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Research article

Global asymptotic stability and qualitative analysis of fuzzy difference equations with composite exponential saturation and parabolic fuzzy parameters

Xiong Xiao[1,2] **and Qianhong Zhang**[1,2,*]

[1] School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guiyang 550025, China

[2] School of Big Data Statistics, Guizhou University of Finance and Economics, Guiyang 550025, China

* **Correspondence:** Email: zqianhong68@163.com.cn; Tel: 085188510734.

Abstract: This paper investigates the dynamic behavior of a class of nonlinear fuzzy difference equations with compound exponential terms. By using g-division, Lyapunov stability analysis, algebraic inequalities, and mathematical induction, the existence and uniqueness of positive fuzzy solutions are proved. In addition, the boundedness and persistence of these solutions are established, along with the global asymptotic stability of the unique positive equilibrium. Finally, numerical simulations are performed to verify the theoretical results. These findings refine the analysis framework for fuzzy systems with compound nonlinear terms. They also have potential applications in biological population control, economic resource allocation, and engineering systems with uncertain parameters.

Keywords: fuzzy difference equations; composite exponential terms; global asymptotic stability; Lyapunov stability; g-division
Mathematics Subject Classification: 39A10

1. Introduction

In the real world, observational data from numerous dynamic processes exist in the form of discrete time series. To explore the global behavior of these dynamic processes, researchers continually seek effective mathematical tools. In this context, difference equations (DEs) have emerged, offering a clear depiction of the evolutionary trajectory of systems over time (see [1–3]). Consequently, the theory of difference equations has been extensively adopted by researchers across various disciplines, such as ecology, economics, genetics, epidemiology, engineering informatics, and sciences (see [4–6]).

However, with the rapid development of science and technology, as well as the increasing complexity of application scenarios, the system modeling challenges encountered in research have become increasingly intricate. The limitations of classical difference equations have gradually become apparent. When systems involve an abundance of ambiguous and uncertain influencing factors, classical difference equations that rely on precise parameters often fall short in providing adequate explanations. It is worth noting that although stochastic models can handle some types of uncertainty, they usually require known probability distributions of noise and assume that random events are independent [7]. In real systems, such as in ecology and economics, uncertainty often arises from subjectivity in human judgment or incomplete information. These fuzzy kinds of information cannot be properly characterized by probability measures. Therefore, it is necessary to introduce fuzzy methods to compensate for the shortcomings of stochastic models [8].

To break through this bottleneck, scholars integrated the fuzzy set theory [9] proposed by Zadeh in 1965 with classical DEs, thereby pioneering fuzzy DEs. Essentially, this approach extends precise parameters or initial conditions in the model into fuzzy sets, enabling the precise characterization of fuzzy and uncertain factors within systems and offering a more comprehensive description of the dynamic behavior of complex systems. As a result, fuzzy DEs have become an effective mathematical tool for representing and handling fuzzy and uncertain information in systems. Subsequently, this paper will briefly review the development history of both classical and fuzzy DEs. With the burgeoning interest in fuzzy set theory, the theoretical value and application potential of fuzzy DEs has become increasingly prominent. They have rapidly become an important emerging branch in the study of discrete dynamical systems (see [10–12]). In addition to theoretical development, fuzzy DEs also show broad application prospects in practical engineering. For example, in the recovery control of islanded microgrids, communication link failures introduce uncertain time delays and data loss. A fully distributed event-triggered secondary control strategy can use fuzzy DEs to model the system dynamics, thereby effectively addressing communication uncertainties and achieving voltage and frequency restoration (see [13, 14]). Such applications further highlight the ability of fuzzy DEs to provide robust solutions in complex uncertain environments [15].

The following is a brief review of the development of classical and fuzzy DEs. In 2001, El-Metwally et al. [16] pioneered the investigation of properties such as periodicity and boundedness in exponential-type difference equations with the following form:

$$z_{n+1} = \omega + \beta z_{n-1} e^{-z_n}, \ n = 0, 1, \cdots,$$

where ω, β, and z_j $(j = -1, 0)$ are positive real numbers ($\Re^+$).

In 2006, Ozturk et al. [17] systematically analyzed the convergence, boundedness, and periodicity of positive solutions (PS) to the exponential-type DEs,

$$z_{n+1} = \frac{\omega + \beta e^{-z_n}}{\psi + z_{n-1}}, \ n = 0, 1, \cdots,$$

where ω, β, ψ, z_0, $z_{-1} \in \Re^+$. Ozturk et al. [18] further advanced the investigation by delving into the boundedness and global behavior of high-order composite exponential-type DEs,

$$z_{n+1} = \frac{\omega e^{-(n z_n + (n-j) z_{n-j})}}{\psi + n z_n + (n-j) z_{n-j}}, \ j \in \{1, 2, \ldots\}, n = 0, 1, \cdots,$$

where ω, ψ, $z_i \in \mathfrak{R}^+$ $(i = -j, -j+1, \cdots, -1, 0)$.

In 2013, Bozkurt et al. [19] systematically analyzed the global behavior of positive solutions and properties, such as semi-cycles for nonlinear difference equations

$$z_{n+1} = \frac{\omega e^{-z_n} + \beta e^{-z_{n-1}}}{\gamma + \omega z_n + \beta z_{n-1}}, \ n = 0, 1, \cdots,$$

where $\omega, \beta, \gamma, z_i \in \mathfrak{R}^+ (i = -1, 0)$.

In 2019, Khan et al. [20] analyzed the global dynamics of three systems of high-order exponential DE, establishing conditions for the uniqueness, stability, and other properties of the PS for each system. In 2024, Babu et al. [21] focused on a discrete exponential population model with mutualistic interactions, investigating the boundedness and global dynamics of system solutions, thereby revealing the influence of mutualism mechanisms on population dynamics.

Regarding fuzzy DEs, in 2010, Stefanidou et al. [22] investigated the existence of both positive fuzzy solutions (PFS) and non-negative equilibria, and the convergence criteria of PFS to the equilibrium for a class of exponential-type fuzzy DEs

$$z_{n+1} = \left(1 - \sum_{k=0}^{j-1} z_{n-k}\right)\left(1 - e^{-\psi z_n}\right), \ n = 0, 1, \ldots, \ j \in \{2, 3, \ldots\},$$

where $z_i (i = -j+1, -j+2, \ldots, 0)$ and ψ are positive fuzzy numbers (FNs). Inspired by this, Zhang et al. [23] systematically analyzed the boundedness and asymptotic behavior of PFS for first-order fuzzy Riccati DE, and presented an example to validate the effectiveness of their conclusions.

Furthermore, in 2020, Zhang et al. [24] conducted an in-depth analysis of the existence of PFS and the global asymptotic behavior for second-order fuzzy exponential models

$$z_{n+1} = \frac{A + B e^{-z_n}}{C + z_{n-1}}, \ n = 0, 1, \ldots,$$

where A, B, C, and z_i $(i = -1, 0)$ are positive FNs. Spurred by studies [22–24], in recent years we have witnessed a growing focus on fuzzy DEs with exponential terms, whose models have become increasingly sophisticated [25].

In 2025, Lin et al. [26] investigated a nonlinear fuzzy difference model with exponential terms

$$z_{n+1} = \frac{C z_n + D e^{-z_n}}{B + A z_{n-1}}, \ n = 0, 1, \ldots,$$

where C, D, B, A, and z_i $(i = -1, 0)$ are positive FNs. They pointed out that this study not only provides insights for the design of robust systems under uncertainty, but also extends the classical stability criteria for nonlinear fuzzy DEs.

In the same year, Saud et al. [27] utilized the g-division to analyze the dynamics of a third-order fuzzy DE with exponential decay

$$z_{n+1} = \frac{\eta e^{-z_n} + \phi e^{-z_{n-1}}}{\xi + z_{n-2}}, \ n = 0, 1, \ldots,$$

where ϕ, η, ξ, and z_i $(i = -2, -1, 0)$ are positive FNs. Additionally, some scholars extend their focus beyond individual equations to investigate the dynamic behavior of two-dimensional fuzzy systems, as reflected in literature [28, 29] and the references therein.

These studies have provided crucial support for the theoretical framework of fuzzy difference equations, yet there remain potential directions for breakthroughs. First, existing models predominantly incorporate a single exponential term, which is insufficient for capturing the complex mechanisms of multi-factor synergistic interactions in practical systems. Second, the types of FN are often limited to triangular FN, lacking flexibility in the representation of uncertainty [30,31]. Third, for fuzzy DEs containing composite exponential terms, there is a lack of systematic theoretical methods to analyze their dynamic behaviors, such as the boundedness of solutions and the verification of local or global stability.

In view of the above shortcomings, this paper proposes a new class of nonlinear fuzzy DEs with composite exponential terms, formulated as

$$x_{n+1} = \frac{A + Bx_n + Ce^{-(x_n+x_{n-1})}}{D + x_{n-1}},\ n = 0, 1, \ldots, \tag{1.1}$$

where the parameters D, A, B, C, and x_i $(i = 0, 1)$ are chosen as positive parabolic FNs. It is worth emphasizing that the composite exponential term $e^{-(x_n+x_{n-1})}$ introduces a unique first-order lag effect by coupling the current and previous states. From a practical perspective, the coupling inside the exponent $e^{-(x_n+x_{n-1})} = e^{-x_n}e^{-x_{n-1}}$ reflects that, for example, the inhibition of population reproduction by accumulated pollutants or the constraint of future growth by past resource consumption requires time accumulation. Putting $x_n + x_{n-1}$ inside the exponent means that the decay is synergistic. The exponential term is significantly compressed only when both states are large at the same time, triggering strong inhibition. This is essentially different from models with separate exponential terms outside the exponent (such as $e^{-x_n} + e^{-x_{n-1}}$), which allow either moment to independently cause saturation. Furthermore, this composite exponential term also introduces a synergistic saturation mechanism. The saturation threshold of the model is determined by $x_n + x_{n-1}$. Unlike a single-exponential model that saturates immediately after x_n exceeds a threshold, in our model, if x_n is very large but x_{n-1} is very small, $x_n + x_{n-1}$ may not reach the saturation bound, and the system response remains unsaturated. Only when both consecutive states are large does the composite exponential term trigger significant negative feedback. This mechanism effectively models the historical cumulative pressure and synergistic saturation characteristics of a system under continuous disturbances. It thus distinguishes itself from traditional single-point strong damping models.

The main innovations and contributions of this paper are as follows:

- We propose a new nonlinear fuzzy DEs model with a composite exponential term.
- We adopt parabolic FNs instead of traditional triangular ones, which significantly improves the flexibility of uncertainty representation and scenario adaptability.
- We systematically establish a theoretical framework for the boundedness and stability of positive fuzzy solutions for model (1.1). By constructing a Lyapunov function, we provide criteria for the uniqueness and global asymptotic behavior of the solutions.

The research framework of this paper is structured as follows. In Section 2, we elaborate on the essential definitions employed throughout the study. In Section 3, we investigate the uniqueness of positive solutions for model (1.1) based on g-division and examine its dynamic behaviors by constructing a Lyapunov function. In Section 4, we visually validate the conclusions derived in the preceding section through two numerical simulations, ensuring the validity of the findings. In Section 5, we provide a concise summary of the principal conclusions drawn from the study.

2. Some definitions

Prior to delving into the core research, this section begins with a review of the definitions and established results employed in this paper. These basic concepts are the foundation for later studies and help readers understand what comes next.

Definition 2.1. [12,32] A function $f_1 : \mathfrak{R} \to [0,1]$ satisfying the following condition (i)–(iv) is termed a fuzzy number.
(i) f_1 is normal, i.e., there exists $t \in \mathfrak{R}$ such that $f_1(t) = 1$;
(ii) f_1 is fuzzy convex, i.e., for any m satisfying $0 \leq m \leq 1$ and $t_1,\ t_2 \in \mathfrak{R}$, the following relation holds:

$$f_1(mt_1 + (1-m)t_2) \geq min(f_1(t_1), f_1(t_2));$$

(iii) f_1 satisfies the upper semi-continuity condition;
(iv) The support of f_1 is compact, where its support is defined as supp$f_1 = \overline{\bigcup_{\alpha\in(0,1]}[f_1]_\alpha} = \overline{\{s \in \mathfrak{R} : f_1(s) > 0\}}$.

For any $0 < \alpha \leq 1$, the set $[\varpi]_\alpha = \{t \in \mathfrak{R} : \varpi(t) \geq \alpha\}$ is referred to as the α-cut of the FN ϖ. The support set of ϖ is defined as supp $\varpi = [\varpi]_0 = \overline{\{t \in \mathfrak{R} \mid \varpi(t) > 0\}}$. The FN ϖ is deemed positive (or negative) if its support supp ϖ is a subset of $(0, +\infty)$ (or supp $\varpi \subset (-\infty, 0)$).In particular, if for all $\alpha \in (0,1]$, the relation $[\varpi]_\alpha = [\varpi, \varpi]$ holds, then $\varpi \in \mathfrak{R}^+$. This case is sometimes referred to in fuzzy number theory as a trivial fuzzy number.

Definition 2.2. [33] Let X be a FN. Then, X can be represented by two functions $X_l(\alpha)$ and $X_r(\alpha)$, where $0 \leq \alpha \leq 1$, satisfying the following conditions (i)–(iii).
(i) $X_l(\alpha)$ and $X_r(\alpha)$ are bounded and left-continuous functions;
(ii) $X_l(\alpha)$ is monotonically non-decreasing, and $X_r(\alpha)$ is monotonically non-increasing;
(iii) $X_l(\alpha) \leq X_r(\mathbf{1})$, $\alpha \in [0,1]$.

Let $\mathfrak{R}_f$ denote the space of FN, defined as

$$\mathfrak{R}_f = \{u \mid u = (u_l(\alpha), u_r(\alpha)), \alpha \in [0,1]\}.$$

Under this representation, $\mathfrak{R}_f^+$ denotes the set of all non-negative FN.

Definition 2.3. [30] A fuzzy number $\tilde{P} = (p_1, p_2, p_3)$ is a parabolic FN if its membership function $F_{\tilde{P}}(t)$ satisfies the following form:

$$F_{\tilde{P}}(t) = \begin{cases} 1 - \left(\frac{p_2 - t}{p_2 - p_1}\right)^2, & p_1 \leq t \leq p_2, \\ 1, & t = p_2, \\ 1 - \left(\frac{t - p_2}{p_3 - p_2}\right)^2, & p_2 \leq t \leq p_3, \\ 0, & \text{otherwise}. \end{cases} \tag{2.1}$$

Furthermore, the α-cut of $\tilde{P}$ is given by $[\tilde{P}]_\alpha = \{t \in \mathfrak{R} : \tilde{P}(t) \geq \alpha\} = \left[p_2 - (p_2 - p_1)\sqrt{1-\alpha}, p_2 + (p_3 - p_2)\sqrt{1-\alpha}\right]$.

Remark 2.1. The α-cut of a parabolic FN is $[\tilde{P}]_\alpha = [p_2-(p_2-p_1)\sqrt{1-\alpha},\ p_2+(p_3-p_2)\sqrt{1-\alpha}]$, where $\sqrt{1-\alpha}$ reflects the non-linearity of the membership function. In the g-division $\kappa = \vartheta \div_g \varpi$, the operation is performed α-level. At each α-level, the division applies linear interval operations (multiplication, inversion, etc.) to the endpoints. However, these endpoints themselves are non-linear functions of α because they contain $\sqrt{1-\alpha}$. Consequently, the endpoints of $[\kappa]_\alpha$ become algebraic combinations of $\sqrt{1-\alpha}$, and the resulting fuzzy number generally does not preserve the parabolic shape. This shows that the g-division works directly on α-cuts and handles the non-linearity of parabolic FNs without assuming any specific analytical form for the membership function of the result.

Definition 2.4. [18] Assume $x_n,\ x \in \Re_f^+$. If x_n satisfies (1.1), then the sequence $\{x_n\}$ is said to be PFS of (1.1). If x satisfies the relation

$$x = \frac{A + Bx + Ce^{-2x}}{D + x},$$

then we call x a non-negative equilibrium of (1.1).

Definition 2.5. [10,32] We define the distance between two FNs in the fuzzy space as follows:

$$D(\eta, \phi) = \sup_{\alpha\in[0,1]} \max\left\{|\eta_{l,\alpha} - \phi_{l,\alpha}|, |\eta_{r,\alpha} - \phi_{r,\alpha}|\right\},$$

where $\eta, \phi \in \Re_f$.

Definition 2.6. [32] If $\lim\limits_{n\to\infty} D(\varpi_n, \varpi) = 0$ holds, then it indicates that $\varpi_n \to \varpi$ as $n \to \infty$, where $\varpi_n,\ \varpi \in \Re_f^+$ and ϖ_n is a sequence of positive FNs.

Definition 2.7. [26,32] Let $\vartheta,\ \varpi \in \Re_f$, $\alpha \in [0,1]$, and $\lambda \in \Re$. Their arithmetic operations are defined as follows.
(i) $[\vartheta + \varpi]_\alpha = [\vartheta_l(\alpha) + \varpi_l(\alpha), \vartheta_r(\alpha) + \varpi_r(\alpha)]$;
(ii) $[\vartheta - \varpi]_\alpha = [\vartheta_l(\alpha) - \varpi_r(\alpha), \vartheta_r(\alpha) - \varpi_l(\alpha)]$;
(iii) $[\lambda\vartheta]_\alpha = \begin{cases} [\lambda\vartheta_l(\alpha), \lambda\vartheta_r(\alpha)], & \text{if } \lambda \geq 0, \\ [\lambda\vartheta_r(\alpha), \lambda\vartheta_l(\alpha)], & \text{if } \lambda < 0. \end{cases}$;
(iv) $[\vartheta\varpi]_\alpha = [\min\{\vartheta_l(\alpha)\varpi_l(\alpha), \vartheta_l(\alpha)\varpi_r(\alpha), \vartheta_r(\alpha)\varpi_l(\alpha), \vartheta_r(\alpha)\varpi_r(\alpha)\},$
$\max\{\vartheta_l(\alpha)\varpi_l(\alpha), \vartheta_l(\alpha)\varpi_r(\alpha), \vartheta_r(\alpha)\varpi_l(\alpha), \vartheta_r(\alpha)\varpi_r(\alpha)\}]$;
(v) $\left[\frac{1}{\varpi}\right]_\alpha = \left[\min\left\{\frac{1}{\varpi_l(\alpha)}, \frac{1}{\varpi_r(\alpha)}\right\}, \max\left\{\frac{1}{\varpi_l(\alpha)}, \frac{1}{\varpi_r(\alpha)}\right\}\right]$, if $0 \notin [\varpi]_\alpha$.

Definition 2.8. [34] Let $\vartheta, \varpi \in \Re_f$ with α-cuts $[\vartheta]_\alpha = [\vartheta_{l,\alpha}, \vartheta_{r,\alpha}]$, $[\varpi]_\alpha = [\varpi_{l,\alpha}, \varpi_{r,\alpha}]$, where $0 \notin [\varpi]_\alpha$, $\forall\alpha \in [0,1]$. Given a fuzzy number $\kappa = \vartheta \div_g \varpi$, its α-cuts $[\kappa]_\alpha = [\kappa_{l,\alpha}, \kappa_{r,\alpha}]$ satisfy the following equivalent relations:

$$[\kappa]_\alpha = [\vartheta]_\alpha \div_g [\varpi]_\alpha \iff \begin{cases} (i) \quad [\vartheta]_\alpha = [\varpi]_\alpha[\kappa]_\alpha, \\ \text{or} \\ (ii) \quad [\varpi]_\alpha = [\vartheta]_\alpha[\kappa]_\alpha^{-1}. \end{cases} \tag{2.2}$$

This division operation is called the g-division (denoted by $\div_g$), where $[\kappa]_\alpha^{-1} = [1/\kappa_{r,\alpha},\ 1/\kappa_{l,\alpha}]$. Furthermore, κ is required to satisfy the conditions that $\kappa_{l,\alpha}$ is non-decreasing, $\kappa_{r,\alpha}$ is non-increasing, and $\kappa_{l,1} \leq \kappa_{r,1}$.

Theorem 2.1. [34] Assume there exists a family of convex, nonempty, and compact subsets in $\mathfrak{R}$, denoted by $\{\kappa_\alpha : \alpha \in [0,1]\}$, such that conditions (i)–(iii) are satisfied.
(i) $\overline{\bigcup \kappa_\alpha} \subset \kappa_0$;
(ii) If $\alpha_1 \leq \alpha_2$, then $\kappa_{\alpha_2} \subset \kappa_{\alpha_1}$;
(iii) If $\alpha_i \uparrow \alpha > 0$, then $\kappa_\alpha = \bigcap_{i\geq 1} \kappa_{\alpha_i}$.
Then, there exists $\vartheta \in \mathfrak{R}_f^n$ such that for all $\alpha \in (0,1]$, its α-cuts set satisfies $[\vartheta]_\alpha = \kappa_\alpha$, while its 0-cuts set satisfies $[\vartheta]_0 = \overline{\bigcup_{0<\alpha\leq 1} \kappa_\alpha} \subset \kappa_0$.

Definition 2.9. [12,24] A sequence of positive FNs (ϖ_n) is said to be persistent (bounded) if it satisfies the following relation:

$$\operatorname{supp} \varpi_n \subset [P, \infty)(\text{resp.}\operatorname{supp} \varpi_n \subset (0, Q]),\ n \in N^+,$$

where $P,\ Q \in \mathfrak{R}^+$. If the condition $\operatorname{supp} \varpi_n \subset [P, Q]$ holds, the sequence is considered boundedly persistent. Furthermore, if $\|\varpi_n\|$ is unbounded, then the sequence is also unbounded.

In the following sections, we will frequently use the α-cut representation of FN. For ease of reading, the main symbols and their meanings are listed in Table 1.

Table 1. Description of important symbols.

Symbol	Description
$[x_n]_\alpha = [x_{n,l,\alpha},\ x_{n,r,\alpha}]$	α-cut of the fuzzy number x_n
$[x_{n,l,\alpha},\ x_{n,r,\alpha}]$	Left and right endpoints of $[x_n]_\alpha$
$[D]_\alpha = [D_{l,\alpha},\ D_{r,\alpha}]$	α-cut of D
$[D_{l,\alpha},\ D_{r,\alpha}]$	Left and right endpoints of $[D]_\alpha$
$[A]_\alpha = [A_{l,\alpha},\ A_{r,\alpha}]$	α-cut of A
$[A_{l,\alpha},\ A_{r,\alpha}]$	Left and right endpoints of $[A]_\alpha$
$[B]_\alpha = [B_{l,\alpha},\ B_{r,\alpha}]$	α-cut of B
$[B_{l,\alpha},\ B_{r,\alpha}]$	Left and right endpoints of $[B]_\alpha$
$[C]_\alpha = [C_{l,\alpha},\ C_{r,\alpha}]$	α-cut of C
$[C_{l,\alpha},\ C_{r,\alpha}]$	Left and right endpoints of $[C]_\alpha$

Other fuzzy numbers are denoted similarly.

3. Key theoretical outcomes

In this section, we present theoretical proofs concerning model (1.1). First, we establish the existence of a unique PFS. Subsequently, for the two cases arising from the $\div_g$ of fuzzy numbers, we study separately several qualitative characteristics of the ordinary difference system in each case. Finally, we conduct an in-depth investigation into the dynamical behaviors of model (1.1) for each of these two cases.

3.1. *Model* (1.1) *admits a unique PFS*

We introduce Lemma 3.1 to facilitate the subsequent proof of the theorem, thereby establishing the existence of a unique PFS for model (1.1).

Lemma 3.1. [32] Assume K is a continuous function $K : (0,+\infty)^3 \to (0,+\infty)$. Then, for any $\varpi,\ \vartheta,\ \mu \in \Re_f^+$, the following relation holds

$$[K(\varpi,\vartheta,\mu)]_\alpha = K([\varpi]_\alpha, [\vartheta]_\alpha, [\mu]_\alpha),\ 0<\alpha\le 1. \tag{3.1}$$

Theorem 3.1. Consider the fuzzy DE (1.1), where D, A, B, C, and x_i $(i=-1,0)$ are all positive FNs. Then, (1.1) admits a unique fuzzy positive solution.

Proof. Given this similarity to the proof of Theorem 3.1 in [11], certain details have been omitted. Under the assumption that a sequence of FNs (x_n) satisfies model (1.1), for $x_i>0$ $(i=-1,0)$, we consider the α-cut, $0<\alpha\le 1$, $n=0,1,2,\cdots$,

$$\begin{aligned}&[x_n]_\alpha = [x_{n,l,\alpha}, x_{n,r,\alpha}],\ [D]_\alpha = [D_{l,\alpha}, D_{r,\alpha}],\ [A]_\alpha = [A_{l,\alpha}, A_{r,\alpha}],\\ &[B]_\alpha = [B_{l,\alpha}, B_{r,\alpha}],\ [C]_\alpha = [C_{l,\alpha}, C_{r,\alpha}].\end{aligned} \tag{3.2}$$

By applying Lemma 3.1, (1.1), and (3.2), we can derive

$$\begin{aligned}[x_{n+1}]_\alpha = [x_{n+1,l,\alpha}, x_{n+1,r,\alpha}] &= \left[\frac{A + Bx_n + Ce^{-(x_n+x_{n-1})}}{D + x_{n-1}}\right]_\alpha\\ &= \frac{[A]_\alpha + [B]_\alpha \times [x_n]_\alpha + [C]_\alpha \times \left[e^{-(x_n+x_{n-1})}\right]_\alpha}{[D]_\alpha + [x_{n-1}]_\alpha}\\ &= \frac{\left[A_{l,\alpha} + B_{l,\alpha}x_{n,l,\alpha} + C_{l,\alpha}e^{-(x_{n,r,\alpha}+x_{n-1,r,\alpha})}, A_{r,\alpha} + B_{r,\alpha}x_{n,r,\alpha} + C_{r,\alpha}e^{-(x_{n,l,\alpha}+x_{n-1,l,\alpha})}\right]}{[D_{l,\alpha} + x_{n-1,l,\alpha}, D_{r,\alpha} + x_{n-1,r,\alpha}]}.\end{aligned}$$

Using $\div_g$, we arrive at the following two cases:
Case I:

$$\begin{aligned}&[x_{n+1}]_\alpha = [x_{n+1,l,\alpha}, x_{n+1,r,\alpha}]\\ &= \left[\frac{A_{l,\alpha} + B_{l,\alpha}x_{n,l,\alpha} + C_{l,\alpha}e^{-(x_{n,r,\alpha}+x_{n-1,r,\alpha})}}{D_{l,\alpha} + x_{n-1,l,\alpha}}, \frac{A_{r,\alpha} + B_{r,\alpha}x_{n,r,\alpha} + C_{r,\alpha}e^{-(x_{n,l,\alpha}+x_{n-1,l,\alpha})}}{D_{r,\alpha} + x_{n-1,r,\alpha}}\right].\end{aligned} \tag{3.3}$$

Case II:

$$\begin{aligned}&[x_{n+1}]_\alpha = [x_{n+1,l,\alpha}, x_{n+1,r,\alpha}]\\ &= \left[\frac{A_{r,\alpha} + B_{r,\alpha}x_{n,r,\alpha} + C_{r,\alpha}e^{-(x_{n,l,\alpha}+x_{n-1,l,\alpha})}}{D_{r,\alpha} + x_{n-1,r,\alpha}}, \frac{A_{l,\alpha} + B_{l,\alpha}x_{n,l,\alpha} + C_{l,\alpha}e^{-(x_{n,r,\alpha}+x_{n-1,r,\alpha})}}{D_{l,\alpha} + x_{n-1,l,\alpha}}\right].\end{aligned} \tag{3.4}$$

Now, considering Case I, i.e., $\frac{A_{l,\alpha}+B_{l,\alpha}x_{n,l,\alpha}+C_{l,\alpha}e^{-(x_{n,r,\alpha}+x_{n-1,r,\alpha})}}{A_{r,\alpha}+B_{r,\alpha}x_{n,r,\alpha}+C_{r,\alpha}e^{-(x_{n,l,\alpha}+x_{n-1,l,\alpha})}} \le \frac{D_{l,\alpha}+x_{n-1,l,\alpha}}{D_{r,\alpha}+x_{n-1,r,\alpha}}$, for $n\ge 0$, $0<\alpha\le 1$, from (1.1), we obtain

$$\begin{cases} x_{n+1,l,\alpha} = \frac{A_{l,\alpha}+B_{l,\alpha}x_{n,l,\alpha}+C_{l,\alpha}e^{-(x_{n,r,\alpha}+x_{n-1,r,\alpha})}}{D_{l,\alpha}+x_{n-1,l,\alpha}},\\ \\ x_{n+1,r,\alpha} = \frac{A_{r,\alpha}+B_{r,\alpha}x_{n,r,\alpha}+C_{r,\alpha}e^{-(x_{n,l,\alpha}+x_{n-1,l,\alpha})}}{D_{r,\alpha}+x_{n-1,r,\alpha}}.\end{cases} \tag{3.5}$$

Therefore, we can simply conclude that for any $(x_{i,l,\alpha}, x_{i,r,\alpha})$, where $i=-1,0$, $\alpha\in(0,1]$, (3.5) has a solution $(x_{n,l,\alpha}, x_{n,r,\alpha})$. Next, we need to prove that a solution x_n of model (1.1), defined by $[x_{n,l,\alpha}, x_{n,r,\alpha}]$, satisfies

$$[x_n]_\alpha = [x_{n,l,\alpha}, x_{n,r,\alpha}], n=0,1,2,\cdots, \tag{3.6}$$

where $0 < \alpha \leq 1$, and x_i $(i = -1, 0)$ are the initial conditions.

Since D, A, B, C, and x_i $(i = -1, 0)$ are positive FNs, it follows that for $n = 1$ and for any $\alpha \in (0, 1]$, we have that the α-cut of $x_1 = \frac{A+Bx_0+Ce^{-(x_0+x_{-1})}}{D+x_{-1}}$ is $[x_{1,l,\alpha}, x_{1,r,\alpha}]$. Therefore,

$$[x_{1,l,\alpha}, x_{1,r,\alpha}] = \left[\frac{A_{l,\alpha} + B_{l,\alpha}x_{0,l,\alpha} + C_{l,\alpha}e^{-(x_{0,r,\alpha}+x_{-1,r,\alpha})}}{D_{l,\alpha} + x_{-1,l,\alpha}}, \right.$$
$$\left.\frac{A_{r,\alpha} + B_{r,\alpha}x_{0,r,\alpha} + C_{r,\alpha}e^{-(x_{0,l,\alpha}+x_{-1,l,\alpha})}}{D_{r,\alpha} + x_{-1,r,\alpha}}\right] = \frac{[A]_\alpha + [B]_\alpha \times [x_0]_\alpha + [C]_\alpha \times \left[e^{-(x_0+x_{-1})}\right]_\alpha}{[D]_\alpha + [x_{-1}]_\alpha}.$$

Using mathematical induction, we first assume that the α-cut of x_k is $[x_{k,l,\alpha}, x_{k,r,\alpha}]$, meaning that $[x_k]_\alpha = [x_{k,l,\alpha}, x_{k,r,\alpha}]$. We now need to prove that the α-cut of $x_{k+1} = \frac{A+Bx_k+Ce^{-(x_k+x_{k-1})}}{D+x_{k-1}}$ is defined by $[x_{k+1,l,\alpha}, x_{k+1,r,\alpha}]$. According to (3.5), for $\alpha \in (0, 1]$, we can derive

$$\begin{aligned}
&[x_{k+1,l,\alpha}, x_{k+1,r,\alpha}]\\
=&\left[\frac{A_{l,\alpha} + B_{l,\alpha}x_{k,l,\alpha} + C_{l,\alpha}e^{-(x_{k,r,\alpha}+x_{k-1,r,\alpha})}}{D_{l,\alpha} + x_{k-1,l,\alpha}}, \frac{A_{r,\alpha} + B_{r,\alpha}x_{k,r,\alpha} + C_{r,\alpha}e^{-(x_{k,l,\alpha}+x_{k-1,l,\alpha})}}{D_{r,\alpha} + x_{k-1,r,\alpha}}\right]\\
=&\frac{\left[A_{l,\alpha} + B_{l,\alpha}x_{k,l,\alpha} + C_{l,\alpha}e^{-(x_{k,r,\alpha}+x_{k-1,r,\alpha})}, A_{r,\alpha} + B_{r,\alpha}x_{k,r,\alpha} + C_{r,\alpha}e^{-(x_{k,l,\alpha}+x_{k-1,l,\alpha})}\right]}{[D_{l,\alpha} + x_{k-1,l,\alpha}, D_{r,\alpha} + x_{k-1,r,\alpha}]}\\
=&\frac{[A]_\alpha + [B]_\alpha \times [x_k]_\alpha + [C]_\alpha \times \left[e^{-(x_k+x_{k-1})}\right]_\alpha}{[D]_\alpha + [x_{k-1}]_\alpha} = \left[\frac{A + Bx_k + Ce^{-(x_k+x_{k-1})}}{D + x_{k-1}}\right]_\alpha.
\end{aligned}\tag{3.7}$$

Therefore, $[x_{k+1,l,\alpha}, x_{k+1,r,\alpha}]$ is the α-cut of $x_{k+1} = \frac{A+Bx_k+Ce^{-(x_k+x_{k-1})}}{D+x_{k-1}}$. Furthermore, for every n and $0 < \alpha \leq 1$, $[x_{n,l,\alpha}, x_{n,r,\alpha}]$ is the α-cut of x_n.

Now we proceed to prove the uniqueness of the fuzzy solution. If model (1.1) has another solution $\bar{x}$ under the initial conditions x_0, x_{-1}, then, following the same proof procedure, we have, for $\alpha \in (0, 1]$,

$$[\bar{x}_n]_\alpha = [x_{n,l,\alpha}, x_{n,r,\alpha}], \ n = 0, 1, 2, \cdots. \tag{3.8}$$

By comparing (3.6) and (3.8), we can derive $[x_n]_\alpha = [\bar{x}_n]_\alpha$. This implies $x_n = \bar{x}_n$.

When Case II occurs, i.e., $\frac{A_{r,\alpha}+B_{r,\alpha}x_{n,r,\alpha}+C_{r,\alpha}e^{-(x_{n,l,\alpha}+x_{n-1,l,\alpha})}}{A_{l,\alpha}+B_{l,\alpha}x_{n,l,\alpha}+C_{l,\alpha}e^{-(x_{n,r,\alpha}+x_{n-1,r,\alpha})}} \leq \frac{D_{r,\alpha}+x_{n-1,r,\alpha}}{D_{l,\alpha}+x_{n-1,l,\alpha}}$, for $n \geq 0$, $0 < \alpha \leq 1$, from (1.1), we obtain

$$\begin{cases}
x_{n+1,l,\alpha} = \frac{A_{r,\alpha}+B_{r,\alpha}x_{n,r,\alpha}+C_{r,\alpha}e^{-(x_{n,l,\alpha}+x_{n-1,l,\alpha})}}{D_{r,\alpha}+x_{n-1,r,\alpha}},\\
\\
x_{n+1,r,\alpha} = \frac{A_{l,\alpha}+B_{l,\alpha}x_{n,l,\alpha}+C_{l,\alpha}e^{-(x_{n,r,\alpha}+x_{n-1,r,\alpha})}}{D_{l,\alpha}+x_{n-1,l,\alpha}}.
\end{cases}\tag{3.9}$$

Therefore, we can simply conclude that for any $(x_{i,l,\alpha}, x_{i,r,\alpha})$, where $i = -1, 0$, $\alpha \in (0, 1]$, (3.9) has a solution $(x_{n,l,\alpha}, x_{n,r,\alpha})$. Next, we need to prove that a solution x_n of model (1.1), defined by $[x_{n,l,\alpha}, x_{n,r,\alpha}]$, satisfies (3.6) under Case II. For $n = 1$, using the α-cut of $x_1 = \frac{A+Bx_0+Ce^{-(x_0+x_{-1})}}{D+x_{-1}}$ and the division rule $\div_g$ under Case II, we obtain

$$[x_1]_\alpha = \left[\frac{A_{r,\alpha} + B_{r,\alpha}x_{0,r,\alpha} + C_{r,\alpha}e^{-(x_{0,l,\alpha}+x_{-1,l,\alpha})}}{D_{r,\alpha} + x_{-1,r,\alpha}}, \frac{A_{l,\alpha} + B_{l,\alpha}x_{0,l,\alpha} + C_{l,\alpha}e^{-(x_{0,r,\alpha}+x_{-1,r,\alpha})}}{D_{l,\alpha} + x_{-1,l,\alpha}}\right] = [x_{1,l,\alpha}, x_{1,r,\alpha}].$$

Thus, (3.6) holds for $n = 1$.

Now assume that for some $k \geq 1$ we have $[x_k]_\alpha = [x_{k,l,\alpha}, x_{k,r,\alpha}]$ and $[x_{k-1}]_\alpha = [x_{k-1,l,\alpha}, x_{k-1,r,\alpha}]$. Then, by (1.1), Lemma 3.1, and the induction hypothesis,

$$[x_{k+1}]_\alpha = \left[\frac{A + Bx_k + Ce^{-(x_k+x_{k-1})}}{D + x_{k-1}}\right]_\alpha = \frac{[A]_\alpha + [B]_\alpha [x_k]_\alpha + [C]_\alpha \left[e^{-(x_k+x_{k-1})}\right]_\alpha}{[D]_\alpha + [x_{k-1}]_\alpha}.$$

Under Case II, the α-cut of the quotient is given by the reversed formula (3.4). Consequently,

$$[x_{k+1}]_\alpha = \left[\frac{A_{r,\alpha} + B_{r,\alpha}x_{k,r,\alpha} + C_{r,\alpha}e^{-(x_{k,l,\alpha}+x_{k-1,l,\alpha})}}{D_{r,\alpha} + x_{k-1,r,\alpha}}, \frac{A_{l,\alpha} + B_{l,\alpha}x_{k,l,\alpha} + C_{l,\alpha}e^{-(x_{k,r,\alpha}+x_{k-1,r,\alpha})}}{D_{l,\alpha} + x_{k-1,l,\alpha}}\right] = [x_{k+1,l,\alpha}, x_{k+1,r,\alpha}].$$

Hence, by mathematical induction, (3.6) holds for all $n \geq 0$ and $\alpha \in (0, 1]$ in Case II.

If $\bar{x}$ is another solution with the same initial conditions, then for each n and α,

$$[\bar{x}_n]_\alpha = [x_{n,l,\alpha}, x_{n,r,\alpha}] = [x_n]_\alpha,$$

so $\bar{x}_n = x_n$.

Since for every n and α either Case I or Case II holds, model (1.1) always admits a unique fuzzy positive solution. This completes the proof. □

3.2. *Analysis of the qualitative characteristics of model* (1.1)

We will conduct an in-depth investigation into the dynamic behavior of model (1.1). First, in accordance with the generalized division ($\div_g$) of FNs, we need to study the two cases mentioned earlier separately, starting with an exploration of the dynamic behavior of the constant difference equation system corresponding to each case.

Next, we will propose several lemmas and provide their proof processes, as these lemmas are of crucial importance when Case I is true.

Lemma 3.2. Consider the following crisp DE system:

$$\begin{cases} u_{n+1} = \dfrac{a_1 + b_1u_n + c_1e^{-(v_n+v_{n-1})}}{d_1 + u_{n-1}}, \\ \\ v_{n+1} = \dfrac{a_2 + b_2v_n + c_2e^{-(u_n+u_{n-1})}}{d_2 + v_{n-1}}, \end{cases} \quad n \in N, \tag{3.10}$$

where v_i, u_i $(i = -1, 0)$, d_j, a_j, b_j, c_j $(j = 1, 2) \in \Re^+$. In the scenario where $b_1 < d_1$, $b_2 < d_2$, the following two propositions hold true.

(i) The PS of system (3.10) are bounded and persistent.

(ii) If condition

$$(d_2 - b_2)(d_1 - b_1) > 4c_1c_2e^{d_1-b_1} \tag{3.11}$$

is satisfied, then system (3.10) has a unique equilibrium $(\bar{u}, \bar{v}) \in \left[0, \frac{a_1+c_1}{d_1-b_1} + u_2\right] \times \left[0, \frac{a_2+c_2}{d_2-b_2} + v_2\right]$. When condition

$$\begin{aligned} &\frac{b_1}{d_1} + \frac{b_2}{d_2} + \frac{b_1b_2 + 4c_1c_2}{d_1d_2} + \frac{(b_2 + d_2)(a_1 + b_1M_u + c_1)}{d_1^2d_2} \\ &+ \frac{(b_1 + d_1)(a_2 + b_2M_v + c_2)}{d_1d_2^2} + \frac{(a_1 + b_1M_u + c_1)(a_2 + b_2M_v + c_2)}{d_1^2d_2^2} < 1 \end{aligned} \tag{3.12}$$

holds, this equilibrium is locally asymptotically stable.

Proof. (i) Now, assuming that any positive solution of (3.10) is (u_n, v_n), since $e^{-x} < 1$ for any $x > 0$, we obtain

$$u_n \leq \frac{a_1 + b_1 u_{n-1} + c_1}{d_1 + u_{n-2}}, \ v_n \leq \frac{a_2 + b_2 v_{n-1} + c_2}{d_2 + v_{n-2}}.$$

Because $u_{n-2} > 0$ and $v_{n-2} > 0$, we have $d_1 + u_{n-2} > d_1$ and $d_2 + v_{n-2} > d_2$. Hence,

$$u_n \leq \frac{a_1 + c_1}{d_1} + \frac{b_1}{d_1} u_{n-1}, \ v_n \leq \frac{a_2 + c_2}{d_2} + \frac{b_2}{d_2} v_{n-1}. \tag{3.13}$$

From (3.10) and (3.13), it follows by mathematical induction that

$$\begin{aligned} u_n &\leq \frac{a_1 + c_1}{d_1} + \frac{b_1}{d_1} u_{n-1} \\ &\leq \frac{a_1 + c_1}{d_1} + \frac{b_1}{d_1}\left(\frac{a_1 + c_1}{d_1} + \frac{b_1}{d_1} u_{n-2}\right) \\ &\leq \frac{a_1 + c_1}{d_1} + \frac{b_1(a_1 + c_1)}{d_1^2} + \frac{b_1^2}{d_1^2}\left(\frac{a_1 + c_1}{d_1} + \frac{b_1}{d_1} u_{n-3}\right) \\ &\leq \cdots \leq \sum_{t=1}^{j} \frac{a_1 + c_1}{d_1}\left(\frac{b_1}{d_1}\right)^{t-1} + \left(\frac{b_1}{d_1}\right)^{j} u_{n-j} \\ &= \frac{a_1 + c_1}{d_1(1 - b_1/d_1)}\left[1 - \left(\frac{b_1}{d_1}\right)^{j}\right] + \left(\frac{b_1}{d_1}\right)^{j} u_{n-j} \\ &\leq \frac{a_1 + c_1}{d_1 - b_1} + u_{n-j}. \end{aligned}$$

The condition $j < n - 1$ can be equivalently stated as $n > j + 1$. Hence, we have

$$u_n \leq \frac{a_1 + c_1}{d_1 - b_1} + u_2 = M_u. \tag{3.14}$$

Similarly, we can derive that

$$v_n \leq \frac{a_2 + c_2}{d_2 - b_2} + v_2 = M_v. \tag{3.15}$$

Moreover, from (3.10), we can derive that

$$\begin{aligned} u_n &= \frac{a_1 + b_1 u_{n-1} + c_1 e^{-(v_{n-1}+v_{n-2})}}{d_1 + u_{n-2}} \geq \frac{a_1 + c_1 e^{-(u_{n-1}+v_{n-2})}}{d_1 + u_{n-2}}, \\ v_n &= \frac{a_2 + b_2 v_{n-1} + l_2 e^{-(u_{n-1}+u_{n-2})}}{d_2 + v_{n-2}} \geq \frac{a_2 + c_2 e^{-(u_{n-1}+u_{n-2})}}{d_2 + v_{n-2}}. \end{aligned} \tag{3.16}$$

Combined with (3.14) and (3.15), we can further conclude that

$$\begin{aligned} u_n &\geq \frac{a_1 + c_1 e^{-(u_{n-1}+v_{n-2})}}{d_1 + u_{n-2}} \geq \frac{a_1 + c_1 e^{-2M_v}}{d_1 + M_u} = m_u, \\ v_n &\geq \frac{a_2 + c_2 e^{-(u_{n-1}+u_{n-2})}}{d_2 + v_{n-2}} \geq \frac{a_2 + c_2 e^{-2M_u}}{d_2 + M_v} = m_v. \end{aligned} \tag{3.17}$$

From (3.14), (3.15), and (3.17), we can obtain

$$m_u \leq u_n \leq M_u,\ m_v \leq v_n \leq M_v.$$

(ii) Now, let us consider the following system:

$$u = \frac{a_1 + b_1 u + c_1 e^{-2v}}{d_1 + u},\ v = \frac{a_2 + b_2 v + c_2 e^{-2u}}{d_2 + v}. \tag{3.18}$$

An equivalent representation of (3.18) is given by

$$u^2 + (d_1 - b_1)u - c_1 e^{-2v} - a_1 = 0,\ v^2 + (d_2 - b_2)v - c_2 e^{-2u} - a_2 = 0. \tag{3.19}$$

Therefore, from (3.19), we may set

$$u = P(v) = \frac{b_1 - d_1 + \sqrt{(d_1 - b_1)^2 + 4(a_1 + c_1 e^{-2v})}}{2}.$$

Additionally, let us set

$$H(v) = v^2 + (d_2 - b_2)v - c_2 e^{-2P(v)} - a_2. \tag{3.20}$$

From (3.20), it is evident that $H(0) < 0$, and

$$H(M_v) = \left(\frac{c_2 + a_2}{d_2 - b_2} + v_2\right)^2 + (d_2 - b_2)v_2 + c_2 - c_2 e^{-2P(M_v)} > 0.$$

Therefore, we can conclude that for $v \in (0, M_v]$, (3.20) admits at least one positive solution $\bar{v}$.

Moreover, from (3.20), we can derive $H'(v)$, and based on (3.11), we obtain

$$\begin{aligned} H'(v) &= 2v + d_2 - b_2 + 2c_2 e^{-2P(v)} P'(v) \\ &= 2v + d_2 - b_2 - 2c_2 e^{-(b_1 - d_1 + \sqrt{(d_1-b_1)^2 + 4(a_1 + c_1 e^{-2v})})} \frac{2c_1 e^{-2v}}{\sqrt{(d_1 - b_1)^2 + 4(a_1 + c_1 e^{-2v})}} \\ &> d_2 - b_2 - \frac{4c_1 c_2 e^{d_1 - b_1}}{d_1 - b_1} > 0. \end{aligned}$$

In summary, when $0 < v < M_v$, the equation $H(v) = 0$ possesses a unique equilibrium $\bar{v}$. Similarly, we can analogously conclude that there exists a unique equilibrium $\bar{u}$ for $0 < u < M_u$.

Now we consider the Jacobian matrix $\Theta_{(\bar{u},\bar{v})}$ of system (3.10) at $(\bar{u}, \bar{v})$, which takes the form

$$\Theta_{(\bar{u},\bar{v})} = \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ 1 & 0 & 0 & 0 \\ \xi_5 & \xi_6 & \xi_7 & \xi_8 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

where $\xi_1 = \frac{b_1}{d_1+\bar{u}}$, $\xi_2 = -\frac{a_1+b_1\bar{u}+c_1e^{-2\bar{v}}}{(d_1+\bar{u})^2}$, $\xi_3 = -\frac{c_1e^{-2\bar{v}}}{d_1+\bar{u}}$, $\xi_4 = -\frac{c_1e^{-2\bar{v}}}{d_1+\bar{u}}$, $\xi_5 = -\frac{c_2e^{-2\bar{u}}}{d_2+\bar{v}}$, $\xi_6 = -\frac{c_2e^{-2\bar{u}}}{d_2+\bar{v}}$, $\xi_7 = \frac{b_2}{d_2+\bar{v}}$, and $\xi_8 = -\frac{a_2+b_2\bar{v}+c_2e^{-2\bar{u}}}{(d_2+\bar{v})^2}$.

Furthermore, we can derive the linearized equation of system (3.10) as

$$\Gamma_{n+1} = \Theta\Gamma_n,$$

where $\Gamma_n = (u_n, u_{n-1}, v_n, v_{n-1})^T$.

Therefore, we can obtain the characteristic equation of $\Theta_{(\bar{u},\bar{v})}$ at $(\bar{u}, \bar{v})$ as

$$\lambda^4 + \eta_1\lambda^3 + \eta_2\lambda^2 + \eta_3\lambda + \eta_4 = 0, \tag{3.21}$$

where $\eta_1 = -\xi_1 - \xi_7$, $\eta_2 = \xi_1\xi_7 - \xi_3\xi_5 - \xi_2 - \xi_8$, $\eta_3 = \xi_2\xi_7 + \xi_1\xi_8 - \xi_4\xi_5 - \xi_3\xi_6$, and $\eta_4 = \xi_2\xi_8 - \xi_4\xi_6$.

Thus, given that (3.12) holds, we obtain the following inequality:

$$\begin{aligned}
\sum_{i=1}^{4}|\eta_i| =&|\xi_1 + \xi_7| + |\xi_1\xi_7 - \xi_3\xi_5 - \xi_2 - \xi_8| + |\xi_2\xi_7 + \xi_1\xi_8 - \xi_4\xi_5 - \xi_3\xi_6| + |\xi_2\xi_8 - \xi_4\xi_6| \\
=&\frac{b_1}{d_1 + \bar{u}} + \frac{b_2}{d_2 + \bar{v}} + \frac{b_1b_2 + 4c_1c_2e^{-2(\bar{u}+\bar{v})}}{(d_1 + \bar{u})(d_2 + \bar{v})} \\
&+ \frac{a_1 + b_1\bar{u} + c_1e^{-2\bar{v}}}{(d_1 + \bar{u})^2} + \frac{a_2 + b_2\bar{v} + c_2e^{-2\bar{u}}}{(d_2 + \bar{v})^2} + \frac{b_2(a_1 + b_1\bar{u} + c_1e^{-2\bar{v}})}{(d_1 + \bar{u})^2(d_2 + \bar{v})} \\
&+ \frac{b_1(a_2 + b_2\bar{v} + c_2e^{-2\bar{u}})}{(d_1 + \bar{u})(d_2 + \bar{v})^2} + \frac{(a_1 + b_1\bar{u} + c_1e^{-2\bar{v}})(a_2 + b_2\bar{v} + c_2e^{-2\bar{u}})}{(d_1 + \bar{u})^2(d_2 + \bar{v})^2} \\
\leq&\frac{b_1}{d_1} + \frac{b_2}{d_2} + \frac{b_1b_2 + 4c_1c_2}{d_1d_2} + \frac{a_1 + b_1M_u + c_1}{d_1^2} + \frac{a_2 + b_2M_v + c_2}{d_2^2} \\
&+ \frac{b_2(a_1 + b_1M_u + c_1)}{d_1^2d_2} + \frac{b_1(a_2 + b_2M_v + c_2)}{d_1d_2^2} + \frac{(a_1 + b_1M_u + c_1)(a_2 + b_2M_v + c_2)}{d_1^2d_2^2} \\
<&1.
\end{aligned}$$

By applying Remark 1.3.1 from reference [35], we have that the eigenvalues satisfy $\lambda_i < 1$ ($i = 1, 2, 3, 4$). Consequently, the equilibrium $(\bar{u}, \bar{v})$ of system (3.10) is locally asymptotically stable. □

Remark 3.1. Inequality (3.11) provides a sufficient condition for the uniqueness of the equilibrium. It implies that the product of the self-regulation strengths $(d_1 - b_1)(d_2 - b_2)$ dominates the coupling term $4c_1c_2$ (with a factor $e^{d_1-b_1}$), thereby ensuring $H'(v) > 0$ and the uniqueness of the equilibrium. Physically, this reflects that the intrinsic decay rates $d_j - b_j$ ($j = 1, 2$) are sufficiently large relative to the coupling strengths c_j.

Lemma 3.3. The equilibrium $(\bar{u}, \bar{v})$ of system (3.10) is globally asymptotically stable if

$$a_1 + b_1M_u + c_1e^{-2m_v} < \bar{u}(d_1 + m_u),\ a_2 + b_2M_v + c_2e^{-2m_u} < \bar{v}(d_2 + m_v). \tag{3.22}$$

Proof. First, we construct a Lyapunov function Δ_n whose specific form is as follows:

$$\Delta_n = \bar{u}\left(\frac{u_n}{\bar{u}} - \ln\frac{u_n}{\bar{u}} - 1\right) + \bar{v}\left(\frac{v_n}{\bar{v}} - \ln\frac{v_n}{\bar{v}} - 1\right). \tag{3.23}$$

Since the inequality $u - \ln u - 1 \geq 0$ holds for all $u > 0$, we can directly conclude that $\Delta_n \geq 0$. Furthermore, based on the inequality $\ln(1+u) \leq u$, we obtain

$$\begin{aligned}
-\ln\frac{u_{n+1}}{u_n} &= \ln\left(1-\left(1-\frac{u_n}{u_{n+1}}\right)\right) \leq -\left(1-\frac{u_n}{u_{n+1}}\right) = -\frac{u_{n+1}-u_n}{u_{n+1}},\\
-\ln\frac{v_{n+1}}{v_n} &= \ln\left(1-\left(1-\frac{v_n}{v_{n+1}}\right)\right) \leq -\left(1-\frac{v_n}{v_{n+1}}\right) = -\frac{v_{n+1}-v_n}{v_{n+1}}.
\end{aligned} \tag{3.24}$$

Based on (3.10), (3.23), and (3.24), we derive the following inequality for the difference $\Delta_{n+1} - \Delta_n$:

$$\begin{aligned}
\Delta_{n+1} - \Delta_n =& \bar{u}\left(\frac{u_{n+1}}{\bar{u}} - \ln\frac{u_{n+1}}{\bar{u}} - 1\right) + \bar{v}\left(\frac{v_{n+1}}{\bar{v}} - \ln\frac{v_{n+1}}{\bar{v}} - 1\right)\\
& - \bar{u}\left(\frac{u_n}{\bar{u}} - \ln\frac{u_n}{\bar{u}} - 1\right) - \bar{v}\left(\frac{v_n}{\bar{v}} - \ln\frac{v_n}{\bar{v}} - 1\right)\\
=& (u_{n+1}-u_n) + (v_{n+1}-v_n) - \bar{u}\ln\frac{u_{n+1}}{u_n} - \bar{v}\ln\frac{v_{n+1}}{v_n}\\
\leq& (u_{n+1}-u_n) + (v_{n+1}-v_n) - \bar{u}\frac{u_{n+1}-u_n}{u_{n+1}} - \bar{v}\frac{v_{n+1}-v_n}{v_{n+1}}\\
=& (u_{n+1}-u_n)\left(1-\frac{\bar{u}}{u_{n+1}}\right) + (v_{n+1}-v_n)\left(1-\frac{\bar{v}}{v_{n+1}}\right)\\
=& (u_{n+1}-u_n)\left(1-\frac{\bar{u}(d_1+u_{n-1})}{a_1+b_1u_n+c_1e^{-(v_{n+1}+v_n)}}\right)\\
& + (v_{n+1}-v_n)\left(1-\frac{\bar{v}(d_2+v_{n-1})}{a_2+b_2v_n+c_2e^{-(u_{n+1}+u_n)}}\right).
\end{aligned} \tag{3.25}$$

Under Condition (3.22), using part (i) of Lemma 3.1, we can further obtain that for $n \geq 0$,

$$\begin{aligned}
\Delta_{n+1} - \Delta_n \leq& (M_u - m_u)\left(\frac{a_1 + b_1M_u + c_1e^{-2m_v} - \bar{u}(d_1+m_u)}{a_1+b_1M_u+c_1e^{-2m_v}}\right)\\
& + (M_v - m_v)\left(\frac{a_2 + b_2M_v + c_2e^{-2m_u} - \bar{v}(d_2+m_v)}{a_2+b_2M_v+c_2e^{-2m_u}}\right) \leq 0.
\end{aligned}$$

Given that $\Delta_n \geq 0$, it follows that $\lim_{n\to\infty}\Delta_n \geq 0$. This in turn implies $\lim_{n\to\infty}(\Delta_{n+1} - \Delta_n) = 0$, consequently leading to $\lim_{n\to\infty}(u_n, v_n) = (\bar{u}, \bar{v})$. Combining this with part (ii) of Lemma 3.1, it follows that $(\bar{u}, \bar{v})$ is globally asymptotically stable.

□

Theorem 3.2. Consider the fuzzy DE (1.1). If, for $\alpha \in (0,1]$,

$$\frac{A_{l,\alpha} + B_{l,\alpha}x_{n,l,\alpha} + C_{l,\alpha}\cdot e^{-(x_{n,r,\alpha}+x_{n-1,r,\alpha})}}{A_{r,\alpha} + B_{r,\alpha}x_{n,r,\alpha} + C_{r,\alpha}\cdot e^{-(x_{n,l,\alpha}+x_{n-1,l,\alpha})}} \leq \frac{D_{l,\alpha} + x_{n-1,l,\alpha}}{D_{r,\alpha} + x_{n-1,r,\alpha}},\ n \in N, \tag{3.26}$$

when $B_{l,\alpha} < D_{l,\alpha}$, $B_{r,\alpha} < D_{r,\alpha}$, the following two propositions hold true.
(i) Each PFS x_n of model (1.1) are bounded and possesses persistence.
(ii) Every PFS x_n of model (1.1) converges to the unique equilibrium point x as $n \to \infty$, provided that, for $\alpha \in (0,1]$,

$$(D_{r,\alpha} - B_{r,\alpha})(D_{l,\alpha} - B_{l,\alpha}) > 4C_{l,\alpha}C_{r,\alpha}\cdot e^{D_{l,\alpha}-B_{l,\alpha}} \tag{3.27}$$

and

$$\begin{aligned} &\frac{B_{l,\alpha}}{D_{l,\alpha}} + \frac{B_{r,\alpha}}{D_{r,\alpha}} + \frac{B_{l,\alpha}B_{r,\alpha} + 4C_{l,\alpha}C_{r,\alpha}}{D_{l,\alpha}D_{r,\alpha}} + \frac{(B_{r,\alpha} + D_{r,\alpha})(A_{l,\alpha} + B_{l,\alpha}M_{u,\alpha} + C_{l,\alpha})}{D_{l,\alpha}^2 D_{r,\alpha}} \\ &+ \frac{(B_{l,\alpha} + D_{l,\alpha})(A_{r,\alpha} + B_{r,\alpha}M_{v,\alpha} + C_{r,\alpha})}{D_{l,\alpha}D_{r,\alpha}^2} + \frac{(A_{l,\alpha} + B_{l,\alpha}M_{u,\alpha} + C_{l,\alpha})(A_{r,\alpha} + B_{r,\alpha}M_{v,\alpha} + C_{r,\alpha})}{D_{l,\alpha}^2 D_{r,\alpha}^2} < 1, \end{aligned} \tag{3.28}$$

where $M_{u,\alpha} = \frac{A_{l,\alpha}+C_{l,\alpha}}{D_{l,\alpha}-B_{l,\alpha}} + x_{2,l,\alpha}$ and $M_{v,\alpha} = \frac{A_{r,\alpha}+C_{r,\alpha}}{D_{r,\alpha}-B_{r,\alpha}} + x_{2,r,\alpha}$.

Proof. (i) According to model (1.1), since $D, A, B, C, x_0, x_{-1} \in \Re_f^+$, there exist positive real numbers $J_D, K_D, J_A, K_A, J_B, K_B, J_C, K_C, J_{-1}, K_{-1}, J_0$, and K_0 such that their α-cuts satisfy

$$\begin{cases} [J_D, K_D] \supseteq [D_{l,\alpha}, D_{r,\alpha}], \ [J_A, K_A] \supseteq [A_{l,\alpha}, A_{r,\alpha}], \\ [J_B, K_B] \supseteq [B_{l,\alpha}, B_{r,\alpha}], \ [J_C, K_C] \supseteq [C_{l,\alpha}, C_{r,\alpha}], \\ [J_{-1}, K_{-1}] \supseteq [x_{-1,l,\alpha}, x_{-1,r,\alpha}], \ [J_0, K_0] \supseteq [x_{0,l,\alpha}, x_{0,r,\alpha}], \end{cases} \quad \alpha \in (0,1]. \tag{3.29}$$

Now, assuming that x_n is a PFS of model (1.1), by applying Lemma 3.2 and referring to (3.26) and (3.29), we obtain

$$\begin{cases} x_{n,l,\alpha} \geq \frac{A_{l,\alpha}+C_{l,\alpha}\cdot e^{-2(\frac{A_{r,\alpha}+C_{r,\alpha}}{D_{r,\alpha}-B_{r,\alpha}}+x_{2,r,0})}}{D_{l,\alpha}+\frac{A_{l,\alpha}+C_{l,\alpha}}{D_{l,\alpha}-B_{l,\alpha}}+x_{2,l,0}} \geq \frac{J_A+J_C\cdot e^{-2(\frac{K_A+K_C}{J_D-K_B}+x_{2,r,0})}}{K_D+\frac{K_A+K_C}{J_D-K_B}+x_{2,l,0}} =: J, \\ x_{n,r,\alpha} \leq \frac{A_{r,\alpha}+C_{r,\alpha}}{D_{r,\alpha}-B_{r,\alpha}} + x_{2,r,0} \leq \frac{K_A+K_C}{J_D-K_B} + x_{2,r,0} =: K. \end{cases} \tag{3.30}$$

From (3.30), it is evident that $[x_{n,l,\alpha}, x_{n,r,\alpha}] \subseteq [J, K]$ holds, which implies that x_n is bounded and persistent.

(ii) Consider a system of the form, for $0 < \alpha \leq 1$,

$$x_{l,\alpha} = \frac{A_{l,\alpha} + B_{l,\alpha}x_{l,\alpha} + C_{l,\alpha}e^{-2x_{r,\alpha}}}{D_{l,\alpha} + x_{l,\alpha}}, \quad x_{r,\alpha} = \frac{A_{r,\alpha} + B_{r,\alpha}x_{r,\alpha} + C_{r,\alpha}e^{-2x_{l,\alpha}}}{D_{r,\alpha} + x_{r,\alpha}}. \tag{3.31}$$

By Lemma 3.2, we have

$$\begin{aligned} \frac{A_{l,\alpha} + C_{l,\alpha} \cdot e^{-2(\frac{A_{r,\alpha}+C_{r,\alpha}}{D_{r,\alpha}-B_{r,\alpha}}+x_{r,0})}}{D_{l,\alpha} + \frac{A_{l,\alpha}+C_{l,\alpha}}{D_{l,\alpha}-B_{l,\alpha}} + x_{l,0}} \leq x_{l,\alpha} \leq \frac{A_{l,\alpha} + C_{l,\alpha}}{D_{l,\alpha} - B_{l,\alpha}} + x_{l,0}, \\ \frac{A_{r,\alpha} + C_{r,\alpha} \cdot e^{-2(\frac{A_{l,\alpha}+C_{l,\alpha}}{D_{l,\alpha}-B_{l,\alpha}}+x_{l,0})}}{D_{r,\alpha} + \frac{A_{r,\alpha}+C_{r,\alpha}}{D_{r,\alpha}-B_{r,\alpha}} + x_{r,0}} \leq x_{r,\alpha} \leq \frac{A_{r,\alpha} + C_{r,\alpha}}{D_{r,\alpha} - B_{r,\alpha}} + x_{r,0}. \end{aligned} \tag{3.32}$$

Assuming that x_n is a PFS of model (1.1) with $[x_n]_\alpha = [x_{n,l,\alpha}, x_{n,r,\alpha}]$, it follows from (3.26) that

$$\begin{cases} x_{n+1,l,\alpha} = \frac{A_{l,\alpha}+B_{l,\alpha}x_{n,l,\alpha}+C_{l,\alpha}e^{-(x_{n,r,\alpha}+x_{n-1,r,\alpha})}}{D_{l,\alpha}+x_{n-1,l,\alpha}}, \\ x_{n+1,r,\alpha} = \frac{A_{r,\alpha}+B_{r,\alpha}x_{n,r,\alpha}+C_{r,\alpha}e^{-(x_{n,l,\alpha}+x_{n-1,l,\alpha})}}{D_{r,\alpha}+x_{n-1,r,\alpha}}, \end{cases} \quad \alpha \in (0,1]. \tag{3.33}$$

Since conditions (3.27) and (3.28) hold, and by employing Lemma 3.2 and Lemma 3.3, we can directly derive that (3.33) has a unique equilibrium $(x_{l,\alpha}, x_{r,\alpha})$, which satisfies

$$\lim_{n\to\infty} x_{n,l,\alpha} = x_{l,\alpha},\ \lim_{n\to\infty} x_{n,r,\alpha} = x_{r,\alpha}. \tag{3.34}$$

According to (3.30) and (3.32), if $0 < \alpha_1 < \alpha_2 < 1$, then

$$0 < x_{l,\alpha_1} \le x_{l,\alpha_2} \le x_{r,\alpha_1} \le x_{r,\alpha_2}. \tag{3.35}$$

Moreover, since $D_{l,\alpha}$, $A_{l,\alpha}$, $B_{l,\alpha}$, $C_{r,\alpha}$, $D_{r,\alpha}$, $A_{r,\alpha}$, $B_{r,\alpha}$, and $C_{r,\alpha}$ are all left-continuous, it follows that $x_{l,\alpha}$ and $x_{r,\alpha}$ in (3.31) are also left-continuous.

From (3.30) and (3.32), we obtain

$$s_1 := \frac{J_A + J_C \cdot e^{-2(\frac{K_A+K_C}{J_D-K_B}+x_{r,0})}}{K_D + \frac{K_A+K_C}{J_D-K_B} + x_{l,0}} \le x_{l,\alpha} \le x_{r,\alpha} \le \frac{K_A + K_C}{J_D - K_B} + x_{r,0} := s_2. \tag{3.36}$$

Clearly, from (3.36), we have $[x_{l,\alpha}, x_{r,\alpha}] \subset [s_1, s_2]$. Furthermore, we obtain $\bigcup_{\alpha\in(0,1]}[x_{l,\alpha}, x_{r,\alpha}] \subset [s_1, s_2]$. Then, $\bigcup_{\alpha\in(0,1]}[x_{l,\alpha}, x_{r,\alpha}]$ is compact, and

$$\bigcup_{\alpha\in(0,1]} [x_{l,\alpha}, x_{r,\alpha}] \subset (0, \infty). \tag{3.37}$$

By applying Definition 2.2 and combining (3.31), (3.35), and (3.37), we conclude that $\exists x \in \Re_f^+$ satisfies the following relation:

$$x = \frac{A + Bx + Ce^{-2x}}{D + x},\ [x]_\alpha = [x_{l,\alpha}, x_{r,\alpha}],\ 0 < \alpha \le 1. \tag{3.38}$$

Suppose $\bar{x}$ is another equilibrium of model (1.1). Then, one has $\bar{x}_{l,\alpha},\ \bar{x}_{r,\alpha} : (0,1) \to (0,\infty)$ satisfies

$$\bar{x} = \frac{A + B\bar{x} + Ce^{-2\bar{x}}}{D + \bar{x}},\ [\bar{x}]_\alpha = [\bar{x}_{l,\alpha}, \bar{x}_{r,\alpha}],\ 0 < \alpha \le 1$$

where

$$\bar{x}_{l,\alpha} = \frac{A_{l,\alpha} + B_{l,\alpha}\bar{x}_{n,l,\alpha} + C_{l,\alpha}e^{-2\bar{x}_{r,\alpha}}}{D_{l,\alpha} + \bar{x}_{l,\alpha}},\ \bar{x}_{r,\alpha} = \frac{A_{r,\alpha} + B_{r,\alpha}\bar{x}_{r,\alpha} + C_{r,\alpha}e^{-2\bar{x}_{l,\alpha}}}{D_{r,\alpha} + \bar{x}_{r,\alpha}}.$$

Therefore, we have $\bar{x}_{l,\alpha} = x_{l,\alpha},\ \bar{x}_{r,\alpha} = x_{r,\alpha}$. It follows that $\bar{x} = x$, thus x is the unique equilibrium of model (1.1).

Finally, from (3.34), we can derive

$$\lim_{n\to\infty} D(x_n, x) = \lim_{n\to\infty} \sup_{\alpha\in[0,1]} \{\max\{|x_{n,l,\alpha} - x_{l,\alpha}|, |x_{n,r,\alpha} - x_{r,\alpha}|\}\} = 0.$$

This implies that every PFS x_n of model (1.1) converges to the unique equilibrium x as $n \to \infty$. □

When Case II occurs, i.e., for $n \in N$, $0 < \alpha \le 1$, $\frac{A_{r,\alpha}+B_{r,\alpha}x_{n,r,\alpha}+C_{r,\alpha}e^{-(x_{n,l,\alpha}+x_{n-1,l,\alpha})}}{A_{l,\alpha}+B_{l,\alpha}x_{n,l,\alpha}+C_{l,\alpha}e^{-(x_{n,r,\alpha}+x_{n-1,r,\alpha})}} \le \frac{D_{r,\alpha}+x_{n-1,r,\alpha}}{D_{l,\alpha}+x_{n-1,l,\alpha}}$, then

$$\begin{cases} x_{n+1,l,\alpha} = \frac{A_{r,\alpha}+B_{r,\alpha}x_{n,r,\alpha}+C_{r,\alpha}e^{-(x_{n,l,\alpha}+x_{n-1,l,\alpha})}}{D_{r,\alpha}+x_{n-1,r,\alpha}}, \\ \\ x_{n+1,r,\alpha} = \frac{A_{l,\alpha}+B_{l,\alpha}x_{n,l,\alpha}+C_{l,\alpha}e^{-(x_{n,r,\alpha}+x_{n-1,r,\alpha})}}{D_{l,\alpha}+x_{n-1,l,\alpha}}. \end{cases} \tag{3.39}$$

Prior to proving Theorem 3.3, we will propose several lemmas along with their proof processes, which will serve as auxiliary tools for the subsequent proof.

Lemma 3.4. Consider the following crisp DE system:

$$\begin{cases} u_{n+1} = \dfrac{a_2 + b_2 v_n + c_2 e^{-(u_n+u_{n-1})}}{d_2 + v_{n-1}}, \\ \\ v_{n+1} = \dfrac{a_1 + b_1 u_n + c_1 e^{-(v_n+v_{n-1})}}{d_1 + u_{n-1}}, \end{cases} \quad n \in N, \tag{3.40}$$

where $v_i,\ u_i (i = -1, 0)$ and $d_i,\ a_i,\ b_i,\ c_i (i = 1, 2)$ are $\Re^+$. In the scenario where $d_1 d_2 > b_1 b_2$, the following two propositions hold true.

(i) The PS of system (3.40) are bounded and persistent.

(ii) (3.40) has a unique positive equilibrium $(\bar{u}, \bar{v}) \in \left[\omega_u = \frac{a_2+c_2e^{-2\Omega_u}}{d_2+\Omega_v}, \Omega_u = \frac{b_2(a_1+c_1)+d_1(a_2+c_2)}{d_1d_2-b_1b_2} + u_2\right] \times \left[\omega_v = \frac{a_1+c_1e^{-2\Omega_v}}{d_1+\Omega_u}, \Omega_v = \frac{d_2(a_1+c_1)+b_1(a_2+c_2)}{d_1d_2-b_1b_2} + v_2\right]$, if

$$\begin{aligned} &((2c_1e^{-2\omega_v} + d_1)\Omega_v - b_1(2c_1e^{-2\Omega_v} + d_1) + a_1 + c_1e^{-2\omega_v} - d_1\omega_v) \\ &\times ((2c_2e^{-2\Xi_{v_1}} + d_2)\Xi_{v_2} - b_2(2c_2e^{-2\Xi_{v_2}} + d_2) + a_2 + c_2e^{-2\Xi_{v_1}} - d_2\Xi_{v_1}) \\ &< (\omega_v - b_1)^2(\Xi_{v_1} - b_2)^2, \end{aligned} \tag{3.41}$$

and

$$\begin{aligned} &((2c_2e^{-2\omega_u} + d_2)\Omega_u - b_2(2c_2e^{-2\Omega_u} + d_2) + a_2 + c_2e^{-2\omega_u} - d_2\omega_u) \\ &\times ((2c_1e^{-2\Xi_{u_1}} + d_1)\Xi_{u_2} - b_1(2c_1e^{-2\Xi_{u_2}} + d_1) + a_1 + c_1e^{-2\Xi_{u_1}} - d_1\Xi_{u_1}) \\ &< (\omega_u - b_2)^2(\Xi_{u_1} - b_1)^2, \end{aligned} \tag{3.42}$$

where $\Xi_{v_1} = \frac{a_1+c_1e^{-2\Omega_v}-d_1\Omega_v}{\Omega_v-b_1}$, $\Xi_{v_2} = \frac{a_1+c_1e^{-2\omega_v}-d_1\omega_v}{\omega_v-b_1}$, $\Xi_{u_1} = \frac{a_2+c_2e^{-2\Omega_u}-d_2\Omega_u}{\Omega_u-b_2}$, and $\Xi_{u_2} = \frac{a_2+c_2e^{-2\omega_u}-d_2\omega_u}{\omega_u-b_2}$.

Proof. (i) From system (3.40), one can easily obtain

$$u_n \leq \frac{a_2 + c_2 + b_2 v_{n-1}}{d_2},\ v_n \leq \frac{a_1 + c_1 + b_1 u_{n-1}}{d_1}.$$

For $2k > n - 1$, using mathematical induction to derive yields the following inequality:

$$\begin{aligned} u_n &\leq \frac{a_2 + c_2}{d_2} + \frac{b_2}{d_2}v_{n-1} \leq \frac{a_2 + c_2}{d_2} + \frac{b_2(a_1 + c_1)}{d_1d_2} + \frac{b_1b_2}{d_1d_2}u_{n-2} \\ &\leq \frac{a_2 + c_2}{d_2} + \frac{b_2(a_1 + c_1)}{d_1d_2} + \frac{b_1b_2(a_2 + c_2)}{d_1d_2^2} + \frac{b_1b_2^2}{d_1d_2^2}v_{n-3} \\ &\leq \frac{a_2 + c_2}{d_2} + \frac{b_2(a_1 + c_1)}{d_1d_2} + \frac{b_1b_2(a_2 + c_2)}{d_1d_2^2} + \frac{b_1b_2^2(a_1 + c_1)}{d_1^2d_2^2} + \frac{b_1^2b_2^2}{d_1^2d_2^2}u_{n-4} \\ &\leq \cdots \leq \frac{a_2 + c_2}{d_2} + \frac{b_1b_2(a_2 + c_2)}{d_1d_2^2} + \frac{b_1^2b_2^2(a_2 + c_2)}{d_1^2d_2^3} + \cdots + \frac{b_1^{k-1}b_2^{k-1}(a_2 + c_2)}{d_1^{k-1}d_2^k} \\ &\quad + \frac{b_2(a_1 + c_1)}{d_1d_2} + \frac{b_1b_2^2(a_1 + c_1)}{d_1^2d_2^2} + \frac{b_1^2b_2^3(a_1 + c_1)}{d_1^3d_2^3} + \cdots + \frac{b_1^{k-1}b_2^k(a_1 + c_1)}{d_1^kd_2^k} + (\frac{b_1b_2}{d_1d_2})^k u_{n-2k} \\ &= \frac{(a_2 + c_2)/d_2}{1 - b_1b_2/d_1d_2}[1 - (\frac{b_1b_2}{d_1d_2})^k] + \frac{b_2(a_1 + c_1)/d_1d_2}{1 - b_1b_2/d_1d_2}[1 - (\frac{b_1b_2}{d_1d_2})^k] + (\frac{b_1b_2}{d_1d_2})^k u_{n-2k} \end{aligned}$$

$$\leq \frac{b_2(a_1+c_1)+d_1(a_2+c_2)}{d_1d_2-b_1b_2}+u_{n-2k}.$$

The condition $2k < n-1$ can be equivalently stated as $n > 2k+1$. Hence, we have

$$u_n \leq \frac{b_2(a_1+c_1)+d_1(a_2+c_2)}{d_1d_2-b_1b_2}+u_2=\Omega_u. \tag{3.43}$$

Similarly, we can derive that

$$v_n \leq \frac{d_2(a_1+c_1)+b_1(a_2+c_2)}{d_1d_2-b_1b_2}+v_2=\Omega_v. \tag{3.44}$$

From (3.40), (3.43), and (3.44), we can further derive that

$$u_n \geq \frac{a_2+c_2e^{-2\Omega_u}}{d_2+\Omega_v}=\omega_u,\ v_n \geq \frac{a_1+c_1e^{-2\Omega_v}}{d_1+\Omega_u}=\omega_v. \tag{3.45}$$

Therefore, from (3.43), (3.44), and (3.45), we can derive that

$$\omega_u \leq u_n \leq \Omega_u,\ \omega_v \leq v_n \leq \Omega_v.$$

(ii) We now consider the following system of equations:

$$u=\frac{a_2+b_2v+c_2e^{-2u}}{d_2+v},\ v=\frac{a_1+b_1u+c_1e^{-2v}}{d_1+u}. \tag{3.46}$$

(3.46) can be equivalently transformed into

$$v=\frac{a_2+c_2e^{-2u}-d_2u}{u-b_2},\ u=\frac{a_1+c_1e^{-2v}-d_1v}{v-b_1}. \tag{3.47}$$

From (3.47), we assume that

$$F(v)=\frac{a_2+c_2e^{-2r(v)}-d_2r(v)}{r(v)-b_2}-v, \tag{3.48}$$

where

$$u=r(v)=\frac{a_1+c_1e^{-2v}-d_1v}{v-b_1},\ v\in[\omega_v,\Omega_v].$$

From (3.48), it directly follows that

$$F(v):\ [\omega_v,\Omega_v]\mapsto[\omega_v,\Omega_v].$$

By differentiating expression (3.48), we obtain

$$\begin{aligned} F'(v)&=-r'(v)\frac{(2c_2e^{-2r(v)}+d_2)(r(v)-b_2)+a_2+c_2e^{-2r(v)}-d_2r(v)}{(r(v)-b_2)^2}-1,\\ r'(v)&=-\frac{(2c_1e^{-2v}+d_1)(v-b_1)+a_1+c_1e^{-2v}-d_1v}{(v-b_1)^2}. \end{aligned} \tag{3.49}$$

Suppose $\bar{v} \in [\omega_v, \Omega_v]$ denotes a solution of $F(v) = 0$. Application of (3.47) and (3.48) then yields

$$a_2 + c_2 e^{-2r(\bar{v})} - d_2 r(\bar{v}) = \bar{v}(r(\bar{v}) - b_2), \tag{3.50}$$

where

$$r(\bar{v}) = \frac{a_1 + c_1 e^{-2\bar{v}} - d_1\bar{v}}{\bar{v} - b_1}. \tag{3.51}$$

Based on (3.49), (3.50), and (3.51), the $F'(v)$ can be expressed in the following form:

$$\begin{aligned} F'(\bar{v}) =& \frac{(2c_1 e^{-2\bar{v}} + d_1)(\bar{v} - b_1) + a_1 + c_1 e^{-2\bar{v}} - d_1\bar{v}}{(\bar{v} - b_1)^2} \\ &\times \left[\frac{2c_2 e^{-2\frac{a_1 + c_1 e^{-2\bar{v}} - d_1\bar{v}}{\bar{v} - b_1}} + d_2}{\frac{a_1 + c_1 e^{-2\bar{v}} - d_1\bar{v}}{\bar{v} - b_1} - b_2} + \frac{a_2 + c_2 e^{-2\frac{a_1 + c_1 e^{-2\bar{v}} - d_1\bar{v}}{\bar{v} - b_1}} - d_2 \frac{a_1 + c_1 e^{-2\bar{v}} - d_1\bar{v}}{\bar{v} - b_1}}{(\frac{a_1 + c_1 e^{-2\bar{v}} - d_1\bar{v}}{\bar{v} - b_1} - b_2)^2} \right] - 1. \end{aligned}$$

Furthermore, under the assumption that condition (3.41) holds, a straightforward calculation yields

$$\begin{aligned} F'(v) \leq& \frac{(2c_1 e^{-2\omega_v} + d_1)\Omega_v - b_1(2c_1 e^{-2\Omega_v} + d_1) + a_1 + c_1 e^{-2\omega_v} - d_1\omega_v}{(\omega_v - b_1)^2} \\ &\times \left[\frac{(2c_2 e^{-2\frac{a_1 + c_1 e^{-2\Omega_v} - d_1\Omega_v}{\Omega_v - b_1}} + d_2)\frac{a_1 + c_1 e^{-2\omega_v} - d_1\omega_v}{\omega_v - b_1} - b_2(2c_2 e^{-2\frac{a_1 + c_1 e^{-2\omega_v} - d_1\omega_v}{\omega_v - b_1}} + d_2)}{(\frac{a_1 + c_1 e^{-2\Omega_v} - d_1\Omega_v}{\Omega_v - b_1} - b_2)^2} \right. \\ &\left. + \frac{a_2 + c_2 e^{-2\frac{a_1 + c_1 e^{-2\Omega_v} - d_1\Omega_v}{\Omega_v - b_1}} - d_2 \frac{a_1 + c_1 e^{-2\Omega_v} - d_1\Omega_v}{\Omega_v - b_1}}{(\frac{a_1 + c_1 e^{-2\Omega_v} - d_1\Omega_v}{\Omega_v - b_1} - b_2)^2} \right] - 1 \\ =& \frac{(2c_1 e^{-2\omega_v} + d_1)\Omega_v - b_1(2c_1 e^{-2\Omega_v} + d_1) + a_1 + c_1 e^{-2\omega_v} - d_1\omega_v}{(\omega_v - b_1)^2} \\ &\times \frac{(2c_2 e^{-2\Xi_{v_1}} + d_2)\Xi_{v_2} - b_2(2c_2 e^{-2\Xi_{v_2}} + d_2) + a_2 + c_2 e^{-2\Xi_{v_1}} - d_2\Xi_{v_1}}{(\Xi_{v_1} - b_2)^2} - 1 < 0. \end{aligned} \tag{3.52}$$

Hence, we conclude that $F(v) = 0$ has a unique equilibrium point, $\bar{v} \in [\omega_v, \Omega_v]$. Employing the same method, we obtain a unique equilibrium point, $\bar{u} \in [\omega_u, \Omega_u]$. □

Lemma 3.5. The equilibrium $(\bar{u}, \bar{v})$ of system (3.40) is locally asymptotically stable if

$$\begin{aligned} &2\frac{c_1 e^{-2\omega_v}}{d_1 + \omega_u} + 2\frac{c_2 e^{-2\omega_u}}{d_2 + \omega_v} + \frac{b_1 b_2 + 4c_1 c_2 e^{-2(\omega_u + \omega_v)}}{(d_1 + \omega_u)(d_2 + \omega_v)} + \frac{b_1(a_2 + b_2\Omega_v + c_2 e^{-2\omega_u})}{(d_1 + \omega_u)(d_2 + \omega_v)^2} \\ &+ \frac{b_2(a_1 + b_1\Omega_u + c_1 e^{-2\omega_v})}{(d_1 + \omega_u)^2(d_2 + \omega_v)} + \frac{(a_1 + b_1\Omega_u + c_1 e^{-2\omega_v})(a_2 + b_2\Omega_v + c_2 e^{-2\omega_u})}{(d_1 + \omega_u)^2(d_2 + \omega_v)^2} < 1. \end{aligned} \tag{3.53}$$

Proof. Suppose $(\bar{u}, \bar{v})$ is the unique equilibrium of the coupled system of Eq (3.40). Then, the Jacobian matrix $\Theta_{(\bar{u},\bar{v})}$ at this point can be expressed as follows:

$$\Theta_{(\bar{u},\bar{v})} = \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ 1 & 0 & 0 & 0 \\ \xi_5 & \xi_6 & \xi_7 & \xi_8 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

where $\xi_1 = -\frac{c_2 e^{-2\bar{u}}}{d_2+\bar{v}}$, $\xi_2 = -\frac{c_2 e^{-2\bar{u}}}{d_2+\bar{v}}$, $\xi_3 = \frac{b_2}{d_2+\bar{v}}$, $\xi_4 = \frac{a_2+b_2\bar{v}+c_2 e^{-2\bar{u}}}{(d_2+\bar{v})^2}$, $\xi_5 = \frac{b_1}{d_1+\bar{u}}$, $\xi_6 = -\frac{a_1+b_1\bar{u}+c_1 e^{-2\bar{v}}}{(d_1+\bar{u})^2}$, $\xi_7 = -\frac{c_1 e^{-2\bar{v}}}{d_1+\bar{u}}$, and $\xi_8 = -\frac{c_1 e^{-2\bar{v}}}{d_1+\bar{u}}$.

Through direct computation, we obtain that the characteristic polynomial of the Jacobian matrix $\Theta_{(\bar{u},\bar{v})}$ at $(\bar{u},\bar{v})$ satisfies

$$\lambda^4 + \eta_1\lambda^3 + \eta_2\lambda^2 + \eta_3\lambda + \eta_4 = 0, \tag{3.54}$$

where $\eta_1 = -\xi_1 - \xi_7$, $\eta_2 = \xi_1\xi_7 - \xi_3\xi_5 - \xi_2 - \xi_8$, $\eta_3 = \xi_2\xi_7 + \xi_1\xi_8 - \xi_4\xi_5 - \xi_3\xi_6$, and $\eta_4 = \xi_2\xi_8 - \xi_4\xi_6$.

Thus, according to Eq (3.53), which holds true, we have

$$\begin{aligned}
\sum_{i=1}^{4}|\eta_i| =& |\xi_1 + \xi_7| + |\xi_1\xi_7 - \xi_3\xi_5 - \xi_2 - \xi_8| + |\xi_2\xi_7 + \xi_1\xi_8 - \xi_4\xi_5 - \xi_3\xi_6| + |\xi_2\xi_8 - \xi_4\xi_6| \\
=& 2\frac{c_1 e^{-2\bar{v}}}{d_1+\bar{u}} + 2\frac{c_2 e^{-2\bar{u}}}{d_2+\bar{v}} + \frac{b_1b_2 + 4c_1c_2 e^{-2(\bar{u}+\bar{v})}}{(d_1+\bar{v})(d_2+\bar{v})} + \frac{b_1(a_2 + b_2\bar{v} + c_2 e^{-2\bar{u}})}{(d_1+\bar{u})(d_2+\bar{v})^2} \\
&+ \frac{b_2(a_1 + b_1\bar{u} + c_1 e^{-2\bar{v}})}{(d_1+\bar{u})^2(d_2+\bar{v})} + \frac{(a_1 + b_1\bar{u} + c_1 e^{-2\bar{v}})(a_2 + b_2\bar{v} + c_2 e^{-2\bar{u}})}{(d_1+\bar{u})^2(d_2+\bar{v})^2} \\
\le& 2\frac{c_1 e^{-2\omega_v}}{d_1+\omega_u} + 2\frac{c_2 e^{-2\omega_u}}{d_2+\omega_v} + \frac{b_1b_2 + 4c_1c_2 e^{-2(\omega_u+\omega_v)}}{(d_1+\omega_u)(d_2+\omega_v)} + \frac{b_1(a_2 + b_2\Omega_v + c_2 e^{-2\omega_u})}{(d_1+\omega_u)(d_2+\omega_v)^2} \\
&+ \frac{b_2(a_1 + b_1\Omega_u + c_1 e^{-2\omega_v})}{(d_1+\omega_u)^2(d_2+\omega_v)} + \frac{(a_1 + b_1\Omega_u + c_1 e^{-2\omega_v})(a_2 + b_2\Omega_v + c_2 e^{-2\omega_u})}{(d_1+\omega_u)^2(d_2+\omega_v)^2} \\
<& 1.
\end{aligned} \tag{3.55}$$

According to (3.55), it can be inferred that the eigenvalues of (3.54) satisfy $\lambda_i < 1, i = 1, 2, 3, 4$, thus demonstrating that the equilibrium $(\bar{u},\bar{v})$ is locally asymptotically stable. □

Theorem 3.3. Consider the fuzzy DE (1.1). If, for $\alpha \in (0, 1]$,

$$\frac{A_{r,\alpha} + B_{r,\alpha}x_{n,r,\alpha} + C_{r,\alpha}e^{-(x_{n,l,\alpha}+x_{n-1,l,\alpha})}}{A_{l,\alpha} + B_{l,\alpha}x_{n,l,\alpha} + C_{l,\alpha}e^{-(x_{n,r,\alpha}+x_{n-1,r,\alpha})}} \le \frac{D_{r,\alpha} + x_{n-1,r,\alpha}}{D_{l,\alpha} + x_{n-1,l,\alpha}}, \ n \in N. \tag{3.56}$$

When $B_{l,\alpha}B_{r,\alpha} < D_{l,\alpha}D_{r,\alpha}$, the following two propositions hold true.

(i) Each PFS x_n of model (1.1) is bounded and possesses persistence.

(ii) Every PFS x_n of model (1.1) converges to the unique equilibrium point x as $n \to \infty$, provided that, for $0 < \alpha \le 1$,

$$\begin{aligned}
&((2C_{l,\alpha}e^{-2\omega_{v,\alpha}} + D_{l,\alpha})\Omega_{v,\alpha} - B_{l,\alpha}(2C_{l,\alpha}e^{-2\Omega_{v,\alpha}} + D_{l,\alpha}) + A_{l,\alpha} + C_{l,\alpha}e^{-2\omega_{v,\alpha}} \\
&- D_{l,\alpha}\omega_{v,\alpha}) \times ((2C_{r,\alpha}e^{-2\Xi_{v_1,\alpha}} + D_{r,\alpha})\Xi_{v_2,\alpha} - B_{r,\alpha}(2C_{r,\alpha}e^{-2\Xi_{v_2,\alpha}} + D_{r,\alpha}) + A_{r,\alpha} \\
&+ C_{r,\alpha}e^{-2\Xi_{v_1,\alpha}} - D_{r,\alpha}\Xi_{v_1,\alpha}) < (\omega_{v,\alpha} - B_{l,\alpha})^2(\Xi_{v_1,\alpha} - B_{r,\alpha})^2,
\end{aligned} \tag{3.57}$$

and

$$\begin{aligned}&((2C_{r,\alpha}e^{-2\omega_{u,\alpha}} + D_{r,\alpha})\Omega_{u,\alpha} - B_{r,\alpha}(2C_{r,\alpha}e^{-2\Omega_{u,\alpha}} + D_{r,\alpha}) + A_{r,\alpha} + C_{r,\alpha}e^{-2\omega_{u,\alpha}}\\ &- D_{r,\alpha}\omega_{u,\alpha}) \times ((2C_{l,\alpha}e^{-2\Xi_{u_1,\alpha}} + D_{l,\alpha})\Xi_{u_2,\alpha} - B_{l,\alpha}(2C_{l,\alpha}e^{-2\Xi_{u_2,\alpha}} + D_{l,\alpha}) + A_{l,\alpha}\\ &+ C_{l,\alpha}e^{-2\Xi_{u_1,\alpha}} - D_{l,\alpha}\Xi_{u_1,\alpha}) < (\omega_{u,\alpha} - B_{r,\alpha})^2(\Xi_{u_1,\alpha} - B_{l,\alpha})^2,\end{aligned} \tag{3.58}$$

where

$$\begin{aligned}\Omega_{u,\alpha} &= \frac{B_{r,\alpha}(A_{l\alpha} + C_{l,\alpha}) + D_{l,\alpha}(A_{r,\alpha} + C_{r,\alpha})}{D_{l,\alpha}D_{r,\alpha} - B_{l,\alpha}B_{r,\alpha}} + x_{2,l,\alpha}, \omega_{u,\alpha} = \frac{A_{r,\alpha} + C_{r,\alpha}e^{-2\Omega_{u,\alpha}}}{D_{r,\alpha} + \Omega_{v,\alpha}},\\ \Omega_{v,\alpha} &= \frac{D_{r,\alpha}(A_{l,\alpha} + C_{l,\alpha}) + B_{l,\alpha}(A_{r,\alpha} + C_{r,\alpha})}{D_{l,\alpha}D_{r,\alpha} - B_{l,\alpha}B_{r,\alpha}} + x_{2,r,\alpha}, \omega_{v,\alpha} = \frac{A_{l,\alpha} + C_{l,\alpha}e^{-2\Omega_{v,\alpha}}}{D_{l,\alpha} + \Omega_{u,\alpha}},\\ \Xi_{v_1,\alpha} &= \frac{A_{l,\alpha} + C_{l,\alpha}e^{-2\Omega_{v,\alpha}} - D_{l,\alpha}\Omega_{v,\alpha}}{\Omega_{v,\alpha} - B_{l,\alpha}},\ \Xi_{v_2,\alpha} = \frac{A_{l,\alpha} + C_{l,\alpha}e^{-2\omega_{v,\alpha}} - D_{l,\alpha}\omega_{v,\alpha}}{\omega_{v,\alpha} - B_{l,\alpha}},\\ \Xi_{u_1,\alpha} &= \frac{A_{r,\alpha} + C_{r,\alpha}e^{-2\Omega_{u,\alpha}} - D_{r,\alpha}\Omega_{u,\alpha}}{\Omega_{u,\alpha} - B_{r,\alpha}},\ \Xi_{u_2,\alpha} = \frac{A_{r,\alpha} + C_{r,\alpha}e^{-2\omega_{u,\alpha}} - D_{r,\alpha}\omega_{u,\alpha}}{\omega_{u,\alpha} - B_{r,\alpha}}.\end{aligned}$$

Proof. (i) According to model (1.1), since D, A, B, C, x_0, $x_{-1} \in \Re_f^+$, there exist positive real numbers J_D, K_D, J_A, K_A, J_B, K_B, J_C, K_C, J_{-1}, K_{-1}, J_0, and K_0 such that their α-cuts satisfy

$$\begin{cases}[J_D, K_D] \supseteq [D_{l,\alpha}, D_{r,\alpha}],\ [J_A, K_A] \supseteq [A_{l,\alpha}, A_{r,\alpha}],\\ [J_B, K_B] \supseteq [B_{l,\alpha}, B_{r,\alpha}],\ [J_C, K_C] \supseteq [C_{l,\alpha}, C_{r,\alpha}], & \alpha \in (0, 1].\\ [J_{-1}, K_{-1}] \supseteq [x_{-1,l,\alpha}, x_{-1,r,\alpha}],\ [J_0, K_0] \supseteq [x_{0,l,\alpha}, x_{0,r,\alpha}],\end{cases} \tag{3.59}$$

Assuming that x_n is a PFS of model (1.1), by applying Lemma 3.4 and referring to (3.39) and (3.56), we obtain

$$\begin{cases}x_{n,l,\alpha} \geq \dfrac{A_{r,\alpha}+C_{r,\alpha}e^{-2(\frac{B_{r,\alpha}(A_{l\alpha}+C_{l,\alpha})+D_{l,\alpha}(A_{r,\alpha}+C_{r,\alpha})}{D_{l,\alpha}D_{r,\alpha}-B_{l,\alpha}B_{r,\alpha}}+x_{2,l,\alpha})}}{D_{r,\alpha}+\frac{D_{r,\alpha}(A_{l,\alpha}+C_{l,\alpha})+B_{l,\alpha}(A_{r,\alpha}+C_{r,\alpha})}{D_{l,\alpha}D_{r,\alpha}-B_{l,\alpha}B_{r,\alpha}}+x_{2,r,\alpha}} \geq \dfrac{J_A+J_Ce^{-2(\frac{K_B(K_A+K_C)+K_D(K_A+K_C)}{J_D^2-K_B^2}+x_{2,l,0})}}{K_D+\frac{K_B(K_A+K_C)+K_D(K_A+K_C)}{J_D^2-K_B^2}+x_{2,r,0}} =: M,\\ x_{n,r,\alpha} \leq \dfrac{D_{r,\alpha}(A_{l,\alpha}+C_{l,\alpha})+B_{l,\alpha}(A_{r,\alpha}+C_{r,\alpha})}{D_{l,\alpha}D_{r,\alpha}-B_{l,\alpha}B_{r,\alpha}} + x_{2,r,\alpha} \leq \dfrac{K_B(K_A+K_C)+K_D(K_A+K_C)}{J_D^2-K_B^2} + x_{2,r,0} =: N.\end{cases} \tag{3.60}$$

From (3.60), it is evident that $[x_{n,l,\alpha}, x_{n,r,\alpha}] \subseteq [M, N]$ holds, which implies that x_n is bounded and persistent.
(ii) Consider a system of the form, for $0 < \alpha \leq 1$,

$$x_{l,\alpha} = \frac{A_{r,\alpha} + B_{r,\alpha}x_{r,\alpha} + C_{r,\alpha}e^{-2x_{l,\alpha}}}{D_{r,\alpha} + x_{r,\alpha}},\ x_{r,\alpha} = \frac{A_{l,\alpha} + B_{l,\alpha}x_{l,\alpha} + C_{l,\alpha}e^{-2x_{r,\alpha}}}{D_{l,\alpha} + x_{l,\alpha}}. \tag{3.61}$$

By Lemma 3.4, we have

$$\begin{aligned}&\frac{A_{r,\alpha} + C_{r,\alpha}e^{-2(\frac{B_{r,\alpha}(A_{l\alpha}+C_{l,\alpha})+D_{l,\alpha}(A_{r,\alpha}+C_{r,\alpha})}{D_{l,\alpha}D_{r,\alpha}-B_{l,\alpha}B_{r,\alpha}}+x_{l,0})}}{D_{r,\alpha} + \frac{D_{r,\alpha}(A_{l,\alpha}+C_{l,\alpha})+B_{l,\alpha}(A_{r,\alpha}+C_{r,\alpha})}{D_{l,\alpha}D_{r,\alpha}-B_{l,\alpha}B_{r,\alpha}} + x_{r,0}} \leq x_{l,\alpha} \leq \frac{B_{r,\alpha}(A_{l\alpha} + C_{l,\alpha}) + D_{l,\alpha}(A_{r,\alpha} + C_{r,\alpha})}{D_{l,\alpha}D_{r,\alpha} - B_{l,\alpha}B_{r,\alpha}} + x_{l,0},\\ &\frac{A_{l,\alpha} + C_{l,\alpha}e^{-2(\frac{D_{r,\alpha}(A_{l,\alpha}+C_{l,\alpha})+B_{l,\alpha}(A_{r,\alpha}+C_{r,\alpha})}{D_{l,\alpha}D_{r,\alpha}-B_{l,\alpha}B_{r,\alpha}}+x_{r,0})}}{D_{l,\alpha} + \frac{B_{r,\alpha}(A_{l\alpha}+C_{l,\alpha})+D_{l,\alpha}(A_{r,\alpha}+C_{r,\alpha})}{D_{l,\alpha}D_{r,\alpha}-B_{l,\alpha}B_{r,\alpha}} + x_{l,0}} \leq x_{r,\alpha} \leq \frac{D_{r,\alpha}(A_{l,\alpha} + C_{l,\alpha}) + B_{l,\alpha}(A_{r,\alpha} + C_{r,\alpha})}{D_{l,\alpha}D_{r,\alpha} - B_{l,\alpha}B_{r,\alpha}} + x_{r,0}.\end{aligned} \tag{3.62}$$

Assuming that x_n is a PFS of model (1.1) with $[x_n]_\alpha = [x_{n,l,\alpha}, x_{n,r,\alpha}]$, it follows from (3.56) that

$$\begin{cases} x_{n+1,l,\alpha} = \frac{A_{r,\alpha}+B_{r,\alpha}x_{n,r,\alpha}+C_{r,\alpha}e^{-(x_{n,l,\alpha}+x_{n-1,l,\alpha})}}{D_{r,\alpha}+x_{n-1,r,\alpha}}, \\ x_{n+1,r,\alpha} = \frac{A_{l,\alpha}+B_{l,\alpha}x_{n,l,\alpha}+C_{l,\alpha}e^{-(x_{n,r,\alpha}+x_{n-1,r,\alpha})}}{D_{l,\alpha}+x_{n-1,l,\alpha}}, \end{cases} \quad \alpha \in (0,1]. \tag{3.63}$$

Since conditions (3.57) and (3.58) hold, and by employing Lemma 3.4 and Lemma 3.5, we can directly derive that (3.63) has a unique equilibrium $(x_{l,\alpha}, x_{r,\alpha})$, which satisfies

$$\lim_{n\to\infty} x_{n,l,\alpha} = x_{l,\alpha}, \ \lim_{n\to\infty} x_{n,r,\alpha} = x_{r,\alpha}. \tag{3.64}$$

According to (3.60) and (3.62), if $0 < \alpha_1 < \alpha_2 < 1$, then

$$0 < x_{l,\alpha_1} \le x_{l,\alpha_2} \le x_{r,\alpha_1} \le x_{r,\alpha_2}. \tag{3.65}$$

Moreover, since $D_{l,\alpha}$, $A_{l,\alpha}$, $B_{l,\alpha}$, $C_{l,\alpha}$, $D_{r,\alpha}$, $A_{r,\alpha}$, $B_{r,\alpha}$, and $C_{r,\alpha}$ are all left-continuous, it follows that $x_{l,\alpha}$ and $x_{r,\alpha}$ in (3.61) are also left-continuous.

From (3.60) and (3.62), we obtain

$$s_1 := \frac{J_A + J_C e^{-2(\frac{K_B(K_A+K_C)+K_D(K_A+K_C)}{J_D^2-K_B^2}+x_{2,l,0})}}{K_D + \frac{K_B(K_A+K_C)+K_D(K_A+K_C)}{J_D^2-K_B^2} + x_{2,r,0}} \le x_{l,\alpha} \le x_{r,\alpha} \le \frac{K_B(K_A+K_C)+K_D(K_A+K_C)}{J_D^2-K_B^2} + x_{r,0} := s_2. \tag{3.66}$$

Clearly, from (3.66), we have $[x_{l,\alpha}, x_{r,\alpha}] \subset [s_1, s_2]$. Furthermore, we obtain $\bigcup_{\alpha\in(0,1]}[x_{l,\alpha}, x_{r,\alpha}] \subset [s_1, s_2]$. Then, $\bigcup_{\alpha\in(0,1]}[x_{l,\alpha}, x_{r,\alpha}]$ is compact, and

$$\bigcup_{\alpha\in(0,1]} [x_{l,\alpha}, x_{r,\alpha}] \subset (0,\infty). \tag{3.67}$$

By applying Definition 2.2 and combining (3.61), (3.65), and (3.67), we conclude that $\exists x \in \Re_f^+$ satisfies the following relation:

$$x = \frac{A + Bx + Ce^{-2x}}{D + x}, \ [x]_\alpha = [x_{l,\alpha}, x_{r,\alpha}], \ 0 < \alpha \le 1. \tag{3.68}$$

Suppose $\bar{x}$ is another equilibrium of model (1.1). Then, one has $\bar{x}_{l,\alpha}, \bar{x}_{r,\alpha} : (0,1) \to (0,\infty)$ satisfies

$$\bar{x} = \frac{A + B\bar{x} + Ce^{-2\bar{x}}}{D + \bar{x}}, \ [\bar{x}]_\alpha = [\bar{x}_{l,\alpha}, \bar{x}_{r,\alpha}], \ 0 < \alpha \le 1$$

where

$$\bar{x}_{l,\alpha} = \frac{A_{r,\alpha} + B_{r,\alpha}\bar{x}_{n,r,\alpha} + C_{r,\alpha}e^{-2\bar{x}_{l,\alpha}}}{D_{r,\alpha} + \bar{x}_{r,\alpha}}, \bar{x}_{r,\alpha} = \frac{A_{l,\alpha} + B_{r,\alpha}\bar{x}_{r,\alpha} + C_{l,\alpha}e^{-2\bar{x}_{r,\alpha}}}{D_{l,\alpha} + \bar{x}_{l,\alpha}}.$$

Therefore, we have $\bar{x}_{l,\alpha} = x_{l,\alpha}$ and $\bar{x}_{r,\alpha} = x_{r,\alpha}$. It follows that $\bar{x} = x$, thus x is the unique equilibrium of model (1.1).

Finally, from (3.64), we can derive

$$\lim_{n\to\infty} D(x_n, x) = \lim_{n\to\infty} \sup_{\alpha\in[0,1]} \{\max\{|x_{n,l,\alpha} - x_{l,\alpha}|, |x_{n,r,\alpha} - x_{r,\alpha}|\}\} = 0.$$

This implies that every PFS x_n of model (1.1) converges to the unique equilibrium x as $n \to \infty$. □

The main theoretical results of model (1.1) under the two g-division cases are summarized in Table 2, providing a clear comparison of the key conditions and conclusions for each case.

Table 2. Comparison of the two cases arising from $\div_g$.

Feature	Case I (Theorem 3.2)	Case II (Theorem 3.3)
Division condition for α-cuts	$\dfrac{N_{l,\alpha}}{N_{r,\alpha}} \leq \dfrac{D_{l,\alpha} + x_{n-1,l,\alpha}}{D_{r,\alpha} + x_{n-1,r,\alpha}}$	$\dfrac{N_{r,\alpha}}{N_{l,\alpha}} \leq \dfrac{D_{r,\alpha} + x_{n-1,r,\alpha}}{D_{l,\alpha} + x_{n-1,l,\alpha}}$
where	$N_{l,\alpha} = A_{l,\alpha} + B_{l,\alpha}x_{n,l,\alpha} + C_{l,\alpha}e^{-(x_{n,r,\alpha}+x_{n-1,r,\alpha})}$ $N_{r,\alpha} = A_{r,\alpha} + B_{r,\alpha}x_{n,r,\alpha} + C_{r,\alpha}e^{-(x_{n,l,\alpha}+x_{n-1,l,\alpha})}$	
Resulting crisp system	$x_{n+1,l,\alpha} = \dfrac{N_{l,\alpha}}{D_{l,\alpha} + x_{n-1,l,\alpha}}$ $x_{n+1,r,\alpha} = \dfrac{N_{r,\alpha}}{D_{r,\alpha} + x_{n-1,r,\alpha}}$	$x_{n+1,l,\alpha} = \dfrac{N_{r,\alpha}}{D_{r,\alpha} + x_{n-1,r,\alpha}}$ $x_{n+1,r,\alpha} = \dfrac{N_{l,\alpha}}{D_{l,\alpha} + x_{n-1,l,\alpha}}$
Bounded and persistent condition	$B_{l,\alpha} < D_{l,\alpha},\ B_{r,\alpha} < D_{r,\alpha}$	$B_{l,\alpha}B_{r,\alpha} < D_{l,\alpha}D_{r,\alpha}$
Unique equilibrium	(3.27) and (3.28)	(3.57) and (3.58)
Equilibrium equation	$x_{l,\alpha} = f_l(x_{l,\alpha}, x_{r,\alpha}),\ x_{r,\alpha} = f_r(x_{l,\alpha}, x_{r,\alpha})$ ((3.31) or (3.61))	
Convergence	$\lim_{n\to\infty} D(x_n, x) = 0$	

General uniqueness of solution (Theorem 3.1): For any of the two cases, model (1.1) has a unique positive fuzzy solution.

4. Numerical examples

In Section 3, a series of arguments were presented to derive theoretical results concerning model (1.1). In this section, two examples will be provided to verify the theoretical findings for the two respective cases of g-division, thereby ensuring the validity of the theoretical conclusions.

Example 4.1. Consider the fuzzy DE (1.1), where D, A, B, C, and x_i ($i = -1, 0$) are parabolic FNs, subject to the following conditions:

$$\begin{cases} A(t) = 1 - \dfrac{(1-t)^2}{0.04},\ 0.8 \le t \le 1.2; \\ B(t) = 1 - \dfrac{(1.5-t)^2}{0.09},\ 1.2 \le t \le 1.8; \\ C(t) = 1 - \dfrac{(0.6-t)^2}{0.01},\ 0.5 \le t \le 0.7; \\ D(t) = 1 - \dfrac{(3.75-t)^2}{0.5625},\ 3 \le t \le 4.5; \\ x_{-1}(t) = \begin{cases} 1 - \left(\dfrac{0.4-t}{0.03}\right)^2,\ 0.37 \le t \le 0.4; \\ 1 - \left(\dfrac{t-0.4}{0.02}\right)^2,\ 0.4 \le t \le 0.42; \end{cases} \\ x_0(t) = \begin{cases} 1 - \left(\dfrac{0.39-t}{0.015}\right)^2,\quad 0.375 \le t \le 0.39; \\ 1 - \left(\dfrac{t-0.39}{0.01}\right)^2,\quad 0.39 \le t \le 0.4; \end{cases} \end{cases} \tag{4.1}$$

From (4.1), we derive that the α-cut is

$$\begin{cases} [A]_\alpha = \left[1 - 0.2\sqrt{1-\alpha},\ 1 + 0.2\sqrt{1-\alpha}\right]; \\ [B]_\alpha = \left[1.5 - 0.3\sqrt{1-\alpha},\ 1.5 + 0.3\sqrt{1-\alpha}\right]; \\ [C]_\alpha = \left[0.6 - 0.1\sqrt{1-\alpha},\ 0.6 + 0.1\sqrt{1-\alpha}\right]; \\ [D]_\alpha = \left[3.75 - 0.75\sqrt{1-\alpha},\ 3.75 + 0.75\sqrt{1-\alpha}\right]; \\ [x_{-1}]_\alpha = \left[0.4 - 0.03\sqrt{1-\alpha},\ 0.4 + 0.02\sqrt{1-\alpha}\right]; \\ [x_0]_\alpha = \left[0.39 - 0.015\sqrt{1-\alpha},\ 0.39 + 0.01\sqrt{1-\alpha}\right]; \end{cases} \qquad \alpha \in [0,1]. \tag{4.2}$$

Therefore, we can obtain

$$\begin{cases} \overline{\bigcup_{\alpha\in(0,1]} [A]_\alpha} = [0.8, 1.2]\,,\ \overline{\bigcup_{\alpha\in(0,1]} [B]_\alpha} = [1.2, 1.8]\,, \\ \overline{\bigcup_{\alpha\in(0,1]} [C]_\alpha} = [0.5, 0.7]\,,\ \overline{\bigcup_{\alpha\in(0,1]} [D]_\alpha} = [3, 4.5]\,, \\ \overline{\bigcup_{\alpha\in(0,1]} [x_{-1}]_\alpha} = [0.37, 0.42]\,,\ \overline{\bigcup_{\alpha\in(0,1]} [x_0]_\alpha} = [0.375, 0.4]\,. \end{cases} \tag{4.3}$$

At present, we take into account the crisp difference equation system

$$x_{n+1,l,\alpha} = \frac{A_{l,\alpha} + B_{l,\alpha}x_{n,l,\alpha} + C_{l,\alpha}e^{-(x_{n,r,\alpha}+x_{n-1,r,\alpha})}}{D_{l,\alpha} + x_{n-1,l,\alpha}},\ x_{n+1,r,\alpha} = \frac{A_{r,\alpha} + B_{r,\alpha}x_{n,r,\alpha} + C_{r,\alpha}e^{-(x_{n,l,\alpha}+x_{n-1,l,\alpha})}}{D_{r,\alpha} + x_{n-1,r,\alpha}}. \tag{4.4}$$

Hence, under the satisfaction of all conditions in Theorem 3.2, part (i) of Theorem 3.2 enables us to easily derive that all PFS x_n of the exponential fuzzy DE (1.1) possess both boundedness and persistence (see Table 3). Additionally, with reference to part (ii) of Theorem 3.2, it follows that the exponential fuzzy DE (1.1) has a unique equilibrium point $\bar{x} = (0.4437, 0.4579, 0.4695)$. From Figures 1 and 2, it is evident that these properties are valid.

Table 3. Variations in the α-cut and their impact on bounded persistent intervals and equilibrium points.

	A_r	B_r	C_r	D_r	$x_{-1,r}$	$x_{0,r}$	x_r	$\bar{x}_r$
	A_l	B_l	C_l	D_l	$x_{-1,l}$	$x_{0,l}$	x_l	$\bar{x}_l$
$\alpha = 0$	1.2000	1.8000	0.7000	4.5000	0.4200	0.4000	(0.2623,0.7037)	0.4695
	0.8000	1.2000	0.5000	3.0000	0.3700	0.3750	(0.2478,0.7222)	0.4437
$\alpha = 0.3$	1.1673	1.7510	0.6837	4.3775	0.4167	0.3984	(0.2615,0.7047)	0.4678
	0.8327	1.2490	0.5163	3.1225	0.3749	0.3775	(0.2495,0.7200)	0.4463
$\alpha = 0.5$	1.1414	1.7121	0.6707	4.2803	0.4141	0.3971	(0.2609,0.7056)	0.4663
	0.8586	1.2879	0.5293	3.2197	0.3788	0.3794	(0.2508,0.7184)	0.4482
$\alpha = 0.8$	1.0894	1.6342	0.6447	4.0854	0.4089	0.3945	(0.2595,0.7074)	0.4634
	0.9106	1.3658	0.5553	3.4146	0.3866	0.3833	(0.2531,0.7155)	0.4520
$\alpha = 1$	1.0000	1.5000	0.6000	3.7500	0.4000	0.3900	(0.2566,0.7111)	0.4579
	1.0000	1.5000	0.6000	3.7500	0.4000	0.3900	(0.2566,0.7111)	0.4579

As clearly illustrated in Figure 1, all positive solutions x_n (with their α-cut $[x_n]_\alpha = [x_{n,l,\alpha}, x_{n,r,\alpha}]$) of the exponential fuzzy DE (1.1), subject to the initial conditions $x_{-1} = (0.37, 0.4, 0.42)$ and $x_0 = (0.375, 0.39, 0.4)$, converge to the unique equilibrium $\bar{x} = (0.4437, 0.4579, 0.4695)$ as $n \to \infty$.

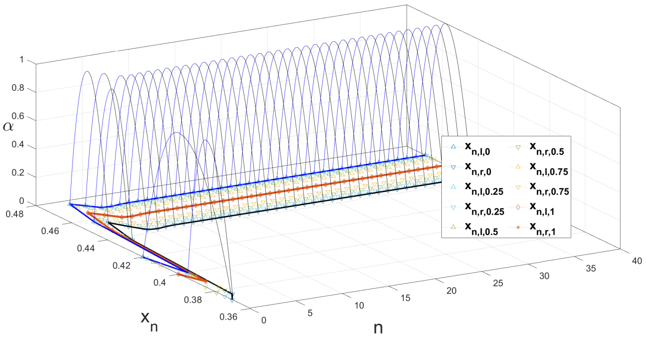
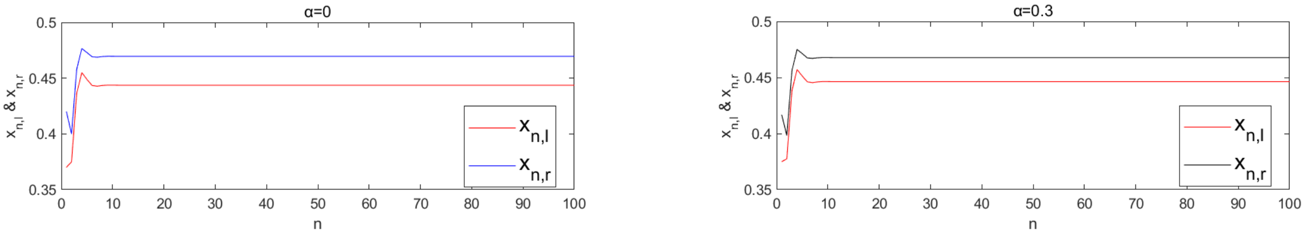
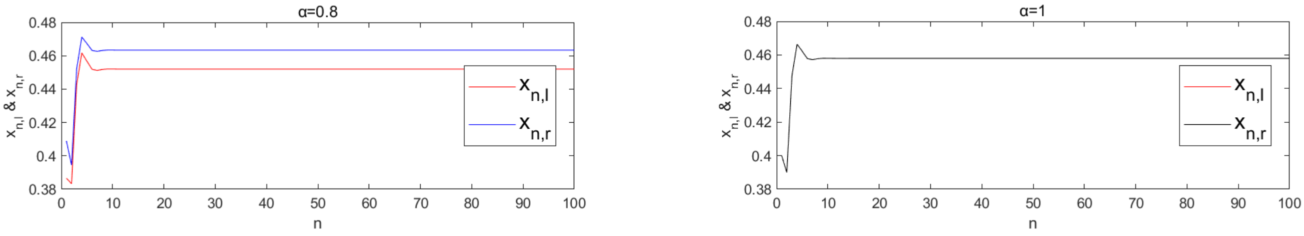


Figure 1. In Case I, the dynamic behavior of the PFS sequence of model (1.1).



(a) $\alpha = 0$ **(b)** $\alpha = 0.3$



(c) $\alpha = 0.8$ **(d)** $\alpha = 1$

Figure 2. Transition in asymptotic behavior of system (4.4) for different α values.

Example 4.2. Similarly, considering (1.1), $D,\ A,\ B,\ C$, and $x_i,\ i = (-1, 0)$ are parabolic FNs, and the membership function satisfies

$$\begin{cases} A(t) & = \begin{cases} 1-\left(\dfrac{3-t}{1.5}\right)^2, & 1.5 \le t \le 3; \\ 1-\left(\dfrac{t-3}{1.2}\right)^2, & 3 \le t \le 4.2; \end{cases} \quad B(t) = \begin{cases} 1-\left(\dfrac{1.7-t}{0.5}\right)^2, & 1.2 \le t \le 1.7; \\ 1-\left(\dfrac{t-1.7}{0.4}\right)^2, & 1.7 \le t \le 2.1; \end{cases} \\ C(t) & = \begin{cases} 1-\left(\dfrac{2.8-t}{1.4}\right)^2, & 1.4 \le t \le 2.8; \\ 1-(t-2.8)^2, & 2.8 \le t \le 3.8; \end{cases} \quad D(t) = \begin{cases} 1-\left(\dfrac{3-t}{0.7}\right)^2, & 2.3 \le t \le 3; \\ 1-\left(\dfrac{t-3}{0.4}\right)^2, & 3 \le t \le 3.4; \end{cases} \\ x_{-1}(t) & = \begin{cases} 1-\left(\dfrac{0.7-t}{0.1}\right)^2, & 0.6 \le t \le 0.7; \\ 1-\left(\dfrac{t-0.7}{0.1}\right)^2, & 0.7 \le t \le 0.8; \end{cases} \quad x_0(t) = \begin{cases} 1-\left(\dfrac{0.6-t}{0.1}\right)^2, & 0.5 \le t \le 0.6; \\ 1-\left(\dfrac{t-0.6}{0.1}\right)^2, & 0.6 \le t \le 0.7; \end{cases} \end{cases} \tag{4.5}$$

From (4.5), we derive that the α-cut is

$$\begin{cases} [A]_\alpha & = \left[3-1.5\sqrt{1-\alpha},\ 3+1.2\sqrt{1-\alpha}\right], \\ [B]_\alpha & = \left[1.7-0.5\sqrt{1-\alpha},\ 1.7+0.4\sqrt{1-\alpha}\right], \\ [C]_\alpha & = \left[2.8-1.4\sqrt{1-\alpha},\ 2.8+\ \sqrt{1-\alpha}\right], \\ [D]_\alpha & = \left[3-0.7\sqrt{1-\alpha},\ 3+0.4\sqrt{1-\alpha}\right], \\ [x_{-1}]_\alpha & = \left[0.7-0.1\sqrt{1-\alpha},\ 0.7+0.1\sqrt{1-\alpha}\right], \\ [x_0]_\alpha & = \left[0.6-0.1\sqrt{1-\alpha},\ 0.6+0.1\sqrt{1-\alpha}\right], \end{cases} \quad \alpha \in [0, 1]. \tag{4.6}$$

Therefore, we can obtain

$$\begin{cases} \overline{\bigcup_{\alpha\in(0,1]}[A]_\alpha} = [1.5, 4.2]\,, \ \overline{\bigcup_{\alpha\in(0,1]}[B]_\alpha} = [1.2, 2.1]\,, \\ \overline{\bigcup_{\alpha\in(0,1]}[C]_\alpha} = [1.4, 3.8]\,, \ \overline{\bigcup_{\alpha\in(0,1]}[D]_\alpha} = [2.3, 3.4]\,, \\ \overline{\bigcup_{\alpha\in(0,1]}[x_{-1}]_\alpha} = [0.6, 0.8]\,, \ \overline{\bigcup_{\alpha\in(0,1]}[x_0]_\alpha} = [0.5, 0.7]\,. \end{cases} \tag{4.7}$$

At present, we take into account the crisp difference equation system

$$x_{n+1,l,\alpha} = \frac{A_{r,\alpha} + B_{r,\alpha}x_{n,r,\alpha} + C_{r,\alpha}e^{-(x_{n,l,\alpha}+x_{n-1,l,\alpha})}}{D_{r,\alpha} + x_{n-1,r,\alpha}}, \ x_{n+1,r,\alpha} = \frac{A_{l,\alpha} + B_{l,\alpha}x_{n,l,\alpha} + C_{l,\alpha}e^{-(x_{n,r,\alpha}+x_{n-1,r,\alpha})}}{D_{l,\alpha} + x_{n-1,l,\alpha}}. \tag{4.8}$$

Hence, under the satisfaction of all conditions in Theorem 3.3, part (i) of Theorem 3.3 enables us to easily derive that all PFS x_n of the exponential fuzzy DE (1.1) possess both boundedness and persistence (see Table 4). Additionally, with reference to part (ii) of Theorem 3.3, it follows that the exponential fuzzy DE (1.1) has a unique equilibrium point $\bar{x} = (0.9241, 1.2599, 1.4668)$. From Figures 3 and 4, it is evident that these properties are valid.

As clearly illustrated in Figure 3, all positive solutions x_n (with their α-cut $[x_n]_\alpha = [x_{n,l,\alpha}, x_{n,r,\alpha}]$) of the exponential fuzzy DE (1.1), subject to the initial conditions $x_{-1} = (0.6, 0.7, 0.8)$ and $x_0 = (0.5, 0.6, 0.7)$, converge to the unique equilibrium $\bar{x} = (0.9241, 1.2599, 1.4668)$ as $n \to \infty$.

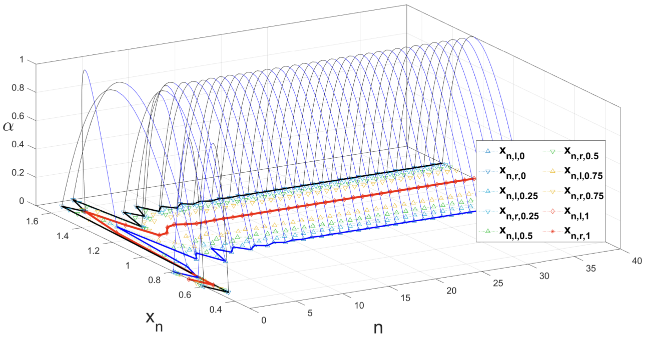
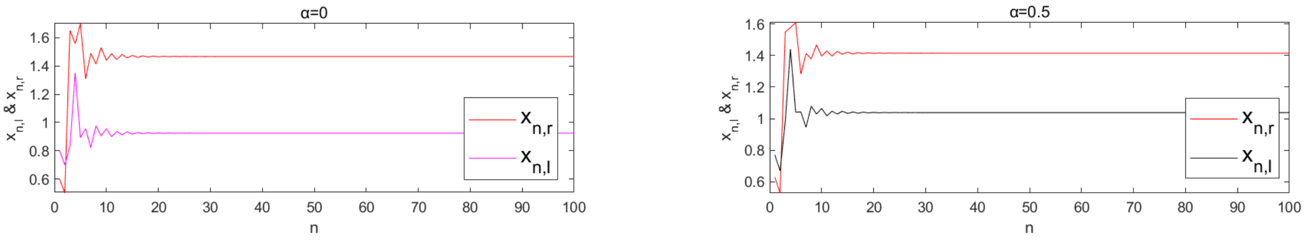
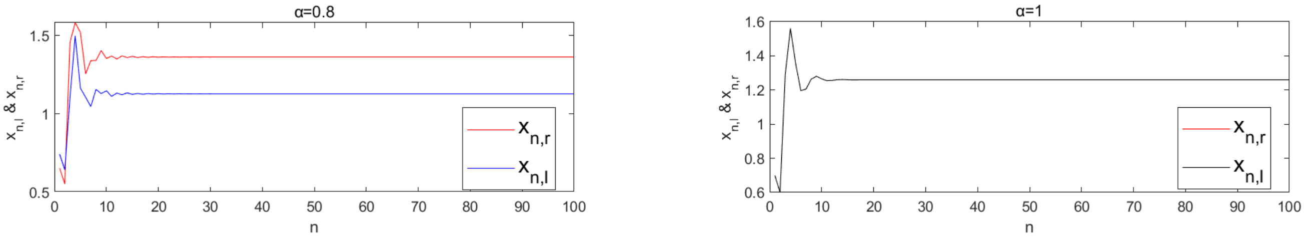


Figure 3. In Case II, the dynamic behavior of the PFS sequence of model (1.1).



(a) $\alpha = 0$ **(b)** $\alpha = 0.5$



(c) $\alpha = 0.8$ **(d)** $\alpha = 1$

Figure 4. Transition in asymptotic behavior of system (4.8) for different α values.

Table 4. Variations in the α-cut and their impact on bounded persistent intervals and equilibrium points.

	A_r	B_r	C_r	D_r	$x_{-1,r}$	$x_{0,r}$	x_r	$\bar{x}_r$
	A_l	B_l	C_l	D_l	$x_{-1,l}$	$x_{0,l}$	x_l	$\bar{x}_l$
$\alpha = 0$	4.2000	2.1000	3.8000	3.4000	0.8000	0.7000	(0.2168,3.6717)	1.4668
	1.5000	1.2000	1.4000	2.3000	0.6000	0.5000	(0.5940,4.6208)	0.9241
$\alpha = 0.25$	4.0392	2.0464	3.6660	3.3464	0.7866	0.6866	(0.2417,3.8335)	1.4442
	1.7010	1.2670	1.5876	2.3938	0.6134	0.5134	(0.5626,4.6468)	0.9786
$\alpha = 0.5$	3.8485	1.9828	3.5071	3.2828	0.7707	0.6707	(0.5286,4.6569)	1.4150
	1.9393	1.3464	1.8101	2.5050	0.6293	0.5293	(0.5286,4.6569)	1.0386
$\alpha = 0.8$	3.5367	1.8789	3.2472	3.1789	0.7447	0.6447	(0.3185,4.2187)	1.3627
	2.3292	1.4764	2.1739	2.6870	0.6553	0.5553	(0.4782,4.6275)	1.1282
$\alpha = 1$	3.0000	1.7000	2.8000	3.0000	0.7000	0.6000	(0.4021,4.4615)	1.2599
	3.0000	1.7000	2.8000	3.0000	0.7000	0.6000	(0.4021,4.4615)	1.2599

Remark 4.1. Figures 1 and 3 illustrate the three-dimensional dynamic behavior of the positive fuzzy solution sequences of model (1.1) under g-division Case I and Case II, respectively. In each figure, the horizontal axis represents the discrete time n, the vertical axis represents the membership degree α, and the upright direction shows the region covered by the left endpoints $x_{n,l,\alpha}$ and right endpoints $x_{n,r,\alpha}$ of the fuzzy number's α-cuts. It can be observed that when $\alpha = 0$ (i.e., the support), the surface is thickest in the upright direction and the band width is the largest, reflecting the maximum uncertainty arising from all possible values of the fuzzy parameters and the initial condition. As α gradually increases (taking values 0.25, 0.75, 1), the surface narrows inward and the width decreases monotonically. When

$\alpha = 1$, the left and right endpoints completely coincide, and the surface degenerates into a crisp curve, corresponding to the deterministic equilibrium point $\bar{x}$ (in Example 4.1, $\bar{x} = (0.4437, 0.4579, 0.4695)$; in Example 4.2, $\bar{x} = (0.9241, 1.2599, 1.4668)$). Figures 2 and 4 further present, for four fixed α-cut levels $\alpha = 0, 0.3, 0.8, 1$, the evolution curves of the left endpoints $x_{n,l,\alpha}$ and right endpoints $x_{n,r,\alpha}$ with respect to n. From each subplot, one can see that for each fixed α, both curves converge to their respective equilibrium limits $x_{l,\alpha}, x_{r,\alpha}$ as n increases. Moreover, $x_{l,\alpha}$ is monotonically nondecreasing and $x_{r,\alpha}$ is monotonically nonincreasing, and the interval width gradually shrinks as α increases. When $\alpha = 1$, the two curves completely coincide and converge to a crisp equilibrium point. These numerical simulation results verify the correctness of Theorem 3.2 and Theorem 3.3.

Remark 4.2. Tables 3 and 4 present, for model (1.1) under Case I and Case II, respectively, the parameters, initial values, and bounded persistence intervals of the solution sequences corresponding to different α values. As α increases (i.e., uncertainty decreases), the support intervals of the parameters and the initial condition gradually shrink. Accordingly, the difference between the left and right endpoints of the equilibrium point becomes smaller. When $\alpha = 1$, the system degenerates to a unique deterministic value, indicating that the fuzzy solution converges to a crisp unique equilibrium point.

5. Conclusions

This study investigates the uniqueness and dynamic behaviors of PFS for model (1.1) using g-division. Under the conditions given in Theorem 3.2 and Theorem 3.3, the PFS are proved to be bounded and persistent, and they converge to a unique equilibrium as $n \to \infty$.

These results are closely related to real-world problems characterized by strong nonlinearity, memory effects, and uncertainty, such as ecosystem modeling, intelligent systems, resource economics, and control theory. More broadly, the methods developed in this paper can be applied to fluid mechanics, biomechanics, aerodynamics, and partial differential equations defined on curved surfaces or domains that are difficult to mesh.

Future research should focus on the following directions. First, extend the model to higher-dimensional systems and time-varying coefficients to more realistically capture scenarios such as spatial interactions or seasonal fluctuations. Second, explore other FN representations (e.g., triangular FN, L-R FN) to achieve a better balance between representation flexibility and computational efficiency. Third, introduce stochastic perturbations to enable the model to more faithfully describe environmental noise.

Author contributions

Xiong Xiao: Writing-original draft, revising and editing; Qianhong Zhang: Writing-review, funding acquisition and guiding the revision of the paper. All authors have read and approved thefinal version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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