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Research article

Optimal investment strategies with derivative trading under 4/2-CIR jump-diffusion stochastic hybrid models

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Abstract: This paper investigates the continuous-time optimal investment strategy for a constant relative risk aversion investor under a novel stochastic hybrid framework: The 4/2-Cox-Ingersoll-Ross (CIR) jump-diffusion stochastic hybrid model. The financial market comprises a money market account, a zero-coupon bond, a stock index, and stock derivatives. Explicit solutions for the optimal strategy are derived using stochastic optimal control theory and the associated Hamilton-Jacobi-Bellman equation under a power utility function. Additionally, we characterize the optimal risk exposure, quantify the suboptimal strategy, and compute the associated utility loss within the 4/2-CIR jump-diffusion stochastic hybrid model. Numerical experiments analyze the impact of key portfolio model parameters on the optimal risk exposure and utility loss. Our results demonstrate that the risk aversion coefficient, investment horizon, equity risk premium, volatility risk premium, interest rate risk premium, and jump intensity significantly influence the optimal risk exposure. Furthermore, the short-sighted losses increase with positive risk premium factors and decrease with negative ones. Crucially, investment decisions derived under the proposed 4/2-CIR jump-diffusion stochastic hybrid model outperform those based on existing 4/2 stochastic volatility and 4/2-CIR stochastic hybrid models.

Keywords: 4/2-CIR jump-diffusion stochastic hybrid model; optimal investment strategy; double exponential distribution; CRRA utility; HJB equation
Mathematics Subject Classification: 91B16, 91G05

1. Introduction

As a fundamental branch of mathematical finance and asset management, portfolio optimization theory is dedicated to establishing a systematic analytical framework and decision-making basis for intertemporal asset allocation under uncertainty. Since the seminal work of Markowitz [1], who proposed the mean-variance portfolio selection model, the field has evolved into a comprehensive and in-depth research system. The theoretical development has consistently been closely aligned with the increasing complexity of financial markets [2], the diversification of derivative instruments [3], and the refinement of risk management practices [4]. Against the backdrop of rising global financial market volatility, intricate interest rate environments, and ceaseless financial innovation, the development of investment strategies that balance robustness, flexibility, theoretical rigor, and practical applicability within realistic, multidimensional stochastic environments encompassing features such as stochastic volatility [5], stochastic interest rates [6], and market frictions [7] has emerged as a cutting-edge research focus shared by academia and industry.

The research on classical portfolio optimization primarily follows two theoretical paradigms: The mean-variance criterion and the expected utility maximization criterion. The mean-variance approach, grounded in the second-order statistical characteristics of asset returns, is both intuitive and effective for capturing the risk-return trade-off and has been extensively applied across static and dynamic frameworks [8–10]. Nevertheless, this criterion exhibits several inherent limitations: First, it relies solely on the first two moments of the distribution, which makes it inadequate for capturing higher-order moment characteristics such as skewness and kurtosis of asset returns, as well as tail risks [11]. Second, the implicit quadratic utility function exhibits a nonmonotonic risk-averse characteristic across the entire scope, which may systematically deviate from investors' actual risk appetite [12]. Third, in dynamic multiperiod decision-making scenarios, this criterion often leads to time inconsistency issues, causing the pre-established strategy to lose its optimality during implementation [10]. In contrast, the expected utility maximization principle is grounded in the von Neumann-Morgenstern axiomatic system. By selecting appropriate utility functions , it can flexibly and consistently characterize investors' heterogeneous risk preferences while naturally ensuring intertemporal consistency in dynamic decision-making [13, 14]. Consequently, when confronting complex market environments, long-term investment planning, or nonlinear friction, the utility maximization criterion has increasingly become the dominant analytical framework, combining theoretical superiority with practical flexibility [15].

In the framework of expected utility maximization, portfolio models have shown a progressive evolution from idealized assumptions to more realistic ones. Early studies mainly relied on geometric Brownian motion and other constant-volatility diffusion processes [13, 16]. To precisely depict time-varying, clustered, and leverage features of volatility, stochastic volatility models have been introduced as core modeling tools. Among these, the Heston [17] model and its extensions (such as the 3/2 model [18] and the 4/2 model [19]) have gained widespread favor due to the tractability of their semianalytical solutions and their superior calibration to volatility smiles. Optimal investment decisions under such models have been explored in depth, across areas such as single-asset allocation [20–23], consumption-investment strategies [24, 25], and reinsurance strategies within insurance contexts [26–28]. Specifically, the 4/2 stochastic volatility model, as a unified framework, provides a powerful analytical tool for examining the effects of volatility risk on asset allocation under a broad parameter regime. For example, in the mean-variance framework, Zhang [28, 29] investigated robust optimal investment strategies for liability managers under ambiguity aversion in a stochastic volatility setting. The studies also examined the impact of derivative trading under coexisting stochastic interest rates and volatilities on asset-liability management performance. They quantified the effects of price and volatility ambiguity aversion on risk exposure and the efficient frontier, while providing analytical solutions for optimal strategies and value functions, offering significant theoretical support for risk management in complex financial environments. Wang et al. [27] developed an analytical framework for investment-reinsurance problems under the 4/2 stochastic volatility model, where they derived explicit solutions to parabolic partial differential equations using parametric kernel and integral transformation techniques, establishing closed-form expressions for efficient strategies and efficient frontiers. On the basis of the utility maximization principle, Cheng and Escobar-Anel [22, 23, 25] examined optimal investment decisions including derivative trading, robust strategies featuring both constant and ambiguity-averse risk attitudes, and the optimal consumption-investment choice; Ma et al. [24] explored optimal consumption-investment problems in the presence of both stochastic volatility and Cox-Ingersoll-Ross (CIR) interest rates. Hata and Yasuda [26] investigated the optimal investment-reinsurance strategy for insurance companies under delay dynamics within finite horizons.

Meanwhile, interest rates, as another key source of risk, exert profound impacts on portfolio decisions for long-term investors. The stochastic evolution of interest rates alters the structure of discount factors and investment opportunity sets, thereby affecting the dynamic path of consumption-investment decisions and the optimal allocation to risky assets. Noh and Kim [30] adopted asymptotic methods within an infinite time horizon to derive optimal investment strategies incorporating stochastic volatility and interest rates, revealing that the strategy's properties are influenced by the correlation between the underlying risk pricing and interest rate dynamics; Escobar et al. [31] addressed ambiguity-averse investors confronting both stochastic volatility and rates, demonstrating the necessity of joint ambiguity modeling, and quantitatively analyzed the differentiated effects of volatility and interest rate ambiguity on optimal allocation and welfare loss; Lin and Riedel [32], in the presence of interest rate uncertainty, constructed a stochastic model with embedded priors to investigate continuous-time consumption-investment problems. Furthermore, in long-term financial decisions such as pension planning and insurance asset-liability management, neglecting interest rates' randomness may lead to significant decision biases [6]. Thus, integrating stochastic interest rate factors into stochastic volatility models to construct dual stochastic volatility-interest rate models has become a key research direction for enhancing the practical explanatory power of these models [24, 28, 30–32].

Faced with a market environment characterized by multidimensional randomness and uncertainty, derivatives, especially options, have acquired increasing strategic importance in portfolio management. Options not only directly hedge directional and volatility risks but also optimize the risk-return efficiency frontier by constructing customized return structures. For example, Wu and Hu [3] incorporated Knightian uncertainty into standard capital structure and real option models, deriving closed-form solutions for optimal capital structure and investment decisions. Wei et al. [7] investigated optimal consumption-investment problems for ambiguity-averse investors with recursive preferences who participate in both the stock and derivatives markets. Their findings demonstrated that implementing fuzzy aversion management for diffusion risks and engaging in derivative trading are crucial for mitigating welfare loss. Egloff et al. [33] explored optimal investment strategies involving variance swaps and equity indices. In a stochastic volatility environment, Li et al. [34] investigated asset-liability management with derivatives using the mean-variance criterion, demonstrating that the effective frontier with derivatives is strictly superior to the no-derivatives scenario. Accordingly, in markets where both volatility and interest rates are stochastic, the introduction of option trading can significantly expand investors' strategic options, enable more refined risk exposure management, and potentially generate considerable utility gains. Moreover, the impact of market jump risks on derivative pricing and portfolio decision-making has also attracted widespread attention [2, 4, 35]. However, the extant literature primarily explores portfolio optimization under stochastic volatility, jump-diffusion, stochastic interest rates, or derivatives trading as standalone or paired features. There remains a notable gap in research on unified models integrating all four key elements, as well as on quantitative analyses of how such enrichment alters optimal risk exposures and certainty-equivalent utility loss.

Against this backdrop, this paper constructs a continuous-time portfolio optimization model that integrates 4/2-type stochastic volatility, double-exponential jump-diffusion, CIR stochastic interest rates, and option trading, and investigates it under the expected utility maximization framework.

To further clarify the economic interpretation of our framework, it is useful to emphasize that the enlargement of the traded asset set by derivative securities should be viewed primarily as a stylized completion mechanism. In continuous-time portfolio optimization with stochastic volatility and other non-traded state variables, such completion devices are frequently used to obtain tractable characterizations of optimal policies and to separate distinct sources of risk exposure. In this sense, our derivative-based formulation is not intended to claim literal market completeness in realistic markets, but rather to provide a theoretically transparent benchmark for understanding how investors optimally allocate wealth when volatility and jump-related risks can be partially or fully spanned by traded contingent claims.

This interpretation is consistent with the broader literature. Ewald [36] studied optimal logarithmic utility and portfolio choice for an insider in a general stochastic volatility market, combining utility maximization methods in incomplete markets with an enlarged information structure. That paper showed how portfolio choice in stochastic volatility environments can be naturally formulated through exposure to additional state variables, even when markets are not complete in the classical sense. In particular, the analysis highlighted the role of information and admissible strategy classes in determining optimal allocations, which is conceptually close to our use of an exposure-based admissible control representation. Relatedly, Poulsen et al. [37] highlighted risk minimization in stochastic volatility models and explicitly investigated the role of derivatives for improving hedging performance under model uncertainty. Their study demonstrated that in incomplete stochastic volatility markets, derivative instruments may substantially reduce hedging errors and can serve as effective spanning devices for volatility-related risk factors. This provides additional motivation for our modeling choice to incorporate derivative trading in the portfolio problem: The derivative is not merely an auxiliary asset, but a mechanism through which otherwise imperfectly hedgeable risks become more directly manageable within the optimization framework.

From the perspective of derivative pricing methodology, our work is also related to the jump-diffusion commodity literature. Ewald and Zou [38] developed quasianalytical pricing formulas for futures and European options in a linear-quadratic jump-diffusion model with stochastic volatility and convenience yield. Their analysis illustrated how stochastic volatility, jump components, and convenience-yield effects can be integrated within a tractable derivative pricing framework under both pricing and physical measures. This is methodologically relevant to the present paper because our derivative dynamics and market-price-of-risk specification likewise rely on a careful distinction between traded risk exposures and the underlying state dynamics. In addition, Ewald et al. [39]

studied the pricing of Asian options in commodity models with stochastic convenience yield and jumps. Their results show how derivative pricing techniques in multifactor jump-diffusion environments can be extended beyond standard European claims to more complex path-dependent structures, while preserving useful analytical and numerical tractability. Although our focus is portfolio optimization rather than exotic option valuation, that line of work reinforces the broader point that derivative-based spanning and pricing in stochastic volatility/jump settings is now well established, especially in commodity and related incomplete-market applications.

Compared with the existing literature, this work makes four main contributions. First, under the assumption of a complete market, a 4/2-CIR jump-diffusion portfolio model is established, which not only integrates the joint effects of stochastic volatility, stochastic interest rates, and jump risk on investment decisions, but also fully characterizes the typical features of asset returns such as non-normality and volatility clustering. Second, stochastic optimal control theory is applied to solve the model, and the analytical solution to the Hamilton-Jacobi-Bellman (HJB) equation, the expression for the optimal risk exposure and the optimal investment strategy are derived under the power utility function. The optimality of the solution under the 4/2-CIR jump-diffusion framework is strictly verified by the martingale method. Third, we analyze the construction and performance of suboptimal strategies within the proposed model, quantitatively evaluating the benefits of our approach relative to simplified benchmarks. Fourth, through systematic numerical simulations and sensitivity analyses, we thoroughly investigate the transmission mechanisms by which key model parameters affect the optimal risk exposures and the extent of utility loss.

The structure of this paper is organized as follows. Section 2 introduces the 4/2-CIR jump-diffusion portfolio model, which incorporates a double-exponential jump-diffusion process to account for the non-normal distribution and volatility clustering of asset returns arising from the interplay of stochastic volatility and stochastic interest rates. Section 3 derives and validates the optimal solution to the HJB equation for the model using dynamic programming principles. Section 4 outlines suboptimal strategies and evaluates the corresponding loss functions within the proposed framework. Section 5 provides a detailed numerical analysis along with the associated results. Finally, Section 6 concludes the paper.

2. Optimization problem description

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ be a complete probability space, where $\{\mathcal{F}_t\}_{t\in[0,T]}$ denotes the right-continuous filtration with left limits generated by a Brownian motion, and $\mathbb{P}$ represents the historical probability measure. The dynamics of the underlying asset price are assumed to be governed by the following stochastic differential equation:

$$\frac{dS_t}{S_t} = \mu dt + (a\sqrt{v_t} + \frac{b}{\sqrt{v_t}})dW_t^{\mathbb{P}} + \eta\sigma_r\sqrt{r_t}dW_{r,t}^{\mathbb{P}} + (e^{\xi} - 1)dN_t^{\mathbb{P}}, \tag{2.1}$$

where μ and η are constants, S_t denotes the stock price at time t, $W_t^{\mathbb{P}}$ and $W_{r,t}^{\mathbb{P}}$ are both standard Brownian motions, and $N_t^{\mathbb{P}}$ is a Poisson process with intensity $\widetilde{\lambda}$. Moreover, $W_{r,t}^{\mathbb{P}}$, $W_t^{\mathbb{P}}$, and $N_t^{\mathbb{P}}$ are assumed to be mutually independent. The random variable ξ is assumed to follow a double-exponential distribution with the density

$$f_\xi(y) = p\eta_1 e^{-\eta_1 y} I_{y\geq 0} + q\eta_2 e^{\eta_2 y} I_{y<0},$$

where $\eta_1 > 1$, $\eta_2 > 0$, and $p + q = 1$.

The dynamics of the volatility and the interest rate are assumed to be governed by the following CIR processes:

$$\begin{cases} dv_t = \zeta_v(\eta_v - v_t)dt + \sigma_v \sqrt{v_t} dW^{\mathbb{P}}_{v,t}, \\ dr_t = \zeta_r(\eta_r - v_t)dt + \sigma_r \sqrt{r_t} dW^{\mathbb{P}}_{r,t}, \end{cases} \tag{2.2}$$

where ζ_v, η_v, and σ_v denote the mean-reversion speed, the long-run mean level, and the volatility of the variance process, respectively, and satisfy the Feller condition $2\zeta_v\eta_v \geq \sigma_v^2$. Similarly, ζ_r, η_r, and σ_r denote the mean-reversion speed, long-run mean level, and volatility of the interest-rate process, respectively, with $2\zeta_r\eta_r \geq \sigma_r^2$. In addition, the Brownian motions $W^{\mathbb{P}}_{v,t}$ and $W^{\mathbb{P}}_t$ possess the correlation coefficient ρ, while $W^{\mathbb{P}}_{v,t}$ and $W^{\mathbb{P}}_{r,t}$ are mutually independent.

Let $\mathbb{Q}$ be an equivalent martingale measure relative to $\mathbb{P}$. Furthermore, let $\nu^p(dy)$ denote the probability distribution of $e^{\xi} - 1$, while $\nu(dy)$ denotes the jump distribution with intensity λ. We then define $\psi(y) = \frac{\lambda}{\lambda}\frac{\nu(dy)}{\nu^p(dy)} - 1, y \in \mathbb{R}$ and introduce the process $X = \{X_t,\ t \in [0, T]\}$ by

$$X_t := \frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\{-(\lambda - \lambda^p)t\}\, \widetilde{X^1_t}\widetilde{X^2_t} \prod_{j=1}^{N_t}(1 + \psi(m_j)),$$

where $m_j = e^{\xi_j} - 1$, while $\widetilde{X^1_t}$ and $\widetilde{X^2_t}$ denote two stochastic exponential martingales, given, respectively, by

$$\widetilde{X^1_t} = \exp\left\{-\int_t^T \overline{\lambda}\sqrt{v_s}dW^{\mathbb{P}}_s - \frac{1}{2}\int_t^T \overline{\lambda}^2 v_s ds\right\}, \quad \widetilde{X^2_t} = \exp\left\{-\int_t^T \frac{\lambda_r}{\sigma_r}\sqrt{r_t}dW^{\mathbb{P}}_{r,s} - \frac{1}{2}\int_t^T \frac{\lambda_r^2}{\sigma_r^2} r_s ds\right\}.$$

where $\overline{\lambda} = \lambda_v\rho + \lambda_s\sqrt{1-\rho^2}$.

If the stochastic exponential martingales $\widetilde{X^1_t}$ and $\widetilde{X^2_t}$ are independent of each other under the measure $\mathbb{P}$, in the sense of sample paths, then

$$\widetilde{X}_t = \widetilde{X^1_t}\widetilde{X^2_t} = \exp\left\{-\int_t^T \overline{\lambda}\sqrt{v_s}dW^{\mathbb{P}}_s - \frac{1}{2}\int_t^T \overline{\lambda}^2 v_s ds - \int_t^T \frac{\lambda_r}{\sigma_r}\sqrt{r_t}dW^{\mathbb{P}}_{r,s} - \frac{1}{2}\int_t^T \frac{\lambda_r^2}{\sigma_r^2} r_s ds\right\},$$

is a martingale.

However, the process X is well defined under the risk-neutral measure if and only if it is a true martingale under $\mathbb{Q}$, ensuring that the martingale property of the discounted underlying asset price process is satisfied

$$u - r_t + \lambda\mathbb{E}(e^{\xi} - 1) - \frac{\lambda_r}{\sigma_r}\sqrt{r_t}\eta\sigma_r\sqrt{r_t} - \overline{\lambda}\sqrt{v_t}\left(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}}\right) = 0,\ \mathbb{P} - a.s.$$

To guarantee the well-posedness of the change of measure, one must require the Novikov condition to be satisfied, namely

$$\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_t^T \overline{\lambda}^2 v_s ds\right\}\right] < \infty, \quad \mathbb{E}\left[\exp\left\{\frac{1}{2}\int_t^T \frac{\lambda_r^2}{\sigma_r^2} r_s ds\right\}\right] < \infty. \tag{2.3}$$

For Eq (2.3) to hold, the following conditions must be satisfied:

$$\frac{1}{2}\overline{\lambda}^2 \leq \frac{\zeta_v^2}{2\sigma_v^2}, \quad \frac{1}{2}\frac{\lambda_r^2}{\sigma_r^2} \leq \frac{\zeta_r^2}{2\sigma_r^2}.$$

The martingale property of the discounted asset price process holds only if

$$\sigma_v^2 \leq 2\zeta_v\eta_v - 2|\sigma_v\rho b|.$$

Moreover, in order to hedge against interest rate risk, the following condition must be satisfied:

$$\lambda_r^2 \geq 2\sigma_r^2,$$

In summary, the well-posedness of the change of measure requires that the following conditions hold:

$$\begin{cases} |\overline{\lambda}| < \dfrac{\zeta_v}{\sigma_v}, \\ \sigma_v^2 \leq 2\zeta_v\eta_v - 2|\sigma_v\rho b|, \\ 2\sigma_r^2 \leq \lambda_r^2 \leq \zeta_r^2. \end{cases} \tag{2.4}$$

An application of Girsanov's theorem yields the Brownian motions W_t, $W_{r,t}$, and $W_{v,t}$, as well as the Poisson process N_t under the measure $\mathbb{Q}$, namely,

$$dW_t = dW_t^{\mathbb{P}} + \overline{\lambda}\sqrt{v_t}dt, \quad dW_{r,t} = dW_{r,t}^{\mathbb{P}} + \frac{\lambda_r}{\sigma_r}\sqrt{r_t}dt, \quad dN_t = dN_t^{\mathbb{P}} - \lambda dt.$$

Accordingly, under the measure $\mathbb{Q}$, the dynamics of the underlying asset price, volatility, and interest rate are given, respectively, by the following stochastic differential equations:

$$\begin{cases} \dfrac{dS_t}{S_t} = (r_t + \overline{\lambda}(av_t + b) + \lambda_r\eta r_t - \lambda\kappa)dt + (a\sqrt{v_t} + \dfrac{b}{\sqrt{v_t}})dW_t + \eta\sigma_r\sqrt{r_t}dW_{r,t} + (e^{\xi} - 1)dN_t, \\ dv_t = \alpha_v(\theta_v - v_t)dt + \sigma_v\sqrt{v_t}dW_{v,t}, \\ dr_t = \alpha_r(\theta_r - r_t)dt + \sigma_r\sqrt{r_t}dW_{r,t}, \end{cases} \tag{2.5}$$

where $\kappa = \mathbb{E}[e^{\xi} - 1] = \frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2 + 1} - 1$, $\alpha_v = \zeta_v + \sigma_v\lambda_v$, $\theta_v = \frac{\zeta_v\eta_v}{\zeta_v + \sigma_v\overline{\lambda}}$, $\alpha_r = \zeta_r + \sigma_r\lambda_r$, $\theta_r = \frac{\zeta_r\eta_r}{\zeta_r + \lambda_r}$, and $dW_t = (\rho dW_{v,t} + \sqrt{1-\rho^2}dW_{s,t})$. Furthermore, $2\alpha_v\theta_v = 2\frac{\zeta_v\eta_v}{\zeta_v + \sigma_v\overline{\lambda}}(\zeta_v + \sigma_v\overline{\lambda}) = 2\zeta_v\eta_v \geq \sigma_v^2$, $2\alpha_r\theta_r = 2\frac{\zeta_r\eta_r}{\zeta_r + \lambda_r}(\zeta_r + \lambda_r) = 2\zeta_r\eta_r \geq \sigma_r^2$. Hence, the Feller condition continues to hold under the measure $\mathbb{Q}$.

If $Z_t = V_t^{-1}$, then the inverse CIR process satisfies the following stochastic differential equation:

$$\mathrm{d}Z_t = \widetilde{\alpha}Z_t(\widetilde{\theta} - Z_t)\mathrm{d}t + \widetilde{\sigma}Z_t^{\frac{3}{2}}\mathrm{d}W_{v,t}, \tag{2.6}$$

where $\widetilde{\alpha} = \alpha\theta - \sigma^2, \widetilde{\theta} = \frac{\alpha}{\alpha\theta - \sigma^2}, \widetilde{\sigma} = -\sigma$.

Because the diffusion term in Eq (2.1) combines a CIR-type component with a $3/2$ components, namely a $1/2$ term together with a $3/2$ term, the resulting specification is commonly termed the $4/2$ stochastic volatility model.

In this paper, we consider a financial market consisting of a money market account, a zero-coupon bond, a stock index, and two derivatives written on the stock index. The market dynamics are driven jointly by a CIR short rate, a 4/2-type stochastic volatility factor, and a double-exponential jump component.

The money market price P_t follows the dynamics given by

$$dP_t = r_t P_t dt, \tag{2.7}$$

where the interest rate r_t is modeled by the CIR model.

We assume that the zero-coupon bond price $B_t^{\bar{T}}$ is given by

$$dB_t^{\bar{T}} = \left[r_t - \lambda_r Z(\bar{T} - t) r_t\right] B_t^{\bar{T}} dt - Z(\bar{T} - t)\sigma_r \sqrt{r_t} B_t^{\bar{T}} dW_{r,t}, \tag{2.8}$$

where $Z(\bar{T} - t) = \frac{2(e^{\zeta t}-1)}{2\zeta+(\zeta+\alpha_r+\lambda_r)(e^{\zeta t}-1)}$ is the decision function, $\zeta = \sqrt{(\alpha_r + \lambda_r)^2 + 2\sigma_r^2}$, λ_r is the interest rate market risk premium, and $\bar{T}$ is the maturity date of the zero-coupon bond.

Assuming that the price of the underlying asset of the equity index derivative follows Eq (2.5) and that the option price O_t at time t is defined as $g(S_t, v_t, r_t, t)$. According to the Itô-Lévy formula, we obtain

$$\begin{aligned} dg =& g_t dt + g_s S_t \left[r_t + r_v \rho(a v_t + b) + \lambda_s \sqrt{1-\rho^2}(a v_t + b) + \eta r_t \lambda_r - \kappa\lambda\right] dt + g_s S_t \eta \sigma_r \sqrt{r_t} dW_{r,t} \\ & + g_s S_t \left(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}}\right)\left(\rho dW_{v,t} + \sqrt{1-\rho^2} dWs, t\right) + g_v(\theta_v - v_t)\alpha_v dt + g_v \sigma_v \sqrt{v_t} dW_{v,t} \\ & + g_r \alpha_r(\theta_r - r_t) dt + g_r \sigma_r \sqrt{r_t} dW_{r,t} + \frac{1}{2} g_{ss} S_t^2 \left[\left(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}}\right)^2 (\rho^2 + 1 - \rho^2)\right] dt \\ & + \frac{1}{2} g_{ss} S_t^2 \eta^2 \sigma_r^2 r_t dt + \frac{1}{2} g_{vv} \sigma_v^2 v_t dt + \frac{1}{2} g_{rr} \sigma_r^2 r_t dt + \frac{1}{2} g_{sv} S_t \left(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}}\right)\rho\sigma_v \sqrt{v_t} dt \\ & + \frac{1}{2} g_{sr} S_t \eta \sigma_r \sqrt{r_t} \sigma_r \sqrt{r_t} dt + \int_R \Delta g dN_t. \end{aligned} \tag{2.9}$$

Moreover, $O_t = g(S_t, v_t, r_t, t)$ satisfies the following partial differential equation

$$\begin{aligned} rg =& g_t + r_t g_s S_t + [\alpha_v(\theta_v - v_t) - \sigma_v \sqrt{v_t}] g_v + \left[\alpha_r(\theta_r - r_t) - \sigma_r \sqrt{r_t} \frac{\lambda_r}{\sigma_r} \sqrt{r_t}\right] g_r + \frac{1}{2} g_{ss} S^2 \eta^2 \sigma_r^2 r_t \\ & + \frac{1}{2} g_{ss} S_t^2 \left[\left(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}}\right)^2 (\rho^2 + 1 - \rho^2)\right] + \frac{1}{2} g_{vv} \sigma_v^2 v_t + \frac{1}{2} g_{sv} S_t \left(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}}\right)\rho\sigma_v \sqrt{v_t} \\ & + \frac{1}{2} g_{rr} \sigma_r^2 r_t + \frac{1}{2} g_{sr} S_t \eta \sigma_r \sqrt{r_t} \sigma_r \sqrt{r_t} - S_t g_s \lambda\kappa + \lambda \int_R \Delta g \phi(\xi) d\xi, \end{aligned} \tag{2.10}$$

Therefore, from Eqs (2.9) and (2.10), we have

$$\begin{aligned} dg = & \left[rg - r_t g_s S_t - [\alpha_v(\theta_v - v_t) - \sigma_v \sqrt{v_t}] g_v - \left[\alpha_r(\theta_r - r_t) - \sigma_r \sqrt{r_t} \frac{\lambda_r}{\sigma_r} \sqrt{r_t}\right] g_r - \frac{1}{2} g_{ss} S^2 \eta^2 \sigma_r^2 r_t \right. \\ & - \frac{1}{2} g_{ss} S^2 \left[\left(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}}\right)^2 (\rho^2 + 1 - \rho^2)\right] - \frac{1}{2} g_{vv} \sigma_v^2 v_t - \frac{1}{2} g_{sv} S_t \left(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}}\right)\rho\sigma_v \sqrt{v_t} \\ & \left. - \frac{1}{2} g_{rr} \sigma_r^2 r_t - \frac{1}{2} g_{sr} S_t \eta \sigma_r \sqrt{r_t} \sigma_r \sqrt{r_t} - S_t g_s \lambda\kappa - \lambda \int_R \Delta g \phi(\xi) d\xi \right] dt + \frac{1}{2} g_{sr} S_t \eta \sigma_r \sqrt{r_t} \sigma_r \sqrt{r_t} dt \end{aligned}$$

$$+ g_s S_t \left[r_t + r_v \rho (a v_t + b) + \lambda_s \sqrt{1-\rho^2}(a v_t + b) + \eta r_t \lambda_r - \kappa \lambda \right] dt + g_s S_t \eta \sigma_r \sqrt{r_t} dW_{r,t}$$
$$+ g_s S_t \left(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) \left(\rho dW_{v,t} + \sqrt{1-\rho^2} dWs,t \right) + g_v(\theta_v - v_t)\alpha_v dt + g_v \sigma_v \sqrt{v_t} dW_{v,t}$$
$$+ g_r \alpha_r (\theta_r - r_t) dt + g_r \sigma_r \sqrt{r_t} dW_{r,t} + \frac{1}{2} g_{ss} S_t^2 \left[\left(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right)^2 (\rho^2 + 1 - \rho^2) \right] dt$$
$$+ \frac{1}{2} g_{ss} S^2 \eta^2 \sigma_r^2 r_t dt + \frac{1}{2} g_{vv} \sigma_v^2 v_t dt + \frac{1}{2} g_{rr} \sigma_r^2 r_t dt + \frac{1}{2} g_{sv} S_t \left(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) \rho \sigma_v \sqrt{v_t} dt + \int_R \Delta g dN_t$$
$$= rg dt + \sigma_v \lambda_v v_t g_v + \lambda_r r_t g_r + \left[g_s S_t \lambda_v \rho (a v_t + b) + g_s S_t \lambda_s \sqrt{1-\rho^2}(a v_t + b) + g_s S_t \eta r_t \lambda_r \right] dt$$
$$- \lambda \int_R \Delta g \phi(\xi) d\xi dt + g_s S_t \left(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) \rho dW_{v,t} + g_s S_t \left(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) \sqrt{1-\rho^2} dW_{s,t}$$
$$+ g_s S_t \eta \sigma_r \sqrt{r_t} dWr,t + g_v \sigma_v \sqrt{v_t} dW_{v,t} + g_r \sigma_r \sqrt{r_t} dW_{r,t} + \int_R \Delta g dN_t. \tag{2.11}$$

Therefore

$$\begin{aligned}
\frac{dO_t}{O_t} =& r_t dt + \frac{1}{O_t} g_v \sigma_v \lambda_v v_t dt + \frac{1}{O_t} g_r \lambda_r r_t dt + \frac{1}{O_t} g_s S_t \lambda_v \rho (a v_t + b) dt + \frac{1}{O_t} g_s \lambda_s (a v_t + b) \sqrt{1-\rho^2} dt \\
&+ \frac{1}{O_t} g_s S_t \eta \lambda_r r_t dt + \frac{1}{O_t} g_s S_t \left(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) \rho dW_{v,t} + \frac{1}{O_t} g_s S_t \left(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}} \right) \sqrt{1-\rho^2} dW_{s,t} \\
&+ \frac{1}{O_t} g_s S_t \eta \sigma_r \sqrt{r_t} dW_{r,t} + \frac{1}{O_t} g_v \sigma_v \sqrt{v_t} dW_{v,t} + \frac{1}{O_t} g_r \sigma_r \sqrt{r_t} dW_{r,t} + \frac{1}{O_t} \int_R \Delta g dN_t \\
&- \frac{1}{O_t} \lambda \kappa g_s S_t dt - \frac{1}{O_t} \lambda \int_R \Delta g \phi(\xi) d\xi dt \\
=& r_t dt + \frac{1}{O_t} g_s S_t \sqrt{1-\rho^2}(a v_t + b) \left(\lambda_s dt + \frac{1}{\sqrt{v_t}} dW_{s,t} \right) + \frac{1}{O_t} \int_R \Delta g dN_t \\
&+ \frac{1}{O_t} g_s S_t \eta \left(\lambda_r r_t dt + \sigma_r \sqrt{r_t} dW_{r,t} \right) + \frac{1}{O_t} g_r \left(\lambda_r r_t dt + \sigma_r \sqrt{r_t} dW_{r,t} \right) \\
&+ \frac{1}{O_t} g_s S_t \rho (a v_t + b) \left(\lambda_v dt + \frac{1}{\sqrt{v_t}} dW_{v,t} \right) - \frac{1}{O_t} \lambda \int_R \Delta g \phi(\xi) d\xi dt \\
&+ \frac{1}{O_t} g_v \sigma_v v_t \left(\lambda_v dt + \frac{1}{\sqrt{v_t}} dW_{v,t} \right),
\end{aligned} \tag{2.12}$$

where $\Delta g = g((1+\kappa)S_t, v_t, r_t, t) - g(S_t, v_t, r_t, t)$.

As discussed above, the derivative security serves an economically important role in our framework. It allows the investor to adjust risk exposures not only with respect to the stock price risk but also with respect to the additional factors embedded in the stochastic hybrid dynamics. This interpretation is consistent with the incomplete-market intuition emphasized by Ewald [38], namely that optimal portfolio selection depends crucially on how investors can access and span latent sources of uncertainty.

In this section, let x_t represent the investor's investment wealth at moment t, and the portfolio's wealth at the initial moment is $x_0 > 0$. The derivatives O_t^1 and O_t^2 should be interpreted as representative liquid instruments or liquid derivative portfolios used to span the targeted risk factors in a frictionless benchmark setting, and the full spanning of jump risk is an idealized complete market assumption introduced for analytical tractability. So our results should therefore be interpreted as a theoretical benchmark rather than as a literal description of all real-world trading environments; in incomplete markets, the strategy derived here is better viewed as a constrained optimum within the span of available tradable instruments.

The proportion of wealth invested in the stock index is $\pi_s(t)$, that in the zero-coupon bond is $\pi_b(t)$, and that in the derivatives is $\pi_{o1}(t)$ and $\pi_{o2}(t)$, respectively. We define the investment strategy of the investor as $(\pi_s(t), \pi_b(t), \pi_{o1}(t), \pi_{o2}(t))$. Accordingly, the dynamics of the wealth process can be expressed as follows:

$$\begin{aligned}\frac{dx_t}{x_t} =&\pi_s(t)\frac{dS_t}{S_t} + \pi_b(t)\frac{dB_t^{\bar{T}}}{B_t^{\bar{T}}} + \pi_{o1}(t)\frac{dO_t^1}{O_t^1} + \pi_{o2}(t)\frac{dO_t^2}{O_t^2}\\ &+ (1 - \pi_s(t) - \pi_b(t) - \pi_{o1}(t) - \pi_{o2}(t))\frac{dP_t}{P_t}.\end{aligned} \tag{2.13}$$

If we combine the differential equations satisfied by P_t, S_t, $B_t^{\bar{T}}$, and O_t, the wealth process x_t satisfies the following stochastic differential equation (SDE):

$$\begin{aligned}\frac{dx_t}{x_t} =& \pi_s(t)(r_t + \lambda_v\rho(av_t + b) + \lambda_s\sqrt{1-\rho^2}(av_t + b) + \eta r_t\lambda_r - \lambda\kappa)dt\\
&+ \pi_s(t)\eta\sigma_r\sqrt{r_t}dW_{r,t} - \pi_b(t)\lambda_r Z(\bar{T} - t)r_t dt + \pi_s(t)\kappa dN_t + \pi_b(t)r_t dt\\
&+ \pi_s(t)(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}})(\rho dW_{v,t} + \sqrt{1-\rho^2}dW_{s,t}) + \pi_{o1}(t)r_t dt + \pi_{o2}(t)r_t dt\\
&- \pi_b(t)Z(\bar{T} - t)\sigma_r\sqrt{r_t}dW_{r,t} + \pi_{o1}(t)\frac{g_s^1}{O_t^1}S_t\sqrt{1-\rho^2}(av_t + b)\lambda_s dt\\
&+ \pi_{o1}(t)\frac{1}{O_t^1}g_s^1 S_t\sqrt{1-\rho^2}(av_t + b)\frac{1}{\sqrt{v_t}}dWw,t + \pi_{o1}(t)\frac{1}{O_t^1}g_r^1\lambda_r r_t dt\\
&+ \pi_{o1}(t)\frac{1}{O_t^1}g_s^1 S_t\eta\left(\lambda_r r_t dt + \sigma_r\sqrt{r_t}dW_{r,t}\right) + \pi_{o1}(t)\frac{g_s^1}{O_t^1}S_t\rho(av_t + b)\lambda_v dt\\
&+ \pi_{o1}(t)\frac{1}{O_t^1}g_r^1\sigma_r\sqrt{r_t}dW_{r,t} + \pi_{o1}(t)\frac{g_s^1}{O_t^1}S_t\rho(av_t + b)\frac{1}{\sqrt{v_t}}dW_{v,t}\\
&+ \pi_{o1}(t)\frac{1}{O_t^1}g_v^1\sigma_v v_t\Big(\lambda_v dt + \frac{1}{\sqrt{v_t}}dW_{v,t}\Big) - \pi_{o1}(t)\frac{1}{O_t^1}\Delta g^1(\lambda dt + dN_t)\\
&+ \pi_{o2}\frac{g_s^2}{O_t^2}S_t\sqrt{1-\rho^2}(av_t + b)\lambda_s dt + \pi_{o2}(t)\frac{g_r^2}{O_t^2}(\lambda_r r_t dt + \sigma_r\sqrt{r_t}dW_{r,t})\\
&+ \pi_{o2}(t)\frac{g_s^2}{O_t^2}S_t\sqrt{1-\rho^2}(av_t + b)\frac{1}{\sqrt{v_t}}dWs,t + \pi_{o2}(t)\frac{g_s^2}{O_t^2}S_t\rho(av_t + b)\\
&\times\left(\lambda_v dt + \frac{1}{\sqrt{v_t}}dW_{v,t}\right) + \pi_{o2}(t)\frac{g_s^2}{O_t^2}S_t\eta\left(\lambda_r r_t dt + \sigma_r\sqrt{r_t}dW_{r,t}\right)\\
&+ \pi_{o2}(t)\frac{g_v^2}{O_t^2}\sigma_v v_t\Big(\lambda_v dt + \frac{1}{\sqrt{v_t}}dW_{v,t}\Big) - \pi_{o2}(t)\frac{1}{O_t^2}\Delta g^2(\lambda dt + dN_t)\\
=&\Big\{r_t + \pi_s(t)\left((\lambda_v\rho + \lambda_s\sqrt{1-\rho^2})(av_t + b) - \lambda\kappa + \eta r_t\lambda_r\right) - \pi_b(t)\lambda_r Z(\bar{T} - t)r_t\Big\}dt\end{aligned}$$

$$
\begin{aligned}
&+\frac{\pi_{o1}(t)}{O_t^1}\left\{g_s^1 S_t(\lambda_s\sqrt{1-\rho^2}+\lambda_v\rho)(av_t+b)+(g_r^1+g_s^1 S_t\eta)\lambda_r r_t+g_v^1\sigma_v v_t\lambda_v-\lambda\Delta g^1\right\}dt\\
&+\frac{\pi_{o2}(t)}{O_t^2}\left\{g_s^2 S_t(\lambda_s\sqrt{1-\rho^2}+\lambda_v\rho)(av_t+b)+(g_r^2+g_s^2 S_t\eta)\lambda_r r_t+g_v^2\sigma_v v_t\lambda_v-\lambda\Delta g^2\right\}dt\\
&+\pi_s(t)\left(a\sqrt{v_t}+\frac{b}{\sqrt{v_t}}\right)\sqrt{1-\rho^2}dW_{s,t}+\pi_{o1}(t)\frac{1}{O_t^1}g_s^1 S_t\sqrt{1-\rho^2}\left(a\sqrt{v_t}+\frac{b}{\sqrt{v_t}}\right)dW_{s,t}\\
&+\pi_{o2}(t)\frac{1}{O_t^2}g_s^2 S_t\sqrt{1-\rho^2}\left(a\sqrt{v_t}+\frac{b}{\sqrt{v_t}}\right)dW_{s,t}+\pi_s(t)\left(a\sqrt{v_t}+\frac{b}{\sqrt{v_t}}\right)\rho dW_{v,t}\\
&+\pi_{o1}(t)\frac{1}{O_t^1}g_s^1 S_t\rho\left(a\sqrt{v_t}+\frac{b}{\sqrt{v_t}}\right)dW_{v,t}+\pi_{o1}\frac{1}{O_t^1}g_v^1\sigma_v\sqrt{v_t}dW_{v,t}+\pi_{o2}\frac{1}{O_t^2}g_v^2\sigma_v\sqrt{v_t}dW_{v,t}\\
&+\pi_{o2}(t)\frac{g_s^2}{O_t^2}S_t\rho\left(a\sqrt{v_t}+\frac{b}{\sqrt{v_t}}\right)dW_{v,t}+\pi_s(t)\eta\sigma_r\sqrt{r_t}dW_{r,t}-\pi_b(t)Z(\bar{T}-t)\sigma_r\sqrt{r_t}dW_{r,t}\\
&+\pi_{o1}(t)\frac{1}{O_t^1}\sqrt{r_t}\sigma_r\left(g_r^1+g_s^1 S_t\eta\right)dW_{r,t}+\pi_{o2}(t)\frac{1}{O_t^2}\sqrt{r_t}\sigma_r\left(g_r^2+g_s^2 S_t\eta\right)dW_{r,t}\\
&+\pi_s(t)\kappa dN_t+\pi_{o1}(t)\frac{1}{O_t^1}\Delta g^1 dN_t+\pi_{o2}(t)\frac{1}{O_t^2}\Delta g^2 dN_t\\
&=r_t dt+\lambda_s v_t\Big[\Big(\pi_s(t)+\pi_{o1}(t)\frac{g_s^1}{O_t^1}S_t+\pi_{o2}(t)\frac{g_s^2}{O_t^2}S_t\Big)\sqrt{1-\rho^2}\Big(a+\frac{b}{v_t}\Big)\Big]dt\\
&+\lambda_v v_t\Big[\Big(\pi_s(t)+\pi_{o1}(t)\frac{g_s^1 S_t}{O_t^1}+\pi_{o2}(t)\frac{g_s^2 S_t}{O_t^2}\Big)\Big(a+\frac{b}{v_t}\Big)\rho+\pi_{o1}(t)\frac{g_v^1\sigma_v}{O_t^1}\\
&+\pi_{o2}(t)\frac{g_v^2\sigma_v}{O_t^2}\Big]dt+\lambda_r r_t\Big[\pi_s(t)\eta-\pi_b(t)Z(\bar{T}-t)+\pi_{o1}(t)\Big(\frac{g_r^1}{O_t^1}+\frac{g_s^1 S_t\eta}{O_t^1}\Big)\\
&+\pi_{o2}(t)\Big(\frac{g_r^2}{O_t^2}+\frac{g_s^2 S_t\eta}{O_t^2}\Big)\Big]dt-\lambda\Big[\pi_s(t)\kappa+\pi_{o1}(t)\frac{\Delta g^1}{O_t^1}+\pi_{o2}(t)\frac{\Delta g^2}{O_t^2}\Big]dt\\
&+\Big[\pi_s(t)\kappa+\pi_{o1}(t)\frac{\Delta g^1}{O_t^1}+\pi_{o2}(t)\frac{\Delta g^2}{O_t^2}\Big]dN_t+\sqrt{v_t}\Big[\Big(\pi_s(t)+\pi_{o1}(t)\frac{g_s^1 S_t}{O_t^1}\\
&+\pi_{o2}(t)\frac{g_s^2 S_t}{O_t^2}\Big)\Big(a+\frac{b}{v_t}\Big)\rho+\pi_{o1}(t)\frac{g_v^1\sigma_v}{O_t^1}+\pi_{o2}(t)\frac{g_v^2\sigma_v}{O_t^2}\Big]dW_{v,t}+\sqrt{r_t}\sigma_r\Big[\pi_s(t)\\
&\eta-\pi_b(t)Z(\bar{T}-t)+\pi_{o1}(t)\Big(\frac{g_r^1}{O_t^1}+\frac{g_s^1 S_t\eta}{O_t^1}\Big)+\pi_{o2}(t)\Big(\frac{g_r^2}{O_t^2}+\frac{g_s^2 S_t\eta}{O_t^2}\Big)\Big]dW_{r,t}\\
&+\sqrt{v_t}\Big[\Big(\pi_s(t)+\pi_{o1}(t)\frac{1}{O_t^1}g_s^1 S_t+\pi_{o2}(t)\frac{1}{O_t^2}g_s^2 S_t\Big)\sqrt{1-\rho^2}\Big(a+\frac{b}{v_t}\Big)\Big]dW_{s,t}\\
&=r_t dt+(\pi_1\lambda_s v_t+\pi_2\lambda_v v_t+\pi_3\lambda_r r_t-\lambda\pi_4)dt+\pi_1\sqrt{v_t}dW_{s,t}+\pi_2\sqrt{v_t}dW_{v,t}\\
&+\pi_4 dN_t+\pi_3\sigma_r\sqrt{r_t}dW_{r,t}\\
&=r_t dt+\pi_1(\lambda_s v_t dt+\sqrt{v_t}dW_{s,t})+\pi_2(\lambda_v v_t dt+\sqrt{v_t}dW_{v,t})\\
&+\pi_3(\lambda_r r_t dt+\sigma_r\sqrt{r_t}dW_{r,t})+\pi_4(dN_t-\lambda dt),
\end{aligned}
\tag{2.14}
$$

where π_1, π_2, π_3, and π_4 denote the exposures of $W_{s,t}$, $W_{v,t}$, $W_{r,t}$, and N_t, respectively, and

$$\begin{pmatrix}\pi_1\\ \pi_2\\ \pi_3\\ \pi_4\end{pmatrix}=\begin{pmatrix}\sqrt{1-\rho^2}(a+\frac{b}{v_t}) & 0 & \frac{g_s^1S_t}{O_t^1}\sqrt{1-\rho^2}(a+\frac{b}{v_t}) & \frac{g_s^2S_t}{O_t^2}\sqrt{1-\rho^2}(a+\frac{b}{v_t})\\ \rho(a+\frac{b}{v_t}) & 0 & \frac{g_s^1S_t\rho}{O_t^1}(a+\frac{b}{v_t})+\frac{g_v^1\sigma_v}{O_t^1} & \frac{g_s^2S_t\rho}{O_t^2}(a+\frac{b}{v_t})+\frac{g_v^2\sigma_v}{O_t^2}\\ \eta & -Z(\bar{T}-t) & \frac{1}{O_t^1}g_r^1+\frac{1}{O_t^1}g_s^1S_t\eta & \frac{1}{O_t^2}g_r^2+\frac{1}{O_t^2}g_s^2S_t\eta\\ \kappa & 0 & \frac{1}{O_t^1}\Delta g^1 & \frac{1}{O_t^2}\Delta g^2\end{pmatrix}\begin{pmatrix}\pi_s(t)\\ \pi_b(t)\\ \pi_{o1}(t)\\ \pi_{o2}(t)\end{pmatrix}. \tag{2.15}$$

Here, G is given by

$$G=\begin{pmatrix}\sqrt{1-\rho^2}(a+\frac{b}{v_t}) & 0 & \frac{g_s^1S_t}{O_t^1}\sqrt{1-\rho^2}(a+\frac{b}{v_t}) & \frac{g_s^2S_t}{O_t^2}\sqrt{1-\rho^2}(a+\frac{b}{v_t})\\ \rho(a+\frac{b}{v_t}) & 0 & \frac{g_s^1S_t\rho}{O_t^1}(a+\frac{b}{v_t})+\frac{g_v^1\sigma_v}{O_t^1} & \frac{g_s^2S_t\rho}{O_t^2}(a+\frac{b}{v_t})+\frac{g_v^2\sigma_v}{O_t^2}\\ \eta & -Z(\bar{T}-t) & \frac{1}{O_t^1}g_r^1+\frac{1}{O_t^1}g_s^1S_t\eta & \frac{1}{O_t^2}g_r^2+\frac{1}{O_t^2}g_s^2S_t\eta\\ \kappa & 0 & \frac{1}{O_t^1}\Delta g^1 & \frac{1}{O_t^2}\Delta g^2\end{pmatrix}. \tag{2.16}$$

Then

$$\begin{pmatrix}\pi_1 & \pi_2 & \pi_3 & \pi_4\end{pmatrix}^{\mathsf{T}}=G\begin{pmatrix}\pi_s(t) & \pi_b(t) & \pi_{o1}(t) & \pi_{o2}(t)\end{pmatrix}^{\mathsf{T}}. \tag{2.17}$$

To ensure the invertibility of G, the following inequality must be satisfied:

$$\begin{vmatrix}\sqrt{1-\rho^2}(a+\frac{b}{v_t}) & 0 & \frac{g_s^1S_t}{O_t^1}\sqrt{1-\rho^2}(a+\frac{b}{v_t}) & \frac{g_s^2S_t}{O_t^2}\sqrt{1-\rho^2}(a+\frac{b}{v_t})\\ \rho(a+\frac{b}{v_t}) & 0 & \frac{g_s^1S_t\rho}{O_t^1}(a+\frac{b}{v_t})+\frac{g_v^1\sigma_v}{O_t^1} & \frac{g_s^2S_t\rho}{O_t^2}(a+\frac{b}{v_t})+\frac{g_v^2\sigma_v}{O_t^2}\\ \eta & -Z(\bar{T}-t) & \frac{1}{O_t^1}g_r^1+\frac{1}{O_t^1}g_s^1S_t\eta & \frac{1}{O_t^2}g_r^2+\frac{1}{O_t^2}g_s^2S_t\eta\\ \kappa & 0 & \frac{1}{O_t^1}\Delta g^1 & \frac{1}{O_t^2}\Delta g^2\end{vmatrix}\neq 0.$$

This leads to the appropriate resolution

$$\left(\frac{\Delta g^1}{O_t^1}-\frac{g_s^1S_t}{O_t^1}\right)\frac{g_v^2}{O_t^2}-\left(\frac{\Delta g^2}{O_t^2}-\frac{g_s^2S_t}{O_t^2}\right)\frac{g_v^1}{O_t^1}\neq 0. \tag{2.18}$$

Assume that within a finite interval $[0,T]$, the investment objective function is defined as follows:

$$J(x_t,v_t,r_t,t)=\mathbb{E}\left[U(x_T)\right],$$

where $U(\cdot)$ denotes the utility function and satisfies $U^{'}(\cdot)>0$, $U^{''}(\cdot)<0$.

The investor's objective is to determine an admissible strategy $(\pi_s^*(t),\pi_b^*(t),\pi_{o1}^*(t),\pi_{o2}^*(t))\in\Pi$ that maximizes the objective function $J(x,v,r,t)$. Accordingly, the definition of an admissible investment strategy is introduced at the outset.

Definition 2.1. *For any given $t\in[0,T]$, the policy $\Pi=(\pi_s(t),\pi_b(t),\pi_{o1}(t),\pi_{o2}(t))$ is deemed to be a permissible investment strategy if it satisfies the following: Π is sequentially measurable with respect to $\mathcal{F}_t$; under Π, the wealth process x_t is non-negative; and for the initial condition $(x_0,v_0,r_0,t_0)\in[0,T]\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+$, Eq (2.13) possesses a unique solution with $\mathbb{E}_{x_0,v_0,r_0,t_0}\left[U(x_t)\right]<\infty$.*

3. Optimal investment strategy based on 4/2-CIR jump-diffusion stochastic hybrid model

The subsequent HJB analysis is conducted for the wealth dynamics expressed in the exposure space, under the admissibility restriction that the quadratic variation of wealth is affine in the state variables v_t and r_t. Hence, the closed-form solution applies to the exposure-based control problem rather than to an unrestricted portfolio problem under the raw 4/2 diffusion coefficient.

On the basis of the optimal investment objective, we define the value function $H(x_t, v_t, r_t, t)$ as follows:

$$H(x_t, v_t, r_t, t) = \sup_{(\pi_s(t),\pi_b(t),\pi_{o1}(t),\pi_{o2}(t))\in\Pi} J(x_t, v_t, r_t, t),$$

where $(\pi_s(t), \pi_b(t), \pi_{o1}(t), \pi_{o2}(t))$ denotes the admissible investment strategy as defined in Definition 2.1.

Proposition 3.0 *Under the exposure representation* $(\pi_1, \pi_2, \pi_3, \pi_4)$*, the investories wealth process satisfies*

$$\begin{aligned}\frac{dx_t}{x_t} &= r_t dt + \pi_1(\lambda_s v_t dt + \sqrt{v_t} dW_{s,t}) + \pi_2(\lambda_v v_t dt + \sqrt{v_t} dW_{v,t}) \\ &\quad + \pi_3(\lambda_r r_t dt + \sigma_r \sqrt{r_t} dW_{r,t}) + \pi_4(dN_t - \lambda dt),\end{aligned}$$

so that its quadratic variation is affine in

$$\frac{dx_t^2}{x_t^2} = (\pi_1^2 + \pi_2^2)v_t dt + \pi_3^2\sigma_r^2 r_t dt.$$

Therefore the HJB equation studied below corresponds to the affine exposure formulation.

Following the dynamic programming principle, the value function $H(x_t, v_t, r_t, t)$ satisfies the following HJB equation:

$$\begin{aligned}0 =& x_t(r_t + \pi_1\lambda_s v_t + \pi_2\lambda_v v_t + \pi_3\lambda_r r_t - \lambda\pi_4)H_x + \frac{1}{2}\sigma_r^2 r_t H_{rr} + \pi_2 x_t \sigma_v v_t H_{xv} \\ &+ \frac{1}{2}\sigma_v^2 v_t H_{vv} + \alpha_r(\theta_r - r_t)H_r + \frac{1}{2}x_t^2\left(\pi_1^2 v_t + \pi_2^2 v_t + \pi_3^2\sigma_r^2 r_t\right)H_{xx} \\ &+ H_t + \alpha_v(\theta_v - v_t)H_v + \pi_3 x_t \sigma_r^2 r_t H_{xr} + \widetilde{\lambda}\Delta H,\end{aligned} \tag{3.1}$$

where H_t, H_x, H_v, H_r and H_{xx}, H_{vv}, H_{rr}, H_{xv}, H_{xr} are first-order and second-order partial derivatives of $H(x_t, v_t, r_t, t)$ with respect to its arguments, respectively; $\Delta H = H(x_t(1 + \pi_4), v_t, r_t, t) - H(x_t, v_t, r_t, t)$ captures the jump component.

Remark 3.1. *For a raw 4/2 diffusion coefficient* $a\sqrt{v_t} + \frac{b}{\sqrt{v_t}}$*, the squared diffusion term contains both* v_t *and* $\frac{1}{v_t}$ *components and is not affine in* v_t*. Therefore, without further restrictions, the corresponding portfolio HJB equation does not need to admit an exponential affine solution. The present paper circumvents this difficulty by formulating the optimization problem in the exposure space and imposing an affine quadratic variation structure on admissible wealth dynamics. Hence, tractability is achieved at the level of the control formulation rather than by claiming an unrestricted affine representation of the raw 4/2 diffusion itself.*

This paper assumes that the investor's utility function takes the following power utility form:

$$U(x) = \frac{x^\delta}{\delta}, \delta < 1, \delta \neq 0. \tag{3.2}$$

The investor's objective is to maximize the expected terminal utility over the admissible class of strategies. By dynamic programming, we derive the associated HJB equation and solve for the value function.

The optimal risk exposure and the optimal investment strategy are then obtained explicitly. The martingale verification argument confirms that the candidate solution is indeed optimal.

The exposure-based interpretation of the optimal strategy is particularly useful here. Rather than viewing derivative positions as purely speculative add-ons, our formulation treats them as instruments that reshape the investor's exposure to volatility, interest-rate, and jump-related risks. This complements the insight of Ewald [38], where the admissible portfolio problem under stochastic volatility is also fundamentally driven by the investor's effective access to the relevant risk factors.

We now present the solution to Eq (3.1) under this power utility framework.

Theorem 3.1. *The solution to problem (3.1) admits the following explicit form:*

$$H(x_t, v_t, r_t, t) = e^{-\beta t}\frac{x^{\delta}}{\delta}\exp\{A(t)v_t + B(t)r_t + D(t)\}, \tag{3.3}$$

with the following corresponding optimal exposures:

$$\begin{cases} \pi_1^* = \dfrac{\lambda_s}{1-\delta} \\ \pi_2^* = \dfrac{\lambda_v}{1-\delta} + \dfrac{\sigma_v}{1-\delta}A(t) \\ \pi_3^* = \dfrac{\lambda_r}{(1-\delta)\sigma_r^2} + \dfrac{1}{1-\delta}B(t) \\ \pi_4^* = \left(\dfrac{\lambda}{\bar{\lambda}}\right)^{\frac{1}{\delta-1}} - 1. \end{cases} \tag{3.4}$$

The optimal investment strategy is given by

$$\begin{pmatrix}\pi_s(t) & \pi_b(t) & \pi_{o1}(t) & \pi_{o2}(t)\end{pmatrix}^{\mathsf{T}} = G^{-1}\begin{pmatrix}\pi_1^* & \pi_2^* & \pi_3^* & \pi_4^*\end{pmatrix}^{\mathsf{T}}, \tag{3.5}$$

where the functions $A(t)$, $B(t)$, and $D(t)$ and the matrix G^{-1} are given by (A.12), (A.14), (A.17), and (A.18) in Appendix A.1, respectively.

Proof. See Appendix A.1. □

The conditions ensuring the well-definedness of the optimal solution are provided below.

Propositionn 3.1. *For any $\delta < 1$ with $\delta \neq 0$, if the following conditions are satisfied:*

$$\begin{cases} \alpha_v^2 + \dfrac{\delta}{1-\delta}\Big[\dfrac{\delta}{1-\delta}\sigma_v^2\lambda_v^2 - 2\sigma_v\lambda_v\alpha_v - \dfrac{1}{1-\delta}\sigma_v^2(\lambda_v^2 + \lambda_s^2)\Big] > 0 \\ \alpha_r^2 - \dfrac{\delta}{1-\delta}\Big[\lambda_r^2 + 2\lambda_r\sigma_r + 2\sigma_r^2\Big] > 0 \end{cases}$$

then the function $H(x_t, v_t, r_t, t)$ constitutes a well-defined solution to the HJB equation (Eq (3.1)).

Proof. See Appendix A.2. □

A verification theorem is provided below, proving that the function $H(x_t, v_t, r_t, t)$ defined in (3.3) is the value function and that the exposure given by Eq (3.4) is the optimal exposure.

Theorem 3.2. *Given the initial conditions* $(x_0, v_0, r_0, t_0) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$*, if the function* $H(x_t, v_t, r_t, t) : [0, T] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *satisfies the following conditions:*

(1) $H(x_t, v_t, r_t, t)$ *is a real-valued and finite function that is first-order differentiable in t and second-order differentiable in v and r;*

(2) $H(x_t, v_t, r_t, t)$ *satisfies Eq (3.1) and admits the representation*

$$H(x_t, v_t, r_t, t) = e^{-\beta t} \frac{x_t^\delta}{\delta} h(v_t, r_t, t)$$

where $h(v_t, r_t, t) = \exp(A(t)v_t + B(t)r_t + D(t))$ *for positive functions, with the terminal condition* $H(x_T, v_T, r_T, T) = e^{-\beta T} \frac{x_T^\delta}{\delta}$*;*

(3) The coefficient function $A(t)$ *in (A.12) of Appendix A.1 satisfies the inequality*

$$-\frac{1}{2}\left[\frac{\sigma_v^2}{(1-\delta)^2} A^2(0) + \frac{2\delta\lambda_v\sigma_v}{(1-\delta)^2} A(0) + \frac{(\lambda_v^2 + \lambda_s^2)\delta^2}{(1-\delta)^2}\right] \geq -\frac{\alpha_v^2}{2\sigma_v^2}; \tag{3.6}$$

(4) The coefficient function $B(t)$ *in (A.14) of Appendix A.1 satisfies the inequality*

$$-\frac{1}{2}\left[\frac{\sigma_r^2}{(1-\delta)^2} B^2(0) + \frac{2\delta\lambda_r}{(1-\delta)^2} B(0) + \frac{\lambda_r^2\delta^2}{(1-\delta)^2\sigma_r^2}\right] \geq -\frac{\alpha_r^2}{2\sigma_r^2}. \tag{3.7}$$

In this case, the function in (3.3) is the value function, the exposure in (3.4) is the optimal exposure, and the investment strategy in (3.5) is the optimal investment strategy.

Proof. See Appendix A.3. □

Theorem 3.1 provides the optimal risk exposure and investment strategy derived from the 4/2-CIR jump-diffusion stochastic portfolio model. Given the market prices of the investor's holdings including bank deposits, interest-rate-sensitive equities, stock indices, and their derivatives, the model parameters can be estimated. Using these estimated parameters together with Eqs (3.4) and (3.5), we derive the investor's optimal risk exposure under the power utility function and the optimal portfolio weights for each asset, thereby facilitating optimal investment decisions. Furthermore, Proposition 3.1 and Theorem 3.2 establish conditions that ensure that the value function $H(x_t, v_t, r_t, t)$ is well-defined and provide solutions to the associated HJB equations. These results characterize the corresponding risk exposures, confirm the optimality of the investment decisions, and furnish the theoretical foundation for applying Theorem 3.1.

Remark 3.2. *When* $\lambda = 0$ *and* $\eta = 0$ *in the 4/2-CIR jump-diffusion stochastic hybrid model, the problem simplifies to the optimal portfolio problem under the 4/2 stochastic volatility model [22]. When* $\lambda = 0$ *and* $\eta \neq 0$*, it simplifies to the optimal investment problem under the 4/2-CIR stochastic hybrid model [24].*

4. Suboptimal strategies and utility loss under the 4/2-CIR jump-diffusion stochastic hybrid model

To assess the economic value of the model's richness, we further construct suboptimal benchmark strategies obtained by ignoring selected state variables or risk channels. We then compute the associated certainty-equivalent utility losses relative to the fully optimal policy.

This subsection analyzes the suboptimal risk exposures $\pi^s = (\pi_1^s, \pi_2^s, \pi_3^s, \pi_4^s)^\top$ that ignore intertemporal maturities and quantifies the associated utility loss.

Let $H^{\pi^s}(x_t, v_t, r_t, t)$ denote the value function corresponding to the suboptimal strategy π^s, which satisfies $H^{\pi^s}(x_t, v_t, r_t, t) \leq H^{\pi^*}(x_t, v_t, r_t, t)$, with equality holding if and only if π^s is optimal. Given that the optimal value function $H^{\pi^*}(x_t, v_t, r_t, t)$ in (3.3), the function $H^{\pi^s}(x_t, v_t, r_t, t)$ satisfies the HJB equation

$$\begin{aligned}
0 =& H_t^{\pi^s} + x_t(r_t + \pi_1^s\lambda_s v_t + \pi_2^s\lambda_v v_t + \pi_3^s\lambda_r r_t - \lambda\pi_4^s)H_x^{\pi^s} \\
&+ \alpha_r(\theta_r - r_t)H_r^{\pi^s} + \frac{1}{2}x_t^2(\pi_1^{s2}v_t + \pi_2^{s2}v_t + \pi_3^{s2}\sigma_r^2 r_t)H_{xx}^{\pi^s} \\
&+ \alpha_v(\theta_v - v_t)H_v^{\pi^s} + \frac{1}{2}\sigma_v^2 v_t H_{vv}^{\pi^s} + \frac{1}{2}\sigma_r^2 r_t H_{rr}^{\pi^s} \\
&+ \pi_2^s x_t\sigma_v v_t H_{xv}^{\pi^s} + \pi_3^s x_t\sigma_r^2 r_t H_{xr}^{\pi^s} + \widetilde{\lambda}\Delta H^{\pi^s},
\end{aligned} \tag{4.1}$$

with the boundary condition $H^{\pi^s}(x_T, v_T, r_T, T) = e^{-\beta T}\frac{x_T^\delta}{\delta}$, and the jump component given by

$$\Delta H^{\pi^s} = H^{\pi^s}(x_t(1+\pi_4^s), v_t, r_t, t) - H^{\pi^s}(x_t, v_t, r_t, t).$$

Propositionn 4.1. *The solution to the partial differential equation (Eq (4.1)) admits the exponential affine representation*

$$H^{\pi^s}(x_t, v_t, r_t, t) = e^{-\beta t}\frac{x_t^\delta}{\delta}\exp\left\{A^s(t)v_t + B^s(t)r_t + D^s(t)\right\}, \tag{4.2}$$

with the suboptimal exposures

$$\begin{cases}
\pi_1^s = \dfrac{\lambda_s}{1-\delta} \\
\pi_2^s = \dfrac{\lambda_v}{1-\delta} \\
\pi_3^s = \dfrac{\lambda_r}{(1-\delta)\sigma_r^2} \\
\pi_4^s = \left(\dfrac{\lambda}{\widetilde{\lambda}}\right)^{\frac{1}{\delta-1}} - 1
\end{cases}. \tag{4.3}$$

The corresponding investment strategies are

$$\begin{pmatrix}\pi_s^s(t) & \pi_b^s(t) & \pi_{o1}^s(t) & \pi_{o2}^s(t)\end{pmatrix}^\top = G^{-1}\begin{pmatrix}\pi_1^s & \pi_2^s & \pi_3^s & \pi_4^s\end{pmatrix}^\top, \tag{4.4}$$

where G^{-1} is given by (A.18) in Appendix A.1. The coefficient functions satisfy

$$A^s(t) = \frac{A_1^s A_2^s(1 - e^{-d_v^s(T-t)})}{A_1^s - A_2^s e^{-d_v^s(T-t)}}, B^s(t) = \frac{B_1^s B_2^s(1 - e^{-d_r^s(T-t)})}{B_1^s - B_2^s e^{-d_r^s(T-t)}},$$

$$D^s(t) = \int_t^T \alpha_v\theta_v A^s(s)ds + \int_t^T \alpha_r\theta_r B^s(s)ds + c(T-t),$$

with the constant c is given by (A.15) in Appendix A.1.

$$A_1^s = \frac{-b_v^s + d_v^s}{a_v^s}, \quad A_2^s = \frac{-b_v^s - d_v^s}{a_v^s}, \quad B_1^s = \frac{-b_r^s + d_r^s}{a_r^s}, \quad B_2^s = \frac{-b_r^s - d_r^s}{a_r^s},$$
$$a_v^s = \sigma_v^2, \quad b_v^s = \frac{\delta\sigma_v}{1-\delta}\lambda_v - \alpha_v, \quad c_v^s = \frac{\delta}{1-\delta}(\lambda_v^2 + \lambda_s^2), \quad d_v^s = \sqrt{b_v^{s2} - a_v^s c_v^s},$$
$$a_r^s = \sigma_r^2, \quad b_r^s = \frac{\delta}{1-\delta}\lambda_r - \alpha_r, \quad c_r^s = 2\delta + \frac{\delta}{1-\delta}\frac{\lambda_r^2}{\sigma_r^2}, \quad d_r^s = \sqrt{b_r^{s2} - a_r^s c_r^s}.$$

The derivation follows the same methodology as Theorem 3.1 and is omitted for brevity.

We quantify economic loss as the percentage of wealth sacrifice L^{π^s} satisfying

$$H^{\pi^*}(x_t(1 - L^{\pi^s}), v_t, r_t, t) = H^{\pi^s}(x_t, v_t, r_t, t),$$

exploiting the exponential affine structure of the value functions. The closed-form loss expression is

$$L^{\pi^s} = 1 - \exp\left\{\frac{1}{\delta}\Big[(A^s(t) - A(t))v_t + (B^s(t) - B(t))r_t + (D^s(t) - D(t))\Big]\right\}. \tag{4.5}$$

Numerical sensitivity analyses quantify the impact of key parameters on the optimal risk exposure and loss function within the proposed 4/2-CIR stochastic hybrid portfolio model. Furthermore, to evaluate the performance of our derived optimal investment strategies, we compute the relative wealth loss for strategies based on the 4/2 stochastic volatility model [22] and the 4/2-CIR stochastic hybrid model [24]. A positive value of this wealth loss metric indicates the superiority of the strategy derived from the 4/2-CIR jump-diffusion stochastic hybrid model.

5. Numerical examples and analysis

In this section, we numerically analyze the optimal risk exposure and utility loss of the 4/2-CIR jump-diffusion stochastic hybrid model with a power utility function and examine the effects of key model parameters on these quantities. The values of the model parameters, unless otherwise stated, are as follows: $\lambda_s = 2.2472, \lambda_v = -6.6932, \sigma_v = 0.2941, \alpha_v = 2.8278, \theta_v = 0.0563, \delta = -0.8123, \lambda_r = -0.1132, \sigma_r = 0.0566, \alpha_r = 0.1300, \theta_r = 0.0025, \lambda = 0.3, \widetilde{\lambda} = 0.1, \beta = 0.5, t = 0, T = 1, v_0 = 0.4, r_0 = 0.03, \rho = -0.2292, \kappa = 5$, and $\eta = -0.5973$. The basic parameters are derived from Escobar et al. [31], and the subjective discount rates are set appropriately.

5.1. The effect of 4/2-CIR jump-diffusion stochastic hybrid model parameters on optimal risk exposure

Figures 1–3 illustrate the impact of the risk aversion factor δ, the investment horizon T, the equity risk premium λ_s, the volatility risk premium λ_v, the interest rate risk premium λ_r, and the jump risk premium λ on the optimal risk exposures $\pi_1^*, \pi_2^*, \pi_3^*$, and π_4^* within the 4/2-CIR jump-diffusion stochastic hybrid model, where the risk factors are $W_{s,t}$, $W_{v,t}$, $W_{r,t}$, and N_t.

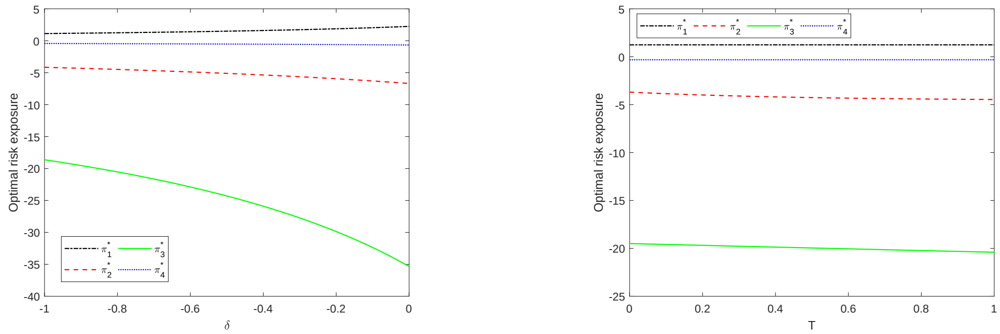

Figure 1. Effect of the parameters δ and T on the optimal risk exposure.

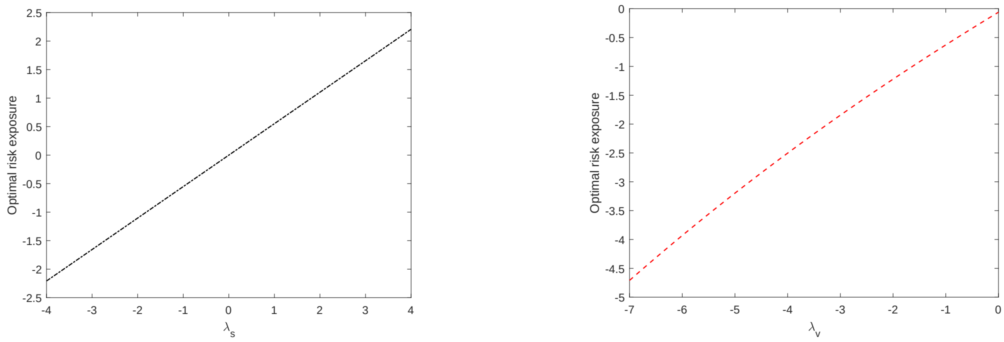

Figure 2. Effect of the parameters λ_s and λ_v on the optimal risk exposure.

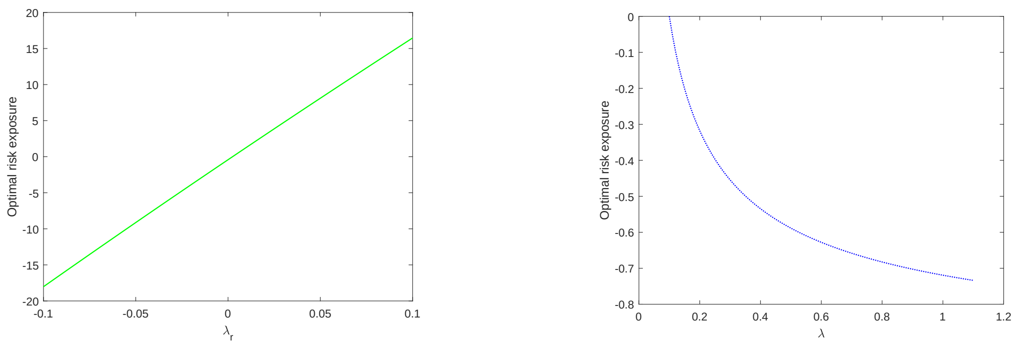

Figure 3. Effect of the parameters λ_r and λ on the optimal risk exposure.

Figure 1 shows that the absolute values of the optimal exposures π_1^*, π_2^*, π_3^*, and π_4^* for the equity, volatility, interest rate, and jump risk factors, respectively, decrease as δ decreases. This is mainly because the smaller δ is, the larger the risk aversion coefficient $1-\delta$ becomes, and a risk-averse investor invests more conservatively to take less risk. In addition, the investment horizon T has some effect on the optimal exposures π_2^* and π_3^* (for volatility and interest rate risk factors), while it has no significant effect on π_1^* and π_4^* (for equity and jump risk factors).

Figure 2 presents the effect of the equity risk premium parameter λ_s on the optimal risk exposure π_1^* and the effect of the volatility risk premium parameter λ_v on the optimal risk exposure π_2^*. Specifically, the optimal exposure π_1^* of the risk factor $W_{s,t}$ increases with the increase in the equity risk premium parameter λ_s; the optimal exposure π_2^* of the risk factor $W_{v,t}$ increases with the increase in the volatility risk premium parameter λ_v. In short, the optimal exposure increases with the increase in the risk premium parameter. When the risk premium parameter is positive and increases, investors become more tolerant of risk and are more willing to adopt aggressive investment strategies to compensate for the increase in risk, thus making the investment riskier and the optimal exposure larger. When the risk premium is negative, the optimal risk exposure is also negative, and the investor adopts a more prudent investment strategy to reduce risk and protect their investment returns.

Figure 3 shows the effects of the interest rate risk premium parameter λ_r and the jump risk premium parameter λ on the optimal exposures π_3^* and π_4^* corresponding to the risk factors $W_{r,t}$ and N_t. First, the optimal exposure π_3^* to $W_{r,t}$ increases with λ_r. When the risk premium parameter is negative, the corresponding optimal risk exposure is also negative, and investors reduce risk through prudent investment strategies. Second, Figure 3 shows that the optimal exposure π_4^* to N_t is negative; that is, in the presence of jump risk, the investor adopts a more robust investment strategy to reduce risk. In addition, the absolute value of the optimal risk exposure increases with the jump's intensity, because the larger the jump's intensity, the greater the investment risk faced by the investor, and thus the optimal risk exposure also increases.

To verify the feasibility of the solution to the HJB equation obtained in this paper, a surface plot of the value function with respect to the stochastic volatility and stochastic interest rate is presented below.

As Figure 4 shows, the value function displays a pronounced nonlinear dependence on the state variables. This nonlinearity arises because the value function is determined not only by the investors' current wealth level but also jointly by stochastic volatility, stochastic interest rates, and the jump-diffusion component. In particular, under the 4/2 volatility specification, the term $a\sqrt{v_t} + \frac{b}{\sqrt{v_t}}$ induces substantial nonlinear effects as v_t varies. As a consequence, the value function surface exhibits a distinct curvature, rather than a simple linear relationship with the state variables.

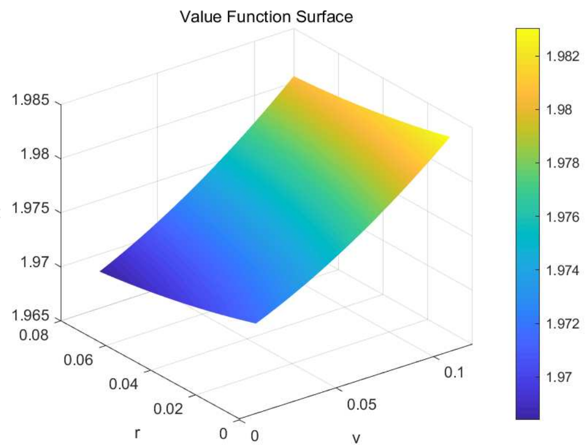

Figure 4. Surface plot of the value function with respect to the stochastic volatility and stochastic interest rate.

From an economic standpoint, the value function characterizes the maximal expected utility attainable by the investor conditional on the current market state. An increase in stochastic volatility generally implies a higher degree of market uncertainty and a less favorable investment opportunity set, thereby tending to reduce the investor's optimal expected utility. Variations in the interest rate affect the value function through two main channels. First, they modify the excess returns and intertemporal investment opportunities associated with risky assets. Second, they influence the present value of future utility via the discounting mechanism. Accordingly, the value function adjusts endogenously to changes in both volatility and interest rate conditions. The state dependence illustrated in Figure 4 therefore suggests that under the joint influence of stochastic volatility and stochastic interest rates, the investor's optimal utility level exhibits significant dynamic heterogeneity.

5.2. The effect of 4/2-CIR jump-diffusion stochastic hybrid model parameters on utility loss

In the financial market investment problem, the absence of an intertemporal hedging strategy may lead to short-sighted behavior, resulting in a suboptimal investment strategy and a loss of utility. Based on the discussion in Section 4, this subsection examines, through numerical computations, how the model's key parameters affect the short-sighted loss.

Figure 5 shows that, for a positive risk premium factor, the short-sighted loss increases monotonically as the risk premium factor rises. This can be attributed to the higher risk tolerance induced by a larger positive risk premium factor, which encourages more aggressive investment behavior and thus amplifies the short-sighted loss. By contrast, Figure 6 and Figure 7 indicate that, for a negative risk premium factor, the short-sighted loss decreases as the risk premium factor increases, reflecting reduced risk tolerance and a shift toward more conservative investment decisions.

To demonstrate the advantages of the investment strategies derived in this work, the 4/2-CIR model is compared with the 4/2 stochastic volatility model and the 4/2-CIR jump-diffusion stochastic hybrid model. The performance of the corresponding investment decisions is evaluated in terms of the resulting utility loss, and the results of the comparison are presented in Figures 8 and 9.

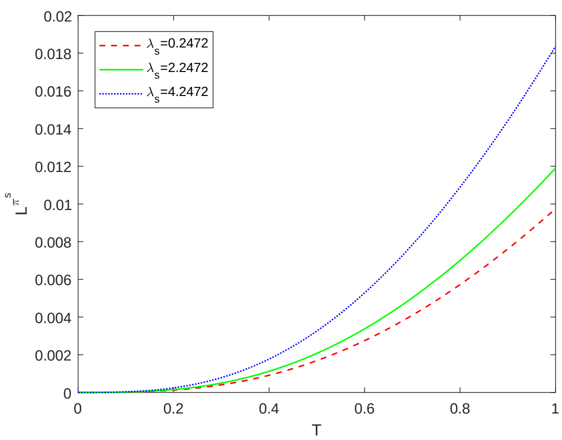

Figure 5. Effect of the parameter λ_s on short-sighted losses.

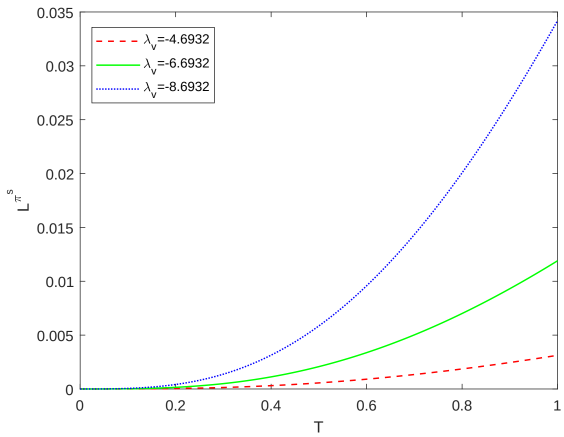

Figure 6. Effect of the parameter λ_v on short-sighted losses.

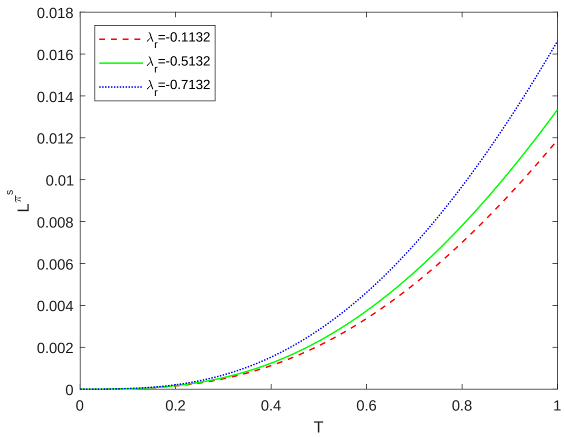

Figure 7. Effect of the parameter λ_r on short-sighted losses.

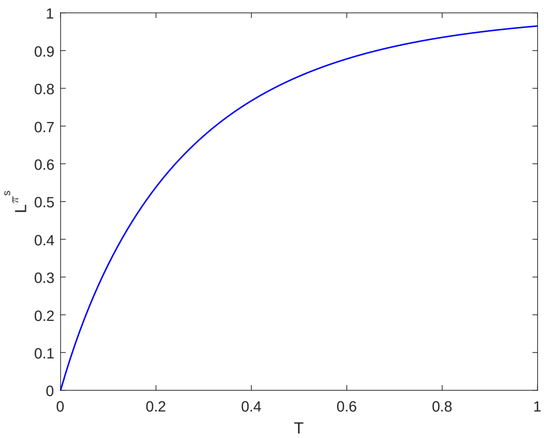

Figure 8. Ignoring losses due to feature hopping.

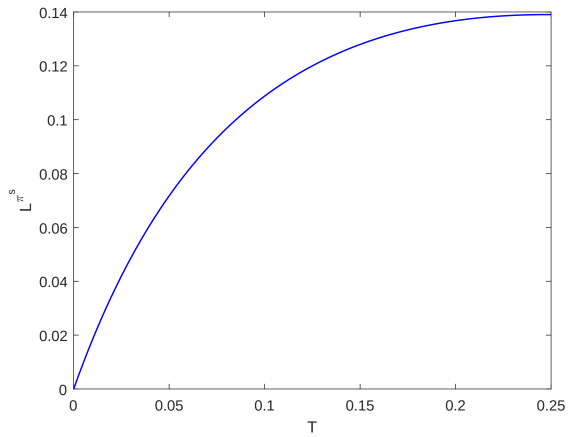

Figure 9. Ignoring losses due to jumps and interest rate randomness.

Figure 8 shows the loss caused by ignoring the jump feature. According to the figure, the loss function is greater than zero, indicating that investing with a strategy that ignores jumps (e.g., derived from the 4/2-CIR stochastic hybrid model without jump-diffusion) leads to a higher loss compared with the strategy obtained from the 4/2-CIR jump-diffusion stochastic hybrid model. Figure 9 shows the loss caused by ignoring the stochastic features of both the jump and the interest rate. The positive value of the loss function in Figure 9 indicates that the investment decision under the 4/2-CIR jump-diffusion stochastic hybrid model is superior to that under the 4/2 stochastic volatility model. In addition, Figures 8 and 9 show that the loss function increases with the investment horizon, indicating that the effects of the asset's jump feature and the interest rate's stochasticity on investment decisions become more significant over time.

We conclude this section by noting that the current numerical analysis is mainly designed to illustrate the behavior of the value function and the sensitivity of the optimal strategy to key parameters. A natural extension for future research would be to implement a Monte Carlo verification of the proposed strategy under the original uncontrolled dynamics, which may provide further evidence of its numerical robustness and practical applicability.

6. Conclusions

Financial asset data exhibit characteristics such as spikes, thick tails, asymmetry, jumps, and volatility clustering. Additionally, market interest rates, influenced by evolving national economic policies and dynamic financial market conditions, behave stochastically rather than remaining constant. Building on existing research, this work addresses the optimal investment decision problem under the 4/2-CIR jump-diffusion stochastic hybrid model by applying the criterion of expected utility maximization. First, a 4/2-CIR jump-diffusion stochastic hybrid portfolio model is constructed, assuming a financial market composed of money market accounts, zero-coupon bonds, stock indices, and stock derivatives. A dynamic equation for the expected utility of terminal wealth is derived. Second, under the framework of the 4/2-CIR jump-diffusion model and assuming a power utility function, the HJB equation governing the value function of the optimal investment problem is formulated. Through dynamic programming and variable substitution, analytical expressions for the optimal risk exposure, optimal value function, and optimal investment strategy are obtained. Third, suboptimal strategies and short-sighted utility losses associated with the 4/2-CIR jump-diffusion stochastic hybrid model are analyzed. Finally, numerical simulations are presented to assess the impact of key model parameters on the optimal risk exposure and short-sighted losses.

The findings of this work can be summarized as follows. (1) A higher risk aversion coefficient correlates with a lower optimal risk exposure, indicating that investors with greater risk aversion tend to adopt more conservative strategies to mitigate risk. (2) The investment horizon exerts a noticeable effect on the optimal risk exposure associated with volatility and interest rate risk factors, but has limited influence on equity and jump risk factors. (3) The optimal risk exposure linked to equity, volatility, and interest rate risk factors increases with the corresponding risk premium parameters. For negative risk premium parameters, the optimal risk exposure is also negative, reflecting investors' shift toward prudent strategies to minimize risk. (4) The optimal risk exposure for jump risk declines as jump intensity increases. Elevated jump intensity amplifies jump-related uncertainty, prompting investors to adopt more conservative strategies, reflected in negative optimal risk exposure values. (5) Myopic loss exhibits divergent behavior with respect to the sign of the risk premium factor: When positive, it increases with the premium factor; when negative, it decreases as the premium factor rises. (6) Comparisons with the investment strategies derived under the 4/2-CIR stochastic hybrid model and the 4/2 stochastic volatility model reveal that the strategy developed using the 4/2-CIR jump-diffusion stochastic hybrid model is superior. This highlights the importance of accurately modeling the stochastic dynamics of asset price trends and market interest rates to enable investors to make optimal decisions and mitigate financial risks effectively.

Finally, although the current paper emphasizes analytical derivation and comparative numerical analysis, an interesting direction for future work would be to perform a Monte Carlo verification of the proposed strategy under the original uncontrolled dynamics. Such a simulation-based exercise could further evaluate the finite-sample performance, robustness, and implementation properties of the strategy in the underlying stochastic environment.

Author contributions

Aiqin Ma: Writing–original draft, Methodology, Validation, Formal analysis, Software, Visualization, Conceptualization, Writing–review and editing; Qingxin Zhang: Conceptualization, Validation, Writing–review and editing; Yubing Wang: Methodology, Formal analysis, Visualization, Software. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence(AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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Appendix A. Proofs

A.1. Proof of optimal investment strategy

Proof of Theorem 3.1. Assume that the form of $H(x_t, v_t, r_t, t)$ is given by

$$H(x_t, v_t, r_t, t) = e^{-\beta t}\frac{x_t^\delta}{\delta}h(v_t, r_t, t), \tag{A.1}$$

where $h(v_T, r_T, T) = 1$.

For notational simplicity, let $H(x_t, v_t, r_t, t) = H$ and $h(v_t, r_t, t) = h$. Therefore, it follows that

$$\begin{gathered}
H_t = -\beta e^{-\beta t}\tfrac{x_t^\delta}{\delta}h + e^{-\beta t}\tfrac{x_t^\delta}{\delta}h_t; \quad H_x = e^{-\beta t}x_t^{\delta-1}h; \quad H_{xx} = (\delta-1)e^{-\beta t}x_t^{\delta-2}h,\\
H_{xv} = e^{-\beta t}x_t^{\delta-1}h_v; \quad H_{vv} = e^{-\beta t}\tfrac{x_t^\delta}{\delta}h_{vv}; \quad H_{rr} = e^{-\beta t}\tfrac{x_t^\delta}{\delta}h_{rr}; \quad H_{xr} = e^{-\beta t}x_t^{\delta-1}h_r,\\
H_v = e^{-\beta t}\tfrac{x_t^\delta}{\delta}h_v; \quad H_r = e^{-\beta t}\tfrac{x_t^\delta}{\delta}h_r; \quad \Delta H = e^{-\beta t}\tfrac{x_t^\delta}{\delta}((1+\pi_4)^\delta - 1)h.
\end{gathered}$$

Substituting these derivatives into Eq (3.1) yields

$$
\begin{aligned}
&-\beta e^{-\beta t}\frac{x_t^\delta}{\delta}h + e^{-\beta t}\frac{x_t^\delta}{\delta}h_t + [r_t + \pi_1\lambda_s v_t + \pi_2\lambda_v v_t + \pi_3\lambda_r r_t - \pi_4\lambda]\, e^{-\beta t}x_t^\delta h \\
&+ \pi_2\sigma_v v_t x_t^\delta e^{-\beta t}h_v + \alpha_r(\theta_r - r_t)e^{-\beta t}\frac{x_t^\delta}{\delta}h_r + \sigma_r^2 r_t e^{-\beta t}\frac{x_t^\delta}{2\delta}h_{rr} + \pi_3\sigma_r^2 r_t x_t^\delta e^{-\beta t}h_r \\
&+ \widetilde{\lambda}e^{-\beta t}\frac{x_t^\delta}{\delta}((1+\pi_4)^\delta - 1)h + \frac{1}{2}(\pi_1^2 v_t + \pi_2^2 v_t + \pi_3^2\sigma_r^2 r_t)(\delta - 1)x_t^\delta e^{-\beta t}h \\
&+ \alpha_v(\theta_v - v_t)e^{-\beta t}\frac{x_t^\delta}{\delta}h_v + \frac{1}{2}\sigma_v^2 v_t e^{-\beta t}\frac{x_t^\delta}{\delta}h_{vv} = 0.
\end{aligned}
$$

This can be simplified to

$$
\begin{aligned}
&-\beta h + [r_t + \pi_1\lambda_s v_t + \pi_2\lambda_v v_t + \pi_3\lambda_r r_t - \pi_4\lambda]\,\delta h + \alpha_v(\theta_v - v_t)h_v + \frac{1}{2}\sigma_v^2 v_t h_{vv} \\
&+ h_t + \frac{1}{2}(\pi_1^2 v_t + \pi_2^2 v_t + \pi_3^2\sigma_r^2 r_t)\delta(\delta - 1)h + \alpha_r(\theta_r - r_t)h_r + \frac{1}{2}\sigma_r^2 r_t h_{rr} \\
&+ \pi_2\sigma_v v_t\delta h_v + \pi_3\sigma_r^2 r_t\delta h_r + \widetilde{\lambda}((1+\pi_4)^\delta - 1)h = 0.
\end{aligned} \tag{A.2}
$$

Taking the first-order conditions with respect to π_1, π_2, π_3, and π_4 gives

$$
\begin{cases}
\lambda_s v_t\delta h + \delta(\delta - 1)v_t\pi_1 h = 0 \\
\lambda_v v_t\delta h + \delta(\delta - 1)v_t\pi_2 h + \sigma_v v_t\delta h_v = 0 \\
\lambda_r r_t\delta h + \delta(\delta - 1)\sigma_r^2\pi_3 r_t h + \sigma_r^2 r_t\delta h_r = 0\text{'} \\
-\lambda\delta h + \widetilde{\lambda}h\delta(1+\pi_4)^{\delta-1} = 0
\end{cases}
$$

which yields the optimal risk exposures

$$
\begin{cases}
\pi_1^* = \dfrac{\lambda_s}{1-\delta} \\
\pi_2^* = \dfrac{\lambda_v h + \sigma_v h_v}{(1-\delta)h} \\
\pi_3^* = \dfrac{\lambda_r h + \sigma_r^2 h_r}{(1-\delta)\sigma_r^2 h} \\
\pi_4^* = \left(\dfrac{\lambda}{\widetilde{\lambda}}\right)^{\frac{1}{\delta-1}} - 1
\end{cases}. \tag{A.3}
$$

Substituting (A.3) back into (A.2) and simplifying leads to

$$
\begin{aligned}
&h_t - \beta h + \left\{r_t + \frac{(\lambda_s^2 + \lambda_v^2)v_t}{1-\delta} + \frac{\sigma_v\lambda_v v_t h_v}{(1-\delta)h} + \frac{\lambda_r^2 r_t}{(1-\delta)\sigma_r^2} + \frac{\lambda_r r_t h_r}{(1-\delta)h}\right\}\delta h \\
&-\frac{1}{2}\left\{\frac{(\lambda_s^2 + \lambda_v^2)v_t}{(1-\delta)} + \frac{2\lambda_s\sigma_v v_t h_v}{(1-\delta)h} + \frac{\sigma_v^2 v_t h_v^2}{(1-\delta)h^2} + \frac{\lambda_r^2\sigma_r^2 r_t}{(1-\delta)\sigma_r^4} + \frac{2\lambda_r\sigma_r^2 r_t h_r}{(1-\delta)\sigma_r^2 h}\right. \\
&\left.-\frac{\sigma_r^2 r_t h_r^2}{(1-\delta)h^2}\right\}\delta h + \alpha_v(\theta_v - v_t)h_v + \alpha_r(\theta_r - r_t)h_r + \frac{\lambda_v h + \sigma_v h_v}{(1-\delta)h}\sigma_v v_t\delta h_v
\end{aligned}
$$

$$
+ \frac{1}{2}\sigma_r^2 r_t h_{rr} + \frac{1}{2}\sigma_v^2 v_t h_{vv} + \frac{\lambda_r h + \sigma_r^2 h_r}{(1-\delta)\sigma_r^2 h}\sigma_r^2 r_t \delta h_r - \Big[\Big(\frac{\lambda}{\widetilde{\lambda}}\Big)^{\frac{1}{\delta-1}} - 1\Big]\lambda\delta h
+ \widetilde{\lambda} h\Big[\Big(\frac{\lambda}{\widetilde{\lambda}}\Big)^{\frac{\delta}{\delta-1}} - 1\Big] = 0,
$$

thus yielding

$$
\begin{aligned}
& h_t + \frac{1}{2}\sigma_v^2 v_t h_{vv} + \Big[\delta r_t - \beta + \frac{\delta(\lambda_s^2 + \lambda_v^2)v_t}{2(1-\delta)} + \frac{\delta\lambda_r^2 r_t}{2(1-\delta)\sigma_r^2} + \widetilde{\lambda}\Big[\Big(\frac{\lambda}{\widetilde{\lambda}}\Big)^{\frac{\delta}{\delta-1}} - 1\Big] \\
& - \delta\lambda\Big[\Big(\frac{\lambda}{\widetilde{\lambda}}\Big)^{\frac{1}{\delta-1}} - 1\Big]\Big]h + \Big[\frac{\delta\sigma_v\lambda_v v_t}{1-\delta} + \alpha_v(\theta_v - v_t)\Big]h_v + \frac{\delta\sigma_r^2 r_t}{2(1-\delta)}\frac{h_r^2}{h} \\
& + \Big[\frac{\delta\lambda_r r_t}{1-\delta} + \alpha_r(\theta_r - r_t)\Big]h_r + \frac{\delta\sigma_v^2 v_t}{2(1-\delta)}\frac{h_v^2}{h} + \frac{1}{2}\sigma_r^2 r_t h_{rr} = 0.
\end{aligned} \tag{A.4}
$$

Furthermore, assume that $h(v_t, r_t, t)$ is of the exponentially affine form

$$
\begin{cases} h(v_t, r_t, t) = \exp(A(t)v_t + B(t)r_t + D(t)) \\ h(v_T, r_T, T) = 1 \end{cases}, \tag{A.5}
$$

where $A(T) = 0, B(T) = 0, D(T) = 0$. This leads to

$$
\begin{aligned}
h_t &= [A'(t)v + B'(t)r + D'(t)]\, h, \\
h_v &= A(t)h, \\
h_{vv} &= A^2(t)h, \\
h_r &= B(t)h, \\
h_{rr} &= B^2(t)h.
\end{aligned}
$$

Substituting these derivatives into Eq (A.4) yields

$$
\begin{aligned}
& A'(t)v_t + B'(t)r_t + D'(t) + r_t\delta - \beta + \frac{1}{2}\frac{\delta(\lambda_s^2 + \lambda_v^2)}{1-\delta}v_t + \frac{1}{2}\frac{\delta\lambda_r^2}{(1-\delta)\sigma_r^2}r_t \\
& + (1-\delta)\frac{\lambda^{\frac{\delta}{\delta-1}}}{\widetilde{\lambda}^{\frac{1}{\delta-1}}} - \widetilde{\lambda} + \lambda\delta + \Big[\frac{\delta}{1-\delta}\sigma_v\lambda_v v_t + \alpha_v(\theta_v - v_t)\Big]A(t) + \frac{1}{2}\sigma_v^2 v_t A^2(t) \\
& + \Big[\frac{\delta}{1-\delta}\lambda_r r_t + \alpha_r(\theta_r - r_t)\Big]B(t) + \frac{1}{2}\sigma_r^2 r_t B^2(t) + \frac{\delta}{2(1-\delta)}\sigma_v^2 v_t A^2(t) \\
& + \frac{1}{2}\frac{\delta}{1-\delta}\sigma_r^2 r_t B^2(t) = 0.
\end{aligned}
$$

This leads to the following three decoupled ordinary differential equations:

$$
\begin{aligned}
& \Big[A'(t) + \frac{1}{2}\frac{\delta}{1-\delta}(\lambda_s^2 + \lambda_v^2) + \Big(\frac{\delta}{1-\delta}\sigma_v\lambda_v - \alpha_v\Big)A(t) + \frac{1}{2}\sigma_v^2 A^2(\tau) + \frac{1}{2}\frac{\delta}{1-\delta} \\
& \times \sigma_v^2 A^2(t)\Big]v_t + \Big[B'(t) + \delta + \frac{1}{2}\frac{\delta}{1-\delta}\frac{\lambda_r^2}{\sigma_r^2} + \Big(\frac{\delta}{1-\delta}\lambda_r - \alpha_r\Big)B(t) + \frac{1}{2}\sigma_r^2 B^2(t) \\
& + \frac{1}{2}\frac{\delta}{1-\delta}\sigma_r^2 B^2(t)\Big]r_t + \Big[D'(t) - \beta + \lambda\delta - \widetilde{\lambda} + (1-\delta)\frac{\lambda^{\frac{\delta}{\delta-1}}}{\widetilde{\lambda}^{\frac{1}{\delta-1}}} + \alpha_v\theta_v A(t) + \alpha_r\theta_r B(t)\Big] = 0.
\end{aligned} \tag{A.6}
$$

The solution to Eq (A.6) reduces to the following three systems of equations

$$\begin{cases} A'(t) + \frac{1}{2}\frac{\delta}{1-\delta}(\lambda_s^2 + \lambda_v^2) + \left(\frac{\delta}{1-\delta}\sigma_v\lambda_v - \alpha_v\right)A(t) + \frac{1}{2}\sigma_v^2 A^2(t) \\ + \frac{1}{2}\frac{\delta}{1-\delta}\sigma_v^2 A^2(t) = 0 \\ A(T) = 0 \end{cases}, \tag{A.7}$$

$$\begin{cases} B'(t) + \delta + \frac{1}{2}\frac{\delta}{1-\delta}\sigma_r^2 B^2(t) + \left(\frac{\delta}{1-\delta}\lambda_r - \alpha_r\right)B(t) + \frac{1}{2}\sigma_r^2 B^2(t) \\ + \frac{1}{2}\frac{\delta}{1-\delta}\frac{\lambda_r^2}{\sigma_r^2} = 0 \\ B(T) = 0 \end{cases}, \tag{A.8}$$

$$\begin{cases} D'(t) - \beta + \lambda\delta - \widetilde{\lambda} + (1-\delta)\frac{\lambda^{\frac{\delta}{\delta-1}}}{\widetilde{\lambda}^{\frac{1}{\delta-1}}} + \alpha_v\theta_v A(t) + \alpha_r\theta_r B(t) = 0 \\ D(T) = 0 \end{cases}. \tag{A.9}$$

Next, we solve the system of equations formed by (A.7)–(A.9).

To solve (A.7), define

$$\begin{aligned} a_v &= \frac{1}{1-\delta}\sigma_v^2, \\ b_v &= \frac{\delta}{1-\delta}\sigma_v\lambda_v - \alpha_v, \\ c_v &= \frac{\delta}{1-\delta}(\lambda_s^2 + \lambda_v^2), \\ d_v &= \sqrt{b_v^2 - a_v c_v}. \end{aligned} \tag{A.10}$$

The equation becomes

$$\begin{cases} A'(t) = -\left[\frac{1}{2}a_v A^2(t) + b_v A(t) + \frac{1}{2}c_v\right]. \\ A(T) = 0 \end{cases} \tag{A.11}$$

Under the condition $b_v^2 - a_v c_v > 0$, the quadratic equation $\frac{1}{2}a_v A^2(t) + b_v A(t) + \frac{1}{2}c_v = 0$ has two distinct root $A_1 = \frac{-b_v + d_v}{a_v}$ and $A_2 = \frac{-b_v - d_v}{a_v}$. The solution is

$$A(t) = \frac{A_1(A_2 - A_1)}{A_1 - A_2 e^{-d_v(T-t)}} + A_1 = \frac{A_1 A_2(1 - e^{-d_v(T-t)})}{A_1 - A_2 e^{-d_v(T-t)}}. \tag{A.12}$$

Similarly, for Eq (A.8), we define

$$\begin{aligned} a_r &= \frac{1}{1-\delta}\sigma_r^2, \\ b_r &= \frac{\delta}{1-\delta}\lambda_r - \alpha_r, \\ c_r &= 2\delta + \frac{\delta}{1-\delta}\frac{\lambda_r^2}{\sigma_r^2}, \\ d_r &= \sqrt{b_r^2 - a_r c_r}. \end{aligned} \tag{A.13}$$

The solution is

$$B(t) = \frac{B_1 B_2 (1 - e^{-d_r(T-t)})}{B_1 - B_2 e^{-d_r(T-t)}}. \tag{A.14}$$

For Eq (A.9), let

$$c = -\beta + \lambda\delta - \widetilde{\lambda} + (1-\delta)\frac{\lambda^{\frac{\delta}{\delta-1}}}{\widetilde{\lambda}^{\frac{1}{\delta-1}}}, \tag{A.15}$$

and the solution is

$$D(t) = \int_t^T \alpha_v \theta_v A(s) ds + \int_t^T \alpha_r \theta_r B(s) ds + c(T-t). \tag{A.16}$$

After evaluating the integrals, we obtain

$$\begin{aligned} D(t) =& \frac{\alpha_v \theta_v (A_1 - A_2)}{d_v} \ln \frac{A_1 - A_2}{A_1 - A_2 e^{-d_v(T-t)}} + \alpha_v \theta_v A_2 (T-t) + c(T-t) \\ &+ \frac{\alpha_r \theta_r (B_1 - B_2)}{d_r} \ln \frac{B_1 - B_2}{B_1 - B_2 e^{-d_r(T-t)}} + \alpha_r \theta_r B_2 (T-t). \end{aligned} \tag{A.17}$$

Therefore, by Eqs (A.1) and (A.5), the HJB Eq (3.1) admits a solution of the following form:

$$H(x_t, v_t, r_t, t) = e^{-\beta t} \frac{x^\delta}{\delta} \exp\left\{A(t) v_t + B(t) r_t + D(t)\right\}.$$

Substituting the expression for $A(t)$, $B(t)$, and $D(t)$ back into the equation, the optimal exposure yields

$$\begin{cases} \pi_1^* = \dfrac{\lambda_s}{1-\delta} \\ \pi_2^* = \dfrac{\lambda_v}{1-\delta} + \dfrac{\sigma_v}{1-\delta} A(t) \\ \pi_3^* = \dfrac{\lambda_r}{(1-\delta)\sigma_r^2} + \dfrac{1}{1-\delta} B(t) \\ \pi_4^* = \left(\dfrac{\lambda}{\widetilde{\lambda}}\right)^{\frac{1}{\delta-1}} - 1 \end{cases}.$$

Finally, the optimal investment strategy is obtained through the transformation

$$\begin{pmatrix} \pi_s(t) & \pi_b(t) & \pi_{o1}(t) & \pi_{o2}(t) \end{pmatrix}^\mathsf{T} = G^{-1} \begin{pmatrix} \pi_1^* & \pi_2^* & \pi_3^* & \pi_4^* \end{pmatrix}^\mathsf{T},$$

where G^{-1} is given by

$$G^{-1} = \begin{pmatrix} E^{1*} - \frac{C^{1*}}{C^{2*}}(1 + E^{1*}B^{1*}) & E^{2*}\left(\frac{C^{1*}B^{1*}}{C^{2*}} - 1\right) & 0 & F^{1*} + \frac{C^{1*}}{C^{2*}}F^{2*} \\ -E^{1*}A^{1*} - \frac{D^{*2}}{C^{2*}}(1 + E^{1*}B^{1*}) & E^{2*}\left(\frac{D^{*2}B^{*1}}{C^{2*}} + A^{1*}\right) & -\frac{1}{Z} & \frac{\eta}{\kappa Z} + \frac{D^{*2}}{C^{2*}}F^{2*} \\ -E^{1*} - \frac{D^{*1}}{C^{2*}}(1 + E^{1*}B^{1*}) & E^{2*}\left(\frac{D^{*1}B^{1*}}{C^{2*}} + 1\right) & 0 & \frac{D^{*2}}{\kappa C^{2*}}F^{2*} \\ \frac{1}{C^{2*}}(1 + E^{1*}B^{1*}) & -E^{2*}\frac{B^{1*}}{C^{2*}} & 0 & -\frac{1}{\kappa C^{2*}}F^{2*} \end{pmatrix}, \tag{A.18}$$

with the coefficients defined as

$$
\begin{aligned}
&A^{1*}=\frac{1}{Z}\left(\frac{g_r^1+g_s^1S_t\eta}{O_t^1}-\frac{\Delta g^1\eta}{\kappa O_t^1}\right),\quad A^{2*}=\frac{1}{Z}\left(\frac{g_r^2+g_s^2S_t\eta}{O_t^2}-\frac{\Delta g^2\eta}{\kappa O_t^2}\right),\\
&B^{1*}=\left(\frac{g_s^1S_t}{O_t^1}-\frac{\Delta g^1}{\kappa O_t^1}\right)F^{1*},\quad B^{2*}=\left(\frac{g_s^2S_t}{O_t^2}-\frac{\Delta g^2}{\kappa O_t^2}\right)F^{1*},\\
&C^{1*}=\frac{O_t^1}{O_t^2}\left(\frac{\Delta g^2}{\Delta g^1}-\frac{g_v^2}{g_v^1}\right),\quad C^{2*}=B^{2*}-\frac{O_t^1}{O_t^2}\frac{g_v^2}{g_v^1}B^{1*},\\
&D^{1*}=\frac{O_t^1}{O_t^2}\frac{g_v^2}{g_v^1},\quad D^{2*}=\frac{O_t^1}{O_t^2}\frac{g_v^2}{g_v^1}A^{1*}-A^{2*},\\
&E^{1*}=\frac{\rho E^{2*}}{\sqrt{1-\rho^2}},\quad E^{2*}=\frac{O_t^1}{g_v^1\sigma_v},\\
&F^{1*}=\frac{O_t^1}{\Delta g^1},\quad F^{2*}=\sqrt{1-\rho^2}\Big(a+\frac{b}{v_t}\Big).
\end{aligned}
\tag{A.19}
$$

□

A.2. Proof of Proposition 3.1

Proof. To establish the well-definedness of the solution function $H(x_t, v_t, r_t, t)$, we must verify that it remains finite and real-valued for all admissible arguments. This requires proving the following.

(1) The function $H(x_t, v_t, r_t, t)$ is real-valued.

For $H(x_t, v_t, r_t, t)$ to be real-valued, the coefficients $A(t)$ and $B(t)$ in its exponential representation must possess real roots. This necessitates the following discriminant conditions:

$$
\begin{cases} b_v^2 - a_v c_v > 0 \\ b_r^2 - a_r c_r > 0 \end{cases}.
$$

Substituting the parameter definitions from Eqs (A.10) and (A.13) yields the explicit conditions

$$
\begin{cases} \alpha_v^2 + \dfrac{\delta}{1-\delta}\Big[\dfrac{\delta}{1-\delta}\sigma_v^2\lambda_v^2 - 2\sigma_v\lambda_v\alpha_v - \dfrac{1}{1-\delta}\sigma_v^2(\lambda_v^2+\lambda_s^2)\Big] > 0 \\ \alpha_r^2 - \dfrac{\delta}{1-\delta}\Big[\lambda_r^2 + 2\lambda_r\sigma_r + 2\sigma_r^2\Big] > 0 \end{cases}.
$$

These inequalities guarantee the existence of real roots for $A(t)$ and $B(t)$, thereby ensuring the real-valued nature of the solution's function.

(2) Finiteness of $H(x_t, v_t, r_t, t)$.

Given the functional form $H(x_t, v_t, r_t, t) = e^{-\beta t}\frac{x_t^\delta}{\delta}h(v_t, r_t, t)$, establishing the finiteness of $H(x_t, v_t, r_t, t)$ reduces to proving the boundedness of $h(v_t, r_t, t)$. This requires verifying that the coefficients $A(t)$, $B(t)$ and $D(t)$ in the representation $h(v_t, r_t, t) = exp(A(t)v_t + B(t)r_t + D(t)))$ remain finite for $t \in [0, T]$.

Case1: For $\delta < 0$, we have $c_v < 0$ and $c_r < 0$. Under these parameter constraints, the coefficients satisfy $A(t) < 0$, $B(t) < 0$, and $D(t) < 0$, ensuring the boundedness of $h(v_t, r_t, t)$.

Case2: When $0 < \delta < 1$, it follows that $c_v > 0$ and $c_r > 0$, implying $A(t) > 0$ and $B(t) > 0$. Finiteness requires the nondegeneracy conditions $A_1 - A_2e^{-d_v(T-t)} \neq 0$ and $B_1 - B_2e^{-d_v(T-t)} \neq 0$.

From the parameter definitions in Equation (A.10), we have

$$a_v c_v = \frac{\delta}{(1-\delta)^2}\sigma_v^2(\lambda_v^2+\lambda_s^2) > 0 \quad (since \quad \delta > 0).$$

This implies

$$\begin{aligned} d_v &= \sqrt{b_v^2 - a_v c_v} < b_v, \\ &\Rightarrow \frac{b_v + d_v}{b_v - d_v} > 1, \\ &\Rightarrow -(b_v - d_v)(1 - \frac{b_v + d_v}{b_v - d_v} e^{-d_v(T-t)}) \neq 0, \\ &\Rightarrow -b_v + d_v + (b_v + d_v)e^{-d_v(T-t)} \neq 0, \\ &\Rightarrow \frac{-b_n + d_v}{a_v} - \frac{-b_v - d_v}{a_v} e^{-d_v(T-t)} \neq 0, \\ &\Rightarrow A_1 - A_2 e^{-d_v(T-t)} \neq 0. \end{aligned}$$

We proceed in a similar fashion for $B(t)$. Under the established conditions, the closed-form expression for $D(t)$ in Eq (A.16) ensures its boundedness over $[0, T]$.

Therefore, for any $\delta < 1$ with $\delta \neq 0$, $h(v_t, r_t, t)$ is bounded. Consequently, $H(x_t, v_t, r_t, t)$ is finite and real-valued. □

A.3. Proof of the verification theorem

Proof of Theorem 3.2. The proof procedure for Conditions (1) and (2) is detailed in Proposition 3.1 and Theorem 3.1. We now prove Conditions (3) and (4).

Assuming $H(x_t, v_t, r_t, t) = e^{-\beta t}\frac{x_t^\delta}{\delta}h(x_t, v_t, r_t, t)$ satisfies the following dynamic differential equation:

$$\begin{aligned} dH_t =& \frac{\partial H}{\partial t}dt + \frac{\partial H}{\partial x_t}dx_t + \frac{\partial H}{\partial v_t}dv_t + \frac{\partial H}{\partial r_t}dr_t + \frac{1}{2}\frac{\partial^2 H}{\partial x_t^2}(dx_t)^2 + \frac{1}{2}\frac{\partial^2 H}{\partial v_t^2}(dv_t)^2 \\ &+ \frac{1}{2}\frac{\partial^2 H}{\partial r_t^2}(dr_t)^2 + \frac{\partial^2 H}{\partial x_t \partial v_t}(dx_t)(dv_t) + \frac{\partial^2 H}{\partial x_t \partial r_t}(dx_t)(dr_t) + \widetilde{\lambda}\Delta H \\ =& e^{-\beta t}\frac{x_t^\delta}{\delta}\left[\delta h(r_t + \pi_1^*\lambda_s v_t + \pi_2^*\lambda_v v_t + \pi_3^*\lambda_r r_t - \pi_4^*\lambda) + \alpha_v(\theta_v - v_t)h_v\right]dt \\ &+ e^{-\beta t}\frac{x_t^\delta}{\delta}\Big[\frac{1}{2}\delta(1-\delta)h((\pi_1^*)^2 v_t + (\pi_2^*)^2 v_t + (\pi_3^*)^2\sigma_r^2 r_t) + \frac{1}{2}\sigma_v^2 v_t h_{vv}\Big]dt \\ &+ e^{-\beta t}\frac{x_t^\delta}{\delta}\left[\alpha_r(\theta_r - r_t)h_r + \delta\pi_2^*\sigma_v v_t h_v + \delta\pi_3^*\sigma_r^2 r_t h_r + \widetilde{\lambda}h(1+\pi_4)^\delta\right]dt \\ &+ e^{-\beta t}\frac{x_t^\delta}{\delta}h\left[\left(\left(\delta\pi_2^* + \frac{\sigma_v h_v}{h}\right)dW_{v,t}\right)\sqrt{v_t} + \left(\delta\pi_3^* + \frac{h_r}{h}\right)\sigma_r\sqrt{r_t}dW_{r,t}\right] \\ &+ e^{-\beta t}\frac{x_t^\delta}{\delta}\Big[h_t dt - \beta h dt + \frac{1}{2}\sigma_r^2 r_t h_{rr}dt - \widetilde{\lambda}h dt + h\delta\pi_1^* dW_{s,t} + h\pi_4^* dN_t\Big]. \end{aligned}$$

From the expressions of $H(x_t, v_t, r_t, t)$ and $h(v_t, r_t, t)$ and Eq (A.2), we derive

$$\frac{dH_t}{H_t} = \delta\pi_1^*\sqrt{v_t}dW_{s,t} + \left(\delta\pi_2^* + \frac{h_v}{h}\sigma_v\right)\sqrt{v_t}dW_{v,t} + \left(\delta\pi_3^* + \frac{h_r}{h}\right)\sigma_r\sqrt{r_t}dW_{r,t} + \pi_4^* dN_t$$

$$=\Big(\frac{\delta\lambda_v}{1-\delta}+\frac{\sigma_v A(t)}{1-\delta}\Big)\sqrt{v_t}dW_{v,t}+\Big(\frac{\delta\lambda_r}{(1-\delta)\sigma_r}+\frac{\sigma_r B(t)}{1-\delta}\Big)\sqrt{r_t}dW_{r,t}$$
$$+\frac{\delta\lambda_s}{1-\delta}\sqrt{v_t}dW_{s,t}+\Big[\Big(\frac{\widetilde{\lambda}}{\lambda}\Big)^{\frac{1}{1-\delta}}-1\Big]dN_t.$$

Define

$$\begin{cases} g_1(t)=\dfrac{\delta\lambda_s}{1-\delta} \\ g_2(t)=\dfrac{\delta\lambda_v}{1-\delta}+\dfrac{\sigma_v A(t)}{1-\delta} \\ g_3(t)=\dfrac{\delta\lambda_r}{(1-\delta)\sigma_r}+\dfrac{\sigma_r B(t)}{1-\delta}, \\ g_4(t)=\left(\dfrac{\widetilde{\lambda}}{\lambda}\right)^{\frac{1}{1-\delta}}-1 \end{cases}$$

which yields

$$\frac{dH_t}{H_t}=g_1(t)\sqrt{v_t}dW_{s,t}+g_2(t)\sqrt{v_t}dW_{v,t}+g_3(t)\sqrt{r_t}dW_{r,t}+g_4(t)dN_t.$$

Let $F(H_t)=\ln(H_t)$. Applying the Lévy-Itô formula

$$\begin{aligned} dF(H_t)=&\frac{\partial F(H_t)}{\partial t}dt+\frac{\partial F(H_t)}{\partial H_t}H_t g_1(t)\sqrt{v_t}dW_{s,t}+\frac{\partial F(H_t)}{\partial H_t}H_t g_2(t)\sqrt{v_t}dW_{v,t} \\ &+\frac{1}{2}\frac{\partial^2 F(H_t)}{\partial H_t^2}H_t^2 g_3^2(t)r_t dt+\int_{R^0}(\ln(H_t+g_4(t)H_t)-\ln(H_t))dN_t \\ &+\int_{R^0}\Big\{(\ln(H_t+g_4(t)H_t)-\ln(H_t))-\frac{\partial F(H_t)}{\partial H_t}H_t g_4(t)\Big\}\nu(d\xi)dt \\ &+\frac{\partial F(H_t)}{\partial H_t}H_t g_3(t)\sqrt{r_t}dW_{r,t}+\frac{1}{2}\frac{\partial^2 F(H_t)}{\partial H_t^2}H_t^2 g_1^2(t)v_t dt+\frac{1}{2}\frac{\partial^2 F(H_t)}{\partial H_t^2}H_t^2 g_2^2(t)v_t dt \\ =&g_1(t)\sqrt{v_t}dW_{s,t}+g_2(t)\sqrt{v_t}dW_{v,t}+g_3(t)\sqrt{r_t}dW_{r,t}-\frac{1}{2}g_1^2(t)v_t dt \\ &-\frac{1}{2}g_3^2(t)r_t dt+\int_{R^0}(\ln(1+g_4(t))-g_4(t))\nu(d\xi)dt-\frac{1}{2}g_2^2(t)v_t dt+\int_{R^0}\ln(1+g_4(t))dN_t. \end{aligned} \tag{A.20}$$

Integrating over $[0,t]$

$$\begin{aligned} F(H_t)-F(H_0)=&\int_0^t g_1(s)\sqrt{v_s}dW_{s,s}+\int_0^t g_2(s)\sqrt{v_s}dW_{v,s}+\int_0^t g_3(s)\sqrt{r_s}dW_{r,s} \\ &+\int_0^t\int_{R^0}(\ln(1+g_4(s))-g_4(s))\nu(d\xi)ds-\int_0^t\frac{r_s}{2}g_3^2(s)ds \\ &-\int_0^t\frac{v_s}{2}(g_1^2(s)+g_2^2(s))ds+\int_0^t\int_{R^0}\ln(1+g_4(s))dN_s. \end{aligned}$$

So

$$H_t=H_0\exp\Big\{\int_0^t g_1(s)\sqrt{v_s}dW_{s,s}+\int_0^t g_2(s)\sqrt{v_s}dW_{v,s}+\int_0^t g_3(s)\sqrt{r_s}dW_{r,s}$$

$$-\frac{1}{2}\int_0^t (g_1^2(s)+g_2^2(s))v_s ds - \frac{1}{2}\int_0^t g_3^2(s) r_s ds\Big\} \exp\{(\ln(1+g_4)-g_4)t\} \times \prod_{i=1}^{N_t} e^{I_i \ln(1+g_4)}.$$

Define

$$\begin{aligned} H_{11} &= \exp\Big\{\int_0^t g_1(s)\sqrt{v_s}dW_{s,s} + \int_0^t g_2(s)\sqrt{v_s}dW_{v,s} - \frac{1}{2}\int_0^t (g_1^2(s)+g_2^2(s))v_s ds\Big\}, \\ H_{22} &= \exp\Big\{\int_0^t g_3(s)\sqrt{r_s}dW_{r,s} - \frac{1}{2}\int_0^t g_3^2(s) r_s ds\Big\}. \end{aligned}$$

If we assume that H_{11} and H_{22} are independent martingales under the measure $\mathbb{Q}$, then

$$\begin{aligned} H_1 = &H_{11} \times H_{22} = \exp\Big\{\int_0^t g_1(s)\sqrt{v_s}dW_{s,s} + \int_0^t g_2(s)\sqrt{v_s}dW_{v,s} - \frac{1}{2}\int_0^t (g_1^2(s)+g_2^2(s))v_s ds\Big\} \\ &\times \exp\Big\{\int_0^t g_3(s)\sqrt{r_s}dW_{r,s} - \frac{1}{2}\int_0^t g_3^2(s) r_s ds\Big\}, \end{aligned}$$

is a martingale, and thus H_t is a martingale.

Next, we verify that H_{11} and H_{22} are martingales.

If

$$\mathbb{E}\left[\exp\left(-\frac{1}{2}\int_0^t \left(g_1^2(s)+g_2^2(s)\right) v_s ds\right)\right] < \infty,$$

then H_{11} is a martingale. Similarly, if

$$\mathbb{E}\left[\exp\left(-\frac{1}{2}\int_0^t g_3^2(s) r_s ds\right)\right] < \infty,$$

then H_{22} is a martingale.

Since v_t and r_t follow CIR processes,

$$\min\left(-\frac{1}{2}\left(g_1^2(t)+g_2^2(t)\right)\right) \geq -\frac{\alpha_v^2}{2\sigma_v^2}, \tag{A.21}$$

$$\min\left(-\frac{1}{2}g_3^2(t)\right) \geq -\frac{\alpha_r^2}{2\sigma_r^2}. \tag{A.22}$$

For Eq (A.21)

$$\begin{aligned} -\frac{1}{2}\left(g_1^2(t)+g_2^2(t)\right) &= -\frac{1}{2}\left(\frac{\lambda_s\delta}{1-\delta}\right)^2 - \frac{1}{2}\left(\frac{\lambda_v\delta}{1-\delta} + \frac{\sigma_v}{1-\delta}A(t)\right)^2 \\ &= -\frac{1}{2}\left(\frac{\sigma_v^2}{(1-\delta)^2}A^2(t) + \frac{2\delta\lambda_v\sigma_v}{(1-\delta)^2}A(t) + \frac{\delta^2(\lambda_v^2+\lambda_s^2)}{(1-\delta)^2}\right). \end{aligned} \tag{A.23}$$

By taking the partial derivative of Eq (A.23) with respect to t, we obtain

$$\left(-\frac{1}{2}(g_1^2(t)+g_2^2(t))\right)' = -\frac{\sigma_v}{(1-\delta)^2}A'(t)\left(\sigma_v A(t) + \delta\lambda_v\right). \tag{A.24}$$

Substituting Eqs (A.10) and (A.12) into (A.11) yields

$$A'(t) = -\frac{2c_v d_v^2 e^{-d_v(T-t)}}{(-b_v + d_v + (b_v + d_v)e^{-d_v(T-t)})^2}.$$

Given that $\delta < 0$, $A'(t) > 0$, and $A(T) = 0$, it follows that $A(t) < 0$ for $t < T$.

$$\left(-\frac{1}{2}\left(g_1^2(t) + g_2^2(t)\right)\right)' > 0.$$

Therefore, Eq (A.21) achieves its minimum at $t = 0$.

For $0 < \delta < 1$, $A^{'}(t) < 0$. Given $A(T) = 0$, this implies that $A(t) > 0$ for $t < T$. Under these conditions, we have

$$\left(-\frac{1}{2}\left(g_1^2(t) + g_2^2(t)\right)\right)' > 0,$$

which implies that Eq (A.21) achieves its minimum at $t = 0$.

Collectively, for $\delta < 1$, and $\delta \neq 0$, Eq (A.21) reaches its minimum at $t = 0$. Consequently, we have

$$\begin{aligned}\min\Big[-\frac{1}{2}(g_1^2(t) + g_2^2(t))\Big] &= -\frac{1}{2}\Big(\frac{\sigma_v^2 A^2(0)}{(1-\delta)^2} + \frac{2\delta\lambda_v\sigma_v A(0)}{(1-\delta)^2} + \frac{\delta^2(\lambda_v^2 + \lambda_s^2)}{(1-\delta)^2}\Big)\\ &\geq -\frac{\alpha_v^2}{2\sigma_v^2}.\end{aligned}$$

A similar analysis yields

$$\min\Big[-\frac{1}{2}g_3^2(t)\Big] = -\frac{1}{2}\Big(\frac{\sigma_r^2 B^2(0)}{(1-\delta)^2} + \frac{2\delta\lambda_r B(0)}{(1-\delta)^2} + \frac{\delta^2\lambda_r^2}{(1-\delta)^2\sigma_r^2}\Big) \geq -\frac{\alpha_r^2}{2\sigma_r^2}.$$

We now demonstrate that $H(x_t, v_t, r_t, t)$ is the optimal value function.

From the preceding proof, $H(x_t, v_t, r_t, t)$ is a martingale. By the martingale property, we have

$$\mathbb{E}[U(x_T^{\pi^*})\big|x_t = x, v_t = v, r_t = r] = H(x_t, v_t, r_t, t).$$

Consider an arbitrary admissible strategy π. Define the process $L_t = \{x_t^{\pi^*}\}^{\delta-1}x_t^{\pi}h(v_t, r_t, t)$. Applying Itô's lemma to the wealth process, volatility dynamics, and interest rate process yields

$$\begin{aligned}dL_t =&(x_t^{\pi^*})^{\delta-1}x_t^{\pi}\Big[h_t + (\delta-1)(r_t + \pi_1^*\lambda_s v_t + \pi_2^*\lambda_v v_t + \pi_3^*\lambda_r r_t - \lambda\pi_4^*)h + hr_t\\ &+ h(\pi_1\lambda_s v_t + \pi_2\lambda_v v_t + \pi_3\lambda_r r_t - \lambda\pi_4) + \frac{h}{2}(\delta-1)(\delta-2)[(\pi_1^*)^2 v_t\\ &+ (\pi_2^*)^2 v_t + (\pi_3^*)^2 r_t] + \frac{h_{vv}}{2}\sigma_v^2 v_t + h_v\alpha_v(\theta_v - v_t) + h_r\alpha_r(\theta_r - r_t)\\ &+ (\delta-1)h[(\pi_1^*\pi_1 + \pi_2^*\pi_2)v_t + \pi_3^*\pi_3\sigma_r^2 r_t] + \frac{h_{rr}}{2}\sigma_r^2 r_t + h_v\pi_2\sigma_v v_t\\ &+ (\delta-1)h_v\pi_2^*\sigma_v v_t + (\delta-1)h_r\pi_3^*\sigma_r^2 r_t + h_r\pi_3\sigma_r^2 r_t\Big]dt\\ &+ (x_t^{\pi^*})^{\delta-1}x_t^{\pi}h(\delta-1)\pi_1^*\sqrt{v_t}dW_{s,t} + (x_t^{\pi^*})^{\delta-1}x_t^{\pi}h(\delta-1)\pi_3^*\sigma_r\sqrt{r_t}dW_{r,t}\end{aligned}$$

$$
\begin{aligned}
&+ (x_t^{\pi^*})^{\delta-1} x_t^{\pi} h_v \sigma_v \sqrt{v_t} dW_{v,t} + (x_t^{\pi^*})^{\delta-1} x_t^{\pi} h_r \sigma_r \sqrt{r_t} dW_{r,t} \\
&+ (x_t^{\pi^*})^{\delta-1} x_t^{\pi} h \pi_1 \sqrt{v_t} dW_{s,t} + (x_t^{\pi^*})^{\delta-1} x_t^{\pi} h \pi_2 \sqrt{v_t} dW_{v,t} \\
&+ (x_t^{\pi^*})^{\delta-1} x_t^{\pi} h \pi_3 \sigma_r \sqrt{r_t} dW_{r,t} + (\widetilde{\lambda} \Delta L_1 dN_t + \widetilde{\lambda} \Delta L_2 dN_t),
\end{aligned}
$$

where the jump components are

$$
\Delta L_1 = ((1+\pi_4^*)^{\delta-1} - 1)(x_t^{\pi^*})^{\delta-1} x_t^{\pi} h, \quad \Delta L_2 = (x_t^{\pi^*})^{\delta-1} x_t^{\pi} \pi_4 h.
$$

The dynamics of L_t can be expressed as

$$
\begin{aligned}
\frac{dL_t}{L_t} =& ((\delta-1)\pi_1^* + \pi_1) \sqrt{v_t} dW_{s,t} + \left(A(t)\sigma_v + (\delta-1)\pi_2^* + \pi_2\right) \sqrt{v_t} dW_{v,t} \\
&+ (B(t) + (\delta-1)\pi_3^* + \pi_3)\sigma_r \sqrt{r_t} dW_{r,t} + \widetilde{\lambda}((1+\pi_4^*)^{\delta-1} - 1 + \pi_4) dN_t \\
=& g_1^{L_t}(t) \sqrt{v_t} dW_{s,t} + g_2^{L_t}(t) \sqrt{v_t} dW_{v,t} + g_3^{L_t}(t) \sigma_r \sqrt{r_t} dW_{r,t} + g_4^{L_t}(t) dN_t,
\end{aligned}
$$

with the coefficient functions

$$
\begin{aligned}
g_1^{L_t}(t) =& ((\delta-1)\pi_1^* + \pi_1), \\
g_2^{L_t}(t) =& (A(t)\sigma_v + (\delta-1)\pi_2^* + \pi_2), \\
g_3^{L_t}(t) =& (B(t) + (\delta-1)\pi_3^* + \pi_3), \\
g_4^{L_t}(t) =& \widetilde{\lambda}((1+\pi_4^*)^{\delta-1} - 1 + \pi_4).
\end{aligned}
$$

Since L_t is a martingale and $h(v_t, r_t, t) = e^{A(t)v_t + B(t)r_t + D(t)}$ for all t, L_t is an supermartingale. Consequently, we have

$$
\begin{aligned}
\mathbb{E}[U(x_T^{\pi}) \big| \mathcal{F}_{\tilde{t}}] &\leq \mathbb{E}[U(x_T^{\pi^*}) \big| \mathcal{F}_{\tilde{t}}] + \mathbb{E}[U_x(x_T^{\pi^*})(x_T^{\pi} - x_T^{\pi^*}) \big| \mathcal{F}_{\tilde{t}}] \\
&\leq \mathbb{E}[U(x_T^{\pi^*}) \big| \mathcal{F}_{\tilde{t}}] + L(\tilde{t}) - \delta \mathbb{E}[H(x_T^{\pi^*}, v_{\tilde{t}}, r_{\tilde{t}}, T) \big| \mathcal{F}_{\tilde{t}}] \\
&= \mathbb{E}[U(x_T^{\pi^*}) \big| \mathcal{F}_{\tilde{t}}],
\end{aligned}
$$

where $H^{\pi}(x_t, v_t, r_t, t) \leq H^{\pi^*}(x_t, v_t, r_t, t)$. Therefore, $H^{\pi^*}(x_t, v_t, r_t, t)$ is the optimal value function for the original problem, and π^* constitutes the optimal investment strategy. □

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