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Research article

Fractional stochastic systems with memory: Existence, Ulam–Hyers stability, and local approximate controllability

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Abstract: This paper studies a class of fractional stochastic integro-differential systems with memory effects and control inputs. The model involves a Caputo fractional derivative of order $\alpha \in (1/2, 1)$, a Volterra-type memory kernel, and stochastic perturbations driven by a Wiener process. Under standard Lipschitz and boundedness assumptions, we establish the existence and uniqueness of mild solutions in the space of mean-square continuous processes via the Banach contraction principle, together with an explicit contraction condition. We further prove Ulam–Hyers stability, providing quantitative bounds that characterize the sensitivity of solutions to perturbations. In addition, we investigate local approximate controllability through a scaled bounded-control approximation framework. We show that, for sufficiently small terminal times and for targets approaching the initial state at the fractional scaling rate $\|x_T - x_0\| \leq \rho T^{\alpha-1/2}$ the controllability of the nonlinear stochastic system can be inferred from that of an associated reduced linear system using controls whose L^2-norms remain uniformly bounded as $T \to 0^+$. A finite-dimensional example is provided to demonstrate that the scaled bounded-control approximation hypothesis can be verified explicitly in a concrete setting and to illustrate the applicability of the controllability framework. The results provide a unified analytical framework for studying well-posedness, stability, and controllability in fractional stochastic systems with memory.

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1. Introduction

Fractional differential equations have emerged as a powerful mathematical framework for modeling dynamical systems with memory and hereditary properties. In many physical and engineering applications, including viscoelasticity, anomalous diffusion, and biological processes, the present state depends not only on the current input but also on the entire history of the system. Fractional-order models therefore provide a more realistic description than their integer-order counterparts; see, for example, [1, 2].

In practical applications, uncertainty is an unavoidable feature due to environmental fluctuations, measurement errors, and external disturbances. This has motivated the development of stochastic differential systems and, more recently, stochastic fractional systems, which combine memory effects with randomness in a unified manner; see [3, 4]. These models have attracted increasing attention due to their ability to capture complex dynamical behaviors that cannot be described within classical frameworks.

A large body of literature is devoted to the analysis of fractional dynamical systems, particularly with regard to existence, uniqueness, and controllability. Controllability of fractional integro-differential systems in Banach spaces was investigated in [5], and further developments for fractional semilinear systems were obtained in [6]. In stochastic settings, approximate controllability and the existence of mild solutions were studied in [7]. In parallel, Ulam–Hyers stability has been widely used to quantify the robustness of solutions under perturbations; see [8–12].

Recent years have witnessed significant progress in the study of more complex fractional stochastic systems. For instance, Sobolev-type stochastic fractional equations driven by fractional Brownian motion were analyzed in [13]. Fractional stochastic systems with generalized memory kernels were investigated in [14, 15]. The approximate controllability of stochastic Hilfer-type systems with delay was studied in [16], and history-dependent stochastic hemivariational inequalities were considered in [17]. Further developments include second-order stochastic equations with jumps [18] and multivalued fractional stochastic systems [19].

Additional analytical frameworks have been developed for fractional systems. Existence theory for ψ-Caputo equations was studied in [20], and boundary value problems for Hilfer and Caputo equations in Banach spaces were addressed in [21]. Multiorder fractional systems were analyzed in [22], and adaptive filtering methods for fractional nonlinear systems were investigated in [23]. Fractional stochastic partial differential equations were studied in [24], and boundary controllability in Hilbert spaces was examined in [25]. Stability properties of stochastic evolution equations with delay were analyzed in [26], and periodic-type behaviors were studied in [27]. Classical analytical tools such as nonlinear integral equations and measures of noncompactness remain essential in this context; see [28, 29]. Applications to financial systems and discrete dynamical models have also been explored in [30, 31], and systems with multiple delays in control were considered in [32].

More recently, attention has been given to broader classes of fractional integro-differential systems involving memory and control. In particular, the stability and controllability of nonlinear Volterra–Fredholm–Hammerstein systems were investigated in [33]. Numerical methods for generalized tempered-type fractional integro-differential equations were developed in [34]. Furthermore, Hyers–Ulam stability and continuous dependence for nonlocal stochastic–integral equations of arbitrary fractional order were established in [35]. These works highlight the growing importance of combining fractional dynamics, stochastic perturbations, and control mechanisms within a unified analytical framework.

Despite these advances, several important challenges remain. First, many studies consider either deterministic fractional systems with memory or stochastic fractional systems without explicit memory effects, and the simultaneous treatment of both features remains comparatively less developed. Second, controllability and stability are often analyzed separately, and unified approaches addressing existence, stability, and controllability within a single framework are still scarce. Third, many controllability results rely on restrictive assumptions and do not explicitly address the boundedness of admissible controls, which is particularly important in stochastic settings. In particular, the bounded-control approximation property required for controllability is rarely analyzed in a rigorous and transparent manner.

Motivated by these observations, the present paper develops a unified framework for fractional stochastic systems with memory effects. We consider a class of stochastic integro-differential equations involving a Caputo fractional derivative of order $\alpha \in (1/2, 1)$, a Volterra-type memory kernel, a control input, and a Wiener stochastic perturbation. The restriction $\alpha > 1/2$ is essential for ensuring the square integrability of the singular kernel arising in the stochastic convolution term, which is required for the application of Itô isometry.

The main contributions of this work are as follows. First, we establish the existence and uniqueness of mild solutions using the Banach contraction principle under an explicit and verifiable condition. Second, we derive Ulam–Hyers stability results providing quantitative robustness estimates. Third, we investigate local approximate controllability under a bounded-control approximation hypothesis with targets satisfying the fractional scaling condition

$$\|x_T - x_0\| \le \rho T^{\alpha-1/2}.$$

This scaling is dictated by the small-time behavior of the fractional controllability operator and ensures the existence of controls whose L^2-norms remain uniformly bounded as $T \to 0^+$. The result clarifies the relationship between the nonlinear stochastic system and its associated reduced linear system. Finally, we present a finite-dimensional example that demonstrates the applicability of the scaled bounded-control approximation hypothesis and verifies explicitly that the controllability framework can be realized in a concrete setting.

The remainder of the paper is organized as follows. Section 2 presents the necessary preliminaries. Section 3 introduces the problem formulation. Section 4 establishes existence and uniqueness. Section 5 is devoted to Ulam–Hyers stability. Section 6 addresses local approximate controllability. Section 7 provides an illustrative example, and Section 8 concludes the paper.

2. Preliminaries

In this section, we recall the functional, fractional, stochastic, and controllability notions used throughout the paper. Throughout the paper, $J = [0, T]$, where $T > 0$, and X denotes a real separable Hilbert space with norm $\|\cdot\|$. Because every Hilbert space is a Banach space, this setting remains compatible with the deterministic fixed-point arguments used below, while providing a standard framework for stochastic integration. This choice also avoids technical complications associated with stochastic integration in general Banach spaces and aligns the abstract framework with the illustrative example presented later.

We denote by $C(J, X)$ the Banach space of all continuous functions from J into X, endowed with the norm

$$\|x\|_C := \sup_{t\in J} \|x(t)\|.$$

The space $C(J, X)$ is recalled here for completeness but will not be used directly in the fixed-point argument. For stochastic processes, we will work in the adapted space $\mathcal{H}_{\mathcal{F}}$ defined below.

2.1. *Fractional calculus*

The following definitions are standard in fractional calculus; see [1, 2].

Definition 2.1. *Let $\alpha > 0$ and $x \in L^1(J, X)$. The Riemann–Liouville fractional integral of order α is defined by*

$$(I^\alpha x)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)\, ds, \qquad t \in J.$$

Definition 2.2. *Let $\alpha \in (0, 1)$ and $x \in AC(J, X)$. The Caputo fractional derivative of order α is defined by*

$$({}^C D_t^\alpha x)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} x'(s)\, ds, \qquad \text{for a.e. } t \in J.$$

The following integral representation is classical; see [1, 2].

Lemma 2.3. *Let $\alpha \in (0, 1)$. Suppose that $x \in AC(J, X)$, $y \in L^1(J, X)$, and*

$${}^C D_t^\alpha x(t) = y(t) \quad \text{for a.e. } t \in J,$$

with $x(0) = x_0$. Then,

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds, \qquad t \in J.$$

We will repeatedly use the following elementary identities. For $\alpha > 0$,

$$\int_0^t (t-s)^{\alpha-1}\, ds = \frac{t^\alpha}{\alpha}, \qquad t \in J,$$

and hence,

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, ds = \frac{t^\alpha}{\Gamma(\alpha+1)}.$$

Moreover, if $\alpha > 1/2$, then

$$\int_0^t (t-s)^{2\alpha-2}\, ds = \frac{t^{2\alpha-1}}{2\alpha-1}, \qquad t \in J,$$

and consequently,

$$\frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{2\alpha-2}\, ds \right)^{1/2} = \frac{t^{\alpha-1/2}}{\Gamma(\alpha)\sqrt{2\alpha-1}}.$$

The restriction $\alpha > 1/2$ is therefore essential in the present stochastic setting, as it guarantees the square integrability of the singular kernel $(t-s)^{\alpha-1}$ appearing in the stochastic convolution term.

2.2. *Stochastic framework*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $W(t)$ be a real-valued standard Wiener process adapted to this filtration.

For an X-valued random variable ξ, we define

$$\|\xi\|_{L^2} := \left(\mathbb{E}\|\xi\|^2\right)^{1/2}.$$

We denote by $C(J, L^2(\Omega, X))$ the space of all stochastic processes $x : J \to L^2(\Omega, X)$ that are mean-square continuous on J, endowed with the norm

$$\|x\|_{C(L^2)} := \sup_{t\in J}\left(\mathbb{E}\|x(t)\|^2\right)^{1/2}.$$

The following definition is standard in the theory of stochastic processes; see [3].

Definition 2.4. *A stochastic process $x : J \to L^2(\Omega, X)$ is said to be mean-square continuous on J if*

$$\lim_{t\to s}\mathbb{E}\|x(t) - x(s)\|^2 = 0$$

for every $s \in J$.

We now introduce the adapted space that will be used as the fixed-point setting throughout the paper.

Definition 2.5. *Define*

$$\mathcal{H}_{\mathcal{F}} := \left\{ x : J \to L^2(\Omega, X) \ : \ \begin{array}{l} x \text{ is mean-square continuous on } J, \\ x \text{ is } \{\mathcal{F}_t\}_{t\geq 0}\text{-adapted} \end{array} \right\},$$

endowed with the norm

$$\|x\|_{\mathcal{H}} := \sup_{t\in J}\left(\mathbb{E}\|x(t)\|^2\right)^{1/2}.$$

Lemma 2.6. *The space $(\mathcal{H}_{\mathcal{F}}, \|\cdot\|_{\mathcal{H}})$ is a Banach space.*

Proof. Let $\{x_n\}_{n\geq 1}$ be a Cauchy sequence in $\mathcal{H}_{\mathcal{F}}$. Because the norm is the supremum norm induced by $L^2(\Omega, X)$, for each fixed $t \in J$, the sequence $\{x_n(t)\}_{n\geq 1}$ is Cauchy in the complete space $L^2(\Omega, X)$. Hence, there exists $x(t) \in L^2(\Omega, X)$ such that

$$x_n(t) \longrightarrow x(t) \quad \text{in } L^2(\Omega, X).$$

Moreover, because $\{x_n\}$ is Cauchy with respect to $\|\cdot\|_{\mathcal{H}}$, the convergence is uniform in $t \in J$, and therefore,

$$\|x_n - x\|_{\mathcal{H}} \longrightarrow 0.$$

We show that $x \in \mathcal{H}_{\mathcal{F}}$. First, the uniform L^2-limit of mean-square continuous processes is mean-square continuous. Indeed, for $s, t \in J$,

$$\begin{aligned}\left(\mathbb{E}\|x(t) - x(s)\|^2\right)^{1/2} \leq &\left(\mathbb{E}\|x(t) - x_n(t)\|^2\right)^{1/2} \\ &+ \left(\mathbb{E}\|x_n(t) - x_n(s)\|^2\right)^{1/2} \\ &+ \left(\mathbb{E}\|x_n(s) - x(s)\|^2\right)^{1/2}.\end{aligned}$$

Given $\varepsilon > 0$, choose n large enough so that the first and third terms are less than $\varepsilon/3$, uniformly in $s, t \in J$. Since x_n is mean-square continuous, the middle term tends to zero as $t \to s$. Hence x is mean-square continuous.

Second, adaptedness is preserved under L^2-limits. For each fixed $t \in J$, the space of $\mathcal{F}_t$-measurable $L^2(\Omega, X)$-valued random variables is closed in $L^2(\Omega, X)$. Since $x_n(t)$ is $\mathcal{F}_t$-measurable for every n and $x_n(t) \to x(t)$ in $L^2(\Omega, X)$, it follows that $x(t)$ is $\mathcal{F}_t$-measurable. Therefore x is $\{\mathcal{F}_t\}_{t\geq 0}$-adapted.

Thus $x \in \mathcal{H}_{\mathcal{F}}$, and $\mathcal{H}_{\mathcal{F}}$ is complete. □

We shall use the following Itô isometry for Hilbert-space-valued stochastic integrals with respect to a real-valued Wiener process; see [3].

Lemma 2.7. *Let* $\phi : J \times \Omega \to X$ *be an adapted, strongly measurable process satisfying*

$$\mathbb{E}\int_0^T \|\phi(s)\|^2\, ds < \infty.$$

Then the stochastic integral $\int_0^t \phi(s)\, dW(s)$ *is well defined in* $L^2(\Omega, X)$*, and for every* $t \in J$*,*

$$\mathbb{E}\left\|\int_0^t \phi(s)\, dW(s)\right\|^2 = \mathbb{E}\int_0^t \|\phi(s)\|^2\, ds.$$

All stochastic integrals appearing throughout this paper are understood in the Itô sense.

In particular, if $x \in \mathcal{H}_{\mathcal{F}}$ and the mapping $\sigma : J \times X \to X$ satisfies a global Lipschitz condition together with

$$\sup_{t\in J} \|\sigma(t, 0)\| < \infty,$$

then

$$\mathbb{E}\int_0^T \|\sigma(s, x(s))\|^2\, ds < \infty.$$

Indeed, if

$$\|\sigma(t, \xi) - \sigma(t, \eta)\| \leq L_\sigma \|\xi - \eta\|, \qquad t \in J, \quad \xi, \eta \in X,$$

then

$$\|\sigma(s, x(s))\|^2 \leq 2L_\sigma^2 \|x(s)\|^2 + 2\|\sigma(s, 0)\|^2,$$

and therefore

$$\mathbb{E}\int_0^T \|\sigma(s, x(s))\|^2\, ds \leq 2L_\sigma^2 T \|x\|_{\mathcal{H}}^2 + 2T \sup_{s\in J} \|\sigma(s, 0)\|^2 < \infty.$$

This observation will be used to justify the stochastic integral appearing in the mild formulation.

2.3. Approximate controllability

We recall the notion of approximate controllability used in fractional control systems; see, for example, [4, 5, 8, 9]. For a given initial state $x_0 \in X$, let $\mathcal{R}(T, x_0)$ denote the reachable set at time T, namely the set of terminal states generated by all admissible controls on J.

Definition 2.8. *The system is said to be approximately controllable on J if*

$$\overline{\mathcal{R}(T, x_0)} = X$$

for every $x_0 \in X$, where the closure is taken in the norm topology of X.

In the stochastic setting of this paper, approximate controllability is understood in the mean-square sense: For every target $x_T \in X$ and every $\varepsilon > 0$, there exists an admissible control u such that the corresponding mild solution x satisfies

$$\left(\mathbb{E}\|x(T) - x_T\|^2\right)^{1/2} < \varepsilon.$$

2.4. Standing convention on the fractional order

Although the deterministic Caputo derivative is meaningful for every $\alpha \in (0, 1)$, the stochastic convolution term in the present model contains the singular kernel $(t-s)^{\alpha-1}$. Therefore, throughout the analysis of the stochastic system, we impose

$$\alpha \in (1/2, 1)\,.$$

This assumption ensures that

$$(t-s)^{\alpha-1} \in L^2(0, t)$$

for every $t \in (0, T]$, which is required for applying the Itô isometry to the stochastic integral.

2.5. A limitation of the bounded-operator framework

Throughout this paper, the linear operator $A : X \to X$ is assumed to be bounded. This assumption is deliberately imposed in order to focus on the combined effects of fractional memory, Volterra-type terms, stochastic perturbations, and control inputs without introducing the additional technical machinery of C_0-semigroups, sectorial operators, or fractional resolvent families. Many partial differential equation (PDE) models naturally lead to unbounded operators; extending the present results to such settings is an important direction for future research.

Finally, generic positive constants denoted by $M_1, M_2, \ldots$ may change from line to line.

3. Problem formulation

In this section, we introduce the class of fractional stochastic systems with memory effects under consideration and state the standing assumptions used throughout the paper.

Let X be a real separable Hilbert space with norm $\|\cdot\|$, and let $J = [0, T]$, where $T > 0$. All stochastic processes considered in this paper are defined on a filtered probability space

$$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$$

satisfying the usual conditions.

We consider the following controlled fractional stochastic system in formal differential form:

$$^{C}D_t^{\alpha} x(t) = Ax(t) + \int_0^t K(t, s)x(s)\, ds + Bu(t) + f(t, x(t)) + \sigma(t, x(t))\, dW(t), \qquad t \in J, \tag{3.1}$$

with initial condition

$$x(0) = x_0. \tag{3.2}$$

Equation (3.1) is understood only as a formal differential representation. Throughout this paper, all solutions are interpreted rigorously through the mild stochastic integral formulation (3.3). The parameter α satisfies

$$\alpha \in (1/2, 1).$$

The restriction $\alpha > 1/2$ ensures that the kernel $(t-s)^{\alpha-1}$ is square integrable on $(0, t)$, which is required for the stochastic integral in (3.3) to be well-defined; see Section 2.

3.1. Mild solution

Using Lemma 2.3, the formal system (3.1)–(3.2) is represented in mild form as

$$\begin{aligned} x(t) = x_0 &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[Ax(s) + \int_0^s K(s,\tau)x(\tau)\, d\tau + Bu(s) + f(s, x(s)) \Big] ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sigma(s, x(s))\, dW(s), \qquad t \in J. \end{aligned} \tag{3.3}$$

Definition 3.1. *A stochastic process $x \in \mathcal{H}_{\mathcal{F}}$ is said to be a mild solution of* (3.1)–(3.2) *if it satisfies the integral equation* (3.3) *for every $t \in J$, almost surely.*

3.2. Assumptions

We impose the following conditions.

(A1) $A : X \to X$ is a bounded linear operator, and there exists $M_A > 0$ such that

$$\|Ax\| \le M_A \|x\|, \qquad x \in X.$$

Remark. This boundedness assumption is imposed to avoid the technical framework of C_0-semigroups, sectorial operators, and fractional resolvent families. Many PDE models involve unbounded operators; extending the present analysis to such settings is an important direction for future research.

(A2) The kernel $K : J \times J \to \mathcal{L}(X)$ is continuous, and there exists $K_0 > 0$ such that

$$\|K(t, s)\| \le K_0, \qquad (t, s) \in J \times J.$$

Remark. The boundedness and continuity of K are imposed to simplify the analysis and to obtain explicit estimates. More general assumptions, such as $K \in L^2(J \times J, \mathcal{L}(X))$, are possible but would require additional technical estimates and are not needed for the main results of this paper.

(A3) The function $f : J \times X \to X$ is continuous and globally Lipschitz in the second variable; that is, there exists $L_f > 0$ such that

$$\|f(t, x) - f(t, y)\| \le L_f \|x - y\|, \qquad t \in J, \quad x, y \in X.$$

(A4) The function $\sigma : J \times X \to X$ is continuous and globally Lipschitz in the second variable; that is, there exists $L_\sigma > 0$ such that

$$\|\sigma(t,x) - \sigma(t,y)\| \le L_\sigma \|x - y\|, \qquad t \in J, \quad x, y \in X,$$

and

$$\sup_{t\in J} \|\sigma(t,0)\|^2 < \infty.$$

Moreover, f and σ are assumed to be Borel measurable, so that for any adapted process x, the compositions $f(\cdot, x(\cdot))$ and $\sigma(\cdot, x(\cdot))$ are progressively measurable and strongly measurable.

(A5) $B : U \to X$ is a bounded linear operator, where U is a separable Hilbert space of controls.

(A6) The control input satisfies $u \in L^2(J, U)$.

(A7) The initial value $x_0 \in L^2(\Omega, X)$ is $\mathcal{F}_0$-measurable.

3.3. Operator formulation

Define the operator $\mathcal{T} : \mathcal{H}_{\mathcal{F}} \to \mathcal{H}_{\mathcal{F}}$ by

$$\begin{aligned}(\mathcal{T}x)(t) = x_0 &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[Ax(s) + \int_0^s K(s,\tau)x(\tau)\,d\tau + Bu(s) + f(s,x(s))\Big]\,ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sigma(s,x(s))\,dW(s), \qquad t \in J.\end{aligned} \tag{3.4}$$

Under assumptions (A1)–(A7), the operator $\mathcal{T}$ is well- defined on $\mathcal{H}_{\mathcal{F}}$. Indeed, if $x \in \mathcal{H}_{\mathcal{F}}$, then the mappings

$$s \mapsto Ax(s), \qquad s \mapsto \int_0^s K(s,\tau)x(\tau)\,d\tau, \qquad s \mapsto Bu(s), \qquad s \mapsto f(s,x(s))$$

are adapted and mean-square integrable on J. Moreover, the stochastic integral term is well-defined by Lemma 2.7. Because the filtration satisfies the usual conditions, and the integrands are adapted, it follows that $(\mathcal{T}x)(t)$ is $\mathcal{F}_t$-measurable for every $t \in J$. A process $x \in \mathcal{H}_{\mathcal{F}}$ is a mild solution of (3.1)–(3.2) if and only if it is a fixed point of $\mathcal{T}$.

4. Existence and uniqueness of mild solutions

In this section, we establish the existence and uniqueness of mild solutions for system (3.1)–(3.2). Throughout this section, Assumptions (A1)–(A7) are satisfied and $\alpha \in (1/2, 1)$.

We work in the Banach space $(\mathcal{H}_{\mathcal{F}}, \|\cdot\|_{\mathcal{H}})$ introduced in Definition 2.5. By Lemma 2.6, this space is complete.

Recall that the operator $\mathcal{T} : \mathcal{H}_{\mathcal{F}} \to \mathcal{H}_{\mathcal{F}}$ is defined by (3.4). A mild solution is a fixed point of $\mathcal{T}$.

4.1. Well-posedness of $\mathcal{T}$

Lemma 4.1. *Under assumptions (A1)–(A7) and $\alpha > 1/2$, the operator $\mathcal{T}$ maps $\mathcal{H}_{\mathcal{F}}$ into itself.*

Proof. Let $x \in \mathcal{H}_{\mathcal{F}}$. Denote

$$r := \|x\|_{\mathcal{H}}.$$

Step 1: Boundedness.

Using Minkowski's inequality in $L^2(\Omega, X)$,

$$\left(\mathbb{E}\|(\mathcal{T}x)(t)\|^2\right)^{1/2} \leq \|x_0\|_{L^2} + I_A(t) + I_K(t) + I_B(t) + I_f(t) + I_\sigma(t).$$

We estimate each term explicitly.

(i) Linear term.

By Assumption (A1),

$$I_A(t) \leq \frac{M_A}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\mathbb{E}\|x(s)\|^2\right)^{1/2} ds.$$

Because

$$\left(\mathbb{E}\|x(s)\|^2\right)^{1/2} \leq r,$$

we obtain

$$I_A(t) \leq \frac{M_A r}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds = \frac{M_A r\, t^\alpha}{\Gamma(\alpha+1)}.$$

(ii) Volterra term.

Using Minkowski's integral inequality and Assumption (A2),

$$\begin{aligned}\left(\mathbb{E}\left\|\int_0^s K(s,\tau)x(\tau)\, d\tau\right\|^2\right)^{1/2} &\leq \int_0^s \|K(s,\tau)\| \left(\mathbb{E}\|x(\tau)\|^2\right)^{1/2} d\tau \\ &\leq K_0 \int_0^s r\, d\tau = K_0 r s.\end{aligned}$$

Hence,

$$I_K(t) \leq \frac{K_0 r}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s\, ds = \frac{K_0 r\, t^{\alpha+1}}{\Gamma(\alpha+2)}.$$

(iii) Control term.

By Assumption (A5) and the Cauchy–Schwarz inequality,

$$\begin{aligned}I_B(t) &\leq \frac{\|B\|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|u(s)\|\, ds \\ &\leq \frac{\|B\|}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{2\alpha-2} ds\right)^{1/2} \|u\|_{L^2(J,U)}.\end{aligned}$$

(iv) Nonlinear term.

By Assumption (A3),

$$\begin{aligned}\left(\mathbb{E}\|f(s,x(s))\|^2\right)^{1/2} &\leq L_f \left(\mathbb{E}\|x(s)\|^2\right)^{1/2} + \|f(s,0)\| \\ &\leq L_f r + \sup_{s\in J} \|f(s,0)\| =: C_f.\end{aligned}$$

Therefore,

$$I_f(t) \le \frac{C_f}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds = \frac{C_f t^\alpha}{\Gamma(\alpha+1)}.$$

(v) Stochastic term.

By Itô isometry (Lemma 2.7),

$$I_\sigma(t) = \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{2\alpha-2} \mathbb{E}\|\sigma(s, x(s))\|^2 ds \right)^{1/2}.$$

Using Assumption (A4),

$$\begin{aligned} \mathbb{E}\|\sigma(s, x(s))\|^2 &\le 2L_\sigma^2 \mathbb{E}\|x(s)\|^2 + 2\|\sigma(s, 0)\|^2 \\ &\le 2L_\sigma^2 r^2 + 2 \sup_{s\in J} \|\sigma(s, 0)\|^2 =: C_\sigma^2. \end{aligned}$$

Because $\alpha > 1/2$,

$$\int_0^t (t-s)^{2\alpha-2} ds = \frac{t^{2\alpha-1}}{2\alpha - 1}.$$

Hence,

$$I_\sigma(t) \le \frac{C_\sigma}{\Gamma(\alpha)} \left(\frac{t^{2\alpha-1}}{2\alpha-1} \right)^{1/2} = \frac{C_\sigma t^{\alpha-1/2}}{\Gamma(\alpha)\sqrt{2\alpha-1}}.$$

All integrals are finite because $\alpha > 1/2$. Consequently,

$$\sup_{t\in J} \left(\mathbb{E}\|(\mathcal{T}x)(t)\|^2 \right)^{1/2} < \infty.$$

Hence, $\mathcal{T}x$ is bounded in the norm of $\mathcal{H}_\mathcal{F}$.

Step 2: Mean-square continuity.

Let $0 \le t_1 < t_2 \le T$. We prove the mean-square continuity of the stochastic term. The deterministic terms are handled similarly using the L^1-continuity of the fractional kernel.

Define

$$S(t) := \int_0^t (t-s)^{\alpha-1} \sigma(s, x(s))\, dW(s).$$

By Itô isometry,

$$\begin{aligned} \mathbb{E}\|S(t_2) - S(t_1)\|^2 \le\ & 2 \int_0^{t_1} \left| (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right|^2 \mathbb{E}\|\sigma(s, x(s))\|^2\, ds \\ & + 2 \int_{t_1}^{t_2} (t_2-s)^{2\alpha-2} \mathbb{E}\|\sigma(s, x(s))\|^2\, ds. \end{aligned}$$

From Step 1,

$$\mathbb{E}\|\sigma(s, x(s))\|^2 \le C_\sigma^2, \qquad s \in J.$$

Therefore,

$$\begin{aligned} \mathbb{E}\|S(t_2) - S(t_1)\|^2 \le\ & 2C_\sigma^2 \int_0^{t_1} \left| (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right|^2 ds \\ & + 2C_\sigma^2 \int_{t_1}^{t_2} (t_2-s)^{2\alpha-2} ds. \end{aligned}$$

The second integral satisfies

$$\int_{t_1}^{t_2} (t_2 - s)^{2\alpha-2} ds = \frac{(t_2 - t_1)^{2\alpha-1}}{2\alpha - 1} \longrightarrow 0 \qquad \text{as } t_2 \to t_1.$$

For the first integral, set

$$r = t_1 - s.$$

Then,

$$(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} = (t_2 - t_1 + r)^{\alpha-1} - r^{\alpha-1}.$$

Hence,

$$\int_0^{t_1} \left|(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right|^2 ds = \int_0^{t_1} \left|(t_2 - t_1 + r)^{\alpha-1} - r^{\alpha-1}\right|^2 dr.$$

Because $\alpha > 1/2$, the function

$$h(r) = r^{\alpha-1}$$

belongs to $L^2(0, T)$. Extending h by zero to $\mathbb{R}$, the continuity of translations in $L^2(\mathbb{R})$ implies

$$\int_0^{t_1} \left|(t_2 - t_1 + r)^{\alpha-1} - r^{\alpha-1}\right|^2 dr \longrightarrow 0 \qquad \text{as } t_2 \to t_1.$$

Thus,

$$\mathbb{E}\|S(t_2) - S(t_1)\|^2 \longrightarrow 0 \qquad \text{as } t_2 \to t_1.$$

Therefore, the stochastic term is mean-square continuous.

For the deterministic terms, the corresponding kernels belong to $L^1(0, T)$, and the integrands are bounded in $L^2(\Omega, X)$ by Step 1. The same splitting argument together with the L^1-continuity of the fractional kernel yields mean-square continuity of the deterministic terms. Consequently, $\mathcal{T}x$ is mean-square continuous.

Step 3: Adaptedness.

For fixed $t \in J$, the map

$$s \mapsto (t - s)^{\alpha-1}\sigma(s, x(s))$$

is adapted because the factor $(t - s)^{\alpha-1}$ is deterministic, and $\sigma(s, x(s))$ is $\mathcal{F}_s$-measurable by the adaptedness of x and the progressive measurability of σ. Hence, the Itô isometry (Lemma 2.7) is applicable.

Because $x \in \mathcal{H}_{\mathcal{F}}$, the process x is adapted. Hence, the mappings

$$s \mapsto Ax(s), \qquad s \mapsto f(s, x(s))$$

are adapted because A is deterministic, and f is measurable.

Next, for each fixed $s \in J$,

$$\int_0^s K(s, \tau)x(\tau)\, d\tau$$

is $\mathcal{F}_s$-measurable because the integrand

$$\tau \mapsto K(s, \tau)x(\tau)$$

is adapted on $[0, s]$, and K is deterministic.

Moreover, Because

$$s \mapsto \sigma(s, x(s))$$

is progressively measurable and square integrable by Step 1, the stochastic integral

$$\int_0^t (t-s)^{\alpha-1} \sigma(s, x(s))\, dW(s)$$

is well-defined and $\mathcal{F}_t$-measurable for every $t \in J$ by Lemma 2.7.

Therefore, every term in the definition of $(\mathcal{T}x)(t)$ is $\mathcal{F}_t$-measurable, which implies that $\mathcal{T}x$ is adapted. Consequently,

$$\mathcal{T}x \in \mathcal{H}_{\mathcal{F}},$$

which completes the proof.

□

4.2. Contraction property

Lemma 4.2. *For any $x, y \in \mathcal{H}_{\mathcal{F}}$, the following estimate holds:*

$$\|\mathcal{T}x - \mathcal{T}y\|_{\mathcal{H}} \leq \Lambda(T) \|x - y\|_{\mathcal{H}},$$

where

$$\Lambda(T) := \frac{(M_A + L_f) T^{\alpha}}{\Gamma(\alpha+1)} + \frac{K_0 T^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{L_\sigma T^{\alpha-1/2}}{\Gamma(\alpha)\sqrt{2\alpha-1}}.$$

Proof. Let $x, y \in \mathcal{H}_{\mathcal{F}}$. For each $t \in J$, subtracting the mild formulations and applying Minkowski's inequality together with Itô isometry yields

$$\left(\mathbb{E}\|(\mathcal{T}x)(t) - (\mathcal{T}y)(t)\|^2\right)^{1/2} \leq J_1(t) + J_2(t) + J_3(t) + J_4(t),$$

where

$$\begin{aligned}
J_1(t) &:= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\mathbb{E}\|A(x(s) - y(s))\|^2\right)^{1/2} ds, \\
J_2(t) &:= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\mathbb{E}\left\|\int_0^s K(s,\tau)(x(\tau) - y(\tau))\, d\tau\right\|^2\right)^{1/2} ds, \\
J_3(t) &:= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\mathbb{E}\|f(s, x(s)) - f(s, y(s))\|^2\right)^{1/2} ds, \\
J_4(t) &:= \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{2\alpha-2} ds\right)^{1/2} \sup_{s \in [0,t]} \left(\mathbb{E}\|\sigma(s, x(s)) - \sigma(s, y(s))\|^2\right)^{1/2}.
\end{aligned}$$

By Assumptions (A1)–(A4),

$$\begin{aligned}
J_1(t) &\le \frac{M_A t^{\alpha}}{\Gamma(\alpha+1)}\|x-y\|_{\mathcal{H}},\\
J_2(t) &\le \frac{K_0 t^{\alpha+1}}{\Gamma(\alpha+2)}\|x-y\|_{\mathcal{H}},\\
J_3(t) &\le \frac{L_f t^{\alpha}}{\Gamma(\alpha+1)}\|x-y\|_{\mathcal{H}},\\
J_4(t) &\le \frac{L_\sigma t^{\alpha-1/2}}{\Gamma(\alpha)\sqrt{2\alpha-1}}\|x-y\|_{\mathcal{H}}.
\end{aligned}$$

Adding the four estimates and taking the supremum over $t \in J$ yields

$$\|\mathcal{T}x - \mathcal{T}y\|_{\mathcal{H}} \le \Lambda(T)\|x-y\|_{\mathcal{H}}.$$

This completes the proof. □

4.3. Existence and uniqueness theorem

Theorem 4.3. *Assume that (A1)–(A7) hold, and* $\alpha \in (1/2, 1)$. *If*

$$\Lambda(T) = \frac{(M_A + L_f)T^{\alpha}}{\Gamma(\alpha+1)} + \frac{K_0 T^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{L_\sigma T^{\alpha-1/2}}{\Gamma(\alpha)\sqrt{2\alpha-1}} < 1,$$

then system (3.1)–(3.2) *admit a unique mild solution in* $\mathcal{H}_{\mathcal{F}}$.

Proof. Lemma 4.1 shows that $\mathcal{T}$ maps $\mathcal{H}_{\mathcal{F}}$ into itself. Lemma 4.2 shows that $\mathcal{T}$ is a contraction on $\mathcal{H}_{\mathcal{F}}$ because $\Lambda(T) < 1$. Because $(\mathcal{H}_{\mathcal{F}}, \|\cdot\|_{\mathcal{H}})$ is complete by Lemma 2.6, the Banach fixed-point theorem guarantees the existence of a unique fixed point $x \in \mathcal{H}_{\mathcal{F}}$ satisfying

$$x = \mathcal{T}x.$$

By Definition 3.1, this fixed point is precisely the unique mild solution of (3.1)–(3.2). □

Remark 4.4. *The condition* $\alpha > 1/2$ *is essential. Indeed,*

$$\int_0^t (t-s)^{2\alpha-2}ds = \frac{t^{2\alpha-1}}{2\alpha-1} < \infty$$

if and only if $\alpha > 1/2$. *This integrability condition is necessary for the stochastic integral term in* (3.3) *to be well- defined in the mean-square sense. If* $\alpha \le 1/2$, *the kernel* $(t-s)^{\alpha-1}$ *fails to belong to* $L^2(0,t)$, *and Itô isometry cannot be applied.*

5. Ulam–Hyers stability

In this section, we investigate the Ulam–Hyers stability of the mild solution to system (3.1)–(3.2). This concept quantifies how small perturbations in the integral formulation affect the solution; see [8, 11] for related results in fractional deterministic and stochastic settings.

Throughout this section, Assumptions (A1)–(A7) hold, and $\alpha \in (1/2, 1)$. Moreover, the contraction condition $\Lambda(T) < 1$ from Theorem 4.3 is assumed whenever needed. We work in the Banach space $(\mathcal{H}_{\mathcal{F}}, \|\cdot\|_{\mathcal{H}})$ introduced in Definition 2.5.

Definition 5.1. *Let* $\varepsilon > 0$*. A stochastic process* $y \in \mathcal{H}_{\mathcal{F}}$ *is called an* ε-approximate solution *of* (3.1)–(3.2) *if there exists* $\Phi \in \mathcal{H}_{\mathcal{F}}$ *such that*

$$\|\Phi\|_{\mathcal{H}} \leq \varepsilon,$$

and y *satisfies, for every* $t \in J$*, almost surely,*

$$\begin{aligned} y(t) = x_0 &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[Ay(s) + \int_0^s K(s,\tau) y(\tau)\, d\tau + Bu(s) + f(s, y(s)) \Big] ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sigma(s, y(s))\, dW(s) + \Phi(t). \end{aligned}$$

Definition 5.2. *System* (3.1)–(3.2) *is said to be* Ulam–Hyers stable *if there exists a constant* $C_{\mathrm{UH}} > 0$ *such that, for every* $\varepsilon > 0$ *and every* ε*-approximate solution* $y \in \mathcal{H}_{\mathcal{F}}$*, there exists an exact mild solution* $x \in \mathcal{H}_{\mathcal{F}}$ *of* (3.1)–(3.2) *satisfying*

$$\|x - y\|_{\mathcal{H}} \leq C_{\mathrm{UH}} \varepsilon.$$

5.1. A priori bound for approximate solutions

Lemma 5.3. *Assume that* $\Lambda(T) < 1$*. If* y *is an* ε*-approximate solution with* $0 < \varepsilon \leq 1$*; then, there exists a constant* $M_y > 0$ *independent of* ε *such that*

$$\|y\|_{\mathcal{H}} \leq M_y.$$

Proof. Let

$$r_y(t) := \left(\mathbb{E}\|y(t)\|^2 \right)^{1/2}, \qquad r_y := \sup_{t \in J} r_y(t) = \|y\|_{\mathcal{H}}.$$

Using the definition of an ε-approximate solution, Minkowski's inequality, Assumptions (A1)–(A4), and the same estimates as in the proof of Lemma 4.1, we obtain

$$\begin{aligned} r_y(t) \leq \|x_0\|_{L^2} &+ \frac{M_A t^{\alpha}}{\Gamma(\alpha+1)} r_y + \frac{K_0 t^{\alpha+1}}{\Gamma(\alpha+2)} r_y + \frac{L_f t^{\alpha}}{\Gamma(\alpha+1)} r_y \\ &+ \frac{L_\sigma t^{\alpha-1/2}}{\Gamma(\alpha)\sqrt{2\alpha-1}} r_y + C_0 + \|\Phi\|_{\mathcal{H}}, \end{aligned}$$

where $C_0 > 0$ depends only on

$$\|B\| \|u\|_{L^2(J,U)}, \qquad \sup_{t \in J} \|f(t,0)\|, \qquad \sup_{t \in J} \|\sigma(t,0)\|,$$

and is independent of ε. Because $\|\Phi\|_{\mathcal{H}} \leq \varepsilon \leq 1$, taking the supremum over $t \in J$ gives

$$r_y \leq \|x_0\|_{L^2} + C_0 + 1 + \Lambda(T) r_y.$$

Because $\Lambda(T) < 1$, we obtain

$$r_y \leq \frac{\|x_0\|_{L^2} + C_0 + 1}{1 - \Lambda(T)} =: M_y.$$

Thus, y is uniformly bounded in $\mathcal{H}_{\mathcal{F}}$, independently of ε. □

5.2. Stability estimate

Theorem 5.4. *Assume that (A1)–(A7) hold, $\alpha \in (1/2, 1)$, and $\Lambda(T) < 1$. Then, system (3.1)–(3.2) is Ulam–Hyers stable. More precisely, for every $\varepsilon > 0$ and every ε-approximate solution $y \in \mathcal{H}_{\mathcal{F}}$, the unique mild solution $x \in \mathcal{H}_{\mathcal{F}}$ satisfies*

$$\|x - y\|_{\mathcal{H}} \leq \frac{1}{1 - \Lambda(T)}\varepsilon.$$

Proof. Let $y \in \mathcal{H}_{\mathcal{F}}$ be an ε-approximate solution, and let $x \in \mathcal{H}_{\mathcal{F}}$ be the unique mild solution guaranteed by Theorem 4.3. Subtracting the mild formulation of x from that of y, we obtain, for every $t \in J$, almost surely,

$$\begin{aligned} y(t) - x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[A(y(s) - x(s)) + \int_0^s K(s,\tau)(y(\tau) - x(\tau))\, d\tau \\ + f(s, y(s)) - f(s, x(s)) \Big] ds \\ + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [\sigma(s, y(s)) - \sigma(s, x(s))]\, dW(s) + \Phi(t). \end{aligned}$$

Applying Minkowski's inequality, the Lipschitz assumptions (A1)–(A4), and the Itô isometry from Lemma 2.7, and proceeding exactly as in the proof of Lemma 4.2, we obtain

$$\left(\mathbb{E}\|y(t) - x(t)\|^2 \right)^{1/2} \leq \Lambda(T)\|y - x\|_{\mathcal{H}} + \|\Phi\|_{\mathcal{H}}.$$

Taking the supremum over $t \in J$ and using $\|\Phi\|_{\mathcal{H}} \leq \varepsilon$, we get

$$\|y - x\|_{\mathcal{H}} \leq \Lambda(T)\|y - x\|_{\mathcal{H}} + \varepsilon.$$

Because $\Lambda(T) < 1$, it follows that

$$\|y - x\|_{\mathcal{H}} \leq \frac{\varepsilon}{1 - \Lambda(T)}.$$

Hence, the system is Ulam–Hyers stable with

$$C_{\mathrm{UH}} = \frac{1}{1 - \Lambda(T)}.$$

□

Remark 5.5. *Lemma 5.3 shows that approximate solutions remain uniformly controlled under the same contraction condition used for well-posedness. The stability estimate itself follows from the global Lipschitz assumptions and the contraction estimate in Lemma 4.2.*

Remark 5.6. *The stability constant C_{UH} depends explicitly on the system parameters through $\Lambda(T)$. In particular, C_{UH} increases as $\Lambda(T)$ approaches 1, which is consistent with the fact that the fixed-point argument degenerates as the contraction constant approaches one.*

6. Local approximate controllability near the initial state

In this section, we establish a local approximate controllability result for system (3.1)–(3.2). The key point is that the reduced fractional controllability operator has a small-time scaling of order $T^{\alpha-1/2}$ when controls are measured in $L^2(0,T;U)$. Therefore, a mathematically consistent local controllability result with uniformly bounded controls must consider targets approaching the initial state at the same fractional scaling rate. Under this scaling, the nonlinear, memory, and stochastic contributions become small in the mean-square sense, and controllability of the nonlinear stochastic system can be inferred from a corresponding local bounded-control approximation property of the reduced linear system.

Throughout this section, Assumptions (A1)–(A7) hold, and $\alpha \in (1/2, 1)$. We work in the Banach space $(\mathcal{H}_{\mathcal{F}}, \|\cdot\|_{\mathcal{H}})$ introduced in Definition 2.5. The contraction constant $\Lambda(T)$ is defined in Theorem 4.3.

6.1. The reduced linear control system

Consider the reduced linear control system

$$^{C}D_t^{\alpha} z(t) = Bu(t), \qquad z(0) = x_0.$$

Its mild solution at time T is

$$z(T) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} Bu(s)\, ds.$$

Define the linear operator

$$\mathcal{W}_T : L^2(0,T;U) \to X$$

by

$$\mathcal{W}_T u := \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} Bu(s)\, ds.$$

By the Cauchy–Schwarz inequality, for every $u \in L^2(0,T;U)$,

$$\begin{aligned} \|\mathcal{W}_T u\| &\le \frac{\|B\|}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \|u(s)\|\, ds \\ &\le \frac{\|B\|}{\Gamma(\alpha)} \left(\int_0^T (T-s)^{2\alpha-2}\, ds \right)^{1/2} \|u\|_{L^2(0,T;U)}. \end{aligned}$$

Because $\alpha > 1/2$,

$$\int_0^T (T-s)^{2\alpha-2}\, ds = \frac{T^{2\alpha-1}}{2\alpha-1}.$$

Therefore,

$$\|\mathcal{W}_T u\| \le \frac{\|B\| T^{\alpha-1/2}}{\Gamma(\alpha)\sqrt{2\alpha-1}} \|u\|_{L^2(0,T;U)}. \tag{6.1}$$

Thus, the norm of $\mathcal{W}_T$ is at most of order $T^{\alpha-1/2}$ as $T \to 0^+$.

6.2. Scaled local bounded-control approximation property

The following assumption is the key controllability hypothesis. In contrast with a fixed-radius local controllability condition, the target radius is scaled by $T^{\alpha-1/2}$, which is the natural small-time scaling of the reduced controllability operator $\mathcal{W}_T$ under L^2-bounded controls.

Assumption 1. *For every $\varepsilon > 0$, there exist constants*

$$\rho = \rho(\varepsilon) > 0, \qquad T_* = T_*(\varepsilon) > 0, \qquad M_u = M_u(\varepsilon) > 0,$$

such that, for every $T \in (0, T_*]$ and every target $x_T \in X$ satisfying

$$\|x_T - x_0\| \leq \rho\, T^{\alpha-1/2},$$

there exists a control $u_T \in L^2(0, T; U)$ satisfying

$$\|x_T - x_0 - \mathcal{W}_T u_T\| < \frac{\varepsilon}{2},$$

and

$$\|u_T\|_{L^2(0,T;U)} \leq M_u.$$

The constant M_u is independent of T for all $0 < T \leq T_*$.

Remark 6.1. *The scaling factor $T^{\alpha-1/2}$ in Assumption 1 is intrinsic to the present fractional stochastic setting. Indeed, Estimate* (6.1) *shows that if $\|u_T\|_{L^2(0,T;U)} \leq M_u$ uniformly in T, then*

$$\|\mathcal{W}_T u_T\| \leq \frac{\|B\| M_u}{\Gamma(\alpha)\sqrt{2\alpha - 1}} T^{\alpha-1/2}.$$

Thus, the displacement generated by uniformly L^2-bounded controls necessarily tends to zero at least at the rate $T^{\alpha-1/2}$ as $T \to 0^+$. Consequently, a local controllability statement with uniformly bounded controls cannot, in general, involve a target ball of fixed radius independent of T. The scaled condition

$$\|x_T - x_0\| \leq \rho T^{\alpha-1/2}$$

is therefore not a technical artifact, but a compatibility condition dictated by the small-time behavior of the fractional controllability operator.

6.3. Uniform bound for the nonlinear solution

For the purpose of the following lemma, define

$$\widetilde{\Lambda}(T) := \frac{(M_A + L_f)T^{\alpha}}{\Gamma(\alpha + 1)} + \frac{K_0 T^{\alpha+1}}{\Gamma(\alpha + 2)} + \frac{\sqrt{2} L_\sigma T^{\alpha-1/2}}{\Gamma(\alpha)\sqrt{2\alpha - 1}}.$$

Because $\alpha > 1/2$, we have

$$\widetilde{\Lambda}(T) \to 0 \qquad \text{as } T \to 0^+.$$

Lemma 6.2. *Assume that (A1)–(A7) hold, and* $\alpha \in (1/2, 1)$. *Let* $u_T \in L^2(0, T; U)$ *satisfy*

$$\|u_T\|_{L^2(0,T;U)} \leq M_u$$

for all sufficiently small $T > 0$. *Let* $x \in \mathcal{H}_{\mathcal{F}}$ *be the mild solution of* (3.1)–(3.2) *corresponding to* u_T. *Then, there exist constants*

$$T_1 > 0, \qquad M_x > 0,$$

independent of T, such that

$$\|x\|_{\mathcal{H}} \leq M_x, \qquad 0 < T \leq T_1.$$

Proof. Let

$$r(t) := \left(\mathbb{E}\|x(t)\|^2\right)^{1/2}, \qquad r := \sup_{t\in[0,T]} r(t) = \|x\|_{\mathcal{H}}.$$

From the mild formulation (3.3) and Minkowski's inequality, for every $t \in [0, T]$,

$$\begin{aligned} r(t) \leq \|x_0\|_{L^2} &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\mathbb{E}\|Ax(s)\|^2\right)^{1/2} ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\mathbb{E}\left\|\int_0^s K(s,\tau)x(\tau)\,d\tau\right\|^2\right)^{1/2} ds \\ &+ \frac{\|B\|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|u_T(s)\|\,ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\mathbb{E}\|f(s,x(s))\|^2\right)^{1/2} ds \\ &+ \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{2\alpha-2} \mathbb{E}\|\sigma(s,x(s))\|^2 ds\right)^{1/2}. \end{aligned}$$

We estimate the terms separately. By (A1),

$$\left(\mathbb{E}\|Ax(s)\|^2\right)^{1/2} \leq M_A \left(\mathbb{E}\|x(s)\|^2\right)^{1/2} \leq M_A r.$$

Thus,

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\mathbb{E}\|Ax(s)\|^2\right)^{1/2} ds \leq \frac{M_A r}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds.$$

For the Volterra term, using Minkowski's integral inequality and (A2), we get

$$\begin{aligned} \left(\mathbb{E}\left\|\int_0^s K(s,\tau)x(\tau)\,d\tau\right\|^2\right)^{1/2} &\leq \int_0^s \|K(s,\tau)\| \left(\mathbb{E}\|x(\tau)\|^2\right)^{1/2} d\tau \\ &\leq K_0 \int_0^s r\,d\tau = K_0 r s. \end{aligned}$$

Hence,

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\mathbb{E}\left\|\int_0^s K(s,\tau)x(\tau)\,d\tau\right\|^2\right)^{1/2} ds \leq \frac{K_0 r}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s\,ds.$$

For the control term, by Cauchy–Schwarz and the bound on u_T,

$$\begin{aligned}\frac{\|B\|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|u_T(s)\|\,ds &\le \frac{\|B\|}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{2\alpha-2}ds\right)^{1/2}\|u_T\|_{L^2(0,T;U)}\\ &\le \frac{\|B\|M_u}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{2\alpha-2}ds\right)^{1/2}.\end{aligned}$$

For the nonlinear term, using (A3),

$$\left(\mathbb{E}\|f(s,x(s))\|^2\right)^{1/2} \le L_f\left(\mathbb{E}\|x(s)\|^2\right)^{1/2} + \|f(s,0)\| \le L_f r + \sup_{\theta\in J}\|f(\theta,0)\|.$$

Therefore,

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\mathbb{E}\|f(s,x(s))\|^2\right)^{1/2}ds \le \frac{L_f r + \sup_{\theta\in J}\|f(\theta,0)\|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds.$$

For the stochastic term, by (A4),

$$\mathbb{E}\|\sigma(s,x(s))\|^2 \le 2L_\sigma^2\mathbb{E}\|x(s)\|^2 + 2\|\sigma(s,0)\|^2 \le 2L_\sigma^2 r^2 + 2\sup_{\theta\in J}\|\sigma(\theta,0)\|^2.$$

Hence,

$$\begin{aligned}&\left(\int_0^t (t-s)^{2\alpha-2}\mathbb{E}\|\sigma(s,x(s))\|^2ds\right)^{1/2}\\ &\le \left(\int_0^t (t-s)^{2\alpha-2}\left[2L_\sigma^2 r^2 + 2\sup_{\theta\in J}\|\sigma(\theta,0)\|^2\right]ds\right)^{1/2}.\end{aligned}$$

Using

$$\sqrt{a^2+b^2} \le a+b, \qquad a,b\ge 0,$$

we obtain

$$\begin{aligned}&\left(\int_0^t (t-s)^{2\alpha-2}\left[2L_\sigma^2 r^2 + 2\sup_{\theta\in J}\|\sigma(\theta,0)\|^2\right]ds\right)^{1/2}\\ &\le \sqrt{2}L_\sigma r\left(\int_0^t (t-s)^{2\alpha-2}ds\right)^{1/2} + \sqrt{2}\sup_{\theta\in J}\|\sigma(\theta,0)\|\left(\int_0^t (t-s)^{2\alpha-2}ds\right)^{1/2}.\end{aligned}$$

Now, evaluate the elementary integrals:

$$\int_0^t (t-s)^{\alpha-1}ds = \frac{t^\alpha}{\alpha},$$

$$\int_0^t (t-s)^{\alpha-1}s\,ds = \frac{t^{\alpha+1}}{\alpha(\alpha+1)},$$

and

$$\left(\int_0^t (t-s)^{2\alpha-2}ds\right)^{1/2} = \frac{t^{\alpha-1/2}}{\sqrt{2\alpha-1}}.$$

Using

$$\frac{1}{\alpha\Gamma(\alpha)} = \frac{1}{\Gamma(\alpha+1)}, \qquad \frac{1}{\alpha(\alpha+1)\Gamma(\alpha)} = \frac{1}{\Gamma(\alpha+2)},$$

we obtain

$$r(t) \le \|x_0\|_{L^2} + C(T) + \widetilde{\Lambda}(T)r,$$

where

$$\begin{aligned} C(T) := \frac{\|B\|M_u T^{\alpha-1/2}}{\Gamma(\alpha)\sqrt{2\alpha-1}} + \frac{T^{\alpha}}{\Gamma(\alpha+1)} \sup_{\theta\in J} \|f(\theta,0)\| \\ + \frac{\sqrt{2}T^{\alpha-1/2}}{\Gamma(\alpha)\sqrt{2\alpha-1}} \sup_{\theta\in J} \|\sigma(\theta,0)\|. \end{aligned}$$

Taking the supremum over $t \in [0,T]$, we get

$$r \le \|x_0\|_{L^2} + C(T) + \widetilde{\Lambda}(T)r.$$

Because

$$\widetilde{\Lambda}(T) \to 0 \qquad \text{as } T \to 0^+,$$

there exists $T_1 > 0$ such that

$$\widetilde{\Lambda}(T) \le 1/2, \qquad 0 < T \le T_1.$$

Therefore,

$$r \le 2\left(\|x_0\|_{L^2} + C(T)\right), \qquad 0 < T \le T_1.$$

Because $\alpha > 1/2$, all powers of T appearing in $C(T)$ are positive. Hence,

$$\sup_{0<T\le T_1} C(T) < \infty.$$

Consequently, there exists $M_x > 0$, independent of T, such that

$$\|x\|_{\mathcal{H}} = r \le M_x, \qquad 0 < T \le T_1.$$

This completes the proof. □

6.4. Local approximate controllability theorem

Theorem 6.3. *Assume that (A1)–(A7) hold, $\alpha \in (1/2, 1)$, and Assumption 1 is satisfied. Then, for every $\varepsilon > 0$, there exist constants*

$$\rho > 0, \qquad T_0 > 0,$$

such that, for every $T \in (0, T_0]$ and every target $x_T \in X$ satisfying

$$\|x_T - x_0\| \le \rho T^{\alpha-1/2},$$

there exists a control $u_T \in L^2(0,T;U)$ such that the corresponding mild solution $x \in \mathcal{H}_{\mathcal{F}}$ satisfies

$$\left(\mathbb{E}\|x(T) - x_T\|^2\right)^{1/2} < \varepsilon.$$

Proof. Fix $\varepsilon > 0$. By Assumption 1, there exist constants

$$\rho > 0, \qquad T_* > 0, \qquad M_u > 0,$$

such that, for every $T \in (0, T_*]$ and every target $x_T \in X$ satisfying

$$\|x_T - x_0\| \le \rho T^{\alpha-1/2},$$

there exists a control $u_T \in L^2(0, T; U)$ satisfying

$$\|x_T - x_0 - \mathcal{W}_T u_T\| < \frac{\varepsilon}{2},$$

and

$$\|u_T\|_{L^2(0,T;U)} \le M_u.$$

Because

$$\Lambda(T) \to 0, \qquad \text{and} \qquad \widetilde{\Lambda}(T) \to 0 \qquad \text{as } T \to 0^+,$$

we may reduce T_*, if necessary, so that

$$\Lambda(T) < 1, \qquad \widetilde{\Lambda}(T) \le 1/2, \qquad 0 < T \le T_*.$$

The condition $\Lambda(T) < 1$ guarantees, by Theorem 4.3, that the nonlinear stochastic system admits a unique mild solution $x \in \mathcal{H}_{\mathcal{F}}$. Moreover, Lemma 6.2 yields a constant $M_x > 0$, independent of T, such that

$$\|x\|_{\mathcal{H}} \le M_x, \qquad 0 < T \le T_*.$$

Evaluating the mild formulation (3.3) at $t = T$, we write

$$x(T) = x_0 + \mathcal{W}_T u_T + R(T),$$

where

$$R(T) = R_1(T) + R_2(T) + R_3(T) + R_4(T),$$

with

$$R_1(T) := \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} A x(s)\, ds,$$

$$R_2(T) := \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \int_0^s K(s,\tau) x(\tau)\, d\tau\, ds,$$

$$R_3(T) := \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, x(s))\, ds,$$

and

$$R_4(T) := \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \sigma(s, x(s))\, dW(s).$$

Using Assumptions (A1)–(A4), the uniform bound $\|x\|_{\mathcal{H}} \le M_x$, and the Itô isometry for $R_4(T)$, we obtain

$$\left(\mathbb{E}\|R_1(T)\|^2\right)^{1/2} \le \frac{M_A M_x T^\alpha}{\Gamma(\alpha+1)},$$

$$\left(\mathbb{E}\|R_2(T)\|^2\right)^{1/2} \le \frac{K_0 M_x T^{\alpha+1}}{\Gamma(\alpha+2)},$$

$$\left(\mathbb{E}\|R_3(T)\|^2\right)^{1/2} \le \frac{\left(L_f M_x + \sup_{\theta\in J}\|f(\theta,0)\|\right)T^{\alpha}}{\Gamma(\alpha+1)},$$

and

$$\left(\mathbb{E}\|R_4(T)\|^2\right)^{1/2} \le \frac{(L_\sigma M_x + \sup_{\theta\in J}\|\sigma(\theta,0)\|)\,T^{\alpha-1/2}}{\Gamma(\alpha)\sqrt{2\alpha-1}}.$$

Consequently, there exist constants $C_1, C_2, C_3 > 0$, independent of T, such that

$$\left(\mathbb{E}\|R(T)\|^2\right)^{1/2} \le C_1 T^{\alpha} + C_2 T^{\alpha+1} + C_3 T^{\alpha-1/2}.$$

Because $\alpha > 1/2$, the right-hand side tends to zero as $T \to 0^+$. Hence,there exists $T_0 \in (0, T_*]$ such that

$$\left(\mathbb{E}\|R(T)\|^2\right)^{1/2} < \frac{\varepsilon}{2}, \qquad 0 < T \le T_0.$$

For every $T \in (0, T_0]$ and every target $x_T \in X$ satisfying

$$\|x_T - x_0\| \le \rho T^{\alpha-1/2},$$

we therefore have

$$\begin{aligned}\left(\mathbb{E}\|x(T) - x_T\|^2\right)^{1/2} &\le \|x_0 + \mathcal{W}_T u_T - x_T\| + \left(\mathbb{E}\|R(T)\|^2\right)^{1/2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This proves the desired local approximate controllability near the initial state. □

Remark 6.4. *Theorem 6.3 is local both in time and in the state space. The scaling condition*

$$\|x_T - x_0\| \le \rho T^{\alpha-1/2}$$

reflects the small-time behavior of the reduced controllability operator $\mathcal{W}_T$. In particular, the result should not be interpreted as approximate controllability for arbitrary fixed targets as $T \to 0^+$. Rather, it states that targets approaching the initial state at the natural fractional scaling rate can be approximated in mean square by controls whose L^2-norms remain uniformly bounded. A concrete finite-dimensional example satisfying Assumption 1 is provided in Section 7.

7. Finite-dimensional controllability example

We now present a finite-dimensional example where Assumption 1 is verified explicitly under the natural fractional scaling

$$\|x_T - x_0\| \le \rho T^{\alpha-1/2}.$$

This example complements the infinite-dimensional example above by showing that the controllability hypothesis used in Theorem 6.3 is not void.

7.1. Setting

Let $X = \mathbb{R}^n$ be equipped with the Euclidean norm, and let the control space be $U = \mathbb{R}^n$. Let $B = I_n$, where I_n denotes the identity matrix. Consider the finite-dimensional fractional stochastic system

$$ {}^C D_t^\alpha x(t) = Ax(t) + \int_0^t K(t,s)x(s)\,ds + u(t) + f(t,x(t)) + \sigma(t,x(t))\,dW(t), \qquad x(0) = x_0, $$

where $A \in \mathbb{R}^{n\times n}$ is a fixed matrix, $K : [0,T] \times [0,T] \to \mathbb{R}^{n\times n}$ is continuous and bounded, and $f, \sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous and globally Lipschitz in the second variable. Then, Assumptions (A1)–(A7) are satisfied.

The associated reduced linear control system is

$$ {}^C D_t^\alpha z(t) = u(t), \qquad z(0) = x_0. $$

Its mild solution at time T is

$$ z(T) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} u(s)\,ds. $$

7.2. Verification of Assumption 1

Let $\varepsilon > 0$ be arbitrary. Choose any $\rho > 0$, set $T_* = 1$, and define

$$ M_u := \alpha\Gamma(\alpha)\rho. $$

For any $0 < T \le T_*$ and any target $x_T \in \mathbb{R}^n$ satisfying

$$ \|x_T - x_0\| \le \rho T^{\alpha-1/2}, $$

define the deterministic control

$$ u_T(s) := \frac{\alpha\Gamma(\alpha)}{T^\alpha}(x_T - x_0), \qquad s \in [0,T]. $$

Then, $u_T \in L^2(0,T;\mathbb{R}^n)$. Moreover,

$$ \begin{aligned} \mathcal{W}_T u_T &= \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} u_T(s)\,ds \\ &= \frac{1}{\Gamma(\alpha)} \left(\int_0^T (T-s)^{\alpha-1}\,ds \right) \frac{\alpha\Gamma(\alpha)}{T^\alpha}(x_T - x_0) \\ &= \frac{1}{\Gamma(\alpha)} \cdot \frac{T^\alpha}{\alpha} \cdot \frac{\alpha\Gamma(\alpha)}{T^\alpha}(x_T - x_0) \\ &= x_T - x_0. \end{aligned} $$

Hence,

$$ \|x_T - x_0 - \mathcal{W}_T u_T\| = 0 < \frac{\varepsilon}{2}. $$

It remains to verify the uniform boundedness of the control. We have

$$\begin{aligned}\|u_T\|_{L^2(0,T;U)} &= \left(\int_0^T \left\|\frac{\alpha\Gamma(\alpha)}{T^\alpha}(x_T - x_0)\right\|^2 ds\right)^{1/2}\\ &= \frac{\alpha\Gamma(\alpha)}{T^\alpha}\|x_T - x_0\|T^{1/2}\\ &= \alpha\Gamma(\alpha)\frac{\|x_T - x_0\|}{T^{\alpha-1/2}}.\end{aligned}$$

Using the target condition

$$\|x_T - x_0\| \leq \rho T^{\alpha-1/2},$$

we obtain

$$\|u_T\|_{L^2(0,T;U)} \leq \alpha\Gamma(\alpha)\rho = M_u.$$

Therefore, M_u is independent of T, and Assumption 1 is satisfied.

Remark 7.1. *The scaling condition*

$$\|x_T - x_0\| \leq \rho T^{\alpha-1/2}$$

exactly compensates for the factor $T^{-(\alpha-1/2)}$ *appearing in the* L^2*-norm of the control. Without this scaling, the control norm would generally blow up as* $T \to 0^+$*. Thus, the example confirms that the scaled version of Assumption 1 is both mathematically consistent and explicitly verifiable.*

7.3. Application of Theorem 6.3

Because Assumptions (A1)–(A7) and Assumption 1 are satisfied, Theorem 6.3 applies. Consequently, for every $\varepsilon > 0$, there exist constants $\rho > 0$ and $T_0 > 0$ such that, whenever

$$0 < T \leq T_0, \qquad \|x_T - x_0\| \leq \rho T^{\alpha-1/2},$$

there exists a control $u_T \in L^2(0, T; \mathbb{R}^n)$ such that the corresponding mild solution satisfies

$$\left(\mathbb{E}\|x(T) - x_T\|^2\right)^{1/2} < \varepsilon.$$

This finite-dimensional example demonstrates that the scaled bounded-control approximation property is not merely formal, but can be verified in a concrete setting.

8. Conclusions

In this paper, we investigated a class of fractional stochastic integro-differential systems involving a Caputo fractional derivative of order $\alpha \in (1/2, 1)$, a Volterra-type memory term, a control input, and stochastic perturbations driven by a Wiener process.

Under standard Lipschitz and boundedness assumptions, we established the existence and uniqueness of mild solutions in the space of mean-square continuous processes by means of the Banach contraction principle. An explicit contraction condition depending on the system parameters and the time horizon was derived. We also proved Ulam–Hyers stability, obtaining quantitative estimates that characterize the robustness of solutions with respect to perturbations.

In addition, we established a local approximate controllability result for sufficiently small terminal times under a scaled bounded-control approximation hypothesis. More precisely, controllability was investigated for targets satisfying

$$\|x_T - x_0\| \le \rho T^{\alpha-1/2},$$

which reflects the intrinsic small-time behavior of the fractional controllability operator. Under this scaling, the corresponding controls remain uniformly bounded in $L^2(0, T; U)$ as $T \to 0^+$. The analysis shows that controllability of the nonlinear stochastic system can be inferred from that of an associated reduced linear system. A finite-dimensional example was presented to verify explicitly that the scaled controllability hypothesis can be realized in a concrete setting and to illustrate the applicability of the controllability framework.

Several limitations of the present study should also be emphasized. First, the analysis was carried out under the assumption that the operator A is bounded, thereby excluding many important partial differential equation models involving unbounded generators. Second, the controllability result is local both in time and in the state space, and it relies on a nontrivial bounded-control approximation property that must be verified separately in applications. In particular, the scaling condition

$$\|x_T - x_0\| \le \rho T^{\alpha-1/2}$$

is necessary in order to maintain uniform boundedness of the controls as $T \to 0^+$, and therefore, the result does not apply to fixed targets independent of T. Third, the memory kernel was assumed to be continuous and bounded, which may be restrictive in certain applications. Finally, the present work is theoretical in nature and does not include numerical simulations.

These observations naturally suggest several directions for future research. Extending the analysis to systems governed by unbounded operators and fractional resolvent families would substantially broaden the applicability of the framework. Another important direction is the investigation of global controllability properties beyond the small-time regime considered here. It would also be interesting to relax the assumptions on the memory kernel and to develop numerical methods for approximating solutions of such fractional stochastic systems. Further extensions may include systems driven by fractional Brownian motion and the study of alternative notions of stability within the same analytical framework.

Overall, the results contribute to the theory of fractional stochastic systems with memory by providing a unified framework in which well-posedness, stability, and controllability are analyzed simultaneously. The explicit estimates and the careful treatment of the controllability hypothesis provide a mathematically consistent basis for further theoretical and applied developments.

Author contributions

Muath Awadalla conceptualized the research idea, developed the mathematical model, conducted the numerical simulations, and prepared the original draft of the manuscript. Maryam G. Alshehri supervised the research, refined the theoretical analysis, verified the stability proofs, and revised the manuscript critically for intellectual content. Both authors read and approved the final version of the paper.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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