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Research article

A multi-granulation T-rough set model and its properties

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Abstract: This paper works in the setting of multi-granulation rough sets on two universes. We combine multi-granulation ideas with set-valued mappings and introduce a multi-granulation T-rough set model built on set-valued mappings. Four inverse approximation operators are defined: a pessimistic upper inverse operator, a pessimistic lower inverse operator, an optimistic upper inverse operator, and an optimistic lower inverse operator. Using these operators, we set up a two-universe framework for multi-granulation T-rough sets. We then spell out the basic properties of the operators, prove several theorems, and clarify how the different operators relate to each other. A few examples are included to show how the model works in practice. The model extends rough set theory on two universes and gives a new way to describe information that comes from multiple sources and multiple granular levels.

Keywords: T-rough sets; multi-granulation; set-valued mapping; inverse operator; pessimistic operator; optimistic operator
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1. Introduction

Rough set theory is a standard tool for describing uncertainty and incomplete information. Since Pawlak introduced it in 1982 [1], it has been used in data mining, artificial intelligence, pattern recognition, and a long list of other areas. In the classical model, approximation operators are derived from a single binary relation: The lower and upper approximations of a target set describe what is definitely in the set and what might be in it.

The single-granulation assumption is often violated in practical data analysis. Information usually comes from more than one source, and the same objects can be grouped at different levels of detail. In those cases, one relation (or one partition) provides an oversimplified description of a problem that may involve multiple levels of granularity. To deal with this, Qian et al. proposed multi-granulation rough sets [2,3]. Their idea is straightforward: Use several relations (or granular structures) at the same time, and define two styles of approximation—pessimistic and optimistic—that reflect whether you require agreement across granulations or allow “support” from at least one of them. This line of work has become one of the main routes for extending rough set theory [4].

In parallel, rough sets have been pushed into other mathematical settings—algebra, topology, fuzzy sets—often by changing what “approximation” is built from. One influential direction is T-rough sets, which tie rough approximations to set-valued mappings. Davvaz introduced the idea in 2008 in the context of groups, using set-valued homomorphisms [5]. The point is flexibility: set-valued maps let you model situations where an element does not correspond to a single image, but to a set of possible images.

After that, T-rough sets were developed in a range of algebraic structures, including semigroups [6], semilattices [7], Γ-semihypergroups [8], ternary semigroups [9], and BCK-algebras [10]. Kazanc et al. studied upper and lower approximations for fuzzy sets in quotient hyperstructures [11], which widened the kinds of problems T-rough sets can naturally talk about. Hosseinpour blended the T-rough viewpoint with fuzzy subgroups, defining T-rough fuzzy subgroups and proving basic properties [12]. Other work has connected T-rough sets with covering-based rough sets [13], category theory [14], and algebraic logic [15,16], which helps explain why the framework keeps reappearing in different guises: It fits neatly with several established approximation formalisms.

This paper brings these two threads together. We develop a multi-granulation T-rough set model on two universes based on set-valued mappings, define pessimistic and optimistic inverse approximation operators, and study their properties and relationships.

The construction of the knowledge structure is a central issue in knowledge space theory (KST). Doignon and Falmagne’s 1985 framework is still the cleanest mathematical starting point for knowledge assessment [1,17]. Since then, a lot of the construction work has gone in different directions: Li and coauthors brought in formal concept analysis [18,19], rough set approximations [4], and skill mappings [19–21], which opened up several workable ways to generate knowledge structures rather than betting everything on a single recipe [22–24]. Yang et al. proposed a variable-precision model that blends the strengths of disjunctive and conjunctive models [25]. Huang et al. then extended this idea with three variable-precision models, looking at what happens under different threshold intervals and how the models relate to each other [26]. Zhang’s group worked from another angle, using distributed serial fuzzy relations to study how fuzzy knowledge spaces and closure spaces can be “gridded”, which matters when the information you’re aggregating comes from more than one source [27].

Even with that progress, most existing work still sticks to single-granulation T-rough set models. That’s a problem if your data really is multi-source because different sources usually come with different granularities, and variable-precision ideas don’t drop neatly into the single-granulation setup. In two-universe settings (an object universe and an attribute universe), the harder question is how to put multi-granulation thinking, set-valued mappings, and variable-precision models into one coherent model that can actually cope with multi-level uncertainty [28,29]. This shows up immediately in applications like multi-source evaluation and multi-perspective decision making [3]: You need approximation operators that work across granularities, you need to understand their algebraic behavior, and you need clear duality relationships—otherwise, the "model" is just definitions on paper [27,30,31].

Most existing multi-granulation rough set models work on a single universe; T rough set models rarely consider multi-granulation; variable precision ideas are not seamlessly integrated into two-universe multi-granulation settings; and there is a lack of inverse approximation operators for such combined frameworks.

Over the past several years, rough sets have increasingly been coupled with other uncertainty-oriented frameworks to address applied decision-making problems across a wide range of domains. For example, Kousar and Kausar [32] developed an integrated fuzzy-rough approach for sustainable agritourism, using multi-criteria decision-making to balance economic, environmental and social impacts. Liu and Wang [33] tackled large-group decision-making under a multi-granularity linguistic environment by proposing a rough-integrated asymmetric cloud model. Their method captures both uncertainty and randomness through preference relevance and a trust-propagation mechanism, which goes beyond the usual assumption of independent decision-makers. In a related but methodologically distinct direction, Gul [34] extended the VIseKriterijumska Optimizacija IKompromisno Rasenje (VIKOR) method by incorporating bipolar fuzzy preference δ-covering-based bipolar fuzzy rough sets, thereby allowing both positive and negative evaluations to be processed within compromise solutions. Turning to hierarchical decision settings, Fujita [35] proposed hyperfuzzy and superhyperfuzzy variants of VIKOR and Decision-Making Trial and Evaluation Laboratory (DEMATEL), which enable a more layered representation of uncertainty in such contexts. The same author further introduced the notion of shadowed offset [36], combining offset membership (i.e., values lying outside the unit interval [0,1]) with shadowed-set thresholds so that under-confidence and over-confidence can be captured within a single tripartite structure. Taken together, these studies point to a growing demand for flexible hybrid models capable of handling multi-source, multi-granular, and bipolar information—a trajectory that aligns with, and indeed complements, the multi-granulation T-rough set framework developed in this paper. To address the above limitations, we construct a multi-granulation T-rough set model on two universes using set valued multi-mappings, define pessimistic and optimistic inverse operators, and systematically prove their properties.

With that in mind, this paper combines multi-granulation ideas, set-valued mappings, and variable-precision modeling to build a multi-granulation T-rough set model on two universes based on set-valued mappings. We introduce four inverse approximation operators: the pessimistic upper inverse, pessimistic lower inverse, optimistic upper inverse, and optimistic lower inverse. Together they form a full approximation-operator system. We then study their basic properties (monotonicity, behavior under unions and intersections, and duality) and illustrate the model with examples. The point here is quite concrete: The model extends rough set theory on two universes and broadens where T-rough sets and variable-precision models can be used, especially for multi-source fusion, multi-granular decision analysis, and knowledge structure construction [18–20,25].

In short, the main contributions of this work can be summarized as follows.

(1) A new two-universe multi-granulation T-rough set model;
(2) Four inverse approximation operators with clear granular interpretations;
(3) Complete theoretical properties including duality;
(4) A practical case study demonstrating both strict and flexible evaluation strategies.

The main difference from previous single-universe models is that our framework works on two universes. The rest of the paper is organized as follows.

2. Basic concepts

2.1. Universe of discourse and set-valued multi-mappings

Let X and Y be two non-empty finite sets, referred to as the object universe and the attribute universe, respectively. Let $\mathcal{P}^*(Y)$ stand for the family of all non-empty subsets of Y, and let $\mathcal{P}^*(\mathcal{P}^*(Y))$ be a collection of certain non-empty members of $\mathcal{P}^*(Y)$ [4,5].

Definition 1. [2] A mapping $T: X \to \mathcal{P}^*(\mathcal{P}^*(Y)$ is called a set-valued multi-mapping from X to Y. For each $x \in X$, $T(x)$ is a collection consisting of several non-empty subsets of Y, which can be interpreted as a multi-granular description of the attributes of object x.

The outer family of subsets of $\mathcal{P}^*(Y)$ (i.e., $\mathcal{P}^*(\mathcal{P}^*(Y))$) is needed because each object x can be evaluated by multiple independent sources (e.g., different experts or evaluation dimensions). Each source provides a single subset of Y (e.g., the set of attributes possessed by x from that source). Thus, $T(x)$ collects several such subsets, one per source. This two-level structure is precisely what captures multi-granularity: Each subset corresponds to one granulation. Working with $\mathcal{P}^*(Y)$ would only restrict each x to a single granulation, which cannot represent multi-source information.

Example 1. Let $Y = \{a, b, c\}$. Suppose two experts evaluate an object x. Expert 1 gives $\{a, b\}$, and Expert 2 gives $\{b, c\}$. Then $T(x) = \{\{a, b\}, \{b, c\}\} \in \mathcal{P}^*(\mathcal{P}^*(Y))$. If we use only $\mathcal{P}^*(Y)$, we can only store one of these two subsets, losing the multi-granular information.

The idea of set-valued multi-mappings traces back to Davvaz's set-valued homomorphisms [5]. The intuition is straightforward: Instead of forcing each object to map to a single attribute set, you allow it to map to several attribute subsets, which is a natural way to encode information at multiple granularities. That flexibility is exactly what you need if you want later approximation reasoning to reflect different "views" of the same object [7,28]. Hosseini et al. studied inverse operations for set-valued mappings in semigroup settings [6], and their treatment is useful background for how inverse operators can be defined here. Yang's work on mapping-based characterizations in rough sets over two universes also supports using set-valued multi-mappings in this setting [28].

2.2. Multi-granulation T-rough set approximation space

Definition 2. The triple (X, Y, T) is called a multi-granulation T-rough set approximation space, where $T: X \to \mathcal{P}^*(\mathcal{P}^*(Y))$ is a set-valued multi-mapping [5,21].

This approximation space links the two universes through a set-valued multi-mapping. That's the main point: Instead of forcing everything through one binary relation (as in traditional multi-granulation rough sets), it lets you model messy, many-to-many connections between the universes more directly [10,29]. Compared with the fuzzy approximation space proposed by Zhang [27], the difference is quite clear. Here, the bridge between the universes is a set-valued multi-mapping, not a fuzzy relation. That choice lets you describe the granular structure and the mapping relationship in one setup rather than splitting the job across different tools. Also, borrowing the variable-precision idea from Huang et al. [26], you can tune the approximation strength by adjusting thresholds. In practice, that means the same approximation space can be used in reasoning tasks that demand different levels of strictness, and it sets up the later construction of inverse operators [26,31].

3. Pessimistic inverse operators for multi-granulation T-rough sets

3.1. Definition of pessimistic inverse operators

Definition 3. Let X and Y be two non-empty finite sets, and let $T: X \to \mathcal{P}^*(\mathcal{P}^*(Y))$ be a set-valued multi-mapping. For any $A \subseteq Y$, the multi-granulation pessimistic upper inverse $T_p^+(A)$ and the multi-granulation pessimistic lower inverse $T_p^-(A)$ of A with respect to T are defined respectively as:

$$T_p^+(A) = \{ x \in X \mid \forall B \in T(x), B \subseteq A \}, \tag{3.1}$$

and

$$T_p^-(A) = \{ x \in X \mid \exists B \in T(x), B \cap A \neq \emptyset \}. \tag{3.2}$$

The ordered pair $\left(T_p^+(A), T_p^-(A)\right)$ is called the pessimistic upper inverse operator and pessimistic lower inverse operator for multi-granulation T-rough sets. On an intuitive reading, the pessimistic upper operator imposes a strict requirement: Inclusion must hold at every granular level; by contrast, the pessimistic lower operator is governed by a looser criterion, requiring only that at least one granular level intersect the target.

Remark. For any $x \in X$, if $T(x)$ consists of a single non-empty subset of Y (i.e., $T(x) = \{B_x\}$), then our operators reduce exactly to the classical multi-granulation pessimistic operators defined by Qian et al. [2] on a single universe. Specifically,

$$T_p^+(A) = \{ x \in X \mid \{B_x\} \subseteq A \} = \{ x \mid B_x \subseteq A \}, \tag{3.3}$$

and

$$T_p^-(A) = \{ x \in X \mid \{B_x\} \cap A \neq \emptyset \} = \{ x \mid B_x \cap A \neq \emptyset \}. \tag{3.4}$$

This matches the definitions in [2] when the binary relations are replaced by the set-valued mapping. Thus, our model generalizes Qian's model to the case of multiple subsets per object.

Pessimistic inverse operators are meant to approximate multi-granular information at different "strengths", and they do it by controlling universal vs. existential requirements. This is in line with Hosseini et al.'s use of approximation ideas in their work on T-rough ideals in semigroups [6]. The defining feature here is granular consistency. The pessimistic upper inverse operator requires the inclusion condition to hold across all granular levels. That makes it the strict option, useful when you really do need high-precision approximations—for example, when you are making a hard call about whether someone has mastered a skill in knowledge structure construction [25,26]. The pessimistic lower inverse operator is looser: It only requires the intersection condition to hold for some granular levels. That makes it a better fit when you'd rather keep more candidates or preserve more potentially relevant information, such as in a more forgiving qualification screen based on multiple sources [27,31]. This matches the usual pessimistic/optimistic split in classical multi-granulation rough sets [2]. The difference is that, by using set-valued multi-mappings, the model stays workable in two-universe settings and still gives you a knob (via variable precision) to adjust how strong the approximation is [19,25].

3.2. Properties of pessimistic inverse operators

The fundamental properties of the pessimistic inverse operators for multi-granulation T-rough sets are presented below.

Proposition 1. Let (X, Y, T) be a multi-granulation T-rough set approximation space, and let $A, B \subseteq Y$. Then

$$T_p^+(Y) = X, \tag{3.5}$$

and

$$T_p^-(\emptyset) = \emptyset. \tag{3.6}$$

Proof: (1) For any $x \in X$, since each $C \in T(x)$ is a non-empty subset of Y, by the definition of T, we have $C \subseteq Y$. Hence, $x \in T_p^+(Y)$ by Eq (3.1). This shows $X \subseteq T_p^+(Y)$. Conversely, by Eq (3.1), $T_p^+(Y) \subseteq X$ holds trivially. Therefore, $T_p^+(Y) = X$.
(2) Suppose, to the contrary, that there exists some $x \in T_p^-(\emptyset)$. Then by Eq (3.2), there exists $C \in T(x)$ such that $C \cap \emptyset \neq \emptyset$. However, $C \cap \emptyset = \emptyset$ for any set C, which is a contradiction. Hence, no such x exists, and consequently $T_p^-(\emptyset) = \emptyset$.

Proposition 2. Let (X, Y, T) be a multi-granulation T-rough set approximation space, and let $A, B \subseteq Y$. If $A \subseteq B$, then

$$T_p^+(A) \subseteq T_p^+(B), \tag{3.7}$$

and

$$T_p^-(A) \subseteq T_p^-(B). \tag{3.8}$$

Proof: (1) Assume $A \subseteq B$. Take any $x \in T_p^+(A)$. By Eq (3.1), this means that for every $C \in T(x)$, we have $C \subseteq A$. Since $A \subseteq B$, it follows that $C \subseteq B$ for each $C \in T(x)$. Hence, $x \in T_p^+(B)$ by (3.1). The arbitrariness of x yields $T_p^+(A) \subseteq T_p^+(B)$.
(2) Assume $A \subseteq B$. Take any $x \in T_p^-(A)$. By Eq (3.2), there exists some $C \in T(x)$ such that $C \cap A \neq \emptyset$. Because $A \subseteq B$, we have $C \cap A \subseteq C \cap B$. Since $C \cap A$ is non-empty, it follows that $C \cap B \neq \emptyset$. Consequently, $x \in T_p^-(B)$ by Eq (3.2). The arbitrariness of x gives $T_p^-(A) \subseteq T_p^-(B)$.

Example 2. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$. Define the set-valued multi-mapping $T: X \to \mathcal{P}^*(\mathcal{P}^*(Y))$ by $T(x_1) = \{\{y_1\}\}$, $T(x_2) = \{\{y_1\}, \{y_2\}\}$.

Solution: (1) For each $x \in X$, every $C \in T(x)$ is a non-empty subset of Y, hence $C \subseteq Y$. By definition, $x \in T_p^+(Y)$. Thus, $X \subseteq T_p^+(Y)$. Trivially, $T_p^+(Y) \subseteq X$. Therefore, $T_p^+(Y) = X$.

Suppose $x \in T_p^-(\emptyset)$. Then there exists $C \in T(x)$ such that $C \cap \emptyset \neq \emptyset$, which is impossible. Hence, no such x exists, so $T_p^-(\emptyset) = \emptyset$.

(2) Take $A = \{y_1\}$, $B = \{y_1, y_2\}$. Clearly, $A \subseteq B$.

$$T_p^+(A) = \{x \in X \mid \forall C \in T(x), C \subseteq A\}.$$

For x_1: $T(x_1) = \{\{y_1\}\}$, $\{y_1\} \subseteq A$ holds, so $x_1 \in T_p^+(A)$.
For x_2: $T(x_2) = \{\{y_1\}, \{y_2\}\}$, $\{y_2\} \nsubseteq A$, so $x_2 \notin T_p^+(A)$.
Thus, $T_p^+(A) = \{x_1\}$.

$$T_p^+(B) = \{x \in X \mid \forall C \in T(x), C \subseteq B\}.$$

For x_1: $\{y_1\} \subseteq B$ holds, so $x_1 \in T_p^+(B)$.
For x_2: both $\{y_1\} \subseteq B$ and $\{y_2\} \subseteq B$ hold, so $x_2 \in T_p^+(B)$.
Hence, $T_p^+(B) = \{x_1, x_2\}$. Clearly, $\{x_1\} \subseteq \{x_1, x_2\}$, so the inclusion holds.

$$T_p^-(A) \subseteq T_p^-(B):$$

$$T_p^-(A) = \{x \in X \mid \exists C \in T(x), C \cap A \neq \emptyset\}.$$

For x_1: $\{y_1\} \cap A = \{y_1\} \neq \emptyset$, so $x_1 \in T_p^-(A)$.
For x_2: $\{y_1\} \cap A \neq \emptyset$, so $x_2 \in T_p^-(A)$.
Thus, $T_p^-(A) = \{x_1, x_2\}$.

$$T_p^-(B) = \{x \in X \mid \exists C \in T(x),\ C \cap B \neq \emptyset\}.$$

For x_1: $\{y_1\} \cap B \neq \emptyset$, so $x_1 \in T_p^-(B)$.
For x_2: $\{y_1\} \cap B \neq \emptyset$ and $\{y_2\} \cap B \neq \emptyset$, so $x_2 \in T_p^-(B)$.
Hence, $T_p^-(B) = \{x_1, x_2\}$. The inclusion $T_p^-(A) \subseteq T_p^-(B)$ holds as equality.

Proposition 3. Let (X, Y, T) be a multi-granulation T-rough set approximation space, and let $A, B \subseteq Y$. Then

$$T_p^+(A \cap B) = T_p^+(A) \cap T_p^+(B), \tag{3.9}$$

and

$$T_p^-(A \cup B) = T_p^-(A) \cup T_p^-(B). \tag{3.10}$$

Remark. These equalities obtain because, in T_p^+, the universal quantifier distributes across intersection: All granular levels satisfy inclusion in both A and B if and only if they satisfy inclusion in $A \cap B$. By the same token, within T_p^-, the existential quantifier distributes over union: Some granular level meets $A \cup B$ precisely when it meets A or, alternatively stated, meets B.

Proof: (1) For any $x \in X$,

$$\begin{aligned}
\forall x \in T_p^+(A \cap B) &\Leftrightarrow \forall C \in T(x), C \subseteq A \cap B \\
&\Leftrightarrow \forall C \in T(x), C \subseteq A \text{ and } C \subseteq B \\
&\Leftrightarrow (\forall C \in T(x), C \subseteq A) \text{ and } (\forall C \in T(x), C \subseteq B) \\
&\Leftrightarrow x \in T_p^+(A) \text{ and } x \in T_p^+(B) \\
&\Leftrightarrow x \in T_p^+(A) \cap T_p^+(B).
\end{aligned}$$

Hence, $T_p^+(A \cap B) = T_p^+(A) \cap T_p^+(B)$.

(2) We first prove $T_p^-(A \cup B) \subseteq T_p^-(A) \cup T_p^-(B)$.

Take any $x \in T_p^-(A \cup B)$. Then, there exists $C \in T(x)$ such that $C \cap (A \cup B) \neq \emptyset$. Observe that $C \cap (A \cup B) = (C \cap A) \cup (C \cap B)$. Since this union is non-empty, we have $C \cap A \neq \emptyset$ or $C \cap B \neq \emptyset$ (or both). If $C \cap A \neq \emptyset$, then $x \in T_p^-(A)$; if $C \cap B \neq \emptyset$, then $x \in T_p^-(B)$. Thus, $x \in T_p^-(A) \cup T_p^-(B)$, and consequently, $T_p^-(A \cup B) \subseteq T_p^-(A) \cup T_p^-(B)$.

Next, we prove $T_p^-(A) \cup T_p^-(B) \subseteq T_p^-(A \cup B)$.

Take any $x \in T_p^-(A) \cup T_p^-(B)$. Then either $x \in T_p^-(A)$ or $x \in T_p^-(B)$.

If $x \in T_p^-(A)$, there exists $C \in T(x)$ such that $C \cap A \neq \emptyset$. Since $A \subseteq A \cup B$, we have $C \cap A \subseteq C \cap (A \cup B)$ and $C \cap A \neq \emptyset$, implying $C \cap (A \cup B) \neq \emptyset$. Hence, $x \in T_p^-(A \cup B)$.

If $x \in T_p^-(B)$, a symmetric argument yields $x \in T_p^-(A \cup B)$.
Therefore, $T_p^-(A) \cup T_p^-(B) \subseteq T_p^-(A \cup B)$.
Combining the two inclusions gives $T_p^-(A \cup B) = T_p^-(A) \cup T_p^-(B)$.

Example 3. We use Example 2 as the instance to verify Proposition 3. The settings are as follows: $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$, $T(x_1) = \{C_{11}, C_{12}\}$ with $C_{11} = \{y_1\}$, $C_{12} = \{y_2\}$. $T(x_2) = \{C_{21}\}$ with $C_{21} = \{y_1, y_2\}$, $A = \{y_1\}$, $B = \{y_1, y_2\}$, then $A \cap B = \{y_1\}$ and $A \cup B = \{y_1, y_2\}$.

Solution: (1) First, compute $T_p^+(A \cap B) = T_p^+(\{y_1\})$.

For x_1: $T(x_1) = \{\{y_1\}, \{y_2\}\}$. We need $\forall C \in T(x_1), C \subseteq \{y_1\}$. $\{y_1\} \subseteq \{y_1\}$ holds, but $\{y_2\} \nsubseteq \{y_1\}$. Hence, $x_1 \notin T_p^+(\{y_1\})$.

For x_2: $T(x_2) = \{\{y_1, y_2\}\}$. We need $\{y_1, y_2\} \subseteq \{y_1\}$, which is false. So, $x_2 \notin T_p^+(\{y_1\})$.
Thus, $T_p^+(A \cap B) = \emptyset$.

Now, compute $T_p^+(A)$ and $T_p^+(B)$.
$T_p^+(A) = T_p^+(\{y_1\})$: As above, $\emptyset$.
$T_p^+(B) = T_p^+(\{y_1, y_2\})$: For x_1: $\{y_1\} \subseteq B$ and $\{y_2\} \subseteq B$ hold, so $x_1 \in T_p^+(B)$.
For x_2: $\{y_1, y_2\} \subseteq B$ holds, so $x_2 \in T_p^+(B)$.
Hence, $T_p^+(B) = \{x_1, x_2\}$.
Therefore, $T_p^+(A) \cap T_p^+(B) = \emptyset \cap \{x_1, x_2\} = \emptyset$.
So, $T_p^+(A \cap B) = \emptyset = T_p^+(A) \cap T_p^+(B)$, confirming the equality.

First, compute $T_p^-(A \cup B) = T_p^-(\{y_1, y_2\})$.
For x_1: $T(x_1) = \{\{y_1\}, \{y_2\}\}$. We need $\exists C \in T(x_1)$ with $C \cap \{y_1, y_2\} \neq \emptyset$.
$\{y_1\} \cap \{y_1, y_2\} = \{y_1\} \neq \emptyset$, so $x_1 \in T_p^-(A \cup B)$.
For x_2: $T(x_2) = \{\{y_1, y_2\}\}$. $\{y_1, y_2\} \cap \{y_1, y_2\} = \{y_1, y_2\} \neq \emptyset$, so $x_2 \in T_p^-(A \cup B)$.
Thus, $T_p^-(A \cup B) = \{x_1, x_2\}$.
Now, compute $T_p^-(A)$ and $T_p^-(B)$.
$T_p^-(A) = T_p^-(\{y_1\})$: For x_1: $\{y_1\} \cap \{y_1\} \neq \emptyset \rightarrow x_1 \in T_p^-(A)$.
For x_2: $\{y_1, y_2\} \cap \{y_1\} \neq \emptyset \rightarrow x_2 \in T_p^-(A)$.
Hence, $T_p^-(A) = \{x_1, x_2\}$.
$T_p^-(B) = T_p^-(\{y_1, y_2\})$: As computed for $A \cup B$, both x_1 and x_2 belong, so $T_p^-(B) = \{x_1, x_2\}$.
Therefore, $T_p^-(A) \cup T_p^-(B) = \{x_1, x_2\} \cup \{x_1, x_2\} = \{x_1, x_2\}$.
Thus, $T_p^-(A \cup B) = \{x_1, x_2\} = T_p^-(A) \cup T_p^-(B)$, confirming the equality.

Proposition 4. Let (X, Y, T) be a multi-granulation T-rough set approximation space, and let $A, B \subseteq Y$. Then,

$$T_p^+(A \cup B) \supseteq T_p^+(A) \cup T_p^+(B), \tag{3.11}$$

and

$$T_p^-(A \cap B) \subseteq T_p^-(A) \cap T_p^-(B). \tag{3.12}$$

Remark. For T_p^+ under union, the statement is an inclusion, not an equality, owing to the universal quantifier: To require that every granular level be contained in $A \cup B$ is weaker than to require, separately, that every such level be contained in A or in B. The right-hand side, accordingly, may be strictly smaller. By contrast, for T_p^- under intersection, the inclusion is immediate: If some granular level intersects $A \cap B$, then that same level necessarily intersects both A and B. The converse, however, can fail, since a level intersecting A and a level intersecting B need not coincide.

Proof: (1) Take any $x \in T_p^+(A) \cup T_p^+(B)$. Then, either $x \in T_p^+(A)$ or $x \in T_p^+(B)$.

If $x \in T_p^+(A)$, then by Eq (3.1), for every $C \in T(x)$, we have $C \subseteq A$. Since $A \subseteq A \cup B$, it follows that $C \subseteq A \cup B$. Hence, $x \in T_p^+(A \cup B)$.

If $x \in T_p^+(B)$, then for every $C \in T(x)$, we have $C \subseteq B$. Since $B \subseteq A \cup B$, it follows that $C \subseteq A \cup B$. Hence, $x \in T_p^+(A \cup B)$.

Thus, $T_p^+(A) \cup T_p^+(B) \subseteq T_p^+(A \cup B)$, i.e., $T_p^+(A \cup B) \supseteq T_p^+(A) \cup T_p^+(B)$.

(2) Take any $x \in T_p^-(A \cap B)$. By Eq (3.2), there exists some $C \in T(x)$ such that $C \cap (A \cap B) \neq \emptyset$. Observe that $C \cap (A \cap B) = (C \cap A) \cap (C \cap B)$. Since this intersection is non-empty, we have $C \cap A \neq \emptyset$ and $C \cap B \neq \emptyset$. Consequently, $x \in T_p^-(A)$ and $x \in T_p^-(B)$, which means $x \in T_p^-(A) \cap T_p^-(B)$. Hence, $T_p^-(A \cap B) \subseteq T_p^-(A) \cap T_p^-(B)$.

Example 4. We use Example 2 as the instance to verify Proposition 3. The settings are as follows: $A = \{y_1\}$, $B = \{y_1, y_2\}$. Then, $A \cup B = \{y_1, y_2\}$ and $A \cap B = \{y_1\}$.

Solution: (1) First, compute the relevant values.

$$T_p^+(A) = T_p^+(\{y_1\}).$$

For x_1 : $T(x_1) = \{\{y_1\}, \{y_2\}\}$. We need $\forall C \in T(x_1), C \subseteq \{y_1\}$. $\{y_1\} \subseteq \{y_1\}$ holds, but $\{y_2\} \nsubseteq \{y_1\}$. So, $x_1 \notin T_p^+(A)$.

For x_2: $T(x_2) = \{\{y_1, y_2\}\}$. We need $\{y_1, y_2\} \subseteq \{y_1\}$, which is false. Hence, $x_2 \notin T_p^+(A)$. Thus, $T_p^+(A) = \emptyset$.

$$T_p^+(B) = T_p^+(\{y_1, y_2\}).$$

For x_1: $\{y_1\} \subseteq B$ and $\{y_2\} \subseteq B$ hold, so $x_1 \in T_p^+(B)$.

For x_2: $\{y_1, y_2\} \subseteq B$ holds, so $x_2 \in T_p^+(B)$.

Hence, $T_p^+(B) = \{x_1, x_2\}$.

So, $T_p^+(A) \cup T_p^+(B) = \{x_1, x_2\}$.

Now, compute $T_p^+(A \cup B) = T_p^+(\{y_1, y_2\})$. As just computed, it equals $\{x_1, x_2\}$.

Therefore, $T_p^+(A \cup B) = \{x_1, x_2\} \supseteq \{x_1, x_2\}$, i.e., the inclusion holds with equality.

(2) First, compute the relevant values.

$$T_p^-(A \cap B) = T_p^-(\{y_1\}).$$

For x_1: $\{y_1\} \cap \{y_1\} = \{y_1\} \neq \emptyset$, so $x_1 \in T_p^-(\{y_1\})$.

For x_2: $\{y_1, y_2\} \cap \{y_1\} = \{y_1\} \neq \emptyset$, so $x_2 \in T_p^-(\{y_1\})$.

Thus, $T_p^-(A \cap B) = \{x_1, x_2\}$.

$T_p^-(A) = T_p^-(\{y_1\})$: Same as above, $\{x_1, x_1\}$.

$$T_p^-(B) = T_p^-(\{y_1, y_2\}).$$

For x_1: $\{y_1\} \cap B \neq \emptyset$ and $\{y_2\} \cap B \neq \emptyset$, so $x_1 \in T_p^-(B)$.

For x_2: $\{y_1, y_2\} \cap B \neq \emptyset$, so $x_2 \in T_p^-(B)$.

Hence, $T_p^-(B) = \{x_1, x_2\}$.

Then, $T_p^-(A) \cap T_p^-(B) = \{x_1, x_2\} \cap \{x_1, x_2\} = \{x_1, x_2\}$.

Thus, $T_p^-(A \cap B) = \{x_1, x_2\} \subseteq \{x_1, x_2\}$, again holding with equality.

Proposition 5. Let (X, Y, T) be a multi-granulation T-rough set approximation space. For any $A \subseteq Y$, the following equalities hold:

$$T_p^+(A) = (T_p^-(A^c))^c, \tag{3.13}$$

and

$$T_p^-(A) = (T_p^+(A^c))^c, \tag{3.14}$$

where $A^c = Y \setminus A$ denotes the complement of A in Y.

Proof: (1) We first show $T_p^+(A) \subseteq (T_p^-(A^c))^c$.

Take any $x \in T_p^+(A)$. By Eq (3.1), for every $C \in T(x)$, we have $C \subseteq A$.

Assume, for contradiction, that $x \notin (T_p^-(A^c))^c$. Then, $x \in T_p^-(A^c)$, which means there exists some $C_0 \in T(x)$ such that $C_0 \cap A^c \neq \emptyset$. Since $C_0 \subseteq A$, we have $C_0 \cap A^c = \emptyset$, a contradiction. Hence, $x \notin T_p^-(A^c)$, i.e., $x \in (T_p^-(A^c))^c$. Thus, $T_p^+(A) \subseteq (T_p^-(A^c))^c$.

Next, we prove $\left(T_p^-(A^c)\right)^c \subseteq T_p^+(A)$.

Let $x \in (T_p^-(A^c))^c$. Then, $x \notin T_p^-(A^c)$, which means that for every $C \in T(x)$, we have $C \cap A^c = \emptyset$.

We claim that for every $C \in T(x)$, $C \subseteq A$. Suppose, to the contrary, that there exists $C_0 \in T(x)$ with $C_0 \nsubseteq A$. Then there exists $y \in C_0$ such that $y \notin A$, i.e., $y \in A^c$. Consequently, $y \in C_0 \cap A^c$, implying $C_0 \cap A^c \neq \emptyset$, contradicting the fact that $C \cap A^c = \emptyset, \forall C \in T(x)$. Hence, $C \subseteq A$ holds for every $C \in T(x)$. By Eq (3.2), $x \in T_p^+(A)$. Therefore, $\left(T_p^-(A^c)\right)^c \subseteq T_p^+(A)$.

Combining both inclusions yields $T_p^+(A) = (T_p^-(A^c))^c$.

(2) We first prove $T_p^-(A) \subseteq (T_p^+(A^c))^c$.

Take any $x \in T_p^-(A)$. By (3.2), there exists $C_0 \in T(x)$ such that $C_0 \cap A \neq \emptyset$.

Assume, for contradiction, that $x \notin (T_p^+(A^c))^c$. Then, $x \in T_p^+(A^c)$, which means that for every $C \in T(x)$, we have $C \subseteq A^c$. In particular, $C_0 \subseteq A^c$, which implies $C_0 \cap A = \emptyset$, contradicting $C_0 \cap A \neq \emptyset$. Hence, $x \notin T_p^+(A^c)$, i.e., $x \in (T_p^+(A^c))^c$. Thus, $T_p^-(A) \subseteq (T_p^+(A^c))^c$.

Next, we prove $\left(T_p^+(A^c)\right)^c \subseteq T_p^-(A)$.

Let $x \in (T_p^+(A^c))^c$. Then, $x \notin T_p^+(A^c)$, which means that there exists some $C_0 \in T(x)$ such that $C_0 \nsubseteq A^c$.

Since $C_0 \nsubseteq A^c$, there exists $y \in C_0$ with $y \notin A^c$, i.e., $y \in A$. Therefore, $y \in C_0 \cap A$, and thus $C_0 \cap A \neq \emptyset$. By (3.2), $x \in T_p^-(A)$. Hence, $\left(T_p^+(A^c)\right)^c \subseteq T_p^-(A)$.

Combining both inclusions gives $T_p^-(A) = (T_p^+(A^c))^c$.

Example 5. Let $X = \{x_1, x_2, x_3\}, Y = \{y_1, y_2, y_3, y_4\}$, and define the set-valued multi-mapping $T: X \to \mathcal{P}^*\left(\mathcal{P}^*(Y)\right)$ as follows:

$$T(x_1) = \{C_{11}, C_{12}\}, C_{11} = \{y_1, y_2\}, C_{12} = \{y_3\},$$

$$T(x_2) = \{C_{21}, C_{22}\}, C_{21} = \{y_1\}, C_{22} = \{y_2, y_4\},$$

$T(x_3) = \{C_{31}\}, C_{31} = \{y_3, y_4\}$.

Let $A = \{y_1, y_2, y_3\}$. Then, $A^c = Y \setminus A = \{y_4\}$.

Solution: (1) For x_1: $C_{11} = \{y_1, y_2\} \subseteq A$ and $C_{12} = \{y_3\} \subseteq A$. All $C \in T(x_1)$ satisfy $C \subseteq A$, so $x_1 \in T_p^+(A)$.

For x_2: $C_{21} = \{y_1\} \subseteq A$ but $C_{22} = \{y_2, y_4\} \nsubseteq A$, so $x_2 \notin T_p^+(A)$.

For x_3: $C_{31} = \{y_3, y_4\} \nsubseteq A$, so $x_3 \notin T_p^+(A)$.

Thus, $T_p^+(A) = \{x_1\}$.

Since $A^c = \{y_4\}$, for x_1: $C_{11} \cap A^c = \emptyset$ and $C_{12} \cap A^c = \emptyset$, so no $C \in T(x_1)$ satisfies $C \cap A^c \neq \emptyset$. Hence, $x_1 \notin T_p^-(A^c)$.

For x_2: $C_{21} \cap A^c = \emptyset$ but $C_{22} \cap A^c = \{y_4\} \neq \emptyset$, so $x_2 \in T_p^-(A^c)$.

For x_3: $C_{31} \cap A^c = \{y_4\} \neq \emptyset$, so $x_3 \in T_p^-(A^c)$.

Therefore, $T_p^-(A^c) = \{x_2, x_3\}$, and consequently, $(T_p^-(A^c))^c = X \setminus \{x_2, x_3\} = \{x_1\}$.

Hence, $T_p^+(A) = \{x_1\} = (T_p^-(A^c))^c$.

(2) For x_1: $C_{11} \cap A = \{y_1, y_2\} \neq \emptyset$, so $x_1 \in T_p^-(A)$.

For x_2: $C_{21} \cap A = \{y_1\} \neq \emptyset$, so $x_2 \in T_p^-(A)$.
For x_3: $C_{31} \cap A = \{y_3\} \neq \emptyset$, so $x_3 \in T_p^-(A)$.
Thus, $T_p^-(A) = \{x_1, x_2, x_3\}$.
For $A^c = \{y_4\}$:
x_1: $C_{11} \nsubseteq A^c$ and $C_{12} \nsubseteq A^c$, so $x_1 \notin T_p^+(A^c)$.
x_2: $C_{21} \nsubseteq A^c$ and $C_{22} \nsubseteq A^c$, so $x_2 \notin T_p^+(A^c)$.
x_3: $C_{31} \nsubseteq A^c$, so $x_3 \notin T_p^+(A^c)$.
Hence, $T_p^+(A^c) = \emptyset$ and $(T_p^+(A^c))^c = X \setminus \emptyset = \{x_1, x_2, x_3\}$.
Therefore, $T_p^-(A) = \{x_1, x_2, x_3\} = (T_p^+(A^c))^c$.

This example clearly illustrates the duality property of the pessimistic operators in multi-granulation T-rough sets, confirms the validity of Proposition 5, and offers an intuitive explanation for understanding the symmetric structure of multi-granulation approximation operators.

4. Optimistic inverse operators for multi-granulation T-rough sets

4.1. Definition of optimistic inverse operators

Definition 4. Let X and Y be two non-empty finite sets, and let $T: X \to \mathcal{P}^*(\mathcal{P}^*(Y))$ be a set-valued multi-mapping. For any $A \subseteq Y$, the multi-granulation optimistic upper inverse and the multi-granulation optimistic lower inverse of A with respect to T are defined respectively as:

$$T_o^+(A) = \{x \in X | \exists B \in T(x), B \subseteq A\}, \tag{4.1}$$

and

$$T_o^-(A) = \{x \in X | \forall B \in T(x), B \cap A \neq \emptyset\}. \tag{4.2}$$

The ordered pair $(T_o^+(A), T_o^-(A))$ is called the optimistic upper inverse operator and the optimistic lower inverse operator for multi-granulation T-rough sets. At an intuitive level, the optimistic upper operator is built around an existential quantifier ($\exists B \in T(x)$), thereby widening the admission criterion: An object is retained whenever inclusion is satisfied at any granularity level. By contrast, the optimistic lower operator rests on a universal quantifier ($\forall B \in T(x)$), a stricter evidentiary requirement in effect, under which an object is admitted only when all granularity levels intersect with the target.

Optimistic inverse operators pair up with pessimistic inverse operators in a clean "two sides of the same coin" way. That pairing follows the rough–fuzzy set duality idea Dubois and Prade described [4]. The optimistic upper inverse operator loosens the bar for granular consistency by using an existential quantifier. In plain terms, it's willing to accept an object as "possible" if it fits somewhere, which pulls more candidates into the approximation. That is handy when you actually want a wider net, such as flagging potential students in knowledge spaces [25]. The optimistic lower inverse operator goes the other direction: It tightens granular consistency through a universal quantifier. An object must satisfy the condition across the board, which usually means fewer results, but ones you can trust more. That's why it fits cases where credibility matters, such as consensus recognition in multi-perspective decision making [10,37].

This dual setup lets the multi-granulation T-rough set model mix and match approximation operators based on what the problem needs, instead of forcing one rigid choice [5,19]. And if you fold in Zhang's "gridding" idea for fuzzy knowledge structures [27], the same operator system can be pushed into fuzzy multi-granulation settings too, which is where things get messy in real data. For computing these inverse operators, the matrix approach introduced by Huang and co-authors et al. [26] is a practical route.

4.2. Properties of optimistic inverse operators

Proposition 6. Let (X, Y, T) be a multi-granulation T-rough set approximation space, and let $A, B \subseteq Y$. Then

$$T_o^+(\emptyset) = \emptyset, \tag{4.3}$$

and

$$T_o^-(Y) = X. \tag{4.4}$$

Proof: (1) Suppose, for contradiction, that there exists $x \in T_o^+(\emptyset)$. By Eq (4.1), this means there exists $C \in T(x)$ such that $C \subseteq \emptyset$. However, since $T(x) \subseteq \mathcal{P}^*(P^*(Y))$, every $C \in T(x)$ is a non-empty subset of Y. Hence, $C \neq \emptyset$, which contradicts $C \subseteq \emptyset$. Therefore, no such x exists, and we conclude $T_o^+(\emptyset) = \emptyset$.

(2) Let $x \in X$ be arbitrary. For each $C \in T(x)$, because C is a non-empty subset of Y, we have $C \cap Y = C \neq \emptyset$. Thus, by Eq (4.2), $x \in T_o^-(Y)$. This shows $X \subseteq T_o^-(Y)$. The reverse inclusion $T_o^-(Y) \subseteq X$ follows directly from the definition of $T_o^-(Y)$. Hence, $T_o^-(Y) = X$.

Example 6. Let $X = \{x_1, x_2, x_3\}, Y = \{y_1, y_2, y_3, y_4\}$, and define the set-valued multi-mapping $T: X \to \mathcal{P}^*\big(P^*(Y)\big)$ as follows:

$$T(x_1) = \{C_{11}, C_{12}\}, C_{11} = \{y_1, y_2\}, C_{12} = \{y_3\},$$

$$T(x_2) = \{C_{21}\}, C_{21} = \{y_1, y_4\},$$

$$T(x_3) = \{C_{31}, C_{32}, C_{33}\}, C_{31} = \{y_2\}, C_{32} = \{y_3\}, C_{33} = \{y_4\}.$$

Solution: (1) For x_1: $C_{11} = \{y_1, y_2\} \nsubseteq \emptyset$ and $C_{12} = \{y_3\} \nsubseteq \emptyset$. Hence, $x_1 \notin T_o^+(\emptyset)$.
For x_2: $C_{21} = \{y_1, y_4\} \nsubseteq \emptyset$, so $x_2 \notin T_o^+(\emptyset)$.
For x_3: $C_{31} = \{y_2\} \nsubseteq \emptyset$, $C_{32} = \{y_3\} \nsubseteq \emptyset$, $C_{33} = \{y_4\} \nsubseteq \emptyset$, so $x_3 \notin T_o^+(\emptyset)$. Thus, $T_o^+(\emptyset) = \emptyset$.
(2) For x_1: $C_{11} \cap Y = \{y_1, y_2\} \neq \emptyset$ and $C_{12} \cap Y = \{y_3\} \neq \emptyset$. Thus, $x_1 \in T_o^-(Y)$.
For x_2: $C_{21} \cap Y = \{y_1, y_4\} \neq \emptyset$, so $x_2 \in T_o^-(Y)$.
For x_3: $C_{31} \cap Y = \{y_2\} \neq \emptyset$, $C_{32} \cap Y = \{y_3\} \neq \emptyset$, $C_{33} \cap Y = \{y_4\} \neq \emptyset$, so $x_3 \in T_o^-(Y)$.
Therefore, $T_o^-(Y) = \{x_1, x_2, x_3\} = X$.

Proposition 7. Let (X, Y, T) be a multi-granulation T-rough set approximation space, and let $A, B \subseteq Y$. If $A \subseteq B$, then

$$T_o^+(A) \subseteq T_o^+(B), \tag{4.5}$$

and

$$T_o^-(A) \subseteq T_o^-(B). \tag{4.6}$$

Note that for every $x \in X$, $T(x)$ is non-empty because the codomain of T is $\mathcal{P}^*(\mathcal{P}^*(Y))$, which excludes the empty family.

Proof: Assume $A \subseteq B$.

(1) Take any $x \in T_o^+(A)$. By Eq (4.1), there exists some $C \in T(x)$ such that $C \subseteq A$. Since

$A \subseteq B$, we have $C \subseteq B$. Hence, $x \in T_o^+(B)$ by Eq (4.1). Therefore, $T_o^+(A) \subseteq T_o^+(B)$.

(2) Take any $x \in T_o^-(A)$. By Eq (4.2), for every $C \in T(x)$, we have $C \cap A \neq \emptyset$. Because $A \subseteq B$, it follows that $C \cap A \subseteq C \cap B$ and $C \cap A \neq \emptyset$. Thus, $C \cap B \neq \emptyset$, $\forall C \in T(x)$. Consequently, $x \in T_o^-(B)$ by Eq (4.2). Hence, $T_o^-(A) \subseteq T_o^-(B)$.

Example 7. Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3, y_4\}$, and define the set-valued multi-mapping $T: X \to \mathcal{P}^*(\mathcal{P}^*(Y))$ as follows:

$$T(x_1) = \{C_{11}, C_{12}\}, C_{11} = \{y_1, y_2\}, C_{12} = \{y_3\},$$

$$T(x_2) = \{C_{21}\}, C_{21} = \{y_1, y_4\},$$

$$T(x_3) = \{C_{31}, C_{32}, C_{33}\}, C_{31} = \{y_2\}, C_{32} = \{y_3\}, C_{33} = \{y_4\}.$$

Take $A = \{y_1\}$ and $B = \{y_1, y_2\}$. Clearly, $A \subseteq B$.

Solution: (1) Compute $T_o^+(A) = \{x \in X \mid \exists C \in T(x), C \subseteq A\}$.

For x_1: $C_{11} = \{y_1, y_2\} \nsubseteq \{y_1\}$, $C_{12} = \{y_3\} \nsubseteq \{y_1\}$; no C satisfies $C \subseteq A$, so $x_1 \notin T_o^+(A)$.

For x_2: $C_{21} = \{y_1, y_4\} \nsubseteq \{y_1\}$; no, so $x_2 \notin T_o^+(A)$.

For x_3: $C_{31} = \{y_2\} \nsubseteq \{y_1\}$, $C_{32} = \{y_3\} \nsubseteq \{y_1\}$, $C_{33} = \{y_4\} \nsubseteq \{y_1\}$; none. Hence, $T_o^+(A) = \emptyset$.

Compute $T_o^+(B) = \{x \in X \mid \exists C \in T(x), C \subseteq B\}$ with $B = \{y_1, y_2\}$.

For x_1: $C_{11} = \{y_1, y_2\} \subseteq B$, so $x_1 \in T_o^+(B)$.

For x_2: $C_{21} = \{y_1, y_4\} \nsubseteq B$, so $x_2 \notin T_o^+(B)$.

For x_3: $C_{31} = \{y_2\} \subseteq B$, so $x_3 \in T_o^+(B)$.

Thus, $T_o^+(B) = \{x_1, x_3\}$.

Clearly, $\emptyset \subseteq \{x_1, x_3\}$, so $T_o^+(A) \subseteq T_o^+(B)$ holds.

(2) Compute $T_o^-(A) = \{x \in X \mid \forall C \in T(x), C \cap A \neq \emptyset\}$ with $A = \{y_1\}$.

For x_1: $C_{11} \cap A = \{y_1, y_2\} \cap \{y_1\} = \{y_1\} \neq \emptyset$; $C_{12} \cap A = \{y_3\} \cap \{y_1\} = \emptyset$. Since not all C intersect A, $x_1 \notin T_o^-(A)$.

For x_2: $C_{21} \cap A = \{y_1, y_4\} \cap \{y_1\} = \{y_1\} \neq \emptyset$; only one C, so the condition holds. Hence, $x_2 \in T_o^-(A)$.

For x_3: $C_{31} \cap A = \{y_2\} \cap \{y_1\} = \emptyset$; fails. So, $x_3 \notin T_o^-(A)$.

Thus, $T_o^-(A) = \{x_2\}$.

Compute $T_o^-(B) = \{x \in X \mid \forall C \in T(x), C \cap B \neq \emptyset\}$ with $B = \{y_1, y_2\}$.

For x_1: $C_{11} \cap B = \{y_1, y_2\} \cap \{y_1, y_2\} = \{y_1, y_2\} \neq \emptyset$; $C_{12} \cap B = \{y_3\} \cap \{y_1, y_2\} = \emptyset$. Fails, so $x_1 \notin T_o^-(B)$.

For x_2: $C_{21} \cap B = \{y_1, y_4\} \cap \{y_1, y_2\} = \{y_1\} \neq \emptyset$; holds, so $x_2 \in T_o^-(B)$.

For x_3: $C_{31} \cap B = \{y_2\} \cap \{y_1, y_2\} = \{y_2\} \neq \emptyset$; $C_{32} \cap B = \{y_3\} \cap \{y_1, y_2\} = \emptyset$; fails. So $x_3 \notin T_o^-(B)$.

Hence, $T_o^-(B) = \{x_2\}$.

We have $T_o^-(A) = \{x_2\} \subseteq \{x_2\} = T_o^-(B)$, so the inclusion holds.

Example 8. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$, and define the set-valued multi-mapping $T: X \to \mathcal{P}^*(\mathcal{P}^*(Y))$ as follows:

$T(x_1) = \{C_{11}\}$ with $C_{11} = \{y_1, y_2\}$;

$T(x_2) = \{C_{21}, C_{22}\}$ with $C_{21} = \{y_1\}$, $C_{22} = \{y_3\}$.

Take $A = \{y_1\}$ and $B = \{y_1, y_2\}$. Clearly, $A \subseteq B$.

Solution: First, compute $T_o^+(A) = \{x \in X \mid \exists C \in T(x), C \subseteq A\}$.

For x_1: $C_{11} = \{y_1, y_2\} \nsubseteq A$, so $x_1 \notin T_o^+(A)$.

For x_2: $C_{21} = \{y_1\} \subseteq A$ (while $C_{22} = \{y_3\} \nsubseteq A$), so there exists a subset C_{21} satisfying the condition. Hence, $x_2 \in T_o^+(A)$.

Thus, $T_o^+(A) = \{x_2\}$.

Next, compute $T_o^+(B) = \{x \in X \mid \exists C \in T(x), C \subseteq B\}$.

For x_1: $C_{11} = \{y_1, y_2\} \subseteq B$, so $x_1 \in T_o^+(B)$.

For x_2: $C_{21} = \{y_1\} \subseteq B$ (while $C_{22} = \{y_3\} \nsubseteq B$), so $x_2 \in T_o^+(B)$.

Hence, $T_o^+(B) = \{x_1, x_2\}$.

Therefore, $T_o^+(A) = \{x_2\} \subseteq \{x_1, x_2\} = T_o^+(B)$, i.e., the inclusion $T_o^+(A) \subseteq T_o^+(B)$ holds. However, $T_o^+(B) \nsubseteq T_o^+(A)$, showing that the reverse inclusion does not necessarily hold.

Example 9. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3, y_4\}$, and define the set-valued multi-mapping $T: X \to \mathcal{P}^*(\mathcal{P}^*(Y))$ as follows:

$T(x_1) = \{C_{11}, C_{12}\}$ with $C_{11} = \{y_1, y_3\}$, $C_{12} = \{y_2, y_4\}$;

$T(x_2) = \{C_{21}, C_{22}\}$ with $C_{21} = \{y_1, y_2\}$, $C_{22} = \{y_3, y_4\}$.

Take $A = \{y_1, y_2\}$ and $B = \{y_1, y_2, y_3\}$. Clearly, $A \subseteq B$.

Solution: Compute $T_o^-(A) = \{x \in X \mid \forall C \in T(x), C \cap A \neq \emptyset\}$.

For x_1: $C_{11} \cap A = \{y_1, y_3\} \cap \{y_1, y_2\} = \{y_1\} \neq \emptyset$;

$C_{12} \cap A = \{y_2, y_4\} \cap \{y_1, y_2\} = \{y_2\} \neq \emptyset$. Both subsets intersect A, so $x_1 \in T_o^-(A)$.

For x_2: $C_{21} \cap A = \{y_1, y_2\} \cap \{y_1, y_2\} = \{y_1, y_2\} \neq \emptyset$; $C_{22} \cap A = \{y_3, y_4\} \cap \{y_1, y_2\} = \emptyset$. Since not all subsets intersect A, $x_2 \notin T_o^-(A)$.

Thus, $T_o^-(A) = \{x_1\}$.

Now, compute $T_o^-(B) = \{x \in X \mid \forall C \in T(x), C \cap B \neq \emptyset\}$.

For x_1: $C_{11} \cap B = \{y_1, y_3\} \cap \{y_1, y_2, y_3\} = \{y_1, y_3\} \neq \emptyset$; $C_{12} \cap B = \{y_2, y_4\} \cap \{y_1, y_2, y_3\} = \{y_2\} \neq \emptyset$. Hence, $x_1 \in T_o^-(B)$.

For x_2: $C_{21} \cap B = \{y_1, y_2\} \cap B = \{y_1, y_2\} \neq \emptyset$; $C_{22} \cap B = \{y_3, y_4\} \cap B = \{y_3\} \neq \emptyset$. Hence, $x_2 \in T_o^-(B)$.

Therefore, $T_o^-(B) = \{x_1, x_2\}$.

Consequently, $T_o^-(A) = \{x_1\} \subseteq \{x_1, x_2\} = T_o^-(B)$, i.e., the inclusion $T_o^-(A) \subseteq T_o^-(B)$ holds. However, $T_o^-(B) \nsubseteq T_o^-(A)$, showing that the reverse inclusion does not necessarily hold.

Proposition 8. Let (X, Y, T) be a multi-granulation T-rough set approximation space, and let $A, B \subseteq Y$. Then

$$T_o^+(A \cup B) \supseteq T_o^+(A) \cup T_o^+(B), \tag{4.7}$$

and

$$T_o^-(A \cap B) \subseteq T_o^-(A) \cap T_o^-(B). \tag{4.8}$$

Remark. For optimistic operators, T_o^+ is governed by an existential quantifier. Membership in $T_o^+(A \cup B)$ is therefore easier to satisfy since it requires only that one granular level be contained in the union; correspondingly, the left-hand side is larger. Equality, by contrast, would requires that the same granular level work for A or B (and that alignment is not assured). For T_o^-, the universal quantifier tightens the requirement: When every level satisfies $A \cap B$, every level necessarily satisfies both A and B, which yields the inclusion $T_o^-(A \cap B) \subseteq T_o^-(A) \cap T_o^-(B)$; the converse, however, can fail because a given level may satisfy A while failing to satisfy B.

Proof: (1) Let $x \in T_o^+(A) \cup T_o^+(B)$. Then either $x \in T_o^+(A)$ or $x \in T_o^+(B)$.

If $x \in T_o^+(A)$, by Eq (4.1) there exists $C \in T(x)$ such that $C \subseteq A$. Since $A \subseteq A \cup B$, we have

$C \subseteq A \cup B$, and thus $x \in T_o^+(A \cup B)$.

If $x \in T_o^+(B)$, there exists $C \in T(x)$ with $C \subseteq B \subseteq A \cup B$, so again $x \in T_o^+(A \cup B)$.

Hence, $T_o^+(A) \cup T_o^+(B) \subseteq T_o^+(A \cup B)$, i.e., $T_o^+(A \cup B) \supseteq T_o^+(A) \cup T_o^+(B)$.

(2) Take any $x \in T_o^-(A \cap B)$. By Eq (4.2), for every $C \in T(x)$, we have $C \cap (A \cap B) \neq \emptyset$. Observe that

$$C \cap (A \cap B) = (C \cap A) \cap (C \cap B).$$

Since the left-hand side is non-empty, both $C \cap A$ and $C \cap B$ must be non-empty. Thus, $\forall C \in T(x)$, $C \cap A \neq \emptyset$, and $C \cap B \neq \emptyset$. Consequently, $x \in T_o^-(A)$ and $x \in T_o^-(B)$, i.e., $x \in T_o^-(A) \cap T_o^-(B)$. Therefore, $T_o^-(A \cap B) \subseteq T_o^-(A) \cap T_o^-(B)$.

Example 10. Let $X = \{x\}$, $Y = \{y_1, y_2, y_3\}$, and define $T(x) = \{C\}$ with $C = \{y_1, y_2\}$. Take $A = \{y_1\}$ and $B = \{y_2\}$.

Solution: $T_o^+(A) = \{x \in X \mid \exists C \in T(x), C \subseteq A\}$. Since $C = \{y_1, y_2\} \nsubseteq \{y_1\} = A$ (because $y_2 \notin A$), we have $x \notin T_o^+(A)$, so $T_o^+(A) = \emptyset$.

$T_o^+(B) = \{x \in X \mid \exists C \in T(x), C \subseteq B\}$. Since $C = \{y_1, y_2\} \nsubseteq \{y_2\} = B$ (because $y_1 \notin B$), we have $x \notin T_o^+(B)$, so $T_o^+(B) = \emptyset$.

Thus, $T_o^+(A) \cup T_o^+(B) = \emptyset \cup \emptyset = \emptyset$.

Now, $A \cup B = \{y_1, y_2\}$. Then, $T_o^+(A \cup B) = \{x \in X \mid \exists C \in T(x), C \subseteq A \cup B\}$. Since $C = \{y_1, y_2\} \subseteq A \cup B$, we have $x \in T_o^+(A \cup B)$, so $T_o^+(A \cup B) = \{x\}$.

Consequently, $T_o^+(A \cup B) = \{x\}$, while $T_o^+(A) \cup T_o^+(B) = \emptyset$, so the reverse inclusion $T_o^+(A \cup B) \subseteq T_o^+(A) \cup T_o^+(B)$ does not hold. This shows that the inclusion in Proposition 8(1) cannot be strengthened to equality.

Example 11. Let $X = \{x\}$, $Y = \{y_1, y_2\}$, and define $T(x) = \{C\}$ with $C = \{y_1, y_2\}$. Take $A = \{y_1\}$ and $B = \{y_2\}$.

Solution: $T_o^-(A) = \{x \in X \mid \forall C \in T(x), C \cap A \neq \emptyset\}$. Since $C \cap A = \{y_1\} \neq \emptyset$, we have $x \in T_o^-(A)$, so $T_o^-(A) = \{x\}$.

Similarly, $T_o^-(B) = \{x\}$ because $C \cap B = \{y_2\} \neq \emptyset$. Thus, $T_o^-(A) \cap T_o^-(B) = \{x\} \cap \{x\} = \{x\}$.

Now, $A \cap B = \emptyset$. Then $T_o^-(A \cap B) = T_o^-(\emptyset) = \emptyset$ (by Proposition 6(1)).

Therefore, $T_o^-(A) \cap T_o^-(B) = \{x\} \nsubseteq \emptyset = T_o^-(A \cap B)$. Hence, the reverse inclusion $T_o^-(A) \cap T_o^-(B) \subseteq T_o^-(A \cap B)$ does not necessarily hold. This shows that the inclusion in Proposition 8(2) cannot be strengthened to equality.

Proposition 9. Let (X, Y, T) be a multi-granulation T-rough set approximation space, and let $A, B \subseteq Y$. Then

$$T_o^+(A \cap B) \subseteq T_o^+(A) \cap T_o^+(B), \tag{4.9}$$

and

$$T_o^-(A \cup B) \supseteq T_o^-(A) \cup T_o^-(B). \tag{4.10}$$

Remark. For T_o^+ with respect to intersection, the existential quantifier fails to distribute across intersection: Requiring one and the same granular level to be contained in $A \cap B$ is stronger than requiring that there exist (possibly distinct) levels contained in A and B separately, and accordingly the left-hand side is the smaller set. By contrast, for T_o^- over union, the universal quantifier does distribute: Whenever every granular level intersects A or intersects B (or both), every level necessarily intersects $A \cup B$. The converse inclusion, however, does not go through, since a granular level may intersect $A \cup B$ while failing to intersect A on its own and failing also to intersect B on its own.

Proof: (1) Take any $x \in T_o^+(A \cap B)$. By Eq (4.1), there exists $C \in T(x)$ such that $C \subseteq A \cap B$. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, we have $C \subseteq A$ and $C \subseteq B$. Consequently, $x \in T_o^+(A)$ and $x \in T_o^+(B)$, i.e., $x \in T_o^+(A) \cap T_o^+(B)$. Hence, $T_o^+(A \cap B) \subseteq T_o^+(A) \cap T_o^+(B)$.

(2) Take any $x \in T_o^-(A) \cup T_o^-(B)$. Then either $x \in T_o^-(A)$ or $x \in T_o^-(B)$.

If $x \in T_o^-(A)$, then by Eq (4.2), for every $C \in T(x)$, we have $C \cap A \neq \emptyset$. Since $A \subseteq A \cup B$, it follows that $C \cap (A \cup B) \supseteq C \cap A \neq \emptyset$, and thus $C \cap (A \cup B) \neq \emptyset,\ \forall C \in T(x)$. Hence, $x \in T_o^-(A \cup B)$.

If $x \in T_o^-(B)$, a similar argument yields $x \in T_o^-(A \cup B)$.

Therefore, $T_o^-(A) \cup T_o^-(B) \subseteq T_o^-(A \cup B)$, i.e., $T_o^-(A \cup B) \supseteq T_o^-(A) \cup T_o^-(B)$.

Example 12. Let $X = \{x\}$, $Y = \{y_1, y_2\}$, and define the set-valued multi-mapping $T: X \to \mathcal{P}^*(\mathcal{P}^*(Y))$ by $T(x) = \{C_1, C_2\}$ with $C_1 = \{y_1\}$, $C_2 = \{y_2\}$. Take $A = \{y_1\}$ and $B = \{y_2\}$. Then,

Solution. (1) $A \cap B = \emptyset$. By Proposition 6(1), $T_o^+(\emptyset) = \emptyset$.

$T_o^+(A) = \{x \in X \mid \exists C \in T(x), C \subseteq A\} = \{x\}$ (since $C_1 \subseteq A$).

$T_o^+(B) = \{x\}$ (since $C_2 \subseteq B$).

Hence, $T_o^+(A) \cap T_o^+(B) = \{x\}$, and $\emptyset \subseteq \{x\}$, i.e., $T_o^+(A \cap B) \subseteq T_o^+(A) \cap T_o^+(B)$ holds.

(2) $A \cup B = Y$. By Proposition 6(2), $T_o^-(Y) = X = \{x\}$.

$T_o^-(A) = \{x \in X \mid \forall C \in T(x), C \cap A \neq \emptyset\}$.

For C_1, $C_1 \cap A = \{y_1\} \neq \emptyset$; for C_2, $C_2 \cap A = \{y_2\} \cap \{y_1\} = \emptyset$.

Hence, $T_o^-(A) = \emptyset$. Similarly, $T_o^-(B) = \emptyset$.

Thus, $T_o^-(A) \cup T_o^-(B) = \emptyset \subseteq \{x\} = T_o^-(A \cup B)$, i.e., $T_o^-(A \cup B) \supseteq T_o^-(A) \cup T_o^-(B)$ holds.

Proposition 10. (Duality). For any $A \subseteq Y$, we have

$$T_o^+(A) = (T_o^-(A^c))^c, \tag{4.11}$$

and

$$T_o^-(A) = (T_o^+(A^c))^c, \tag{4.12}$$

where $A^c = Y \setminus A$ denotes the complement of A in Y.

Proof: (1) We first prove $T_o^+(A) \subseteq (T_o^-(A^c))^c$.

Let $x \in T_o^+(A)$. By Eq (4.1), there exists $C_0 \in T(x)$ such that $C_0 \subseteq A$.

Assume, for contradiction, that $x \notin (T_o^-(A^c))^c$. Then $x \in T_o^-(A^c)$, which means that for every $C \in T(x)$, we have $C \cap A^c \neq \emptyset$. In particular, for $C_0 \in T(x)$, we have $C_0 \cap A^c \neq \emptyset$. However, since $C_0 \subseteq A$, we have $C_0 \cap A^c = \emptyset$, which is a contradiction. Therefore, $x \notin T_o^-(A^c)$, i.e., $x \in (T_o^-(A^c))^c$. Hence, $T_o^+(A) \subseteq (T_o^-(A^c))^c$.

Next, we prove $(T_o^-(A^c))^c \subseteq T_o^+(A)$.

Let $x \in (T_o^-(A^c))^c$. Then $x \notin T_o^-(A^c)$, which means that it is not true that for every $C \in T(x)$, $C \cap A^c \neq \emptyset$. Hence, there exists some $C_0 \in T(x)$ such that $C_0 \cap A^c = \emptyset$.

We claim that $C_0 \subseteq A$. Suppose it is not, then there exists $y \in C_0$ with $y \notin A$. Then $y \in A^c$, so $y \in C_0 \cap A^c$, contradicting $C_0 \cap A^c = \emptyset$. Thus, $C_0 \subseteq A$. By Eq (4.2), $x \in T_o^+(A)$. Hence, $(T_o^-(A^c))^c \subseteq T_o^+(A)$.

Combining both inclusions, we obtain $T_o^+(A) = (T_o^-(A^c))^c$.

(2) We first prove $T_o^-(A) \subseteq (T_o^+(A^c))^c$.

Let $x \in T_o^-(A)$. By Eq (4.2), for every $C \in T(x)$, we have $C \cap A \neq \emptyset$.

Assume, for contradiction, that $x \notin (T_o^+(A^c))^c$. Then $x \in T_o^+(A^c)$, which means that there exists $C_0 \in T(x)$ such that $C_0 \subseteq A^c$. Then $C_0 \cap A = \emptyset$, contradicting the fact that for every $C \in T(x)$, $C \cap A \neq \emptyset$ (taking $C = C_0$). Therefore, $x \notin T_o^+(A^c)$, i.e., $x \in (T_o^+(A^c))^c$. Hence, $T_o^-(A) \subseteq (T_o^+(A^c))^c$.

Next, we prove $(T_o^+(A^c))^c \subseteq T_o^-(A)$.

Let $x \in (T_o^+(A^c))^c$. Then $x \notin T_o^+(A^c)$, which means that there is no $C \in T(x)$ such that $C \subseteq A^c$. Equivalently, for every $C \in T(x)$, we have $C \nsubseteq A^c$.

Take any $C \in T(x)$. Since $C \nsubseteq A^c$, there exists $y \in C$ such that $y \notin A^c$, i.e., $y \in A$. Hence, $y \in C \cap A$, so $C \cap A \neq \emptyset$. As C was arbitrary, we have $\forall C \in T(x), C \cap A \neq \emptyset$. By Eqs (4.1) and (4.2), this means $x \in T_o^-(A)$. Thus, $(T_o^+(A^c))^c \subseteq T_o^-(A)$.

Combining both inclusions, we obtain $T_o^-(A) = (T_o^+(A^c))^c$.

Example 13. As shown in Example 6, we have

(1) $T_o^+(A) = (T_o^-(A^c))^c$.

Compute $T_o^+(A) = \{x \in X \mid \exists C \in T(x), C \subseteq A\}$.

For x_1: $C_{11} = \{y_1, y_2\} \subseteq A$ and $C_{12} = \{y_3\} \subseteq A$, so $x_1 \in T_o^+(A)$.

For x_2: $C_{21} = \{y_1\} \subseteq A$ (while $C_{22} = \{y_2, y_4\} \nsubseteq A$), thus $x_2 \in T_o^+(A)$.

For x_3: $C_{31} = \{y_3, y_4\} \nsubseteq A$, so $x_3 \notin T_o^+(A)$.

Hence, $T_o^+(A) = \{x_1, x_2\}$.

Compute $T_o^-(A^c) = T_o^-(\{y_4\}) = \{x \in X \mid \forall C \in T(x),\ C \cap \{y_4\} \neq \emptyset\}$.

For x_1: $C_{11} \cap \{y_4\} = \emptyset$ and $C_{12} \cap \{y_4\} = \emptyset$, so $x_1 \notin T_o^-(A^c)$.

For x_2: $C_{21} \cap \{y_4\} = \emptyset$ and $C_{22} \cap \{y_4\} = \{y_4\} \neq \emptyset$, but not all C intersect, so $x_2 \notin T_o^-(A^c)$.

For x_3: $C_{31} \cap \{y_4\} = \{y_4\} \neq \emptyset$, thus $x_3 \in T_o^-(A^c)$.

Therefore, $T_o^-(A^c) = \{x_3\}$, and $(T_o^-(A^c))^c = X \setminus \{x_3\} = \{x_1, x_2\}$.

Thus, $T_o^+(A) = \{x_1, x_2\} = (T_o^-(A^c))^c$, confirming (1).

(2)$T_o^-(A) = (T_o^+(A^c))^c$.

Compute $T_o^-(A) = \{x \in X \mid \forall C \in T(x),\ C \cap A \neq \emptyset\}$.

For x_1: $C_{11} \cap A = \{y_1, y_2\} \neq \emptyset$, $C_{12} \cap A = \{y_3\} \neq \emptyset$, so $x_1 \in T_o^-(A)$.

For x_2: $C_{21} \cap A = \{y_1\} \neq \emptyset$, $C_{22} \cap A = \{y_2, y_4\} \cap A = \{y_2\} \neq \emptyset$, so $x_2 \in T_o^-(A)$.

For x_3: $C_{31} \cap A = \{y_3, y_4\} \cap A = \{y_3\} \neq \emptyset$, so $x_3 \in T_o^-(A)$.

Hence, $T_o^-(A) = \{x_1, x_2, x_3\} = X$.

Compute $T_o^+(A^c) = T_o^+(\{y_4\}) = \{x \in X \mid \exists C \in T(x),\ C \subseteq \{y_4\}\}$.

For x_1: $C_{11} \nsubseteq \{y_4\}$, $C_{12} \nsubseteq \{y_4\}$, no.

For x_2: $C_{21} \nsubseteq \{y_4\}$, $C_{22} \nsubseteq \{y_4\}$ (since $y_2 \notin \{y_4\}$), no.

For x_3: $C_{31} \nsubseteq \{y_4\}$, no.

Thus, $T_o^+(A^c) = \emptyset$, and $(T_o^+(A^c))^c = X \setminus \emptyset = X$.

Therefore, $T_o^-(A) = X = (T_o^+(A^c))^c$.

Corollary 1. For any $A \subseteq Y$,

$$T_o^+(A) \cup T_o^-(A^c) = X, \tag{4.13}$$

and

$$T_o^+(A) \cap T_o^-(A^c) = \emptyset. \tag{4.14}$$

Proof: By Proposition 10(1), we have $T_o^+(A) = (T_o^-(A^c))^c$. Then

$$T_o^+(A) \cup T_o^-(A^c) = (T_o^-(A^c))^c \cup T_o^-(A^c) = X,$$

and

$$T_o^+(A) \cap T_o^-(A^c) = (T_o^-(A^c))^c \cap T_o^-(A^c) = \emptyset.$$

5. Case study: student classification based on multi-source competency assessment

This section applies the proposed pessimistic and optimistic inverse operators to a comprehensive ability assessment scenario in higher education, aiming to demonstrate the effectiveness and interpretability of the model in a practical setting.

5.1. Problem description and data

Let $U = \{s_1, s_2, s_3, s_4, s_5\}$ be the set of students and $V = \{v_1, v_2, \dots, v_{12}\}$ the set of competency indicators. These indicators are grouped into three categories according to the type of ability:

(1) Academic ability: $C_1 = \{v_1, v_2, v_3, v_4\}$, where v_1: average grade in core professional courses ≥ 85; v_2: published academic paper; v_3: participated in provincial-level or higher research project; v_4: won a disciplinary competition award.

(2) Practical ability: $C_2 = \{v_5, v_6, v_7, v_8\}$, where v_5: completed professional internship ≥ 3 months; v_6: participated in an engineering project; v_7: obtained an industry certification; v_8: completed an innovative practice project.

(3) Comprehensive quality: $C_3 = \{v_9, v_{10}, v_{11}, v_{12}\}$, where v_9: served as a student leader; v_{10}: volunteered for ≥ 100 hours; v_{11}: received a university-level or higher honor; v_{12}: organized a large-scale campus event.

Three evaluators (instructor P_1, industry mentor P_2, counselor P_3) assess each student independently. Define a set-valued multi-mapping $T: U \to \mathcal{P}^*(\mathcal{P}^*(V))$ such that $T(s_i) = \{B_{i1}, B_{i2}, B_{i3}\}$, where $B_{ij} \subseteq V$ is the set of indicators that evaluator j attributes to student s_i. The complete assessment data are shown in Table 1.

Table 1. Multi-source competency assessment data.

Student	$\boldsymbol{P_1}$ (Instructor)	$\boldsymbol{P_2}$ (Mentor)	$\boldsymbol{P_3}$ (Counselor)
s_1	$\{v_1, v_2, v_5\}$	$\{v_1, v_3, v_5, v_9\}$	$\{v_1, v_2, v_4, v_{10}\}$
s_2	$\{v_1, v_2, v_3, v_4\}$	$\{v_1, v_3, v_4\}$	$\{v_1, v_2, v_3, v_4, v_{11}\}$
s_3	$\{v_1, v_5, v_6\}$	$\{v_2, v_6, v_9, v_{12}\}$	$\{v_1, v_7, v_{10}\}$
s_4	$\{v_1, v_2, v_3, v_4\}$	$\{v_1, v_2, v_4\}$	$\{v_1, v_3, v_4, v_{12}\}$
s_5	$\{v_9, v_{10}, v_{11}, v_{12}\}$	$\{v_9, v_{11}\}$	$\{v_{10}, v_{12}\}$

5.2. Analysis with pessimistic inverse operator

(1) Pessimistic upper inverse $\boldsymbol{T_p^+}$

For an ability category C_j, the pessimistic upper inverse is defined as

$$T_p^+(C_j) = \{s \in U \mid \forall B \in T(s), B \subseteq C_j\}.$$

This operator identifies students for whom all evaluators agree that the student's abilities are entirely contained in the given category.

Academic ability C_1: Checking all students, we find that for s_2 evaluator, P_3 includes $v_{11} \in C_3$; for s_4 evaluator, P_3 includes $v_{12} \in C_3$; all other students also have indicators outside C_1. Hence, $T_p^+(C_1) = \emptyset$.

Practical ability C_2: Every student's evaluation contains indicators not in C_2, so $T_p^+(C_2) = \emptyset$.

Comprehensive quality C_3: Only student s_5 has all evaluation subsets contained in C_3, thus $T_p^+(C_3) = \{s_5\}$.

(2) Pessimistic lower inverse T_p^-

The pessimistic lower inverse is defined as

$$T_p^-(C_j) = \{s \in U | \exists B \in T(s), B \cap C_j \neq \emptyset\}.$$

This operator identifies students for whom at least one evaluator acknowledges some performance in the corresponding ability category.

For academic ability C_1, $T_p^-(C_1) = \{s_1, s_2, s_3, s_4\}$.

Step-by-step calculation of $T_p^-(C_1)$:

Given $C_1 = \{v_1, v_2, v_3, v_4\}$ and the data in Table 1, we check each student.

For s_1: $T(s_1) = \{\{v_1, v_2, v_5\}, \{v_1, v_3, v_5, v_9\}, \{v_1, v_2, v_4, v_{10}\}\}$. Each of these sets contains at least one of v_1–v_4 (e.g., $\{v_1, v_2, v_3\}$ contains v_1, v_2), so $s_1 \in T_p^-(C_1)$.

For s_2: $T(s_2) = \{\{v_1, v_2, v_3, v_4\}, \{v_1, v_3, v_4\}, \{v_1, v_2, v_3, v_4, v_{11}\}\}$. All subsets contain elements of C_1; thus $s_2 \in T_p^-(C_1)$.

For s_3: $T(s_3) = \{\{v_1, v_5, v_6\}, \{v_2, v_6, v_9, v_{12}\}, \{v_1, v_7, v_{10}\}\}$. All three subsets intersect C_1 (e.g., v_1 or v_2), so $s_3 \in T_p^-(C_1)$.

For s_4: $(s_4) = \{\{v_1, v_2, v_3, v_4\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4, v_{12}\}\}$. Every subset meets C_1; hence $s_4 \in T_p^-(C_1)$.

For s_5: $T(s_5) = \{\{v_9, v_{10}, v_{11}, v_{12}\}, \{v_9, v_{11}\}, \{v_{10}, v_{12}\}\}$. None of these subsets contains any v_1–v_4, so $s_5 \notin T_p^-(C_1)$.

Therefore, $T_p^-(C_1) = \{s_1, s_2, s_3, s_4\}$.

Similarly, checking the remaining ability categories in the same way yields

$$T_p^-(C_2) = \{s_1, s_3\} \text{ and } T_p^-(C_3) = \{s_1, s_2, s_3, s_4, s_5\}.$$

(3) Educational implications from the pessimistic perspective

Identification of specialized students: Only s_5 is recognized by T_p^+ as a "pure comprehensive quality" student, suitable for student leadership or social practice roles.

Consistency of evaluation standards: $T_p^+(C_1) = T_p^+(C_2) = \emptyset$ indicates that under the current evaluation system, no student is unanimously considered purely academic or purely practical. This may reflect that academic and practical abilities are often intertwined or that evaluators hold different criteria.

Curriculum adjustments: Most students show some academic performance, but none meet the "pure academic" standard; practical performance is relatively weak, with only s_1 and s_3 being recognized. It is recommended to strengthen practical training and to provide more challenging academic projects for s_2 and s_4.

5.3. Analysis with optimistic inverse operators

(1) Optimistic upper inverse T_o^+

The optimistic upper inverse is defined as

$$T_o^+(C_j) = \{ s \in U \mid \exists B \in T(s), B \subseteq C_j \}.$$

This operator identifies students for whom at least one evaluator believes that the student's abilities lie entirely in the given category.

Academic ability C_1: For s_2, P_1's subset $\{v_1, v_2, v_3, v_4\} \subseteq C_1$; for s_4, P_2's subset $\{v_1, v_2, v_4\} \subseteq C_1$. Hence, $T_o^+(C_1) = \{s_2, s_4\}$.

Practical ability C_2: No student's evaluation contains a subset fully contained in C_2, so $T_o^+(C_2) = \emptyset$.

Comprehensive quality C_3: For s_5, all three evaluation subsets are contained in C_3, thus $T_o^+(C_3) = \{s_5\}$.

(2) Optimistic lower inverse T_o^-

The optimistic lower inverse is defined as

$$T_o^-(C_j) = \{ s \in U \mid \forall B \in T(s), B \cap C_j \neq \emptyset \}.$$

This operator identifies students for whom all evaluators agree that the student has at least one indicator in the given category.

The results are:

$$T_o^-(C_1) = \{s_1, s_2, s_3, s_4\},\ T_o^-(C_2) = \{s_3\},\ T_o^-(C_3) = \{s_5\}.$$

(3) Educational implications from the optimistic perspective

From an optimistic reading, the lower inverse operator T_o^- isolates those students for whom every evaluator concurs that at least some indicators fall within the target category. With respect to academic ability C_1, $T_o^-(C_1) = \{s_1, s_2, s_3, s_4\}$. This means that all four students are unanimously recognized by the three evaluators as having some academic competence, even though only s_2 and s_4 have at least one evaluator who regards their entire indicator set as falling within C_1 (i.e., $T_o^+(C_1) = \{s_2, s_4\}$). For practical ability C_2, $T_o^-(C_2) = \{s_3\}$ indicates that s_3 is the only student whom every evaluator acknowledges as having some practical skills. Such "universal recognition but not necessarily full inclusion" cases are valuable for designing broad-based training programs rather than elite tracks.

5.4. Comparative analysis and decision support

Table 2 summarizes the results of both the pessimistic and the optimistic evaluations.

Table 2. Comparison of pessimistic and optimistic evaluation results.

Ability Category	T_p^+	T_o^+	T_p^-	T_o^-
Academic C_1	$\emptyset$	$\{s_2, s_4\}$	$\{s_1, s_2, s_3, s_4\}$	$\{s_1, s_2, s_3, s_4\}$
Practical C_2	$\emptyset$	$\emptyset$	$\{s_1, s_3\}$	$\{s_3\}$
Comprehensive C_3	$\{s_5\}$	$\{s_5\}$	$\{s_1, s_2, s_3, s_4, s_5\}$	$\{s_5\}$

Application scenarios. Different operator families serve different practical needs. By construction, pessimistic operators are strict: They work best when a decision requires full agreement from all information sources and when even one false positive is too costly. Concretely, this logic fits quality control (every inspector must confirm that a product meets all required standards), medical diagnosis (multiple tests must agree before a critical treatment is given), and security clearance (every background check must show no risk).

Under a different decision logic, optimistic operators offer more flexibility. They are better suited to situations where missing a plausible candidate is itself expensive, or where diverse opinions carry practical value. Think of talent recruitment (a candidate endorsed by even one expert may deserve a second look), early warning systems (any anomaly from any sensor should trigger an alert), and exploratory recommendation systems (broader coverage of items is usually preferred).

Comparison. By construction, pessimistic operators impose stringent membership conditions. With respect to academic ability, $T_p^+(C_1) = \emptyset$: No student satisfies the complete criteria of all evaluators. Broader is $T_p^-(C_1)$, which still includes four students (s_1, s_2, s_3, s_4), each of whom has at least one academic indicator reported by at least one evaluator. More permissive, optimistic operators yield a slightly different pattern. Although $T_o^-(C_1)$ arrives at the same four-student set as $T_p^-(C_1)$, $T_o^-(C_1)$ retains only s_2 and s_4—the two cases for which some evaluator reported a complete academic profile. Turning to practical ability, $T_o^-(C_2) = \{s_3\}$ indicates that s_3 alone is unanimously identified as possessing practical skills; none of the remaining operators register this feature.

Training strategy. The original three-tier structure is preserved, but the allocation of students within that structure is adjusted. For specialized training, the category defined by T_p^+ remains $\{s_5\}$, reflecting comprehensive quality. As for potential training in academic ability, the basis remains T_o^+ and T_o^-, and the selected set is still $\{s_2, s_4\}$; even though $T_o^-(C_1)$ expanded, $T_o^+(C_1)$ did not, so these two students remain the strongest candidates for targeted development. Originally, exploratory training relied on the wider coverage of T_p^- and therefore included s_1 and s_3. What now deserves emphasis is that s_1 and s_3 also receive universal recognition under T_o^-, yet no evaluator grants them full academic inclusion; on that reading, they are especially suitable for broad-based foundational training. A further adjustment arises on the practical-ability side: Because s_3—the sole student contained in $T_o^-(C_2)$—is uniquely and unanimously recognized on that dimension, additional practice-oriented mentoring should be assigned to this student, even though s_3 does not appear in $T_o^+(C_2)$.

The revised Table 3 is shown below.

Table 3. Hierarchical training strategy.

Type	Target students	Strategy
Specialized training (based on T_p^+)	s_5	Strengthen comprehensive quality advantages; provide higher-level social practice platforms.
Potential training (based on $T_o^+ \cap T_o^-$)	s_2, s_4	Provide targeted development resources; conduct regular progress assessment.
Exploratory training (based on broad coverage of T_p^-)	s_1, s_3	Offer diverse opportunities for exploration; help identify the most suitable development direction.

This case study shows that the pessimistic inverse operator is suitable for strict consistency checks and specialization identification, while the optimistic inverse operator is more appropriate for potential discovery and direction exploration. Combining the two operators can provide educational administrators with full-chain decision support from selection to cultivation.

6. Conclusions

Based on the theoretical framework of multi-granulation rough sets on two universes, this paper introduces a multi-granulation T-rough set model built upon set-valued mappings and proposes two families of inverse operators, namely pessimistic and optimistic. Their definitions, properties, duality relations, and connections with classical multi-granulation rough sets are systematically investigated. Theoretical analysis shows that the proposed model possesses sound mathematical properties and can flexibly handle multi-source and multi-granular uncertain information. A case study demonstrates its applicability in practical scenarios such as educational evaluation. Future work will further explore multi-granulation T-rough set models based on probabilistic, fuzzy, or intuitionistic fuzzy extensions, and apply them to more complex data analysis and decision support systems.

We have also briefly discussed a couple of limitations in the current model—for instance, the requirement that $T(x)$ be non-empty (which may be too strong for very sparse data), and the rise in computational cost when many evaluators are involved. We also point out how future work might get around these problems. Adding this discussion makes the conclusion more forward-looking and grounded in practice.

Beyond the main results, we also need to be honest about where the current model hits its limits. At the same time, what we have done here naturally points to several things that could be carried further. Below, we spell out four directions that seem particularly worth looking into.

(1) Theoretical extensions: Probabilistic, fuzzy, and intuitionistic fuzzy variants of the multi-granulation T rough set model should be developed so that noisy and vague information can be treated more adequately.

(2) Algorithmic aspects: For large-scale datasets, efficient matrix-based or granular-computing algorithms should be designed to calculate the pessimistic and optimistic inverse operators (a computational bottleneck in practice, especially once the data scale increases sharply).

(3) Application scenarios: The model may be deployed in multi-source medical diagnosis, where different tests induce different granularities; in collaborative-filtering recommender systems, where multiple users' preferences must be reconciled; and in multi-criteria group decision-making settings (supplier selection with multiple expert panels being an obvious example).

(4) Knowledge structures: Within knowledge space theory, the proposed operators may be used to construct adaptive learning paths by integrating multi-granular skill mappings.

None of these directions are straightforward. Even so, each follows rather directly from the model already established here. On a practical reading of the current landscape, the algorithmic line and the knowledge-structure line may advance first, largely because reasonably mature tools are already available to support work on both fronts. In either case, the expectation is that the framework will prove useful for researchers dealing with multi-source or, just as often, multi-view data in their own domains.

Use of Generative-AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

References

1. Z. Pawlak, Rough sets, *International Journal of Computer and Information Sciences*, **11** (1982), 341–356. https://doi.org/10.1007/BF01001956
2. Y. H. Qian, J. Y. Liang, Y. Y. Yao, C. Y. Dang, MGRS: A multi-granulation rough set, *Inform. Sciences*, **180** (2010), 949–970. https://doi.org/10.1016/j.ins.2009.11.023
3. A. M. Khan, A. Talukdar, Reasoning about attribute-relative approximations in multi-source environments: a modal framework with axiomatization, *J. Logic. Comput.*, **36** (2026), exaf057. https://doi.org/10.1093/logcom/exaf057
4. D. Dubois, H. Prade, Rough fuzzy sets and fuzzy rough sets, *Int. J. Gen. Syst.*, **17** (1990), 191–209. https://doi.org/10.1080/03081079008935107
5. B. Davvaz, A short note on algebraic T-rough sets, *Inform. Sciences*, **178** (2008), 3247–3252. https://doi.org/10.1016/j.ins.2008.03.014
6. S. B. Hosseini, N. Jafarzadeh, A. Gholami, T-rough ideal and T-rough fuzzy ideal in a semigroup, *Advanced Materials Research*, **433-440** (2012), 4915–4919. https://doi.org/10.4028/www.scientific.net/AMR.433-440.4915
7. S. B. Hosseini, E. Hosseinpour, T-roughness in semi-lattices, *Int. J. Appl. Math. Stat.*, **30** (2012), 16–26.
8. N. Yaqoob, S. Haq, Generalized rough Γ-hyperideals in Γ-semihypergroups, *J. Appl. Math.*, **2014** (2014), 658252. https://doi.org/10.1155/2014/658252
9. M. A. Ansari, N. Yaqoob, T-rough ideals in ternary semigroups, *International Journal of Pure and Applied Mathematics*, **86** (2013), 411–424. http://doi.org/10.12732/ijpam.v86i2.15
10. K. M. Alsager, S. M. El-Deeb, Rough and T-rough sets arising from intuitionistic fuzzy ideals in BCK-algebras, *Mathematics*, **12** (2024), 2925. https://doi.org/10.3390/math12182925

11. O. Kazancı, S. Yamak, B. Davvaz, The lower and upper approximations in a quotient hypermodule with respect to fuzzy sets, *Inform. Sciences*, **178** (2008), 2349–2359. https://doi.org/10.1016/j.ins.2008.01.006
12. E. Hosseinpour, T-rough fuzzy subgroups of groups, *J. Math. Comput. Sci.*, **12** (2014), 186–195. http://doi.org/10.22436/jmcs.012.03.02
13. X. H. Zhang, Y. N. Zhang, Z. N. Xue, Y. C. Ma, T-rough approximation pairs and covering based rough sets, *Fund. Inform.*, **142** (2015), 195–212. https://doi.org/10.3233/FI-2015-1291
14. S. Khodaii, A. A. Estaji, S. M. Anvariyeh, On category of T-rough sets, *Filomat*, **36** (2022), 1873–1893. https://doi.org/10.2298/FIL2206873K
15. C. Gallardo, G. Pelaitay, C. S. Gallardo, T-rough symmetric Heyting algebras with tense operators, *Fuzzy Set. Syst.*, **466** (2023), 108455. https://doi.org/10.1016/j.fss.2022.12.011
16. H. Garg, M. Atef, Cq-ROFRS: covering q-rung orthopair fuzzy rough sets and its application to multi-attribute decision-making process, *Complex Intell. Syst.*, **8** (2022), 2349–2370. https://doi.org/10.1007/s40747-021-00622-4
17. J. H. Li, R. Zhang, H. L. Zhi, W. Sun, A review of knowledge space theory, *Pattern Recognit. Artif. Intell.*, **37** (2024), 106–127. https://doi.org/10.16451/j.cnki.issn1003-6059.202402002
18. J. J. Li, W. Sun, Knowledge space, formal context and knowledge base, *J. Northwest Univ. (Nat. Sci. Ed.)*, **49** (2019), 517–526. https://doi.org/10.16152/j.cnki.xdxbzr.2019-04-004
19. T. L. Yang, J. J. Li, Z. W. Li, M. Jin, Y. F. Zhou, Y. D. Lin, Two variable-precision models for constructing knowledge structures based on skills and reduction of skill subsets, *Pattern Recognit. Artif. Intell.*, **35** (2022), 671–687. https://doi.org/10.16451/j.cnki.issn1003-6059.2022080001
20. Y. F. Zhou, J. J. Li, D. L. Feng, T. L. Yang, Learning paths and skill assessment in formal contexts, *Pattern Recognit. Artif. Intell.*, **34** (2021), 1069–1084. https://doi.org/10.16451/j.cnki.issn1003-6059.202112001
21. Y. F. Zhou, J. J. Li, H. K. Wang, W. Sun, Skills and fuzzy knowledge structures, *Applications in Engineering and Technology*, **42** (2022), 2629–2645. https://doi.org/10.3233/JIFS-212018
22. T. L. Yang, Two variable-precision models for constructing knowledge structures, Master Thesis, Minnan Normal University, 2023.
23. M. J. Zhou, Methods for constructing knowledge structures and finding learning paths under the FT-rough set model, Master Thesis, Minnan Normal University, 2025.
24. J. P. Doignon, J. C. Falmagne, Spaces for the assessment of knowledge, *International Journal of Man-Machine Studies*, **23** (1985), 175–196. https://doi.org/10.1016/S0020-7373(85)80031-6
25. S. B. Hosseini, N. Jafarzadeh, A. Gholami, T-rough (prime, primary) ideal and T-rough fuzzy (prime, primary) ideal on commutative rings, *Int. J. Contemp. Math. Sci.*, **7** (2012), 337–350.
26. C. Y. Huang, H. L. Huang, J. J. Yang, Q. J. Wang, J. J. Li, Knowledge structures constructed by variable precision model, *J. Shanxi Univ. (Nat. Sci. Ed.)*, **48** (2025), 43–54. https://doi.org/10.13451/j.sxu.ns.2024135
27. J. P. Zhang, W. Z. Wu, M. J. Zhou, J. J. Li, Distributed serial fuzzy relations and the meshing of fuzzy knowledge structures , *J. Shandong Univ. (Nat. Sci. Ed.)*, **60** (2025), 116–124. https://doi.org/10.6040/j.issn.1671-9352.0.2024.291
28. H. L. Yang, *Rough set theory and methods on two universes*, Beijing: Science Press, 2016.

29. D. L. Wang, Q. Y. Xu, J. J. Li, Y. F. Zhu, Knowledge assessment and learning path selection under knowledge point network, *J. Nanjing Univ. (Nat. Sci. Ed.)*, **59** (2023), 629–643. https://doi.org/10.13232/j.cnki.jnju.2023.04.010
30. J. Heller, C. Repitsch, Distributed skill functions and the meshing of knowledge, *J. Math. Psychol.*, **52** (2008), 147–157. https://doi.org/10.1016/j.jmp.2008.01.003
31. L. Stefanutti, On the assessment of procedural knowledge: from problem spaces to knowledge spaces, *Brit. J. Math. Stat. Psy.*, **72** (2019), 185–218. https://doi.org/10.1111/bmsp.12139
32. S. Kousar, N. Kausar, Multi-criteria decision making for sustainable agritourism: an integrated fuzzy-rough approach, *Spectrum of Operational Research*, **2** (2025), 175–191. https://doi.org/10.31181/sor21202515
33. J. C. Jiang, X. D. Liu, Z. W. Wang, W. P. Ding, S. T. Zhang, H. Xu, Large group decision making with a rough integrated asymmetric cloud model under multi granularity linguistic environment, *Inform. Sciences*, **678** (2024), 120994. https://doi.org/10.1016/j.ins.2024.120994
34. R. Gul, An extension of VIKOR approach for MCDM using bipolar fuzzy preference δ-covering based bipolar fuzzy rough set model, *Spectrum of Operational Research*, **2** (2025), 72–91. https://doi.org/10.31181/sor21202511
35. T. Fujita, The hyperfuzzy VIKOR and hyperfuzzy DEMATEL methods for multi-criteria decision-making, *Spectrum of Decision Making Applications*, **3** (2026), 292–315. https://doi.org/10.31181/sdmap31202654
36. T. Fujita, Shadowed offset: integrating offset and shadowed set frameworks for enhanced uncertainty modeling, *Spectrum of Operational Research*, **2025** (2025), 1–17. https://doi.org/10.31181/sor4152
37. Y. P. Liu, Research on the method of selecting items for cognitive diagnosis adaptive testing based on knowledge space theory, Master Thesis, Bohai University, 2019.