathcal{T}u)(x_n) - (\mathcal{T}u)(a)\right| = \left|\int_a^{x_n} f(t, u(t))\, d_{Q,e}t\right| \le \int_a^{x_n} \left|f(t, u(t))\right| d_{Q,e}t \le M \int_a^{x_n} 1\, d_{Q,e}t.$$

By Definition 4.1,

$$\int_a^{x_n} 1\, d_{Q,e}t = \frac{\langle (I - Q)(x_n - a), e \rangle}{1 - q}.$$

Since $x_n \to a$, one has $\langle (I-Q)(x_n-a), e\rangle \to 0$, and therefore

$$\int_a^{x_n} 1\, d_{Q,e}t \to 0 \quad \text{as } n \to \infty.$$

Consequently, $(\mathcal{T}u)(x_n) \to (\mathcal{T}u)(a) = u_0$, which proves that $\mathcal{T}u$ is continuous at a. Consequently, $\mathcal{T}u \in X$ (see Remark 4.2), that is, $\mathcal{T} : X \to X$ is well-defined.

Let $u, v \in X$ and fix $x \in \Gamma_Q(b)$. Using the Lipschitz condition and the monotonicity of the (a, Q, e)-integral, we obtain

$$\begin{aligned} |(\mathcal{T}u)(x) - (\mathcal{T}v)(x)| &= \left| \int_a^x \Big(f(t,u(t)) - f(t,v(t)) \Big)\, d_{Q,e}t \right| \\ &\le \int_a^x \Big| f(t,u(t)) - f(t,v(t)) \Big|\, d_{Q,e}t \\ &\le \int_a^x \varphi(t)\, |u(t)-v(t)|\, d_{Q,e}t \\ &\le \|u-v\|_\infty \int_a^x \varphi(t)\, d_{Q,e}t. \end{aligned}$$

Since $(\mathcal{T}u)(a) = (\mathcal{T}v)(a) = u_0$, taking the supremum over $x \in \{a\} \cup \Gamma_Q(b)$ yields

$$\|\mathcal{T}u - \mathcal{T}v\|_\infty \le \left(\sup_{x\in\Gamma_Q(b)} \int_a^x \varphi(t)\, d_{Q,e}t \right) \|u-v\|_\infty.$$

By assumption, the constant in parentheses is strictly less than 1, hence $\mathcal{T}$ is a contraction on the Banach space $(X, \|\cdot\|_\infty)$. By the Banach fixed point theorem, $\mathcal{T}$ admits a unique fixed point $u \in X$. In particular,

$$u(x) = u_0 + \int_a^x f(t,u(t))\, d_{Q,e}t, \qquad x \in \Gamma_Q(b).$$

By Lemma 5.10, u is the unique solution of (5.8) in the sense of Definition 5.8. □

Remark 5.12. Condition (5.10) is a sufficient condition ensuring that the integral operator used in the proof is a contraction. It should be viewed as a convenient Banach-type sufficient condition, not as an optimal uniqueness condition. In particular, the bound may be relaxed in special situations, but this would require arguments adapted to the particular structure of f, rather than the general contraction argument used here.

6. Conclusions

In this paper, we developed a matrix-induced quantum calculus on discrete forward orbits generated by affine contractions. This framework extends one-dimensional q-calculus to higher dimensions while preserving a transparent geometric interpretation, driven by the affine dynamics associated with the map $T(x) = a + Q(x-a)$ and the Q-admissible direction e. As applications, we established a Hermite–Hadamard-type inequality for convex functions under a natural segment-preserving assumption on Q, derived a Poincaré-type inequality on $\Gamma_Q(b)$, and proved existence and uniqueness for a nonlinear first-order problem governed by the matrix-induced directional derivative.

The proposed framework opens several directions for further research. Possible extensions include the development of higher-order and fractional matrix-induced operators, as well as the study of boundary value problems driven by ${}_aD_{Q,e}$ and its iterates. Another natural direction is the investigation of variational principles and spectral problems associated with matrix-induced quantum operators on forward orbits.

Author contributions

All authors contributed equally to this work.

Use of Generative-AI tools declaration

The authors declare that no generative AI tools were used in the development of this manuscript.

Acknowledgments

The second author was supported by the Ongoing Research Funding Program, (ORF-2026-4), King Saud University, Riyadh, Saudi Arabia.

Conflict of interest

The authors declare no conflict of interest in this work.

References

1. F. H. Jackson, On q-definite integrals, *Quart. J. Pure Appl. Math.,* **41** (1910), 193–203.
2. F. H. Jackson, q-Difference equations, *Amer. J. Math.,* **32** (1910), 305–314. http://dx.doi.org/10.2307/2370183
3. G. E. Andrews, *q-Hypergeometric and related functions*, in: NIST Handbook of Mathematical Functions, U.S. Department of Commerce, Washington, DC, USA, 2010, 419–433.
4. Y. Özkan, S. Korkmaz, E. Deniz, The monotony of the q-Bessel functions, *J. Math. Anal. Appl.,* **549** (2025), 129439. http://dx.doi.org/10.1016/j.jmaa.2025.129439
5. R. Pérez-Marco, On the definition of higher Gamma functions, *Constr. Approx.,* **61** (2025), 413–443. http://dx.doi.org/10.1007/s00365-023-09674-w
6. H. Chen, G. Filipuk, Y. Chen, Nonlinear difference equations arising from the generalized Stieltjes–Wigert and q-Laguerre weights, *Math. Methods Appl. Sci.,* **41** (2018), 2442–2465. http://dx.doi.org/10.1002/mma.4751
7. N. Joshi, T. L. Latimer, Asymptotics of discrete q-Freud II orthogonal polynomials from the q-Riemann–Hilbert problem, *Nonlinearity,* **36** (2023), 3969–4006. http://dx.doi.org/10.1088/1361-6544/acdbb3

8. T. Lasic Latimer, Asymptotics for multiple q-orthogonal polynomials from the Riemann–Hilbert problem, *J. Differ. Equ. Appl.,* **31** (2025), 1457–1486. http://dx.doi.org/10.1080/10236198.2025.2554148

9. I. Bouzida, Characterization theorems for the B_q-binomial and the q-Poisson distributions, *J. Math. Phys. Anal. Geom.,* **18** (2022), 182–193. http://dx.doi.org/10.15407/mag18.02.182

10. B. Imed, M. Afif, Z. Mouna, Estimation parameters for the binomial q-distribution, *Commun. Statist. Theory Methods,* **50** (2020), 5101–5113. http://dx.doi.org/10.1080/03610926.2020.1725825

11. H. Zhong, L. Zhao, q-binomial identities finder, *Adv. Appl. Math.,* **172** (2026), 102965. http://dx.doi.org/10.1016/j.aam.2025.102965

12. C. Voigt, R. Yuncken, *Complex semisimple quantum groups and representation theory*, Lecture Notes in Mathematics, Vol. 2264, Springer, Cham, 2020. http://dx.doi.org/10.1007/978-3-030-52463-0

13. A. Khan, Z. Abbas, M. Qasim, M. Mursaleen, Approximation by modified Lupaş operators based on (p, q)-integers, *J. Comput. Anal. Appl.,* **29** (2021), 922–933.

14. M. J. Mohammed, A. Ghafarpanah, S. Etemad, S. K. Ntouyas, J. Tariboon, On the existence of (p, q)-solutions for the post-quantum Langevin equation: A fixed-point-based approach, *Axioms,* **14** (2025), 474. http://dx.doi.org/10.3390/axioms14060474

15. A. E. Moreka, S. Kumar, M. Mursaleen, On wavelets Kantorovich–Baskakov operators and approximation properties, *J. Inequal. Appl.,* **2023** (2023), 134. http://dx.doi.org/10.1186/s13660-023-03045-6

16. M. H. Annaby, Z. S. Mansour, *q-Fractional calculus and equations*, Lecture Notes in Mathematics, Vol. 2056, Springer, Berlin, Germany, 2012. http://dx.doi.org/10.1007/978-3-642-30898-7

17. M. Houas, M. E. Samei, Existence and stability of solutions for linear and nonlinear damping of q-fractional Duffing–Rayleigh problem, *Mediterr. J. Math.,* **20** (2023), 148. http://dx.doi.org/10.1007/s00009-023-02355-9

18. Z. Qin, S. Sun, Positive solutions for fractional (p, q)-difference boundary value problems, *J. Appl. Math. Comput.,* **68** (2022), 2571–2588. https://doi.org/10.1007/s12190-021-01630-w

19. I. Bouzida, M. Zitouni, H. Eleuch, A new characterization and estimation framework for the Gamma-Lindley quantum distribution and applications in q-Schrödinger equation, *J. Math. Comput. Sci.,* **42** (2026), 288–307. https://dx.doi.org/10.22436/jmcs.042.03.02

20. T. Ernst, *A Comprehensive treatment of q-calculus*, Springer, Basel, Switzerland, 2012. http://dx.doi.org/10.1007/978-3-0348-0431-8

21. V. Kac, P. Cheung, *Quantum calculus*, Springer, New York, NY, USA, 2002. http://dx.doi.org/10.1007/978-1-4613-0071-7

22. J. Tariboon, S. K. Ntouyas, Quantum calculus on finite intervals and applications to impulsive difference equations, *Adv. Differ. Equ.,* **2013** (2013), 282. http://dx.doi.org/10.1186/1687-1847-2013-282

23. I. Area, N. Atakishiyev, E. Godoy, J. Rodal, Linear partial q-difference equations on q-linear lattices and their bivariate q-orthogonal polynomial solutions, *Appl. Math. Comput.,* **223** (2013), 520–536. http://dx.doi.org/10.1016/j.amc.2013.08.018

24. Z. G. Liu, A q-extension of a partial differential equation and the Hahn polynomials, *Ramanujan J.,* **38** (2015), 481–501. http://dx.doi.org/10.1007/s11139-014-9632-1

25. S. K. Mishra, M. E. Samei, S. K. Chakraborty, B. Ram, On q-variant of Dai–Yuan conjugate gradient algorithm for unconstrained optimization problems, *Nonlinear Dyn.,* **104** (2021), 2471–2496. http://dx.doi.org/10.1007/s11071-021-06378-3

26. J. Tariboon, S. K. Ntouyas, Quantum integral inequalities on finite intervals, *J. Inequal. Appl.,* **2014** (2014), 121. http://dx.doi.org/10.1186/1029-242X-2014-121

27. N. Alp, M. Z. Sarikaya, M. Kunt, I. Işcan, q-Hermite–Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions, *J. King Saud Univ. Sci.,* **30** (2018), 193–203. http://dx.doi.org/10.1016/j.jksus.2016.09.007

28. S. Bermudo, P. Kórus, J. N. Valdés, On q-Hermite–Hadamard inequalities for general convex functions, *Acta Math. Hungar.,* **162** (2020), 364–374. http://dx.doi.org/10.1007/s10474-020-01025-6

29. S. B. Akbar, M. Abbas, H. Budak, Generalization of quantum calculus and corresponding Hermite–Hadamard inequalities, *Anal. Math. Phys.,* **14** (2024), 99. http://dx.doi.org/10.1007/s13324-024-00960-9

30. J. L. Cardoso, E. M. Shehata, Hermite–Hadamard inequalities for quantum integrals: A unified approach, *Appl. Math. Comput.,* **463** (2024), 128345. http://dx.doi.org/10.1016/j.amc.2023.128345

31. S. K. Mishra, R. Sharma, J. Bisht, Quantum analogue of Hermite–Hadamard type inequalities for strongly convex functions, *Rend. Circ. Mat. Palermo,* **74** (2025), 29. http://dx.doi.org/10.1007/s12190-024-02135-y

32. M. Toseef, Z. Zhang, M. A. Ali, On q-Hermite–Hadamard–Mercer and midpoint-Mercer inequalities for general convex functions with their computational analysis, *Int. J. Geom. Methods M.,* **22** (2025), 2450319. http://dx.doi.org/10.1142/S0219887824503195

AIMS Press