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Research article

Some more generalized integration operators on general function spaces

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Abstract: Let $H(\mathbb{D})$ be the class of all analytic functions on $\mathbb{D}$ and $\vec{h} = (h_0, h_1, \dots, h_{n-1})$ with $h_k \in H(\mathbb{D})$ for $k = 0, 1, \dots, n-1$. In this paper, we study a complex integration operator

$$(T^{(n)}_{\vec{h}} f)(z) = I^n(h^{(n)}_0 f + h^{(n-1)}_1 f' + \cdots + h'_{n-1} f^{(n-1)})(z), \quad f \in H(\mathbb{D}),$$

where I is the classical integration operator

$$(If)(z) = \int_0^z f(w)dw$$

and I^n is the nth iteration of I. We characterize the bounded and compact operators $T^{(n)}_{\vec{h}}$ on $F(p, q, s)$ by using some characterizations of the space.

Keywords: $F(p, q, s)$ space; integration operator; boundedness; compactness
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1. Introduction

Let $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the boundary of $\mathbb{D}$, $H(\mathbb{D})$ be the class of all analytic functions on $\mathbb{D}$, and $dA(z) = \frac{1}{\pi}dxdy$ be the normalized Lebesgue area measure on $\mathbb{D}$.

For $a \in \mathbb{D}$, let

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}$$

be the special Möbius map of $\mathbb{D}$, which interchanges the points 0 and a. For $0 < p < +\infty$, $-2 < q <$

$+\infty$, $0 \le s < +\infty$, and $q + s > -1$, the space $F(p, q, s)$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{sF(p,q,s)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q (1 - |\varphi_a(z)|^2)^s dA(z) < +\infty.$$

If we further assume that $1 \le p < +\infty$, then $F(p, q, s)$ is a Banach space with the norm

$$\|f\|_{F(p,q,s)} = |f(0)| + \|f\|_{sF(p,q,s)}.$$

We need to assume that $q + s > -1$ since otherwise $F(p, q, s)$ becomes the space of constant functions. The spaces $F(p, q, s)$ were introduced by Zhao in [25]. Zhao obtained numerous results; for example, $F(p, q, s)$ is not separable, and $F(p, q, s)$ is a subset of $\mathcal{B}^{\frac{q+2}{p}}$. Besides, we see that for different parameter values p, q, and s, they coincide with some classical function spaces. For instance, $F(2, 1, 0)$ is the Hardy space H^2, $F(p, p, 0)$ is the Bergman space A^p, $F(2, 0, s)$ is the Q_s space, and $F(2, 0, 1)$ is the BMOA space, the space of analytic functions with bounded mean oscillation.

For $\alpha > 0$, the weighted Bloch space $\mathcal{B}^\alpha$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < +\infty.$$

$\mathcal{B}^\alpha$ is a Banach space with the norm $\|\cdot\|_{\mathcal{B}^\alpha}$. The little weighted Bloch space $\mathcal{B}^\alpha_0$ consists of all $f \in \mathcal{B}^\alpha$ such that

$$\lim_{|z|\to 1} (1 - |z|^2)^\alpha |f'(z)| = 0.$$

If $\alpha = 1$, then $\mathcal{B}^\alpha$ and $\mathcal{B}^\alpha_0$ are called the Bloch space and the little Bloch space, respectively, denoted by $\mathcal{B}$ and $\mathcal{B}_0$. There has been a great deal of research on these spaces. For some good references, see, for example, [28]. See [6] for an overview of the Bloch spaces and their connection to other function spaces.

For $I \subseteq \mathbb{T}$, let $S(I)$ denote the Carleson box based on the arc I, that is,

$$S(I) = \left\{ z \in \mathbb{D} : 1 - |I| \le |z| < 1 \text{ and } \frac{z}{|z|} \in I \right\},$$

where $|I|$ denotes the arc length of the arc I. We can consider the introduction of the Carleson box to be related to the bounded (compact) Carleson measures (see, for example, [8, 9]).

Next, we need to introduce some related operators. For $h \in H(\mathbb{D})$, the integration operator I_h and its companion operator J_h on $H(\mathbb{D})$ are defined by

$$(I_h f)(z) = \int_0^z f(w)h'(w)dw \quad \text{and} \quad (J_h f)(z) = \int_0^z f'(w)h(w)dw.$$

The operators I_h and J_h have been studied extensively on analytic function spaces (see, for example, [1, 2, 7, 8]). They contain some well-known operators such as the integral operator I_h with $h(z) = z$ and the Cesáro operator I_h with $h(z) = \log(1/(1 - z))$. The multiplication operator M_h on $H(\mathbb{D})$ is defined by

$$(M_h f)(z) = h(z)f(z).$$

The relationship between these three operators is that

$$(I_h f)(z) + (J_h f)(z) = (M_h f)(z) - f(0)h(0).$$

Let $\vec{a} = (a_0, a_1, \ldots, a_{n-1}) \in \mathbb{C}^n$ and $h \in H(\mathbb{D})$. Chalmoukis in [4] introduced the following operator on $H(\mathbb{D})$:

$$(I_{\vec{a},h} f)(z) = I^n(a_0 h^{(n)} f + a_1 h^{(n-1)} f' + \cdots + a_{n-1} h' f^{(n-1)})(z),$$

where I is the integration operator

$$(If)(z) = \int_0^z f(w)dw$$

and I^n is the nth iteration of I, and he completely characterized the bounded and compact operators $I_{\vec{a},h} : H^p \to H^q$ for $0 < p \le q < +\infty$. In [5], Chalmoukis et al. continuously characterized these properties for the case $0 < q < p < +\infty$.

Motivated by the operator $I_{\vec{a},h}$, Arroussi et al. introduced the operator

$$(I_{\vec{h}}^{(n)} f)(z) = I^n(h_0 f + h_1 f' + \cdots + h_{n-1} f^{(n-1)})(z) \tag{1.1}$$

in [3], where $\vec{h} = (h_0, h_1, \ldots, h_{n-1})$ with $h_k \in H(\mathbb{C})$ for $k = 0, 1, \ldots, n-1$, and they characterized the boundedness and compactness of the operator between Fock spaces. As a simplified version of operator $I_{\vec{h}}^{(n)}$, Qian and Zhu in [15] studied the operator

$$(T_h^{n,k} f)(z) = I^n(h^{(n-k)} f^{(k)})(z)$$

from Hardy spaces H^p into tent spaces. Zhu and Qiu in [26] studied the operator $T_h^{n,k}$ from Besov spaces into general function spaces.

Using the operators $T_h^{n,k}$, a complex integration operator considered in this paper is defined by

$$(T_{\vec{h}}^{(n)} f)(z) = I^n(h_0^{(n)} f + h_1^{(n-1)} f' + \cdots + h'_{n-1} f^{(n-1)})(z),$$

which is essentially operator $I_{\vec{h}}^{(n)}$ in (1.1). It is easy to see that $T_{\vec{h}}^{(n)}$ is a sum of the operators $T_{h_k}^{n,k}$, that is,

$$T_{\vec{h}}^{(n)} = \sum_{k=0}^{n-1} T_{h_k}^{n,k}.$$

Because of this relationship, we have reason to believe that it is meaningful to study the operators $T_{\vec{h}}^{(n)}$ between function spaces. To this end, we characterize the bounded and compact operators $T_{\vec{h}}^{(n)}$ on $F(p,q,s)$ in this paper. Actually, we prove that $T_{\vec{h}}^{(n)}$ is bounded on $F(p,q,s)$ if and only if each $T_{h_k}^{n,k}$ is bounded on $F(p,q,s)$ if and only if each h_k belongs to $\mathcal{B}$. We also prove that $T_{\vec{h}}^{(n)}$ is compact on $F(p,q,s)$ if and only if each $T_{h_k}^{n,k}$ is compact on $F(p,q,s)$ if and only if each h_k belongs to $\mathcal{B}_0$.

As usual, we use the uppercase letter C to denote positive numbers, and they may vary in different situations. The notation $a \lesssim b$ (resp. $a \gtrsim b$) means that there is a positive number C such that $a \le Cb$ (resp. $a \ge Cb$). If $a \lesssim b$ and $b \gtrsim a$, then we write $a \asymp b$.

2. Auxiliary lemmas

First, we need the following result (see Lemma 4.2.2 in [28]).

Lemma 2.1. *Let* $z \in \mathbb{D}$, $t > -1$, *and* $c > 0$. *Then it holds that*

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^t}{|1-\bar{z}w|^{t+2+c}} dA(w) \asymp \frac{1}{(1-|z|^2)^c}.$$

For a fixed $w \in \mathbb{D}$ and $j \in \mathbb{N}_0$, set

$$k_{w,j}(z) = \frac{(1-|w|^2)^{j+1}}{(1-\overline{w}z)^{j+\frac{q+2}{p}}}, \quad z \in \mathbb{D}. \tag{2.1}$$

We will prove that the functions $k_{w,j}$ belong to $F(p,q,s)$. Through a review of the literature, we find that Zhao in [24] provided a class of functions in $F(p,q,s)$ but not the functions $k_{w,j}$.

Lemma 2.2. *Let* $0 < p < +\infty$, $-2 < q < +\infty$, $0 \le s < +\infty$, $q+s > -1$, $w \in \mathbb{D}$, *and* $j \in \mathbb{N}_0$. *Then the functions* $k_{w,j}$ *belong to* $F(p,q,s)$.

Moreover,

$$\sup_{w\in\mathbb{D}} \|k_{w,j}\|_{sF(p,q,s)} \lesssim 1. \tag{2.2}$$

Proof. By the definition, we have

$$k'_{w,j}(z) = (j + \frac{q+2}{p}) \frac{(1-|w|^2)^{j+1}\overline{w}}{(1-\overline{w}z)^{j+1+\frac{q+2}{p}}}, \quad z \in \mathbb{D},$$

from which together with Lemma 2.1, it follows that

$$\begin{aligned} \|k_{w,j}\|^p_{sF(p,q,s)} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |k'_{w,j}(z)|^p (1-|z|^2)^q (1-|\varphi_a(z)|^2)^s dA(z) \\ &\le \int_{\mathbb{D}} |k'_{w,j}(z)|^p (1-|z|^2)^q dA(z) \\ &\lesssim (1-|w|^2)^{p(j+1)} \int_{\mathbb{D}} \frac{(1-|z|^2)^q}{|1-\overline{w}z|^{p(j+1)+q+2}} dA(z) \\ &\lesssim 1. \end{aligned} \tag{2.3}$$

It follows from (2.3) that the functions $k_{w,j}$ belong to $F(p,q,s)$ and (2.2) holds. □

Not only that, but we also see that $k_{w,j} \to 0$ uniformly on any compact subset of $\mathbb{D}$ as $|w| \to 1$. Next, we will use the functions $k_{w,j}$ to obtain some new functions by the methods and techniques described in [21] and used frequently, for example, in [10, 11, 19, 20].

Lemma 2.3. *Let* $0 < p < +\infty$*,* $-2 < q < +\infty$*,* $0 \leq s < +\infty$*,* $q + s > -1$*, and* $w \in \mathbb{D}$*. Then for each fixed* $k \in \{0, 1, \dots, n-1\}$*, there exist constants* $c_{k,0}$*,* $c_{k,1}$*, …,* $c_{k,n-1}$ *such that the function*

$$f_{w,k}(z) = \sum_{j=0}^{n-1} c_{k,j} k_{w,j}(z) \tag{2.4}$$

satisfies

$$f_{w,k}^{(k)}(w) = \frac{\overline{w}^k}{(1 - |w|^2)^{k + \frac{q+2}{p} - 1}} \quad \text{and} \quad f_{w,k}^{(j)}(w) = 0 \tag{2.5}$$

for $j \in \{0, 1, \dots, n-1\} \backslash \{k\}$*.*

Moreover,

$$\sup_{w \in \mathbb{D}} \left\| f_{w,k} \right\|_{sF(p,q,s)} \lesssim 1. \tag{2.6}$$

Proof. Let $a := \frac{q+2}{p}$. If $f_{w,k}$ in (2.4) satisfies the conditions in (2.5), then by a calculation, we have

$$\begin{gathered} c_{k,0} + c_{k,1} + \cdots + c_{k,n-1} = 0, \\ a c_{k,0} + (a+1) c_{k,1} + \cdots + (a+n-1) c_{k,n-1} = 0, \\ \vdots \\ \prod_{j=0}^{k-2} (a+j) c_{k,0} + \prod_{j=0}^{k-2} (a+1+j) c_{k,1} + \cdots + \prod_{j=0}^{k-2} (a+n-1+j) c_{k,n-1} = 0, \\ \prod_{j=0}^{k-1} (a+j) c_{k,0} + \prod_{j=0}^{k-1} (a+1+j) c_{k,1} + \cdots + \prod_{j=0}^{k-1} (a+n-1+j) c_{k,n-1} = 1, \\ \prod_{j=0}^{k} (a+j) c_{k,0} + \prod_{j=0}^{k} (a+1+j) c_{k,1} + \cdots + \prod_{j=0}^{k} (a+n-1+j) c_{k,n-1} = 0, \\ \vdots \\ \prod_{j=0}^{n-1} (a+j) c_{k,0} + \prod_{j=0}^{n-1} (a+1+j) c_{k,1} + \cdots + \prod_{j=0}^{n-1} (a+n-1+j) c_{k,n-1} = 0. \end{gathered} \tag{2.7}$$

By [21], the determinant of the system (2.7) is different from zero. From this and (2.2), the lemma, and asymptotic inequality (2.6) hold. □

The following result is Theorem 3.2 in [16] and is also Theorem 3.13 in [24].

Lemma 2.4. *Let* $f \in H(\mathbb{D})$*,* $0 < p < +\infty$*,* $-2 < q < +\infty$*, and* $0 \leq s < +\infty$*. Let* $n \in \mathbb{N}$ *and* $q + s > -1$*, or* $n = 0$ *and* $q + s - p > -1$*. Then the following statements are equivalent.*

(a)

$$\|f\|_{sF(p,q,s)}^p = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q (1 - |\varphi_a(z)|^2)^s dA(z) < +\infty.$$

(b)

$$\|f\|_{s_1F(p,q,s)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|f^{(n)}(z)\right|^p (1-|z|^2)^{np-p+q}(1-|\varphi_a(z)|^2)^s dA(z) < +\infty.$$

(c)

$$\|f\|_{s_2F(p,q,s)} = \sup_{I\subset\mathbb{T}} \frac{1}{|I|^s} \int_{S(I)} |f^{(n)}(z)|^p (1-|z|^2)^{np-p+q+s} dA(z) < +\infty.$$

Moreover, when $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$,

$$\|f\|^p_{sF(p,q,s)} \asymp \|f\|_{s_1F(p,q,s)} \asymp \|f\|_{s_2F(p,q,s)}.$$

The proof of the next result is similar to that of Lemma 2.4 in [15]. So, we omit the proof here.

Lemma 2.5. *Let* $n\in\mathbb{N}$, $0<p<+\infty$, $-2<q<+\infty$, $0\le s<+\infty$, *and* $q+s>-1$. *If* $f\in F(p,q,s)$, *then*

$$|f^{(n)}(z)| \lesssim \frac{\|f\|_{sF(p,q,s)}}{(1-|z|^2)^{n+\frac{q+2}{p}-1}}.$$

The next lemma comes from Proposition 7 and Proposition 8 in [27].

Lemma 2.6. *Let* $\alpha>0$, $n\in\mathbb{N}$, *and* $g\in H(\mathbb{D})$. *Then the following statements hold.*

(a) $g\in\mathcal{B}^\alpha$ *if and only if*

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha-1+n}|g^{(n)}(z)| < +\infty,$$

and the following asymptotic relation holds:

$$\|g\|_{\mathcal{B}^\alpha} \asymp \sum_{l=0}^{n-1}|g^{(l)}(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha-1+n}|g^{(n)}(z)|.$$

(b) $g\in\mathcal{B}^\alpha_0$ *if and only if*

$$\lim_{|z|\to1}(1-|z|^2)^{\alpha-1+n}|g^{(n)}(z)| = 0.$$

Lemma 2.7. *Let* $h\in H(\mathbb{D})$, $0<p<+\infty$, $-2<q<+\infty$, $0\le s<+\infty$, $q+s-p>-1$, $n\in\mathbb{N}$, *and* $0\le k\le n-1$. *Then the operator* $T^{n,k}_h$ *is bounded on* $F(p,q,s)$ *if and only if* $h\in\mathcal{B}$.

Moreover, if $h(0)=h'(0)=\cdots=h^{(n-k-1)}(0)=0$, *then*

$$\|T^{n,k}_h\|_{F(p,q,s)\to F(p,q,s)} \asymp \|h\|_{\mathcal{B}}. \tag{2.8}$$

Proof. Let the operator $T_h^{n,k}$ be bounded on $F(p,q,s)$. For $w \in \mathbb{D}$, let $k_{w,0}$ be the function defined in (2.1) for $j = 0$. From the boundedness of $T_h^{n,k}$ and Lemma 2.5, it follows that

$$\frac{\|T_h^{n,k}k_{w,0}\|_{sF(p,q,s)}}{(1-|w|^2)^{n+\frac{q+2}{p}-1}} \gtrsim \left|(T_h^{n,k}k_{w,0})^{(n)}(w)\right| \gtrsim \frac{|w|^k}{(1-|w|^2)^{k+\frac{q+2}{p}-1}}|h^{(n-k)}(w)|,$$

which shows

$$\sup_{|w|>1/2}(1-|w|^2)^{n-k}|h^{(n-k)}(w)| \lesssim \|T_h^{n,k}k_{w,0}\|_{sF(p,q,s)} \lesssim \|T_h^{n,k}\|_{F(p,q,s)\to F(p,q,s)} < +\infty. \tag{2.9}$$

Set $f_k(z) = z^k/k!$, $z \in \mathbb{D}$. Then $f_k \in F(p,q,s)$. From this fact and the boundedness of $T_h^{n,k}$ on $F(p,q,s)$, we have that $T_h^{n,k}f_k \in F(p,q,s)$. By Lemma 2.5, we obtain

$$\left|h^{(n-k)}(z)f_k^{(k)}(z)\right| = \left|(T_h^{n,k}f_k)^{(n)}(z)\right| \lesssim \frac{\|T_h^{n,k}f_k\|_{sF(p,q,s)}}{(1-|z|^2)^{n+\frac{q+2}{p}-1}} \lesssim \frac{\|T_h^{n,k}\|_{F(p,q,s)\to F(p,q,s)}}{(1-|z|^2)^{n+\frac{q+2}{p}-1}}$$

for each $z \in \mathbb{D}$, from which it follows that

$$\sup_{|z|\le 1/2}(1-|z|^2)^{n-k}\left|h^{(n-k)}(z)\right| \lesssim \sup_{|z|\le 1/2}\frac{\|T_h^{n,k}\|_{F(p,q,s)\to F(p,q,s)}}{(1-|z|^2)^{k+\frac{q+2}{p}-1}} \lesssim \|T_h^{n,k}\|_{F(p,q,s)\to F(p,q,s)}. \tag{2.10}$$

From (2.9) and (2.10), we obtain

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{n-k}\left|h^{(n-k)}(z)\right| \lesssim \|T_h^{n,k}\|_{F(p,q,s)\to F(p,q,s)} < +\infty. \tag{2.11}$$

Hence, it follows from Lemma 2.6 that $h \in \mathcal{B}$.

Conversely, assume that $h \in \mathcal{B}$. Let $f \in F(p,q,s)$. Since $(T_h^{n,k}f)^{(j)}(0) = 0$ for $j = 0,1,\dots,n-1$, from Lemma 2.4 and Lemma 2.6, we have

$$\begin{aligned}
\|T_h^{n,k}f\|_{F(p,q,s)}^p &\lesssim \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|(T_h^{n,k}f)^{(n)}(z)|^p(1-|z|^2)^{np-p+q}(1-|\varphi_a(z)|^2)^s dA(z)\\
&= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|h^{(n-k)}(z)f^{(k)}(z)|^p(1-|z|^2)^{np-p+q}(1-|\varphi_a(z)|^2)^s dA(z)\\
&\le \|h\|_{\mathcal{B}}^p\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f^{(k)}(z)|^p(1-|z|^2)^{kp-p+q}(1-|\varphi_a(z)|^2)^s dA(z)\\
&= \|h\|_{\mathcal{B}}^p\|f\|_{s_1F(p,q,s)}^p\\
&\lesssim \|h\|_{\mathcal{B}}^p\|f\|_{sF(p,q,s)}^p\\
&\le \|h\|_{\mathcal{B}}^p\|f\|_{F(p,q,s)}^p,
\end{aligned}$$

which shows that the operator $T_h^{n,k}$ is bounded on $F(p,q,s)$, and

$$\|T_h^{n,k}\|_{F(p,q,s)\to F(p,q,s)} \lesssim \|h\|_{\mathcal{B}}. \tag{2.12}$$

If $h(0) = h'(0) = \cdots = h^{(n-k-1)}(0)$, then from (2.11) and (2.12), we get (2.8). □

To study the compactness of the operators, some applicable characterizations have been already obtained (see [12, 13, 17, 18, 22, 23]). The lemma below gives a characterization of the compactness of $T_{\vec{h}}^{(n)}$ on $F(p,q,s)$.

Lemma 2.8. *Let* $0 < p < +\infty$, $-2 < q < +\infty$, $0 \le s < +\infty$, $q+s-p > -1$, $n \in \mathbb{N}$, $\vec{h} = (h_0, h_1, \dots, h_{n-1})$ *with* $h_k \in H(\mathbb{D})$ *for* $k = 0, 1, \dots, n-1$ *and the operator* $T_{\vec{h}}^{(n)}$ *be bounded on* $F(p,q,s)$*. Then* $T_{\vec{h}}^{(n)}$ *is compact on* $F(p,q,s)$ *if and only if* $\|T_{\vec{h}}^{(n)} f_j\|_{F(p,q,s)} \to 0$ *as* $j \to +\infty$ *whenever* $(f_j)_{j\in\mathbb{N}}$ *is bounded in* $F(p,q,s)$ *and* $f_j \to 0$ *uniformly on any compact subset of* $\mathbb{D}$ *as* $j \to +\infty$.

Proof. Assume that for any bounded sequence $(f_j)_{j\in\mathbb{N}} \subset F(p,q,s)$ that converges to zero uniformly on compact subsets of $\mathbb{D}$, it follows that $\|T_{\vec{h}}^{(n)} f_j\|_{F(p,q,s)} \to 0$ as $j \to +\infty$. By Corollary 2.8 in [25], $F(p,q,s) \subset \mathcal{B}^{\frac{q+2}{p}}$, i.e., $\|f\|_{\mathcal{B}^{\frac{q+2}{p}}} \lesssim \|f\|_{F(p,q,s)}$. From this inequality and the estimations in $\mathcal{B}^{\frac{q+2}{p}}$ (see [14]), it follows that there exists a positive constant C independent of $f \in \mathcal{B}^{\frac{q+2}{p}}$ and $z \in \mathbb{D}$ such that

$$|f(z)| \le C\|f\|_{F(p,q,s)}.$$

Let $\sup_{j\in\mathbb{N}} \|f_j\|_{F(p,q,s)} = M$. Then, we get that $|f_j(z)| \le CM$ on $\{z : |z| \le r\}$ for each $j \in \mathbb{N}$. Since every compact subset K of $\mathbb{D}$ is contained in $\{z : |z| \le r\}$ for some $r > 0$, $(f_j)_{j\in\mathbb{N}}$ is uniformly bounded on every compact subset of $\mathbb{D}$. Hence, it follows from Montel's theorem that there is a subsequence $(f_{j_k})_{k\in\mathbb{N}}$ of $(f_j)_{j\in\mathbb{N}}$, which converges uniformly on every compact subset of $\mathbb{D}$ to an analytic function f. Since $f'_{j_k}(z) \to f'(z)$ uniformly on compacts of $\mathbb{D}$ as $k \to +\infty$, we have that $|f'_{j_k}(z)| \to |f'(z)|$ on each $z \in \mathbb{D}$ as $k \to +\infty$. By Fatou's lemma, we get that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q (1-|\varphi_a(z)|^2)^s dA(z) \le \liminf_{k\to+\infty} \|f_{j_k}\|^p_{sF(p,q,s)} \le M^p,$$

from which together with Lemma 2.4, we get that $f \in F(p,q,s)$. Then $(f_{j_k} - f)_{k\in\mathbb{N}}$ is a bounded sequence in $F(p,q,s)$ and $f_{j_k} - f \to 0$ uniformly on any compact subset of $\mathbb{D}$ as $k \to +\infty$. It follows from the assumption that $\|T_{\vec{h}}^{(n)}(f_{j_k} - f)\|_{F(p,q,s)} = 0$ as $k \to +\infty$, which shows that the operator $T_{\vec{h}}^{(n)}$ is compact on $F(p,q,s)$.

Conversely, assume that the operator $T_{\vec{h}}^{(n)}$ is compact on $F(p,q,s)$. Let B be the closed unit ball in $F(p,q,s)$. Then, the compactness of $T_{\vec{h}}^{(n)}$ shows that the set $T_{\vec{h}}^{(n)}(B)$ is a relatively compact subset of $F(p,q,s)$. Let $(f_j)_{j\in\mathbb{N}} \subset B$ be a sequence converging to zero uniformly on compacts of $\mathbb{D}$ as $j \to +\infty$. By the Cauchy's estimate, we have that $f_j^{(k)} \to 0$ uniformly on compacts of $\mathbb{D}$ as $j \to +\infty$ for each $k \in \mathbb{N}_0$. We need to prove that $\|T_{\vec{h}}^{(n)} f_j\|_{F(p,q,s)} \to 0$ as $j \to +\infty$. To this end, it is enough to prove that the zero function is a unique limit point of the sequence $(T_{\vec{h}}^{(n)} f_j)_{j\in\mathbb{N}}$. By the compactness of $T_{\vec{h}}^{(n)}$ on $F(p,q,s)$, there exists a function $f \in F(p,q,s)$ such that

$$\lim_{j\to+\infty} \|T_{\vec{h}}^{(n)} f_j - f\|_{F(p,q,s)} = 0. \tag{2.13}$$

From this and Lemma 2.5, it follows that

$$\left|h_0^{(n)}(z) f_j(z) + h_1^{(n-1)}(z) f_j'(z) + \cdots + h_{n-1}'(z) f_j^{(n-1)}(z) - f^{(n)}(z)\right| \to 0 \tag{2.14}$$

as $j \to +\infty$ for each $z \in \mathbb{D}$. By using the uniform convergence on compacts of the sequence $(f_j^{(k)})_{j\in\mathbb{N}}$ in (2.14), we get that $f^{(n)}(z) = 0$ for all $z \in \mathbb{D}$.

Furthermore, from the definition of $T_{\vec{h}}^{(n)}$, we have that $(T_{\vec{h}}^{(n)} f_j)^{(k)}(0) = 0$ for each $k = 0, 1, \dots, n-1$. Hence, from (2.13) and Lemma 2.5, it follows that $f^{(k)}(0) = 0$ for each $k = 0, 1, \dots, n-1$. From this and the Taylor formula for the function f, it follows that $f \equiv 0$. This shows that the zero function is a unique limit of $(T_{\vec{h}}^{(n)} f_j)_{j\in\mathbb{N}}$. □

Lemma 2.9. *Let $h \in H(\mathbb{D})$, $0 < p < +\infty$, $-2 < q < +\infty$, $0 \le s < +\infty$, $q + s - p > -1$, $n \in \mathbb{N}$, and $0 \le k \le n-1$. Then the operator $T_h^{n,k}$ is compact on $F(p, q, s)$ if and only if $h \in \mathcal{B}_0$.*

Proof. Let the operator $T_h^{n,k}$ be compact on $F(p, q, s)$. Then, it is bounded. Let $(k_{w_j,0})_{j\in\mathbb{N}}$, where $w_j \in \mathbb{D}$ satisfies that $|w_j| \to 1$ as $j \to +\infty$, be the sequence defined by

$$k_{w_j,0}(z) = \frac{1 - |w_j|^2}{(1 - \overline{w}_j z)^{\frac{q+2}{p}}}, \quad z \in \mathbb{D}.$$

By Lemma 2.2, the sequence $(k_{w_j,0})_{j\in\mathbb{N}}$ is bounded on $F(p, q, s)$ and converges to zero uniformly on compact subsets of $\mathbb{D}$ as $j \to +\infty$. Then, by Lemma 2.8, we have

$$\lim_{j\to+\infty} \|T_h^{n,k} k_{w_j,0}\|_{F(p,q,s)} = 0. \tag{2.15}$$

By the boundedness of $T_h^{n,k}$ and (2.9), we have

$$(1 - |w_j|^2)^{n-k} |h^{(n-k)}(w_j)| \lesssim \|T_h^{n,k} k_{w_j,0}\|_{F(p,q,s)}$$

for $|w_j| \ge 1/2$. From this and (2.15), it follows that

$$\lim_{j\to+\infty} (1 - |w_j|^2)^{n-k} |h^{(n-k)}(w_j)| = 0$$

for the sequence $(w_j)_{j\in\mathbb{N}} \subset \mathbb{D}$ satisfying the condition $|w_j| \to 1$ as $j \to +\infty$. By the arbitrariness of the sequence $(w_j)_{j\in\mathbb{N}} \subset \mathbb{D}$, we get

$$\lim_{|z|\to 1} (1 - |z|^2)^{n-k} |h^{(n-k)}(z)| = 0, \tag{2.16}$$

which is equivalent to $h \in \mathcal{B}_0$ (see Lemma 2.6).

Now, we assume that $h \in \mathcal{B}_0$, and then (2.16) holds. Hence, for every $\varepsilon > 0$, there is an $r \in (0, 1)$ such that

$$(1 - |z|^2)^{n-k} |h^{(n-k)}(z)| < \varepsilon \tag{2.17}$$

for $r < |z| < 1$.

Let $(f_j)_{j\in\mathbb{N}} \subset F(p, q, s)$ be a bounded sequence converging to zero uniformly on compacts of $\mathbb{D}$ as $j \to +\infty$. Then, the sequence $(f_j^{(k)})_{j\in\mathbb{N}}$ also converges to zero uniformly on compacts of $\mathbb{D}$ as $j \to \infty$. So, for every $\varepsilon > 0$, there is a $k_0 \in \mathbb{N}$ such that

$$|f_j^{(k)}(z)| < \varepsilon \tag{2.18}$$

for $j \geq k_0$ and every $|z| \leq r$, where r is chosen in (2.17). By the continuousness of $|h^{(n-k)}(z)|(1-|z|^2)^{np-p+q}$ on $\overline{r\mathbb{D}}$, we get that

$$M = \max_{z\in\overline{r\mathbb{D}}} |h^{(n-k)}(z)|^p(1-|z|^2)^{np-p+q} < +\infty.$$

Since $(T_h^{n,k} f_j)^{(l)}(0) = 0$ for $l = 0, 1, \dots, n-1$, by Lemma 2.4, we have

$$\|T_h^{n,k} f_j\|^p_{F(p,q,s)} \asymp \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(T_h^{n,k} f_j)^{(n)}(z)|^p(1-|z|^2)^{np-p+q}(1-|\varphi_a(z)|^2)^s dA(z). \tag{2.19}$$

Then, from (2.17), (2.18), (2.19), and Lemma 2.4, it follows that

$$\begin{aligned}\|T_h^{n,k} f_j\|^p_{F(p,q,s)} &\lesssim \sup_{a\in\mathbb{D}} \int_{\overline{r\mathbb{D}}} |h^{(n-k)}(z) f_j^{(k)}(z)|^p(1-|z|^2)^{np-p+q}(1-|\varphi_a(z)|^2)^s dA(z)\\ &\quad + \sup_{a\in\mathbb{D}} \int_{\mathbb{D}\setminus\overline{r\mathbb{D}}} |h^{(n-k)}(z) f_j^{(k)}(z)|^p(1-|z|^2)^{np-p+q}(1-|\varphi_a(z)|^2)^s dA(z)\\ &\lesssim \int_{\overline{r\mathbb{D}}} |h^{(n-k)}(z)|^p(1-|z|^2)^{np-p+q} dA(z)\varepsilon^p + \|f_j\|^p_{F(p,q,s)}\varepsilon^p\\ &\lesssim \Big(M + \sup_{j\in\mathbb{N}} \|f_j\|^p_{F(p,q,s)}\Big)\varepsilon^p.\end{aligned} \tag{2.20}$$

From (2.20) and since ε is an arbitrary positive number, we get that

$$\lim_{j\to+\infty} \|T_h^{n,k} f_j\|_{F(p,q,s)} = 0,$$

from which together with Lemma 2.8, it follows that $T_h^{n,k}$ is compact on $F(p, q, s)$. □

3. Boundedness and compactness

First, we characterize the boundedness of the operator $T^{(n)}_{\vec{h}}$ on $F(p, q, s)$.

Theorem 3.1. *Let* $0 < p < +\infty$, $-2 < q < +\infty$, $0 \leq s < +\infty$, $q + s - p > -1$, $n \in \mathbb{N}$, *and* $\vec{h} = (h_0, h_1, \dots, h_{n-1})$ *with* $h_k \in H(\mathbb{D})$ *for* $k = 0, 1, \dots, n-1$. *Then, the operator* $T^{(n)}_{\vec{h}}$ *is bounded on* $F(p, q, s)$ *if and only if the operators* $T^{n,k}_{h_k}$, $k = 0, 1, \dots, n-1$ *are bounded on* $F(p, q, s)$.

If the operator $T^{(n)}_{\vec{h}}$ *is bounded on* $F(p, q, s)$, *then*

$$\|T^{(n)}_{\vec{h}}\|_{F(p,q,s)\to F(p,q,s)} \asymp \sum_{k=0}^{n-1} \sup_{z\in\mathbb{D}} (1-|z|^2)^{n-k}|h_k^{(n-k)}(z)|. \tag{3.1}$$

Moreover, if $h_k(0) = h'_k(0) = \cdots = h_k^{(n-k-1)}(0) = 0$ *for each* $k = 0, 1, \dots, n-1$, *then*

$$\|T^{(n)}_{\vec{h}}\|_{F(p,q,s)\to F(p,q,s)} \asymp \sum_{k=0}^{n-1} \|h_k\|_{\mathcal{B}}. \tag{3.2}$$

Proof. Assume that the operator $T^{(n)}_{\vec{h}}$ is bounded on $F(p,q,s)$. For $w \in \mathbb{D}$ and $k \in \{0,1,\dots,n-1\}$, let $f_{w,k}$ be the function defined in Lemma 2.3. Then, it follows from Lemma 2.5 that

$$\begin{aligned}\frac{\|T^{(n)}_{\vec{h}} f_{w,k}\|_{sF(p,q,s)}}{(1-|w|^2)^{n+\frac{q+2}{p}-1}} &\gtrsim \left|\left(I^n(h_0^{(n)} f_{w,k} + h_1^{(n-1)} f'_{w,k} + \cdots + h_k^{(n-k)} f^{(k)}_{w,k} + \cdots + h'_{n-1} f^{(n-1)}_{w,k})\right)^{(n)}(w)\right| \\ &= \frac{|w|^k}{(1-|w|^2)^{k+\frac{q+2}{p}-1}} |h_k^{(n-k)}(w)|,\end{aligned}$$

which shows

$$\sup_{|w|>1/2} (1-|w|^2)^{n-k} |h_k^{(n-k)}(w)| \lesssim \|T^{(n)}_{\vec{h}}\|_{F(p,q,s)\to F(p,q,s)} < \infty. \tag{3.3}$$

Set $f_0(z) = 1$. Then, by Lemma 2.5,

$$\left|h_0^{(n)}(z)\right| = \left|(T^{(n)}_{\vec{h}} f_0)^{(n)}(z)\right| \lesssim \frac{\|T^{(n)}_{\vec{h}}\|_{F(p,q,s)\to F(p,q,s)}}{(1-|z|^2)^{n+\frac{q+2}{p}-1}},$$

which implies

$$(1-|z|^2)^{n+\frac{q+2}{p}-1} \left|h_0^{(n)}(z)\right| \lesssim \|T^{(n)}_{\vec{h}}\|_{F(p,q,s)\to F(p,q,s)},$$

and then

$$\sup_{|z|\le 1/2} (1-|z|^2)^n \left|h_0^{(n)}(z)\right| \lesssim \|T^{(n)}_{\vec{h}}\|_{F(p,q,s)\to F(p,q,s)}. \tag{3.4}$$

From (3.4), it follows that

$$\sup_{|z|\le 1/2} (1-|z|^2)^{n-1} |h_0^{(n)}(z)| \le \frac{4}{3} \sup_{|z|\le 1/2} (1-|z|^2)^n |h_0^{(n)}(z)| \lesssim \|T^{(n)}_{\vec{h}}\|_{F(p,q,s)\to F(p,q,s)}. \tag{3.5}$$

Consider $f_1(z) = z$. By Lemma 2.5, we have

$$\left|h_0^{(n)}(z) z + h_1^{(n-1)}(z)\right| = \left|(T^{(n)}_{\vec{h}} f_1)^{(n)}(z)\right| \lesssim \frac{\|T^{(n)}_{\vec{h}}\|_{F(p,q,s)\to F(p,q,s)}}{(1-|z|^2)^{n+\frac{q+2}{p}-1}}.$$

From this and (3.5), we get

$$\sup_{|z|\le 1/2} (1-|z|^2)^{n-1} \left|h_1^{(n-1)}(z)\right| \lesssim \|T^{(n)}_{\vec{h}}\|_{F(p,q,s)\to F(p,q,s)}.$$

Now, assume that we have proved

$$\sup_{|z|\le 1/2} (1-|z|^2)^{n-j} |h_j^{(n-j)}(z)| \lesssim \|T^{(n)}_{\vec{h}}\|_{F(p,q,s)\to F(p,q,s)} \tag{3.6}$$

for $j = 0,1,\dots,k-1$, where k is some number $2 \le k < n-1$. Set

$$f_k(z) = \frac{z^k}{k!}, \quad z \in \mathbb{D}.$$

Then, we have

$$\left|h_0^{(n)}(z)\frac{z^k}{k!}+h_1^{(n-1)}(z)\frac{z^{k-1}}{(k-1)!}+\cdots+h_k^{(n-k)}(z)\right|=\left|(T_{\vec{h}}^{(n)}f_k)^{(n)}(z)\right|\lesssim\frac{\|T_{\vec{h}}^{(n)}\|_{F(p,q,s)\to F(p,q,s)}}{(1-|z|^2)^{n+\frac{q+2}{p}-1}} \tag{3.7}$$

for $z\in\mathbb{D}$. From (3.7) and the assumption (3.6), we obtain

$$\sup_{|z|\le 1/2}(1-|z|^2)^{n-k}\left|h_k^{(n-k)}(z)\right|\lesssim\|T_{\vec{h}}^{(n)}\|_{F(p,q,s)\to F(p,q,s)}. \tag{3.8}$$

This shows that (3.6) holds for $k=0,1,\dots,n-1$. From (3.3) and (3.8), it follows that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{(n-k)}\left|h_k^{(n-k)}(z)\right|\lesssim\|T_{\vec{h}}^{(n)}\|_{F(p,q,s)\to F(p,q,s)}. \tag{3.9}$$

By Lemma 2.7, the operator $T_{h_k}^{n,k}$ is bounded on $F(p,q,s)$.

Conversely, assume that each operator $T_{h_k}^{n,k}$ is bounded on $F(p,q,s)$. Then, it is clear that the operator $T_{\vec{h}}^{(n)}$ is bounded on $F(p,q,s)$. It is easy to see that

$$\|T_{h_k}^{n,k}\|_{F(p,q,s)\to F(p,q,s)}\lesssim\sup_{z\in\mathbb{D}}(1-|z|^2)^{n-k}\left|h_k^{(n-k)}(z)\right|. \tag{3.10}$$

Since

$$T_{\vec{h}}^{(n)}=\sum_{k=0}^{n-1}T_{h_k}^{n,k},$$

we get

$$\|T_{\vec{h}}^{(n)}\|_{F(p,q,s)\to F(p,q,s)}\lesssim\sum_{k=0}^{n-1}\sup_{z\in\mathbb{D}}(1-|z|^2)^{n-k}\left|h_k^{(n-k)}(z)\right|. \tag{3.11}$$

From (3.9) and (3.11), the relation (3.1) follows, and then from (3.1) and Lemma 2.6, the relation (3.2) also follows. □

Now, we characterize the compactness of the operator $T_{\vec{h}}^{(n)}$ on $F(p,q,s)$.

Theorem 3.2. *Let* $0<p<+\infty$, $-2<q<+\infty$, $0\le s<+\infty$, $q+s-p>-1$, $n\in\mathbb{N}$, *and* $\vec{h}=(h_0,h_1,\dots,h_{n-1})$ *with* $h_k\in H(\mathbb{D})$ *for* $k=0,1,\dots,n-1$. *Let* $h_k(0)=h_k'(0)=\cdots=h_k^{(n-k-1)}(0)=0$ *for each* $k=0,1,\dots,n-1$. *Then, the operator* $T_{\vec{h}}^{(n)}$ *is compact on* $F(p,q,s)$ *if and only if the operators* $T_{h_k}^{n,k}$ *for* $k=0,1,\dots,n-1$ *are compact on* $F(p,q,s)$.

Proof. Let the operator $T_{\vec{h}}^{(n)}$ be compact on $F(p,q,s)$. Then, it is bounded. Let $(w_j)_{j\in\mathbb{N}}\subset\mathbb{D}$ be such that $|w_j|\to 1$ as $j\to+\infty$, and $k\in\{0,1,\dots,n-1\}$. Then, the sequence $(f_{w_j,k})_{j\in\mathbb{N}}$ defined in Lemma 2.2 is bounded in $F(p,q,s)$ and converges uniformly to zero on compacts of $\mathbb{D}$ as $j\to+\infty$. By the compactness of $T_{\vec{h}}^{(n)}$, Lemma 2.5, and Lemma 2.8, we have

$$\lim_{j\to+\infty}(1-|w_j|^2)^{n-k}|h_k^{(n-k)}(w_j)|\lesssim\lim_{j\to+\infty}\|T_{\vec{h}}^{(n)}f_{w_j,k}\|_{sF(p,q,s)}=0,$$

from which and by the arbitrariness of $(w_j)_{j\in\mathbb{N}}$, it follows that

$$\lim_{|z|\to 1}(1-|z|^2)^{n-k}|h_k^{(n-k)}(z)|=0.$$

This and Lemma 2.6 show that $h_k \in \mathscr{B}_0$. Then, it follows from Lemma 2.9 that the operator $T_{h_k}^{n,k}$ is compact on $F(p,q,s)$.

Conversely, assume that each operator $T_{h_k}^{n,k}$ is compact on $F(p,q,s)$. Since

$$T_{\vec{h}}^{(n)}=\sum_{k=0}^{n-1}T_{h_k}^{n,k},$$

we get that $T_{\vec{h}}^{(n)}$ is compact on $F(p,q,s)$. □

4. Conclusions

In the paper, we investigate the operator $T_{\vec{h}}^{(n)}$ on the space $F(p,q,s)$. By using the characterizations of the space, we establish the boundedness and compactness of this operator. We hope this research will further stimulate scholarly interest.

Use of Generative-AI tools declaration

The author(s) declare(s) they have used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that he has no conflict of interest.

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