



Research article

Dynamics of a stochastic single species model with mean-reverting Ornstein-Uhlenbeck and Allee effect

Jin Kang¹, Rong Liu^{2,*} and Xuhui Shen²

¹ Department of General Education Courses, Shanxi Open University, Taiyuan 030027, China

² School of Applied Mathematics, Shanxi University of Finance and Economics, Taiyuan 030006, China

* **Correspondence:** Email: rliu29@sxufe.edu.cn.

Abstract: This paper studies the dynamics of a stochastic single species model with Ornstein-Uhlenbeck process and Allee effect. First, we establish the existence of a unique global solution and obtain moment estimates. Then, sufficient conditions for the existence of a unique stationary distribution of the model and for the extinction of species are derived. The results show that a weaker attack rate guarantees a unique stationary distribution, whereas a stronger attack rate leads to species extinction. Moreover, we obtain an approximate expression for the stationary density around the stochastic quasi-positive equilibrium by solving the associated Fokker-Planck equation. Finally, some numerical simulations are presented.

Keywords: Ornstein-Uhlenbeck process; Allee effect; density function; stationary distribution; extinction

Mathematics Subject Classification: 60H10, 92D25, 93E15

1. Introduction

The Allee effect, named for ecologist Warder Clyde Allee, denotes to a positive correlation between population size (or density) and the mean individual fitness with a population or species [1]. Recently, there has been a rich literature on the population dynamics with Allee effect. To name a few, see [2–4] and the references therein. In [4], the authors applied Poincaré-Bendixson theory to analyze the following single species model with a component Allee effect:

$$\frac{dx}{dt} = x \left[r - cx - \frac{a}{1 + hax} \right] \triangleq xg(x), \tag{1.1}$$

where x denotes the population size or density and $g(x)$ is called the per capita growth function. c is the strength of intraspecific competition and r represents the intrinsic growth rate. Under conditions of

predation satiation, a and h are the attack rate and handling time of predator, respectively. Taylor and Hastings [5] pointed out that if $g(x)$ has a positive slope at small population sizes, that is, $g'(0) > 0$ ($a > \sqrt{\frac{c}{h}}$), model (1.1) has an Allee effect.

However, populations are inherently influenced by the environmental noise. Ripa and Lundberg [6] noted that the additional noise consists of white noise and color noise. White noise is uncorrelated and each randomly drawn noise value is entirely independent of the previous one. Color noise, in contrast, is temporally autocorrelated, meaning that the random values in the noise process depend on those that came before.

Up to now, there have been many literatures on stochastic models with white noise; see [7–9] for stochastic population models and [10–12] for stochastic epidemic models. Inspired by the above work, we suppose that the intrinsic growth rate r is influenced by white noise. That is, $r \rightarrow r + \sigma \dot{B}(t)$, in which σ^2 represents the intensity of white noise $\dot{B}(t)$, and $B(t)$ denotes a standard Brownian motion on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ under the usual hypotheses. Hence, we arrive at the following stochastic single species model with white noise:

$$dx(t) = x(t) \left[r - cx(t) - \frac{a}{1 + hax(t)} \right] dt + \sigma x(t) dB(t). \quad (1.2)$$

Liu and Liu [13] introduced jumps into model (1.2) and discussed the persistence and stability in distribution of a stochastic model with jumps and Allee effect. Yu et al. [14] introduced the Markov chain into model (1.2) with $c = \frac{r}{K}$ and discussed the ergodicity and persistence for a stochastic model with Allee effect and regime switching.

In this context, we will discuss the influences of color noise to the species in model (1.1). Inspired by [15–17], we replace the Brownian motion with mean-reverting Ornstein-Uhlenbeck process and obtain a stochastic single species model with color noise. Unlike white noise, which always assumes that environmental fluctuations at different time points are mutually independent and ignores temporal autocorrelation (often referred to as memory), the Ornstein-Uhlenbeck process is a mean-reverting stochastic process that captures the tendency of environmental conditions to persist over short time scales and revert to a long-term average. This property makes the Ornstein-Uhlenbeck process more biologically realistic for modeling population growth rates under temporally correlated environmental disturbances. Assume that growth rate r in model (1.1) satisfies the mean-reverting Ornstein-Uhlenbeck process, i.e., it is the solution of the Langevin equation (see [18, Example 5.2]):

$$dr(t) = -\alpha(r(t) - \bar{r})dt + \beta dB(t), \quad (1.3)$$

where α, β , and \bar{r} are positive constants, and $B(t)$ is the Brownian motion. \bar{r} is the average growth rate. α and β are reversion speed and volatility intensity, respectively. By [18, Example 5.2], we have

$$r(t) = e^{-\alpha t} r(0) + \bar{r}(1 - e^{-\alpha t}) + \beta \int_0^t e^{-\alpha(t-s)} dB(s),$$

from which one can deduce that $r(t)$ is normally distributed, i.e., $r(t) \sim \mathbf{N}(\mathbb{E}[r(t)], \text{Var}[r(t)])$ for any $t \geq 0$, with $\mathbb{E}[r(t)] = e^{-\alpha t} r(0) + \bar{r}(1 - e^{-\alpha t})$ and $\text{Var}[r(t)] = \frac{\beta^2}{2\alpha} (1 - e^{-2\alpha t})$. Observe that $\lim_{t \rightarrow 0^+} \mathbb{E}[r(t)] = r(0)$ and $\lim_{t \rightarrow 0^+} \text{Var}[r(t)] = 0$. From [16], this is consistent with the continuous disturbance property of environmental noise. Hence, the novel stochastic single species model with Ornstein-Uhlenbeck process and Allee effect takes the form

$$\begin{cases} dx(t) = x(t) \left[r(t) - cx(t) - \frac{a}{1+hx(t)} \right] dt, \\ dr(t) = -\alpha(r(t) - \bar{r})dt + \beta dB(t), \end{cases} \quad (1.4)$$

with initial condition $(x(0), r(0)) \in \mathbb{R}_+ \times \mathbb{R} = (0, +\infty) \times (-\infty, +\infty)$.

As we know, most existing Allee effect models rely on white noise perturbations, where the theoretical framework is well established. However, few studies consider the mean-reverting Ornstein-Uhlenbeck process to model colored noise. Although [16] also studies a stochastic population model with Ornstein-Uhlenbeck process and the Allee effect, their Allee effect takes a polynomial form $r + \rho P - \theta P^2$, whereas in our work the Allee effect arises from a Holling type II predator-prey interaction, taking the rational form $\frac{a}{1+hx}$. This structural difference leads to qualitatively distinct dynamical behaviors. More importantly, [16] concludes that Ornstein-Uhlenbeck noise has no effect on persistence (requiring only $\bar{r} > 0$), while our persistence condition in Theorem 4.1, $a < \bar{r} - \beta / \sqrt{2\alpha}$, explicitly depends on the reversion speed α and volatility intensity β —a dependence not seen in white-noise models or in [16]. Furthermore, our extinction condition in Theorem 5.1 is independent of Ornstein-Uhlenbeck parameters, in contrast to the dependence reported in [16].

The structure of this paper is as follows: Section 2 is devoted to the existence of a unique global solution to model (1.4). Section 3 provides the moment estimates of the solution. Section 4 focuses on the stationary distribution: By using classical Lyapunov function methods, we first establish sufficient conditions for its existence, and by solving the Fokker-Planck equation, we obtain an approximate expression for its probability density function. In Section 5, sufficient criteria for the extinction of the species in model (1.4) are obtained. Finally, some numerical simulations are presented to support the theoretical results.

2. Global existence and uniqueness

In this section, we prove the existence of a unique global solution $(x(t), r(t))$ to (1.4), with $x(t) > 0$. With this aim, we first introduce the following basic inequalities.

Lemma 2.1 (see [15]). *The following inequalities hold.*

(i) *For any $a, b \geq 0$ and any $p > 1, q > 1, \frac{1}{p} + \frac{1}{q} = 1$, we have*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \quad (2.1)$$

(ii) *For any $p \geq 1$, we have*

$$\left| \sum_{i=1}^n a_i \right|^p \leq n^{p-1} \sum_{i=1}^n |a_i|^p. \quad (2.2)$$

Theorem 2.1. *For any initial data $(x(0), r(0)) \in \mathbb{R}_+ \times \mathbb{R}$, model (1.4) admits a unique global solution $(x(t), r(t))$ for all $t \geq 0$, which remains in $\mathbb{R}_+ \times \mathbb{R}$ almost surely.*

Proof. Since the coefficients of (1.4) are locally Lipschitz, there exists a unique local solution $(x(t), r(t))$ on $t \in [0, \tau_e)$ almost surely for any initial data $(x(0), r(0)) \in \mathbb{R}_+ \times \mathbb{R}$, where τ_e denotes the explosion time. To establish global existence, it suffices to prove that $\tau_e = \infty$ almost surely.

To this end, choose $n_0 \geq 1$ sufficiently large such that both $x(0)$ and $e^{r(0)}$ all lie within the interval $[\frac{1}{n_0}, n_0]$. For each integer $n \geq n_0$, we define the stopping time as

$$\tau_n = \inf \left\{ t \in [0, \tau_e) : \min \{x(t), e^{r(t)}\} \leq \frac{1}{n} \text{ or } \max \{x(t), e^{r(t)}\} \geq n \right\}.$$

Here throughout this paper, we denote by \emptyset the empty set and adopt the convention $\inf \emptyset = \infty$. It is straightforward to verify that $\{\tau_n\}$ is nondecreasing as $n \rightarrow \infty$. Define $\tau_\infty = \lim_{n \rightarrow \infty} \tau_n$, which satisfies $\tau_\infty \leq \tau_e$ almost surely. If it can be shown that $\tau_\infty = \infty$ almost surely, then $\tau_e = \infty$ and $(x(t), r(t)) \in \mathbb{R}_+ \times \mathbb{R}$ almost surely for all $t \geq 0$.

We now prove that $\tau_\infty = \infty$ almost surely. Assume that the assertion is false, then there exist $T > 0$ and $\varepsilon \in (0, 1)$ such that $\mathbb{P}\{\tau_\infty \leq T\} > \varepsilon$. Let

$$\Omega_n = \left\{ \omega \in \Omega : \tau_n = \tau_n(\omega) \leq T \right\} \text{ for any } n \geq n_0.$$

Hence, $\mathbb{P}(\Omega_n) \geq \varepsilon$ for any $n \geq n_0$. Consider the C^2 -function $V : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$ defined by

$$V(x, r) = x - 1 - \ln x + \frac{r^4}{4}.$$

It follows from the Itô formula that

$$dV(x, r) = LV(x, r)dt + \beta r^3 dB(t), \quad (2.3)$$

where

$$\begin{aligned} LV(x, r) &= (x-1) \left[r - cx - \frac{a}{1+hax} \right] - \alpha(r-\bar{r})r^3 + \frac{3}{2}\beta^2 r^2 \\ &= -cx^2 + cx - \frac{ax}{1+hax} + \frac{a}{1+hax} + rx - r - \alpha r^4 + \alpha \bar{r} r^3 + \frac{3}{2}\beta^2 r^2 \\ &\leq -cx^2 + cx + a + rx - r - \alpha r^4 + \alpha \bar{r} r^3 + \frac{3}{2}\beta^2 r^2. \end{aligned}$$

From (2.1) in Lemma 2.1, it follows that

$$rx \leq \frac{|r|^3}{3} + \frac{2}{3}x^{\frac{3}{2}}.$$

Thus, we have

$$\begin{aligned} LV &\leq -cx^2 + \frac{2}{3}x^{\frac{3}{2}} + cx + a - \alpha r^4 + \alpha \bar{r} r^3 + \frac{|r|^3}{3} + \frac{3}{2}\beta^2 r^2 - r \\ &\leq \sup_{x \in \mathbb{R}_+} \left[-cx^2 + \frac{2}{3}x^{\frac{3}{2}} + cx + a \right] + \sup_{r \in \mathbb{R}} \left[-\alpha r^4 + \alpha \bar{r} r^3 + \frac{|r|^3}{3} + \frac{3}{2}\beta^2 r^2 - r \right] \\ &\leq H, \end{aligned} \quad (2.4)$$

where H is a positive constant independent of the initial condition. By integrating both sides of (2.3) from 0 to $\tau_n \wedge T$, using (2.4), and taking expectations yields

$$\mathbb{E}V(x(\tau_n \wedge T), r(\tau_n \wedge T)) = V(x(0), r(0)) + \mathbb{E} \int_0^{\tau_n \wedge T} LV(x(s), r(s)) ds$$

$$\begin{aligned} &\leq V(x(0), r(0)) + H\mathbb{E}(\tau_n \wedge T) \\ &\leq V(x(0), r(0)) + HT. \end{aligned}$$

Observe that for every $\omega \in \Omega_n$, at least one of $x(\tau_n, \omega)$ and $e^{r(\tau_n, \omega)}$ is equal to n or $\frac{1}{n}$. Consequently,

$$V(x(0), r(0)) + HT \geq \mathbb{E}\left[I_{\Omega_n}(\omega)V(x(\tau_n, \omega), r(\tau_n, \omega))\right] \geq \mathbb{P}(\Omega_n)\left[(n-1-\ln n) \wedge \left(\frac{1}{n}-1+\ln n\right) \wedge \frac{1}{4}(\ln n)^4\right],$$

where I_{Ω_n} denotes the indicator function of Ω_n . Taking the limit as $n \rightarrow \infty$ leads to the contradiction

$$\infty > V(x(0), r(0)) + HT = \infty.$$

Hence, we must have $\tau_\infty = \infty$ almost surely. The proof is complete. \square

3. The p th moment boundedness

This section establishes the boundedness of the p th moment of the solution to model (1.4) by using a stochastic Lyapunov method.

Theorem 3.1. *Let $(x(0), r(0)) \in \mathbb{R}_+ \times \mathbb{R}$ and $p \geq 2$ be an integer. Then there exists a constant $K(p) > 0$ such that the solution $(x(t), r(t))$ to model (1.4) satisfies*

$$\limsup_{t \rightarrow \infty} \mathbb{E}\left[\frac{x^p(t)}{p} + \frac{r^{2p}(t)}{2p}\right] \leq K(p).$$

Proof. For $p \geq 2$, the Lyapunov function is defined as

$$V(x, r) = \frac{x^p}{p} + \frac{r^{2p}}{2p}.$$

Then, taking expectations and using Itô's formula to $e^{\theta t}V(x, r)$, we have

$$\mathbb{E}[e^{\theta t}V(x(t), r(t))] = V(x(0), r(0)) + \mathbb{E} \int_0^t L[e^{\theta s}V(x(s), r(s))] ds, \quad (3.1)$$

where θ satisfies $0 < \theta < 2\alpha p$. We note that

$$\begin{aligned} L[e^{\theta t}V(x, r)] &= \theta e^{\theta t}V(x, r) + e^{\theta t}LV(x, r) \\ &= e^{\theta t} \left\{ \frac{\theta}{p}x^p + \frac{\theta}{2p}r^{2p} + x^p \left[r - cx - \frac{a}{1+hax} \right] - \alpha(r - \bar{r})r^{2p-1} + \frac{2p-1}{2}\beta^2 r^{2p-2} \right\} \\ &\leq e^{\theta t} \left\{ \frac{\theta}{p}x^p + \frac{\theta}{2p}r^{2p} + rx^p - cx^{p+1} - \alpha(r - \bar{r})r^{2p-1} + \frac{2p-1}{2}\beta^2 r^{2p-2} \right\} \\ &= e^{\theta t} \left\{ -cx^{p+1} + \frac{\theta}{p}x^p + \left(\frac{\theta}{2p} - \alpha \right) r^{2p} + \alpha \bar{r} r^{2p-1} + \frac{2p-1}{2}\beta^2 r^{2p-2} + rx^p \right\}. \end{aligned}$$

Moreover, we use the inequality (2.1) in Lemma 2.1 to get

$$rx^p \leq \frac{1}{p + \frac{3}{2}}|r|^{p+\frac{3}{2}} + \frac{1}{\frac{p+\frac{3}{2}}{p+\frac{1}{2}}}x^{p \cdot \frac{p+\frac{3}{2}}{p+\frac{1}{2}}} = \frac{1}{p + \frac{3}{2}}|r|^{p+\frac{3}{2}} + \frac{p + \frac{1}{2}}{p + \frac{3}{2}}x^{p+\frac{p}{p+\frac{1}{2}}}.$$

Thus, from $0 < \theta < 2\alpha p$ and $p \geq 2$, we can obtain

$$\begin{aligned} L[e^{\theta t} V(x, r)] &\leq e^{\theta t} \left\{ -cx^{p+1} + \frac{p+\frac{1}{2}}{p+\frac{3}{2}} x^{p+\frac{p}{p+\frac{1}{2}}} + \frac{\theta}{p} x^p + \left(\frac{\theta}{2p} - \alpha \right) r^{2p} + \frac{1}{p+\frac{3}{2}} |r|^{p+\frac{3}{2}} + \alpha \bar{r} r^{2p-1} + \frac{2p-1}{2} \beta^2 r^{2p-2} \right\} \\ &\leq e^{\theta t} \left\{ \sup_{x \in \mathbb{R}_+} \left[-cx^{p+1} + \frac{p+\frac{1}{2}}{p+\frac{3}{2}} x^{p+\frac{p}{p+\frac{1}{2}}} + \frac{\theta}{p} x^p \right] \right. \\ &\quad \left. + \sup_{r \in \mathbb{R}} \left[\left(\frac{\theta}{2p} - \alpha \right) r^{2p} + \frac{1}{p+\frac{3}{2}} |r|^{p+\frac{3}{2}} + \alpha \bar{r} r^{2p-1} + \frac{2p-1}{2} \beta^2 r^{2p-2} \right] \right\} \\ &\doteq H(p) e^{\theta t}. \end{aligned}$$

This, together with (3.1), yields

$$\mathbb{E}[e^{\theta t} V(x(t), r(t))] \leq V(x(0), r(0)) + \frac{H(p)(e^{\theta t} - 1)}{\theta},$$

which implies

$$\mathbb{E}[V(x(t), r(t))] \leq e^{-\theta t} V(x(0), r(0)) + \frac{H(p)}{\theta} (1 - e^{-\theta t}).$$

Taking the limit as $t \rightarrow \infty$ yields

$$\limsup_{t \rightarrow \infty} \mathbb{E}[V(x(t), r(t))] \leq \frac{H(p)}{\theta} \doteq K(p).$$

The proof is complete. \square

4. On the existence of a stationary distribution and its density function

This section establishes sufficient conditions for the existence of a stationary distribution of model (1.4) and derives an approximate expression for its probability density function.

4.1. On the existence of a stationary distribution

In this subsection, we establish sufficient conditions ensuring the existence of a stationary distribution of the model, which describes the weak persistence of species $x(t)$ in model (1.4). Denote $X(t)$ as a homogeneous Markov process defined on \mathbb{R}^d , governed by the following stochastic differential equation:

$$dX(t) = b(X(t))dt + \sum_{i=1}^k \sigma_i(X(t))dB_i(t), \quad X(0) = X_0. \quad (4.1)$$

Lemma 4.1 (see Khasminskii [19]). *Assume that the vectors $b(X)$, $\sigma_i(X)$ ($i = 1, 2, \dots, k$) ($t \geq 0, X \in \mathbb{R}^d$) are continuous functions with respect to X .*

(A1) *There is a positive constant B such that*

$$|b(X_1) - b(X_2)| + \sum_{i=1}^k |\sigma_i(X_1) - \sigma_i(X_2)| \leq B|X_1 - X_2|,$$

$$|b(X)| + \sum_{i=1}^k |\sigma_i(X)| \leq B(1 + |X|).$$

(A2) There exists a non-negative function $V(X) \in C^2(\mathbb{R}^d)$ such that

$$LV(X) \leq -1 \text{ for all } X \text{ outside some compact set exists.}$$

Then, the solution $X(t)$ of Eq (4.1) defines a stationary Markov process.

Remark 4.1. From [20, Remark 5], condition (A1) in Lemma 4.1 is to ensure the existence and uniqueness of the solution to the model, which we have proved in Theorem 2.1.

Theorem 4.1. Let $(x(0), r(0)) \in \mathbb{R}_+ \times \mathbb{R}$ be any initial value. If $a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}}$, then model (1.4) has a stationary distribution $\pi(\cdot)$.

Proof. Define a C^2 -function $V_1 : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$V_1(x, r) = -\ln x + \frac{\delta(r - \bar{r})^2}{2\alpha},$$

where $\delta = \frac{\sqrt{2\alpha}}{2\beta} > 0$. With the help of the Itô's formula, we get

$$\begin{aligned} LV_1 &= -\left[r - cx - \frac{a}{1 + hax}\right] + \frac{\delta}{\alpha}(r - \bar{r})[-\alpha(r - \bar{r})] + \frac{\delta\beta^2}{2\alpha} \\ &\leq -r + cx + a - \delta(r - \bar{r})^2 + \frac{\delta\beta^2}{2\alpha} \\ &\leq cx + a - \bar{r} + \frac{\delta\beta^2}{2\alpha} + |r - \bar{r}| - \delta(r - \bar{r})^2 \\ &\leq cx + a - \bar{r} + \frac{\delta\beta^2}{2\alpha} + \frac{1}{4\delta} \\ &= cx + a - \bar{r} + \frac{\beta}{\sqrt{2\alpha}}. \end{aligned}$$

Further, let C^2 -function $V_2 : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$

$$V_2(x, r) = x + \frac{(r - \bar{r})^4}{4}.$$

It follows from the Itô's formula and (2.1) in Lemma 2.1 that

$$\begin{aligned} LV_2 &= x\left[r - cx - \frac{a}{1 + hax}\right] + (r - \bar{r})^3(-\alpha(r - \bar{r})) + \frac{3\beta^2}{2}(r - \bar{r})^2 \\ &= rx - cx^2 - \frac{ax}{1 + hax} - \alpha(r - \bar{r})^4 + \frac{3\beta^2}{2}(r - \bar{r})^2 \\ &= (r - \bar{r})x - cx^2 + \bar{r}x - \frac{ax}{1 + hax} - \alpha(r - \bar{r})^4 + \frac{3\beta^2}{2}(r - \bar{r})^2 \\ &\leq \frac{|r - \bar{r}|^3}{3} + \frac{2}{3}x^{\frac{3}{2}} - cx^2 + \bar{r}x - \alpha(r - \bar{r})^4 + \frac{3\beta^2}{2}(r - \bar{r})^2. \end{aligned}$$

Let

$$V_0(x, r) = M_0 V_1(x, r) + V_2(x, r) = M_0 \left[-\ln x + \frac{\delta(r - \bar{r})^2}{2\alpha} \right] + x + \frac{(r - \bar{r})^4}{4},$$

where $M_0 > 0$ is sufficiently large and satisfies

$$-M_0 \left[\bar{r} - a - \frac{\beta}{\sqrt{2\alpha}} \right] + \sup_{(x,r) \in \mathbb{R}_+ \times \mathbb{R}} \left\{ -\frac{c}{2}x^2 + \frac{2}{3}x^{\frac{3}{2}} + \bar{r}x - \frac{\alpha}{2}(r - \bar{r})^4 + \frac{|r - \bar{r}|^3}{3} + \frac{3\beta^2}{2}(r - \bar{r})^2 \right\} \leq -2.$$

It is evident that

$$\liminf_{k \rightarrow +\infty, (x,r) \in \mathbb{R}_+ \times \mathbb{R} \setminus U_k} V_0(x, r) = +\infty,$$

where $U_k = (\frac{1}{k}, k) \times (-k, k)$. Then, together with the continuity of function $V_0(x, r)$, one can yield that $V_0(x, r)$ has a minimum point (\bar{x}_0, \bar{r}_0) in the interior of $\mathbb{R}_+ \times \mathbb{R}$. We then define a nonnegative C^2 -function $V: \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$ by

$$V(x, r) = V_0(x, r) - V_0(\bar{x}_0, \bar{r}_0).$$

It is easy to show that

$$\begin{aligned} LV = LV_0 &\leq M_0 \left[cx + a - \bar{r} + \frac{\beta}{\sqrt{2\alpha}} \right] + \frac{|r - \bar{r}|^3}{3} + \frac{2}{3}x^{\frac{3}{2}} - cx^2 + \bar{r}x - \alpha(r - \bar{r})^4 + \frac{3\beta^2}{2}(r - \bar{r})^2 \\ &= -M_0 \left[\bar{r} - a - \frac{\beta}{\sqrt{2\alpha}} \right] - cx^2 + \frac{2}{3}x^{\frac{3}{2}} + \bar{r}x - \alpha(r - \bar{r})^4 + \frac{|r - \bar{r}|^3}{3} + \frac{3\beta^2}{2}(r - \bar{r})^2 + M_0 cx \\ &\leq +M_0 cx - \frac{c}{2}x^2 - \frac{\alpha}{2}(r - \bar{r})^4 - M_0 \left[\bar{r} - a - \frac{\beta}{\sqrt{2\alpha}} \right] \\ &\quad + \sup_{(x,r) \in \mathbb{R}_+ \times \mathbb{R}} \left\{ -\frac{c}{2}x^2 + \frac{2}{3}x^{\frac{3}{2}} + \bar{r}x - \frac{\alpha}{2}(r - \bar{r})^4 + \frac{|r - \bar{r}|^3}{3} + \frac{3\beta^2}{2}(r - \bar{r})^2 \right\} \\ &\leq -2 + M_0 cx - \frac{c}{2}x^2 - \frac{\alpha}{2}(r - \bar{r})^4 \\ &\triangleq F(x, r). \end{aligned}$$

Moreover, we have

$$LV(x, r) \leq \begin{cases} F(0, r) \leq -2, & \text{as } x \rightarrow 0^+, \\ F(+\infty, r) \rightarrow -\infty, & \text{as } x \rightarrow +\infty, \\ F(x, -\infty) \rightarrow -\infty, & \text{as } r \rightarrow -\infty, \\ F(x, +\infty) \rightarrow -\infty, & \text{as } r \rightarrow +\infty. \end{cases}$$

Hence, one can find $\varrho > 0$ (sufficiently small) such that

$$LV(x, r) \leq -1 < 0 \quad \text{for all } (x, r) \in \mathbb{R}_+ \times \mathbb{R} \setminus D_\varrho,$$

where

$$D_\varrho = \left\{ (x, r) \in \mathbb{R}_+ \times \mathbb{R} : x \in \left[\varrho, \frac{1}{\varrho} \right], r \in \left[-\frac{1}{\varrho}, \frac{1}{\varrho} \right] \right\}.$$

Therefore, the condition (A2) in Lemma 4.1 is fulfilled. It follows that model (1.4) admits a stationary distribution $\pi(\cdot)$ on $\mathbb{R}_+ \times \mathbb{R}$ whenever $a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}}$. This completes the proof. \square

4.2. Probability density function

According to Theorem 4.1, model (1.4) admits a stationary distribution $\pi(\cdot)$ whenever $a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}}$. In what follows, we investigate the probability density function of $\pi(\cdot)$. Now, we define a stochastic quasi-positive equilibrium $E^* = (x^*, r^*)$, given by the following equations:

$$\begin{cases} r^* - cx^* - \frac{a}{1+hx^*} = 0, \\ r^* - \bar{r} = 0. \end{cases}$$

It follows that $r^* = \bar{r}$. From Kang and Udiani [1], if $a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}} < \bar{r}$, equation $r^* - cx^* - \frac{a}{1+hx^*} = 0$ admits a unique solution $x^* = \frac{(har^*-c)+\sqrt{(har^*+c)^2-4a^2hc}}{2hac} > 0$. Thus, $E^* = (x^*, r^*) \in \mathbb{R}_+^2$ when $a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}}$. In addition, it can be noticed that x^* is a positive constant that is independent of the reversion speed α and the volatility intensity β . Setting $u = x - x^*$ and $v = r - r^*$, the linearized equation of model (1.4) around E^* can be written as

$$\begin{cases} du(t) = [-a_{11}u(t) + a_{12}v(t)]dt, \\ dv(t) = -a_{22}v(t)dt + \beta dB(t), \end{cases} \quad (4.2)$$

where

$$a_{11} = r^* - \frac{a(1+2hax^*)}{(1+hax^*)^2} > 0, \quad a_{12} = x^* > 0, \quad a_{22} = \alpha > 0.$$

Theorem 4.2. *If $a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}}$, then the solution (u, v) of system (4.2) has a distribution that admits a density function*

$$\Phi(u, v) = (2\pi)^{-1} |\Sigma|^{-\frac{1}{2}} e^{-\frac{1}{2}(u,v)\Sigma^{-1}(u,v)^T}.$$

Here, Σ denotes the covariance matrix of (u, v) , which satisfies

$$\Sigma = (\sigma_{ij})_{2 \times 2} = \begin{bmatrix} \frac{a_{12}^2\beta^2}{2a_{11}a_{22}(a_{11}+a_{22})} & \frac{a_{12}\beta^2}{2a_{22}(a_{11}+a_{22})} \\ \frac{a_{12}\beta^2}{2a_{22}(a_{11}+a_{22})} & \frac{\beta^2}{2a_{22}} \end{bmatrix},$$

where $\Sigma^{-1} = \frac{\Sigma^*}{|\Sigma|}$ and $|\Sigma| = \det(\Sigma)$, and the adjoint matrix is $\Sigma^* = \begin{bmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{21} & \sigma_{11} \end{bmatrix}$.

Proof. Let $X = (u, v)^T$, $A = \begin{bmatrix} -a_{11} & a_{12} \\ 0 & -a_{22} \end{bmatrix}$, and $\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & \beta \end{bmatrix}$. Then we can equivalently rewrite system (4.2) as $dX(t) = AX(t)dt + \Lambda dB(t)$. From Gardiner [21], system (4.2) has a unique transient probability density function $\Phi(u, v)$, which can be determined by the Fokker-Planck equation

$$\frac{\partial[(-a_{11}u + a_{12}v)\Phi]}{\partial u} + \frac{\partial[(-a_{22}v)\Phi]}{\partial v} - \frac{\beta^2}{2} \frac{\partial^2 \Phi}{\partial v^2} = 0. \quad (4.3)$$

Note that the diffusion matrix Λ is a constant matrix. As pointed out by Roozen [22], $\Phi(u, v)$ can be approximated by the following quasi-Gaussian distribution:

$$\Phi(u, v) = C \exp \left\{ -\frac{1}{2}(u, v)M(u, v)^T \right\}. \quad (4.4)$$

Here, C is a positive constant determined by the normalization condition $\int_{\mathbb{R}^2} \Phi(u, v) du dv = 1$, and $M = (m_{ij})_{2 \times 2}$ is a symmetric matrix.

Taking (4.4) into (4.3), by denoting the row vectors of A and M as A_i and M_i ($i = 1, 2$), respectively, one can obtain that

$$\begin{aligned} 0 &= \Phi \left[-a_{11} - a_{22} + \frac{\beta^2}{2} m_{22} \right] + \Phi \left[(-a_{11}u + a_{12}v)(-M_1X) + (-a_{22}v)(-M_2X) - \frac{\beta^2}{2} (-M_2X)(-M_2X) \right] \\ &= \Phi \left[-a_{11} - a_{22} + \frac{\beta^2}{2} m_{22} \right] + \Phi \left[A_1X(-M_1X) + A_2X(-M_2X) - \frac{\beta^2}{2} (-M_2X)(-M_2X) \right] \\ &= \Phi \left[-a_{11} - a_{22} + \frac{\beta^2}{2} m_{22} \right] - \frac{1}{2} \Phi \left[\sum_{i=1}^2 [(M_iX)^T(A_iX) + (A_iX)^T(M_iX)] + \beta^2 (M_2X)^T(M_2X) \right] \\ &= \Phi \left[-a_{11} - a_{22} + \frac{\beta^2}{2} m_{22} \right] - \frac{1}{2} \Phi X^T [(M^T A + A^T M) + M^T \Lambda^2 M] X. \end{aligned}$$

Noting that $M = (m_{ij})_{2 \times 2}$ is a symmetric matrix, we have

$$0 = \Phi \left[-a_{11} - a_{22} + \frac{\beta^2}{2} m_{22} \right] - \frac{1}{2} \Phi X^T [(MA + A^T M) + M \Lambda^2 M] X. \quad (4.5)$$

From the arbitrariness of X , condition (4.5) holds if and only if

$$\beta^2 m_{22} = 2(a_{11} + a_{22}) \quad \text{and} \quad M \Lambda^2 M + MA + A^T M = 0.$$

From [23], one can obtain that M is invertible and unique. Thus, $M \Lambda^2 M + MA + A^T M = 0$ can be equivalent of $M \Lambda^2 + MAM^{-1} + A^T = 0$. A straightforward calculation gives

$$\beta^2 m_{22} = \text{trace}(M \Lambda^2) = \text{trace}(-MAM^{-1} - A^T) = -2\text{trace}(A) = 2(a_{11} + a_{22}).$$

Thus, we only need to solve equation $M \Lambda^2 + MAM^{-1} + A^T = 0$ to obtain M . Defining $\Sigma = M^{-1}$, we see that the equation $M \Lambda^2 + MAM^{-1} + A^T = 0$ is equivalent to

$$\Lambda^2 + A\Sigma + \Sigma A^T = 0. \quad (4.6)$$

Direct calculation yields

$$\Sigma = (\sigma_{ij})_{2 \times 2} = \begin{bmatrix} \frac{a_{12}^2 \beta^2}{2a_{11}a_{22}(a_{11}+a_{22})} & \frac{a_{12}\beta^2}{2a_{22}(a_{11}+a_{22})} \\ \frac{a_{12}\beta^2}{2a_{22}(a_{11}+a_{22})} & \frac{\beta^2}{2a_{22}} \end{bmatrix}.$$

It is easy to show that $\sigma_{11} = \frac{a_{12}^2 \beta^2}{2a_{11}a_{22}(a_{11}+a_{22})} > 0$ and

$$\det(\Sigma) = \begin{vmatrix} \frac{a_{12}^2 \beta^2}{2a_{11}a_{22}(a_{11}+a_{22})} & \frac{a_{12}\beta^2}{2a_{22}(a_{11}+a_{22})} \\ \frac{a_{12}\beta^2}{2a_{22}(a_{11}+a_{22})} & \frac{\beta^2}{2a_{22}} \end{vmatrix} = \frac{a_{12}^2 \beta^4}{4a_{11}a_{22}(a_{11} + a_{22})^2} > 0.$$

Therefore, Σ in Eq (4.6) is a positive definite, implying M is positive definite. It follows that (u, v) has a unique normal distribution $\mathbb{N}(0, 0, \sigma_{11}, \sigma_{22}, \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}})$ with a well-defined density function

$$\Phi(u, v) = \frac{1}{2\pi} \frac{1}{\sqrt{\sigma_{11}\sigma_{22}(1 - \frac{\sigma_{12}^2}{\sigma_{11}\sigma_{22}})}} \exp \left\{ -\frac{1}{2(1 - \frac{\sigma_{12}^2}{\sigma_{11}\sigma_{22}})} \left[\frac{u^2}{\sigma_{11}} + \frac{v^2}{\sigma_{22}} - \frac{2\sigma_{12}uv}{\sigma_{11}\sigma_{22}} \right] \right\}$$

$$\begin{aligned}
&= \frac{1}{2\pi} \frac{1}{\sqrt{\sigma_{11}\sigma_{22} - \sigma_{12}^2}} \exp \left\{ -\frac{1}{2} \frac{1}{\sigma_{11}\sigma_{22} - \sigma_{12}^2} [\sigma_{22}u^2 + \sigma_{11}v^2 - 2\sigma_{12}uv] \right\} \\
&= (2\pi)^{-1} |\Sigma|^{-\frac{1}{2}} e^{-\frac{1}{2}(u,v)\Sigma^{-1}(u,v)^T}.
\end{aligned}$$

The proof is complete. \square

Remark 4.2. From Theorem 4.2, when $a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}}$, u and v in system (4.2) admits a unique normal distribution $\mathbb{N}(0, \sigma_{11})$ and $\mathbb{N}(0, \sigma_{22})$. Thus, u and v have the following normal density functions:

$$\phi_1(u) = \frac{1}{\sqrt{2\pi\sigma_{11}}} \exp \left\{ -\frac{u^2}{2\sigma_{11}} \right\}, \quad \phi_2(v) = \frac{1}{\sqrt{2\pi\sigma_{22}}} \exp \left\{ -\frac{v^2}{2\sigma_{22}} \right\}.$$

Remark 4.3. In view of the relation of model (1.4) and system (4.2), from Theorem 4.2, when $a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}}$, around E^* , the stationary distribution $\pi(\cdot)$ is approximately given by the following normal density function:

$$\Phi(x, r) = (2\pi)^{-1} |\Sigma|^{-\frac{1}{2}} e^{-\frac{1}{2}(x-x^*, r-r^*)\Sigma^{-1}(x-x^*, r-r^*)^T}.$$

Accordingly, around x^* and r^* , the species $x(t)$ and the variable $r(t)$ are approximately distributed according to the following normal density functions:

$$\phi_1(x) = \frac{1}{\sqrt{2\pi\sigma_{11}}} \exp \left\{ -\frac{(x-x^*)^2}{2\sigma_{11}} \right\}, \quad \phi_2(r) = \frac{1}{\sqrt{2\pi\sigma_{22}}} \exp \left\{ -\frac{(r-r^*)^2}{2\sigma_{22}} \right\}.$$

5. Extinction

Theorem 4.1 indicates that model (1.4) admits a stationary distribution when $a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}}$, which means weak persistence of the species $x(t)$. The next theorem establishes sufficient criteria for species extinction in model (1.4), both with and without the Allee effect.

Lemma 5.1. For any $r(0) \in \mathbb{R}$, denote by $r(t)$ the solution of Eq (1.3) with initial value $r(0)$. Then,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t r(s) ds = \bar{r}. \quad (5.1)$$

Proof. By [18, Example 5.2], the solution of (1.3) has the form

$$r(t) = e^{-\alpha t} r(0) + \bar{r}(1 - e^{-\alpha t}) + \beta \int_0^t e^{-\alpha(t-s)} dB(s). \quad (5.2)$$

It is clear that $M(t) \doteq \beta \int_0^t e^{-\alpha(t-s)} dB(s)$ is a local martingale with $M(0) = 0$. Note that the quadratic variation of $M(t)$ is $\langle M, M \rangle_t = \beta^2 \int_0^t e^{-2\alpha(t-s)} ds = \frac{\beta^2}{2\alpha} (1 - e^{-2\alpha t})$. Thus, we have

$$\lim_{t \rightarrow \infty} \langle M, M \rangle_t = \lim_{t \rightarrow \infty} \frac{\beta^2}{2\alpha} (1 - e^{-2\alpha t}) = \frac{\beta^2}{2\alpha}.$$

From the strong law of large numbers for local martingales, it follows that

$$\lim_{t \rightarrow \infty} \frac{M(t)}{t} = \lim_{t \rightarrow \infty} \frac{1}{t} \left[\beta \int_0^t e^{-\alpha(t-s)} dB(s) \right] = 0.$$

This, together with (5.2), can yield

$$\lim_{t \rightarrow \infty} \frac{r(t)}{t} = \lim_{t \rightarrow \infty} \frac{1}{t} \left[e^{-\alpha t} r(0) + \bar{r}(1 - e^{-\alpha t}) + \beta \int_0^t e^{-\alpha(t-s)} dB(s) \right] = 0. \quad (5.3)$$

Integrating both sides of (1.3) from 0 to t yields

$$r(t) = r(0) - \alpha \int_0^t (r(s) - \bar{r}) ds + \beta B(t) = r(0) + \alpha \bar{r} t - \alpha \int_0^t r(s) ds + \beta B(t),$$

which implies

$$\frac{1}{t} \int_0^t r(s) ds = \bar{r} + \frac{1}{\alpha} \left[\frac{r(0)}{t} - \frac{r(t)}{t} + \frac{\beta B(t)}{t} \right].$$

Note that $\lim_{t \rightarrow \infty} \frac{B(t)}{t} = 0$. Then, from (5.3), it follows that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t r(s) ds = \bar{r} + \frac{1}{\alpha} \lim_{t \rightarrow \infty} \left[\frac{r(0)}{t} - \frac{r(t)}{t} + \frac{\beta B(t)}{t} \right] = \bar{r}.$$

The proof is complete. \square

Theorem 5.1. Denote by $(x(t), r(t))$ the solution of model (1.4) with initial value $(x(0), r(0)) \in \mathbb{R}_+ \times \mathbb{R}$. If either of the following holds:

- (i) $a \leq \sqrt{\frac{c}{h}}$ and $a > \bar{r}$ (without the Allee effect);
- (ii) $a > \sqrt{\frac{c}{h}}$ and $a > \frac{c}{h} \frac{1}{2\sqrt{\frac{c}{h} - \bar{r}}} > 0$ (with the Allee effect),

then,

$$\lim_{t \rightarrow \infty} x(t) = 0 \text{ a.s.}$$

Hence, the species in model (1.4) goes extinct almost surely.

Proof. Let $V : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ be a C^2 -function

$$V(x, r) = \ln x + \frac{\delta(r - \bar{r})^2}{2\alpha},$$

where $\delta = \frac{\sqrt{2\alpha}}{2\beta} > 0$. From the Itô's formula, it follows that

$$d \left[\ln x(t) + \frac{\delta(r - \bar{r})^2}{2\alpha} \right] = LV(x(t), r(t)) dt + \frac{\delta\beta}{\alpha} (r(t) - \bar{r}) dB(t),$$

where

$$LV = \left[r - cx - \frac{a}{1 + hax} \right] - \delta(r - \bar{r})^2 + \frac{\delta\beta^2}{2\alpha}$$

$$\begin{aligned}
&= \bar{r} - cx - \frac{a}{1+hax} - \delta(r - \bar{r})^2 + (r - \bar{r}) + \frac{\delta\beta^2}{2\alpha} \\
&\leq \bar{r} - cx - \frac{a}{1+hax} + \frac{1}{4\delta} + \frac{\delta\beta^2}{2\alpha} \\
&= \bar{r} - cx - \frac{a}{1+hax} + \frac{\beta}{\sqrt{2\alpha}}.
\end{aligned}$$

Thus,

$$\ln x(t) + \frac{\delta(r - \bar{r})^2}{2\alpha} = \int_0^t LV(x(s), r(s)) ds + \frac{\delta\beta}{\alpha} \int_0^t (r(s) - \bar{r}) dB(s) + \left[\ln x_0 + \frac{\delta(r_0 - \bar{r})^2}{2\alpha} \right],$$

which implies

$$\ln x(t) \leq \int_0^t \left[\bar{r} - cx(s) - \frac{a}{1+hax(s)} + \frac{\beta}{\sqrt{2\alpha}} \right] ds + \int_0^t \frac{\delta\beta}{\alpha} (r(s) - \bar{r}) dB(s) + \ln x(0). \quad (5.4)$$

Consider $f(x) = -cx - \frac{a}{1+hax}$ on $[0, +\infty)$. By direct computation, we obtain $f'(x) = \frac{g(x)}{(hax+1)^2}$, where $g(x) = -ch^2a^2x^2 - 2chax + ha^2 - c$. The roots of the quadratic equation $g(x) = 0$ can be easily determined as

$$x_1 = \frac{c + \sqrt{cha^2}}{-cha}, \quad x_2 = \frac{c - \sqrt{cha^2}}{-cha}.$$

We now distinguish two cases:

Case 1: Condition (i) holds. That is, $a \leq \sqrt{\frac{c}{h}}$. Under this assumption, we have $x_1 < 0$ and $x_2 \leq 0$. Then, $g(x) < 0$ for all $x \in (0, +\infty)$, which implies that $f'(x) < 0$ on $(0, \infty)$. It follows that $f(x) < f(0) = -a$ for all $x \in (0, +\infty)$. Therefore, from (5.4), we obtain

$$\frac{1}{t} \ln x(t) \leq \frac{1}{t} \int_0^t r(s) ds - a + \frac{\ln x(0)}{t}.$$

With (5.1), one yields

$$\limsup_{t \rightarrow \infty} \left[\frac{1}{t} \ln x(t) \right] \leq \bar{r} - a < 0 \quad \text{a.s.}$$

Case 2: Condition (ii) holds. If $a > \sqrt{\frac{c}{h}}$, then $x_1 < 0$ and $x_2 > 0$. Therefore, function $f(x)$ attains its maximum at $x = x_2$, and $f_{\max} = \frac{c}{ha} - 2\sqrt{\frac{c}{h}}$. Now, from (5.4), it follows that

$$\frac{1}{t} \ln x(t) \leq \frac{1}{t} \int_0^t r(s) ds + \frac{c}{ha} - 2\sqrt{\frac{c}{h}} + \frac{\ln x(0)}{t}.$$

With (5.1), we have

$$\limsup_{t \rightarrow \infty} \left[\frac{1}{t} \ln x(t) \right] \leq \bar{r} + \frac{c}{ha} - 2\sqrt{\frac{c}{h}} \quad \text{a.s.}$$

From $a > \frac{c}{h} \frac{1}{2\sqrt{\frac{c}{h} - \bar{r}}} > 0$, it follows that $\bar{r} + \frac{c}{ha} - 2\sqrt{\frac{c}{h}} < 0$. Hence, the desired results follow. This completes the proof. \square

Remark 5.1. It follows from $a > \sqrt{\frac{c}{h}}$ that $\bar{r} + \frac{c}{ha} - 2\sqrt{\frac{c}{h}} < \bar{r} - \sqrt{\frac{c}{h}}$. This means that if $a > \sqrt{\frac{c}{h}}$ and $\sqrt{\frac{c}{h}} > \bar{r}$, the species $x(t)$ in model (1.4) will become extinct. Thus, we can conclude that if $\bar{r} < \min\{a, \sqrt{\frac{c}{h}}\}$, then the species will become extinct. Furthermore, the stochastic perturbation arising from the Ornstein-Uhlenbeck process has no impact on the extinction of the species.

6. Numerical simulations

To verify our theoretical results, we use Milstein's higher-order method [24] to perform numerical simulations in this section.

Example 6.1. To verify Theorem 4.1, we set the parameters as $\bar{r} = 0.8$, $c = 0.005$, $a = 0.5$, $h = 0.04$, $\alpha = 0.5$, and $\beta = 0.01$. It is clear that $0.5 = a < \bar{r} - \frac{\beta}{\sqrt{2}\alpha} = 0.79$. Hence, by Theorem 4.1, model (1.4) admits a stationary distribution $\pi(\cdot)$. By direct calculation, we have $E^* = (x^*, r^*) = (132.6209, 0.8)$ and

$$\Sigma = (\sigma_{ij})_{2 \times 2} = \begin{bmatrix} \frac{a_{12}^2 \beta^2}{2a_{11}a_{22}(a_{11}+a_{22})} & \frac{a_{12}\beta^2}{2a_{22}(a_{11}+a_{22})} \\ \frac{a_{12}\beta^2}{2a_{22}(a_{11}+a_{22})} & \frac{\beta^2}{2a_{22}} \end{bmatrix} = \begin{bmatrix} 2.9334 & 0.0125 \\ 0.0125 & 0.0001 \end{bmatrix}.$$

According to Remark 4.3, around the equilibrium E^* , the distribution $\pi(\cdot)$ can be approximated by the normal density function $\Phi(x, r) = (2\pi)^{-1} |\Sigma|^{-\frac{1}{2}} e^{-\frac{1}{2}(x-x^*, r-r^*)\Sigma^{-1}(x-x^*, r-r^*)^T}$ (see Figure 1). Model (1.4) thus admits a stationary distribution $\pi(\cdot)$, reflecting the persistence of the species x (see Figure 2). Furthermore, the distributions of the species $x(t)$ around x^* and the variable $r(t)$ around r^* are well approximated by normal density functions

$$\phi_1(x) = \frac{1}{\sqrt{2\pi\sigma_{11}}} e^{-\frac{(x-x^*)^2}{2\sigma_{11}}}, \quad \phi_2(r) = \frac{1}{\sqrt{2\pi\sigma_{22}}} e^{-\frac{(r-r^*)^2}{2\sigma_{22}}}.$$

Figure 3 shows the marginal density functions $\phi_1(x)$ and $\phi_2(r)$ of $\Phi(x, r)$ along with the frequency histogram fitting curves for the species $x(t)$ and the variable $r(t)$ at $t = 1000$. It is evident from Figure 3 that the marginal densities $\phi_1(x)$ and $\phi_2(r)$ closely correspond to their respective histogram fitting curves. This indicates that the true density function of the distribution $\pi(\cdot)$ is well approximated by $\Phi(x, r)$.

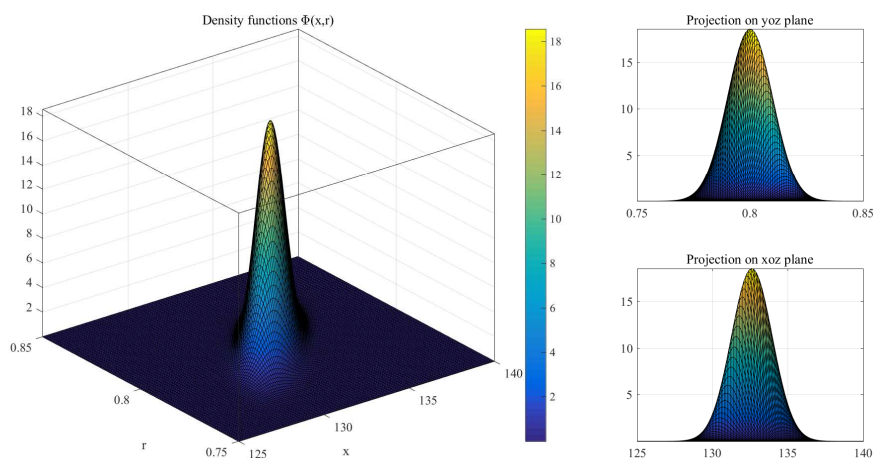


Figure 1. Normal density function $\Phi(x, r)$ and its projection on yoz plane and xoz plane.

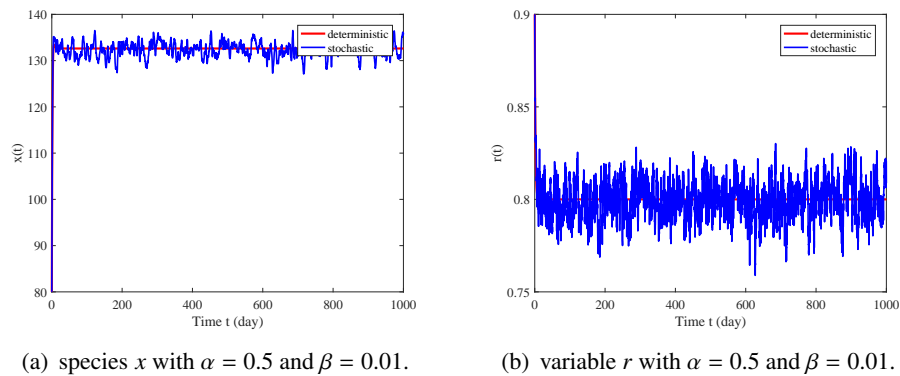


Figure 2. Numerical simulations of model (1.4) with $\alpha = 0.5$ and $\beta = 0.01$, and of deterministic model. (Color figure online)

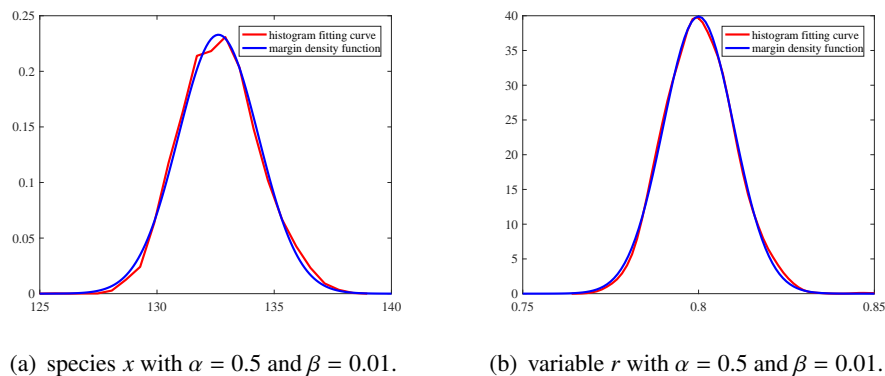


Figure 3. Marginal density functions and corresponding frequency histogram fitting curves of x and r for model (1.4), with $\alpha = 0.5$ and $\beta = 0.01$. (Color figure online)

Example 6.2. Setting $\bar{r} = 0.8$, $c = 0.005$, $a = 0.5$, and $h = 0.04$, we investigate the effect of the reversion speed α on the stationary distribution of the species $x(t)$. Specifically, we consider $\alpha = 0.3$ and 0.9 , while keeping volatility intensity $\beta = 0.01$ fixed. It is easy to see that the condition of Theorem 4.1 holds ($0.5 = a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}} \approx 0.7871$ and $0.5 = a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}} \approx 0.7925$). Compared with Figures 4(a) and 4(b), the reversion speed α has little effect on the persistence of the species. It follows from Figures 4(c) and 4(d) that the distribution of species $x(t)$ is around the deterministic steady state. This is because the positive equilibrium state is independent of the reversion speed α . Moreover, compared with Figures 4(e) and 4(f), we can see that the density distribution scope becomes smaller with the reversion speed α increasing. This is consistent with the conclusion that the variance σ_{11} of species $x(t)$ is monotonically decreasing in α .

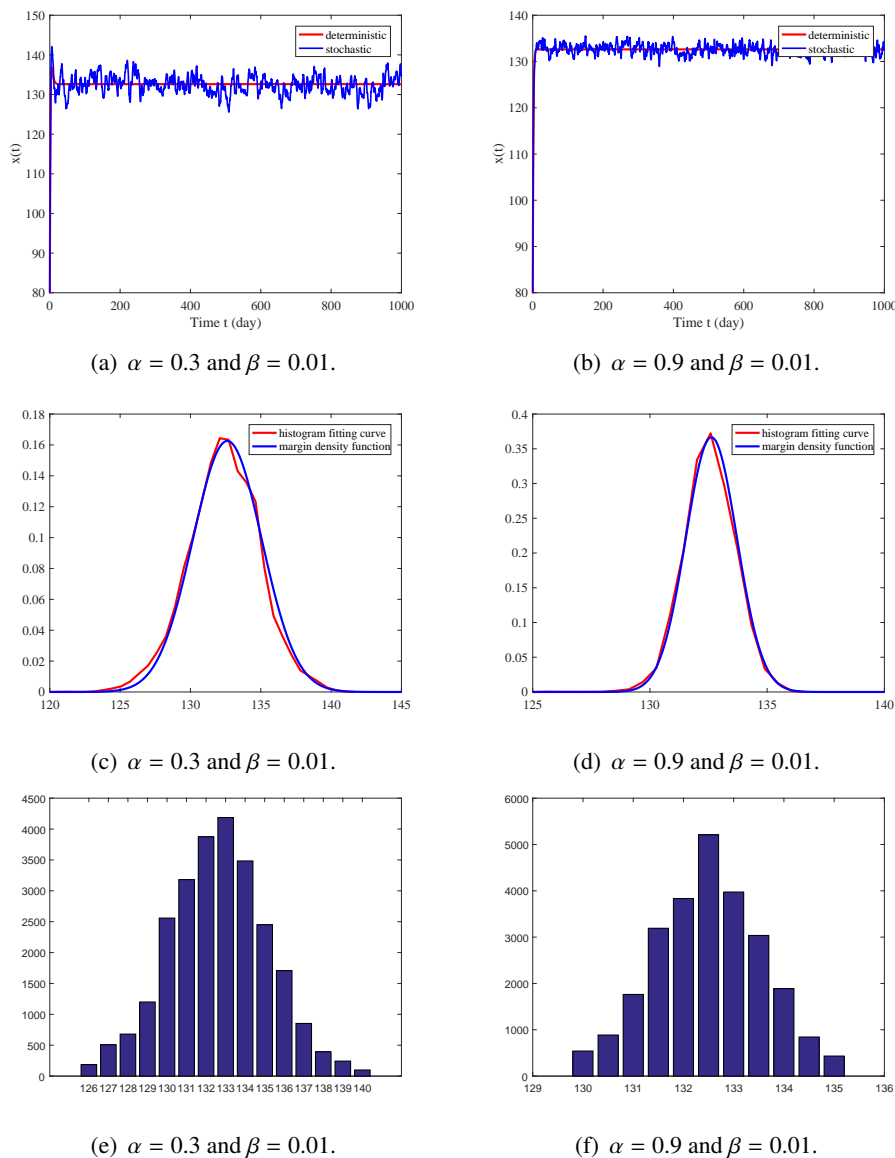


Figure 4. (a) and (b): The path of $x(t)$ for model (1.4) and its corresponding deterministic model. (c) and (d): The marginal density functions and frequency histogram fitting curves of $x(t)$ in model (1.4). (e) and (f): The histogram of $x(t)$ in model (1.4). (Color figure online)

Example 6.3. Setting $\bar{r} = 0.8$, $c = 0.005$, $a = 0.5$, and $h = 0.04$, we investigate the effect of volatility intensity β on the stationary distribution of species $x(t)$. Specifically, we choose $\beta = 0.01$ and 0.03 , keeping $\alpha = 0.4$ fixed. It is straightforward to verify that Theorem 4.1 is satisfied, namely, $0.5 = a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}} \approx 0.7888$ and $0.5 = a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}} \approx 0.7665$. Compared with Figures 5(a) and 5(b), the fluctuation of $x(t)$ around the steady state becomes larger with β increasing. From Figures 5(c) and 5(d), the distribution of species $x(t)$ is around the steady state. This is because the equilibrium state is independent of volatility intensity β . Compared with Figures 5(e) and 5(f), it can be seen that the density distribution scope becomes larger with volatility intensity β increasing. This is consistent with the conclusion that the variance σ_{11} is monotonically increasing with respect to volatility intensity β .

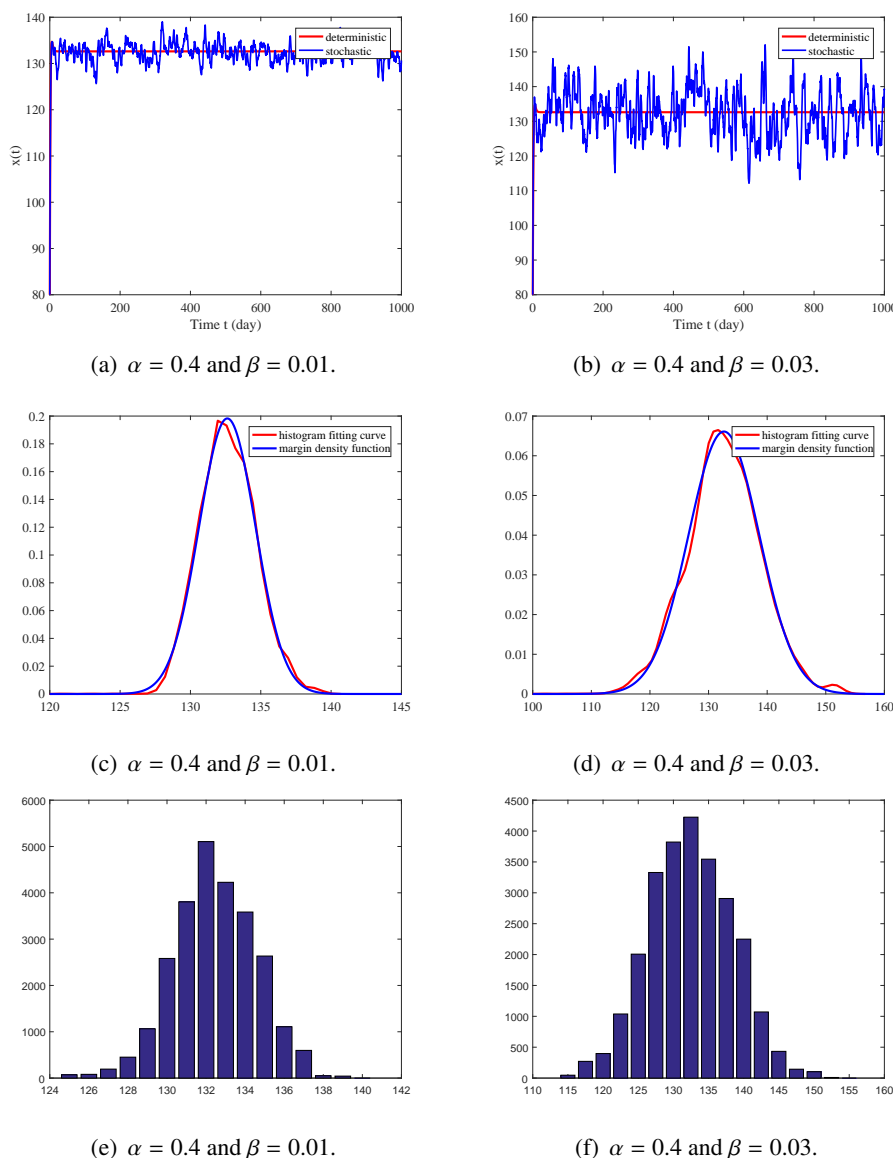


Figure 5. (a) and (b): The path of $x(t)$ for model (1.4) and its corresponding deterministic model. (c) and (d): The marginal density functions and frequency histogram fitting curves of x in model (1.4). (e) and (f): The histogram of species $x(t)$ in model (1.4). (Color figure online)

Example 6.4. To verify Theorem 5.1, take $c = 0.005$ and $h = 0.04$. It is clear that $\sqrt{\frac{c}{h}} \approx 0.3536$. On the one hand, take $a = 0.35$ and $\bar{r} = 0.34$. By a simple computation, we have $0.35 = a \leq \sqrt{\frac{c}{h}} \approx 0.3536$ and $0.35 = a > \bar{r} = 0.34$. This implies that the condition (i) of Theorem 5.1 is fulfilled. Then, from Theorem 5.1, the species $x(t)$ in model (1.4) will become extinct. In the left of Figure 6, we give the numerical results of $x(t)$ with different reversion speed α and volatility intensity β . On the other hand, take $a = 0.354$ and $\bar{r} = 0.34$. By a simple computation, we have $0.354 = a > \sqrt{\frac{c}{h}} \approx 0.3536$ and $0.354 = a > \frac{c}{h} \frac{1}{2\sqrt{\frac{c}{h}-\bar{r}}} \approx 0.3405$. Thus, the condition (ii) of Theorem 5.1 is fulfilled. Then, from Theorem 5.1, the species $x(t)$ in model (1.4) will become extinct. In the right of Figure 6, we give the

numerical results of $x(t)$ with different reversion speed α and volatility intensity β .

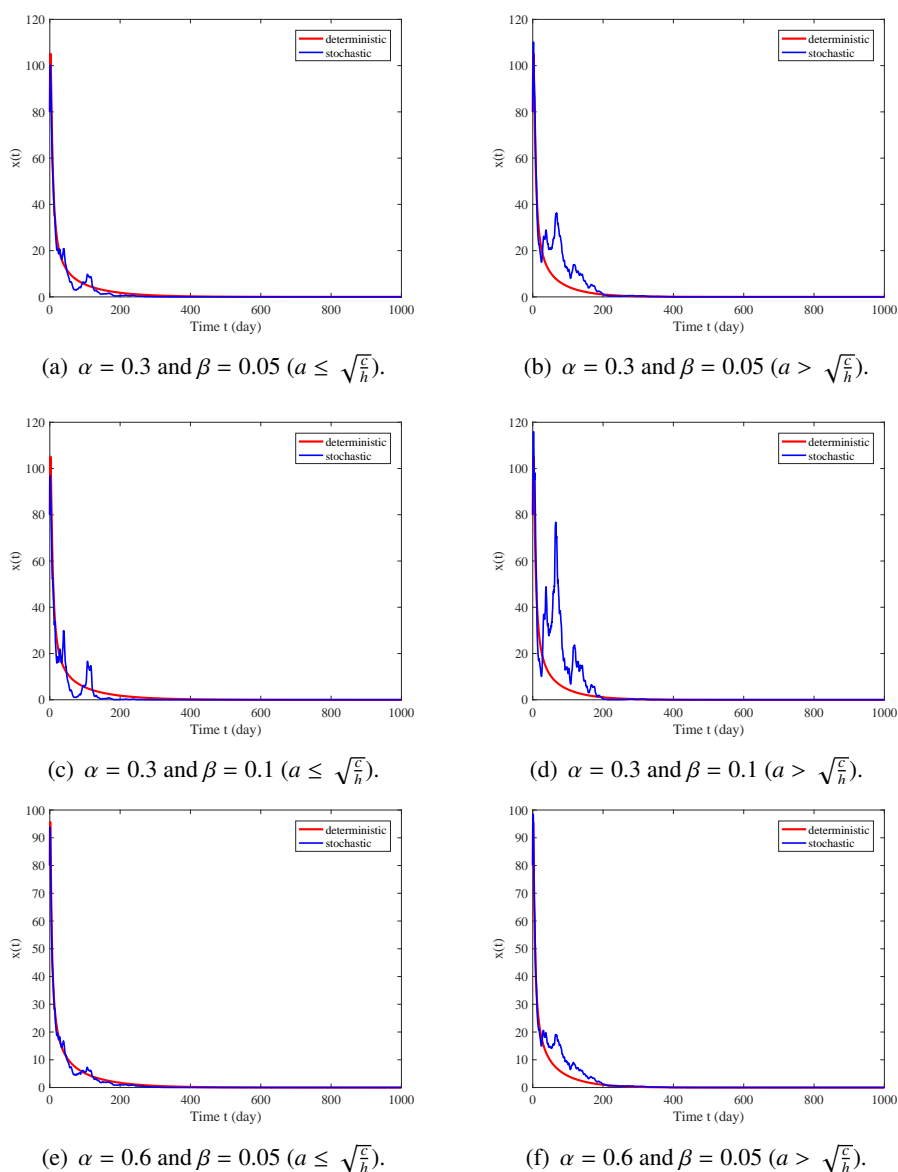


Figure 6. Numerical simulations of stochastic model (1.4) and of the deterministic model. The numerical solutions of $x(t)$ in model (1.4) with or without the Allee effect are given in the left and right parts of the figure. (Color figure online)

On the one hand, from Theorems 4.1 and 5.1, without the Allee effect (that is, $a \leq \sqrt{\frac{c}{h}}$), if $a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}}$, the species $x(t)$ in the model is weakly persistent; while if $a > \bar{r}$, the species $x(t)$ in model (1.4) will become extinct. In Example 6.5, we discuss the influence of parameters \bar{r} , α , and β on the dynamics of species $x(t)$ in the model when $a \leq \sqrt{\frac{c}{h}}$.

Example 6.5. Take $c = 0.0144$, $h = 0.04$, and $a = 0.54$. It is easy to show that $0.54 = a < \sqrt{\frac{c}{h}} = 0.6$. This means that there is no Allee effect (see Figure 7).

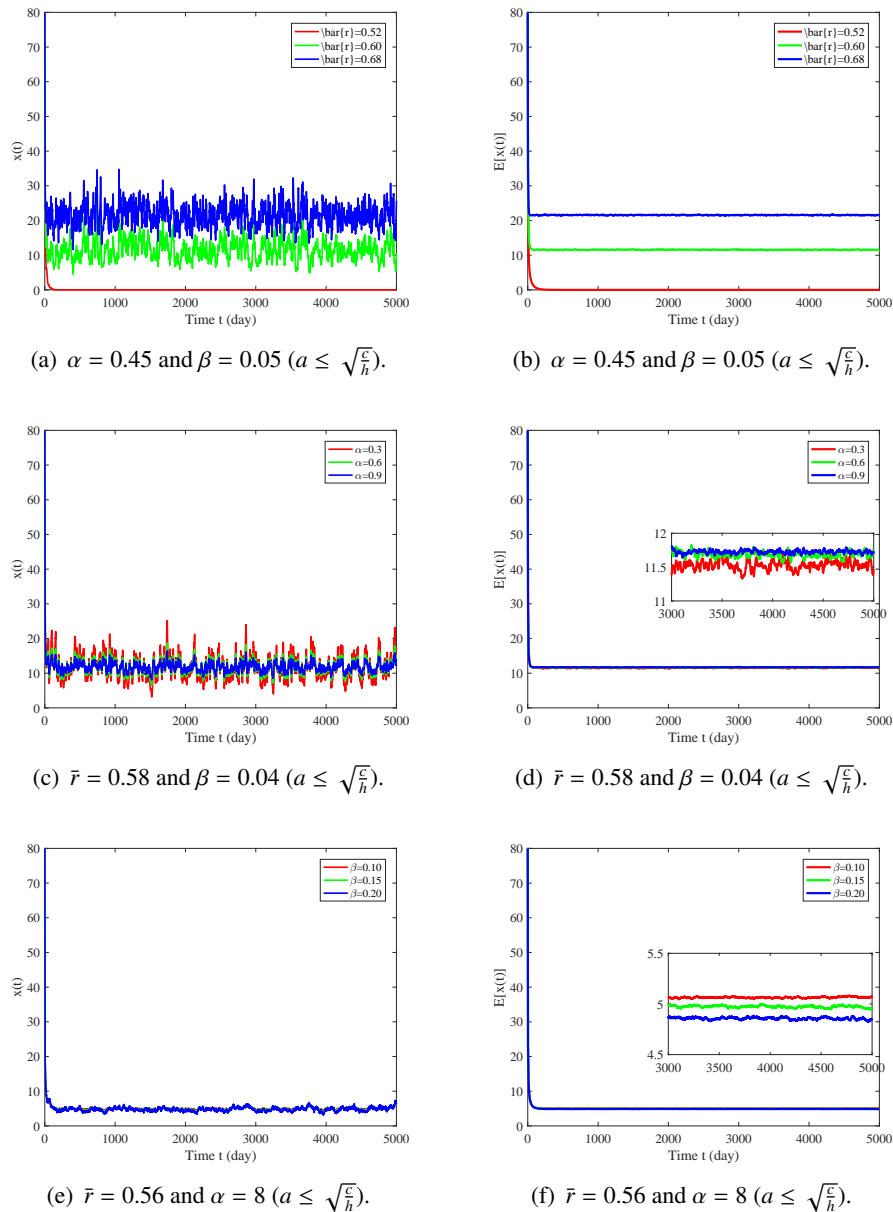


Figure 7. Numerical simulations of $x(t)$ and $\mathbb{E}[x(t)]$ in stochastic model (1.4) without the Allee effect. (Color figure online)

- (i) To show the effect of average growth rate \bar{r} , we choose $\bar{r} = 0.52, 0.60,$ and 0.68 , keeping reversion speed $\alpha = 0.45$ and volatility intensity $\beta = 0.05$ fixed. Figures 7(a) and 7(b) show, respectively, the numerical simulations of $x(t)$ and $\mathbb{E}[x(t)]$ with $\bar{r} = 0.52, 0.60,$ and 0.68 . It can be seen from Figure 7(a) that the species $x(t)$ will become extinct when $\bar{r} = 0.52 < a$ and the species $x(t)$ can be persistent when $\bar{r} = 0.60$ ($0.54 = a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}} \approx 0.5473$) and $\bar{r} = 0.68$ ($0.54 = a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}} \approx 0.6273$).
- (ii) To show the effect of reversion speed α , we choose $\alpha = 0.3, 0.6,$ and 0.9 , keeping average growth rate $\bar{r} = 0.6$ and volatility intensity $\beta = 0.04$ fixed. Figures 7(c) and 7(d) show, respectively, the numerical simulations of $x(t)$ and $\mathbb{E}[x(t)]$ with $\alpha = 0.3, 0.6,$ and 0.9 . From Theorems 4.1, the

species $x(t)$ is weakly persistent when $\alpha = 0.3$ ($0.54 = a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}} \approx 0.5484$), $\alpha = 0.6$ ($0.54 = a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}} \approx 0.5635$), and $\alpha = 0.9$ ($0.54 = a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}} \approx 0.5702$) (see Figure 7(c)). Moreover, as can be seen from Figure 7(d), with the increase of reversion speed α , $\mathbb{E}[x(t)]$ increases.

- (iii) To show the effect of volatility intensity β , we choose $\beta = 0.10, 0.15$, and 0.20 , keeping average growth rate $\bar{r} = 0.56$ and reversion speed $\alpha = 8$ fixed. Figures 7(e) and 7(f) show, respectively, the numerical simulations of $x(t)$ and $\mathbb{E}[x(t)]$ with $\beta = 0.01, 0.05$, and 0.09 . From Theorems 4.1, the species $x(t)$ is weakly persistent when $\beta = 0.10$ ($0.54 = a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}} \approx 0.5750$), $\beta = 0.15$ ($0.54 = a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}} \approx 0.5625$), and $\beta = 0.20$ ($0.54 = a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}} \approx 0.5500$) (see Figure 7(e)). Moreover, as can be seen from Figure 7(f), with the increase of volatility intensity β , $\mathbb{E}[x(t)]$ decreases.

On the other hand, it is easy to verify that $\frac{c}{h} \frac{1}{2\sqrt{\frac{c}{h}-\bar{r}}} \geq \bar{r}$ and the equation holds when $\bar{r} = \sqrt{\frac{c}{h}}$. Moreover, from Theorems 4.1 and 5.1, with the Allee effect (that is, $a > \sqrt{\frac{c}{h}}$), if $a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}}$, the species $x(t)$ in the model is weakly persistent; while if $a > \frac{c}{h} \frac{1}{2\sqrt{\frac{c}{h}-\bar{r}}} > 0$, the species $x(t)$ in model (1.4) will become extinct. In Example 6.6, we discuss the influence of parameters \bar{r} , α , and β on the dynamics of species $x(t)$ in the model when $a > \sqrt{\frac{c}{h}}$.

Example 6.6. Take $c = 0.0144$, $h = 0.04$, and $a = 0.62$. It is clear that $0.62 = a > \sqrt{\frac{c}{h}} = 0.6$. This means that there exists the Allee effect (see Figure 8).

- (i) To show the effect of average growth rate \bar{r} , we choose $\bar{r} = 0.60, 0.66$, and 0.72 , keeping reversion speed $\alpha = 0.45$ and volatility intensity $\beta = 0.05$ fixed. Figures 8(a) and 8(b) show, respectively, the numerical simulations of $x(t)$ and $\mathbb{E}[x(t)]$ with $\bar{r} = 0.60, 0.66$, and 0.72 . If $\bar{r} = 0.66$ and 0.72 , then $0.62 = a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}} \approx 0.6073$ and $0.62 = a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}} \approx 0.6673$. Thus, the species $x(t)$ in the model is weakly persistent. If $\bar{r} = 0.60$, then $0.62 = a > \frac{c}{h} \frac{1}{2\sqrt{\frac{c}{h}-\bar{r}}} = 0.6$, which means that the species in the model will become extinct (see Figure 8(a)).
- (ii) To show the effect of reversion speed α , we choose $\alpha = 0.3, 0.6$, and 0.9 , keeping average growth rate $\bar{r} = 0.68$ and volatility intensity $\beta = 0.04$ fixed. Figures 8(c) and 8(d) show, respectively, the numerical simulations of $x(t)$ and $\mathbb{E}[x(t)]$ with $\alpha = 0.3, 0.6$, and 0.9 . From Theorems 4.1, the species $x(t)$ is weakly persistent when $\alpha = 0.3$ ($0.62 = a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}} \approx 0.6284$), $\alpha = 0.6$ ($0.62 = a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}} \approx 0.6435$), and $\alpha = 0.9$ ($0.62 = a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}} \approx 0.6502$) (see Figure 8(c)). Moreover, as can be seen from Figure 8(d), with the increase of reversion speed α , the mathematical expectation $\mathbb{E}[x(t)]$ increases.
- (iii) To show the effect of volatility intensity β , we choose $\beta = 0.10, 0.15$, and 0.20 , keeping average growth rate $\bar{r} = 0.68$ and reversion speed $\alpha = 8$ fixed. Figures 8(e) and 8(f) show, respectively, the numerical simulations of $x(t)$ and $\mathbb{E}[x(t)]$ with $\beta = 0.10, 0.15$, and 0.20 . The species $x(t)$ is weakly persistent when $\beta = 0.10$ ($0.62 = a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}} \approx 0.6500$), $\beta = 0.15$ ($0.54 = a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}} \approx 0.5825$), and $\beta = 0.20$ ($0.54 = a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}} \approx 0.5500$) (see Figure 8(e)). Moreover, one can see from Figure 8(f), with the increase of volatility intensity β , the mathematical expectation $\mathbb{E}[x(t)]$ decreases.

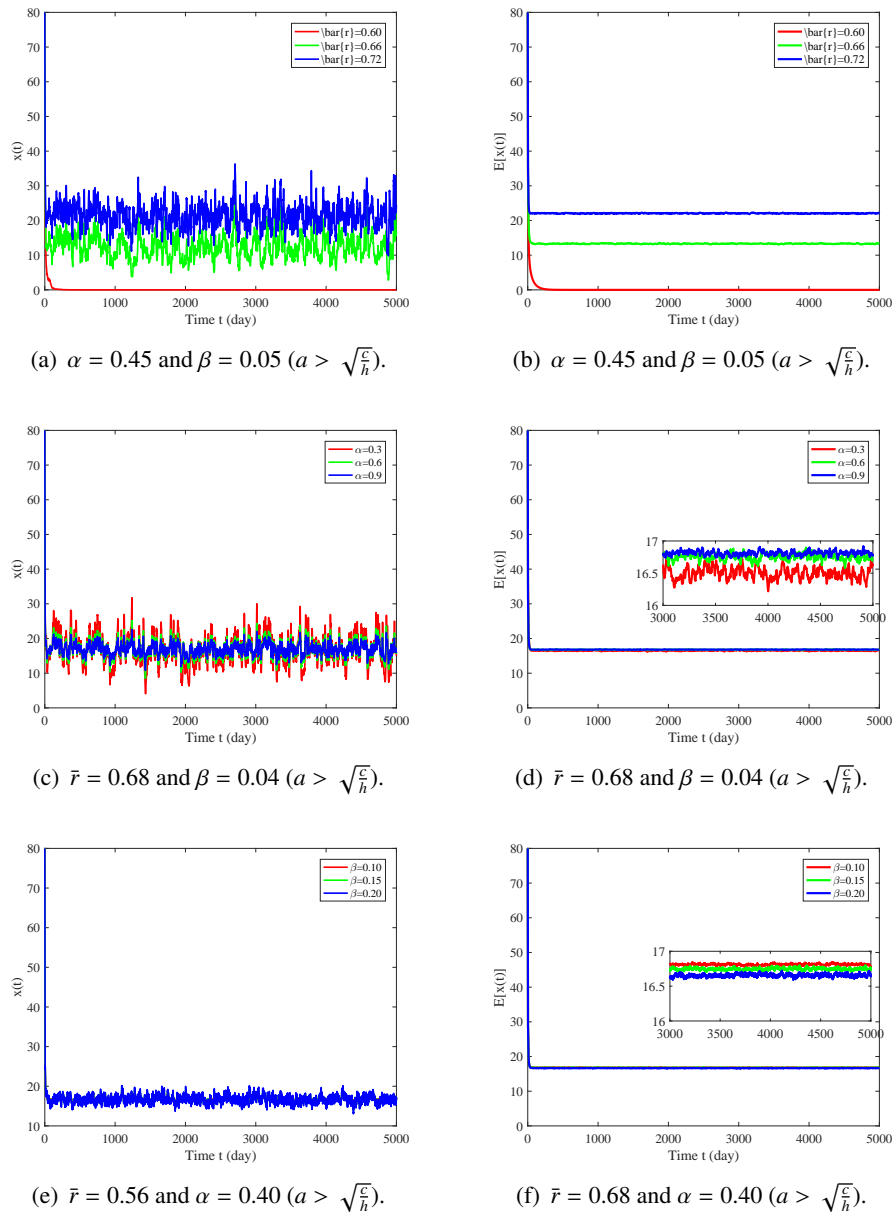


Figure 8. Numerical simulations of $x(t)$ and $\mathbb{E}[x(t)]$ in stochastic model (1.4) with the Allee effect. (Color figure online)

7. Conclusions

This paper is devoted to the study the dynamic behavior of a stochastic single species model with the mean-reverting Ornstein-Uhlenbeck process and Allee effect. We first prove the global existence and uniqueness of the solution in Theorem 2.1. Theorem 3.1 gives the corresponding moment estimates of the solution. In Theorem 4.1, sufficient conditions ensuring the existence of a unique stationary distribution are established, which are determined by the reversion speed α and the volatility intensity β . Furthermore, Theorem 5.1 provides the sufficient conditions for the extinction of species $x(t)$ in

model (1.4), which also depends on α and β . More precisely,

- (i) From Theorem 4.1, if $a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}}$, model (1.4) has a unique stationary distribution $\pi(\cdot)$, which reflects the weak persistence of the species. Moreover, from Remark 4.3, the stationary distribution $\pi(\cdot)$ around E^* approximately has the normal density function $\Phi(x, r)$, and the species $x(t)$ in model (1.4) around x^* approximately has the normal density function

$$\phi_1(x) = \frac{1}{\sqrt{2\pi\sigma_{11}}} \exp\left\{-\frac{(x-x^*)^2}{2\sigma_{11}}\right\},$$

where $\sigma_{11} = \frac{a_{12}^2\beta^2}{2a_{11}a_{22}(a_{11}+a_{22})}$. It can be seen from the expression of σ_{11} that the variance $\text{Var}[x(t)] = \sigma_{11}$ increases with the reversion speed α decreasing or the volatility intensity β increasing. Thus, with the decrease of reversion speed α or the increase of volatility intensity β , the range of the density function of x becomes more dispersed (see Figures 4(e), 4(f), 5(e), and 5(f)).

- (ii) From Theorem 5.1, in the absence of the Allee effect, if $a \leq \sqrt{\frac{c}{h}}$ and $a > \bar{r}$, the species $x(t)$ in model (1.4) will become extinct (see Figure 6(a)); otherwise, in the presence of the Allee effect, if $a > \sqrt{\frac{c}{h}}$ and $a > \frac{c}{h} \frac{1}{2\sqrt{\frac{c}{h}-\bar{r}}} > 0$, the species $x(t)$ in model (1.4) will become extinct (see Figure 6(b)).

From the condition $a < \bar{r} - \frac{\beta}{\sqrt{2\alpha}}$, it can be seen that larger reversion speed and smaller volatility intensity are conducive to the survival of the species.

Outlook. Our analysis in this paper provides persistence conditions for $a < \bar{r} - \beta/\sqrt{2\alpha}$ and extinction for $a > \bar{r}$. However, there remains an interval not covered by the theoretical framework, namely $\bar{r} - \beta/\sqrt{2\alpha} < a < \bar{r}$. Exploring the dynamical behavior in this regime, possibly involving non-stationary or transient dynamics, is left for future work.

Author contributions

Jin Kang: Writing-review & editing; Rong Liu: Writing-original draft; Xuhui Shen: Software, Numerical simulation. All authors have reviewed and approved the final manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Nos. 12501620 and 12001341) and the Natural Science Foundation of Shanxi (Nos. 202403021221214 and 202503021211057).

Conflict of interest

The authors declare there are no conflicts of interest.

References

1. Y. Kang, O. Udiani, Dynamics of a single species evolutionary model with Allee effects, *J. Math. Anal. Appl.*, **418** (2014), 492–515. <https://doi.org/10.1016/j.jmaa.2014.03.083>
2. W. Allee, *Animal aggregations: a study in general sociology*, Chicago: University of Chicago Press, 1931. <https://doi.org/10.5962/bhl.title.7313>
3. S. Biswas, Md. Saifuddin, S. Sasmal, S. Samanta, N. Pal, F. Ababneh, et al., A delayed prey-predator system with prey subject to the strong Allee effect and disease, *Nonlinear Dyn.*, **84** (2016), 1569–1594. <https://doi.org/10.1007/s11071-015-2589-9>
4. J. Cushing, The evolutionary dynamics of a population model with a strong Allee effect, *Math. Biosci. Eng.*, **12** (2015), 643–660. <https://doi.org/10.3934/mbe.2015.12.643>
5. C. Taylor, A. Hastings, Allee effects in biological invasions, *Ecol. Lett.*, **8** (2005), 895–908. <https://doi.org/10.1111/j.1461-0248.2005.00787.x>
6. J. Ripa, P. Lundberg, Noise colour and the risk of population extinctions, *Proc. Biol. Sci.*, **263** (1996), 1751–1753. <https://doi.org/10.1098/rspb.1996.0256>
7. S. Zhang, T. Zhang, S. Yuan, Dynamics of a stochastic predator-prey model with habitat complexity and prey aggregation, *Ecol. Complex.*, **45** (2021), 100889. <https://doi.org/10.1016/j.ecocom.2020.100889>
8. R. Liu, G. Liu, Complex dynamics and optimal harvesting for a stochastic food-web model with intraguild predation and time delays, *Int. J. Biomath.*, **15** (2022), 2250050. <https://doi.org/10.1142/S1793524522500504>
9. X. Yi, R. Liu, Y. Wang, Dynamics and control for a stochastic giving up smoking model, *AIMS Mathematics*, **10** (2025), 26484–26510. <https://doi.org/10.3934/math.20251164>
10. Y. Wang, D. Jiang, T. Hayat, A. Alsaedi, Stationary distribution of an HIV model with general nonlinear incidence rate and stochastic perturbations, *J. Franklin. I.*, **356** (2019), 6610–6637. <https://doi.org/10.1016/j.jfranklin.2019.06.035>
11. P. Wu, C. Fang, Spatiotemporal dynamics of syphilis in Xinjiang via a demographic-geographic data-validated reaction diffusion model, *J. Math. Phys.*, **66** (2025), 062704. <https://doi.org/10.1063/5.0273893>
12. B. Zhou, D. Jiang, Y. Dai, T. Hayat, A. Alsaedi, Stationary distribution and probability density function of a stochastic SVIS epidemic model with standard incidence and vaccination strategies, *Chaos Soliton. Fract.*, **143** (2021), 110601. <https://doi.org/10.1016/j.chaos.2020.110601>
13. R. Liu, G. Liu, Dynamics of a stochastic population model with Allee effect and jumps, *Math. Model. Nat. Phenom.*, **17** (2022), 1. <https://doi.org/10.1051/mmnp/2022002>
14. X. Yu, S. Yuan, T. Zhang, Persistence and ergodicity of a stochastic single species model with Allee effect under regime switching, *Commun. Nonlinear Sci.*, **59** (2018), 359–374. <https://doi.org/10.1016/j.cnsns.2017.11.028>
15. Q. Yang, X. Zhang, D. Jiang, Dynamical behaviors of a stochastic food chain system with Ornstein-Uhlenbeck process, *J. Nonlinear Sci.*, **32** (2022), 34. <https://doi.org/10.1007/s00332-022-09796-8>

16. B. Zhou, D. Jiang, T. Hayat, Analysis of a stochastic population model with mean-reverting Ornstein-Uhlenbeck process and Allee effects, *Commun. Nonlinear Sci.*, **111** (2022), 106450. <https://doi.org/10.1016/j.cnsns.2022.106450>
17. X. Mu, D. Jiang, T. Hayat, A. Alsaedi, Y. Liao, A stochastic turbidostat model with Ornstein-Uhlenbeck process: dynamics analysis and numerical simulations, *Nonlinear Dyn.*, **107** (2022), 2805–2817. <https://doi.org/10.1007/s11071-021-07093-9>
18. X. Mao, *Stochastic differential equations and applications*, Chichester: Horwood Publishing Limited, 2007. <https://doi.org/10.1533/9780857099402>
19. R. Khasminskii, *Stochastic stability of differential equations*, Berlin: Springer, 2011. <https://doi.org/10.1007/978-3-642-23280-0>
20. D. Xu, Y. Huang, Z. Yang, Existence theorems for periodic Markov process and stochastic functional differential equations, *Discrete Cont. Dyn.*, **24** (2009), 1005–1023. <https://doi.org/10.3934/dcds.2009.24.1005>
21. C. Gardiner, *Handbook of stochastic methods: for physics, chemistry and the natural sciences*, Berlin: Springer, 1983.
22. H. Roizen, An asymptotic solution to a two-dimensional exit problem arising in population dynamics, *SIAM J. Appl. Math.*, **49** (1989), 1793–1810. <https://doi.org/10.1137/0149110>
23. Z. Ma, Y. Zhou, C. Li, *Qualitative and stability methods for ordinary differential equations (Chinese)*, Beijing: Science Press, 2015.
24. D. Higham, An algorithmic introduction to numerical simulation of stochastic differential equations, *SIAM Rev.*, **43** (2001), 525–546. <https://doi.org/10.1137/S0036144500378302>



AIMS Press

©2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<https://creativecommons.org/licenses/by/4.0>)