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Research article

Li–Yau gradient estimates of weighted nonlinear diffusion equations under geometric flow

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Abstract: In the present study, we establish novel gradient bounds reminiscent of Li–Yau inequalities for strictly positive solutions of a specific category of weighted nonlinear diffusion equations incorporating a potential term, formulated as

$$\partial_t u(z,t) = \Delta_\phi u^p(z,t) + Cu^q(z,t), \quad (z,t) \in M \times [0,T],$$

where the underlying space is a weighted Riemannian manifold $(M^n, g(t), e^{-\phi}\mathrm{d}v)$ evolving under a geometric flow governed by $\frac{\partial g}{\partial t} = 2h(t)$. As a direct consequence, we further establish associated Harnack-type inequalities for such evolving settings using partial differential equations.

Keywords: Harnack inequality; Li–Yau-type gradient estimate; geometric flow; partial differential equations; Ricci curvature
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1. Introduction

The present work aims to investigate space-time gradient bounds for positive solutions to weighted porous medium-type equations (WPMEs) of the form

$$\partial_t u(z,t) = \Delta_\phi u^p(z,t) + Cu^q(z,t), \quad (z,t) \in M \times [0,T], \tag{1.1}$$

where the underlying space is a weighted Riemannian manifold $(M^n, g, e^{-\phi}dv)$ undergoing a geometric evolution governed by the partial differential equations

$$\frac{\partial}{\partial t}g_{ij} = 2h_{ij}(t), \quad \frac{\partial}{\partial t}\phi = \Delta\phi. \tag{1.2}$$

Here, h represents a general symmetric $(0, 2)$-tensor field on the manifold M that varies with time, whereas C, p, and q are fixed scalar parameters. The time interval $[0, T]$ corresponds to the maximal existence interval for the flow. Several prominent geometric evolution equations fall under the broader category of curvature-driven flows. Among them, the Ricci flow [1] stands out, formulated via the deformation tensor $h = -\text{Ric}$, where Ric denotes the Ricci curvature tensor. Another example is the Yamabe flow [2], described by the evolution law $h = -\frac{1}{2}Rg$, with R representing the scalar curvature. The Ricci–Bourguignon flow [3] generalizes this by introducing a real parameter ρ into the equation $h = -\text{Ric} + \rho Rg$. Additionally, the extended Ricci flow [4] incorporates a dynamic scalar field ϕ and is governed by $h = -\text{Ric} + \alpha\nabla\phi \otimes \nabla\phi$, where the coupling coefficient $\alpha(t)$ is a non-increasing function of time.

Consider a complete Riemannian manifold (M^n, g), where $\text{Ric} \geq -Kg$ for some non-negative constant K. Our attention is directed toward analyzing positive solutions of the classical heat equation on such a geometric background:

$$\frac{\partial u}{\partial t} = \Delta u. \tag{1.3}$$

Li and Yau [5] obtained a fundamental gradient estimate of the form

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} \leq \frac{n\alpha^2 K}{2(\alpha - 1)} + \frac{n\alpha^2}{2t}, \tag{1.4}$$

valid for any constant $\alpha > 1$. Subsequent refinements and extensions have appeared in the literature: Davies [6] improved the constant in (1.4), Li and Yau [5] derived more refined estimates involving hyperbolic trigonometric functions, and Hamilton [7] produced related gradient inequalities with exponential factors depending on K and t. Chen and Xiong [8] further developed gradient estimates tailored to doubly nonlinear diffusion equations of the form $u_t = \Delta_p u^\gamma$.

Regarding porous medium and p-Laplace equations, numerous works (e.g., [9–14]) have studied gradient bounds for positive solutions, providing applications across geometric analysis and nonlinear partial differential equations.

More generally, nonlinear reaction-diffusion equations

$$\partial_t u(z, t) = \Delta_p u^\gamma(z, t) + cu^q(z, t), \tag{1.5}$$

on Riemannian manifolds were analyzed in [15], where c and q are constants, and the parameters satisfy $\gamma > 0$, $p > 1$. Equation (1.5) encapsulates a broad spectrum of significant diffusion models. For instance, it reduces to the classical heat equation under the parameter choices $p = 2$, $\gamma = 1$, and $c = 0$. Setting $p = 2$, $\gamma > 1$, and $c = 0$ yields the porous medium equation, whereas the regime $0 < \gamma < 1$ with $p = 2$ and $c = 0$ characterizes the fast diffusion equation. When $\gamma = 1$ and $c = 0$, the equation simplifies to the well-known parabolic p-Laplace equation. Additionally, the formulation with $c = 0$ covers the class of doubly nonlinear diffusion equations. Notably, the equation also specializes to the p-Laplacian Lichnerowicz equation when $\gamma = 1$. These nonlinear diffusions are intimately connected with geometric flows: For instance, when $c = 0$, choosing $p = 2$ and $\gamma = \frac{n-2}{n+3}$ corresponds to the Yamabe flow, $p = 1$, $\gamma = 1$ relates to the inverse mean curvature flow [12], and the limit $\gamma \to 1$ with $p = 2$ connects to logarithmic fast diffusion relevant to the Ricci flow on surfaces.

In the following, we consider u to be a strictly smooth and positive solution of the WPME (1.1). Denoting by $d(y, z, t)$, the geodesic distance between points $y, z \in M$ with respect to the evolving metric $g(t)$, for a fixed base point $z_0 \in M$ and radius $R > 0$, we define the parabolic neighborhood

$$Q_{2R,T} := \{(z, t) \in M^n \times [0, T] \mid 2R \geq d(z, z_0, t)\}.$$

Our primary objective is to establish new gradient estimates for solutions u within this evolving geometric and weighted setting, generalizing and extending previously known results.

The structure of this document is outlined as follows: Section 2 introduces the necessary preliminaries, including essential definitions, notations, and foundational lemmas that support the subsequent analysis. In Section 3, we present the main results, providing new space-time gradient estimates for positive solutions of the WPME evolving under geometric flows. Section 4 is dedicated to the detailed proofs of these results, employing advanced methods from geometric analysis and nonlinear partial differential equation theory. In conclusion, the paper recaps of the findings and explores potential avenues for future research.

2. Preliminaries

We begin by recalling fundamental concepts and notations that underpin the analysis carried out in this work.

Let (M^n, g) be a complete Riemannian manifold equipped with a smooth weight function $\phi \in C^2(M)$. The associated weighted manifold is then described by the triple $(M^n, g, e^{-\phi} dv)$, where dv denotes the Riemannian volume element induced by the metric g. The weighted (or Witten or drift) Laplace operator is determined by

$$\Delta_\phi := \Delta - \nabla\phi \cdot \nabla,$$

where Δ is the Laplacian corresponding to g, and ∇ denotes the Levi–Civita connection.

The weighted manifold is naturally associated with the Bakry–Émery Ricci tensor, which generalizes the classical Ricci curvature by incorporating the Hessian of the weight function ϕ:

$$\mathrm{Ric}_\phi := \mathrm{Ric} + \mathrm{Hess}(\phi),$$

where $\mathrm{Hess}(\phi)$ denotes the Hessian of the function ϕ taken relative to the metric g.

For a fixed integer $m > n$, one further defines the m-dimensional Bakry–Émery tensor as

$$\mathrm{Ric}_\phi^{m-n} := \mathrm{Ric} + \mathrm{Hess}(\phi) - \frac{\nabla\phi \otimes \nabla\phi}{m - n}.$$

This tensor plays a crucial role in comparison geometry and functional inequalities on weighted manifolds.

In the framework of this paper, the underlying metric and weight functions evolve in time according to the coupled system in the term of partial differential equations

$$\frac{\partial}{\partial t} g_{ij} = 2h_{ij}(t), \quad \frac{\partial}{\partial t}\phi = \Delta\phi. \tag{2.1}$$

This general formulation encompasses important flows. For instance,

- The Ricci flow: $h = -\mathrm{Ric}$;
- The Yamabe flow: $h = -\frac{1}{2}Rg$;
- The Ricci–Bourguignon flow: $h = -\mathrm{Ric} + \rho Rg$ for constant ρ;
- The extended Ricci flow: $h = -\mathrm{Ric} + \alpha(t)\nabla\phi \otimes \nabla\phi$ with a non-increasing function $\alpha(t)$.

Gradient estimates form a backbone in the analysis of nonlinear parabolic partial differential equations on manifolds. The seminal work of Li and Yau [5] proved gradient bounds for positive solutions of the heat equation on complete manifolds with Ricci curvature bounded from below. These results have been refined and extended to cover nonlinear diffusion equations, weighted manifolds, and evolving geometric backgrounds.

In particular, under appropriate curvature and flow conditions, one can derive differential Harnack inequalities and gradient bounds that depend on curvature lower bounds, flow tensors, and parameters of the nonlinear equation. These inequalities serve as crucial tools to study regularity, long-time behavior, and uniqueness of solutions. Throughout this paper, we denote by $d(y, z, t)$ the geodesic distance between points $y, z \in M$ with respect to the time-dependent metric $g(t)$. For any fixed point $z_0 \in M$ and radius $R > 0$, the parabolic neighborhood

$$Q_{2R,T} := \{(z, t) \in M \times [0, T] : 2R \geq d(z, z_0, t)\}$$

is used as the domain for local gradient estimates. We also adopt the standard convention that constants denoted by $C, c_0, c_1, \ldots$ may vary from line to line but depend only on fixed geometric quantities and parameters of the equation.

3. Main results

In the following, we establish novel gradient estimates for positive solutions of the WPME under an evolving geometric flow on weighted Riemannian manifolds. These estimates generalize classical results by incorporating time-dependent metrics and weights, providing tools for analyzing the behavior of solutions in dynamic geometric settings. As an important application, we derive a Harnack-type inequality that relates solution values at different points and times, reflecting the interplay between geometry and diffusion.

Theorem 3.1. *Consider a complete weighted Riemannian manifold* $(M, g(0), e^{-\phi_0}dv)$ *where the metric* $g(t)$ *and the weight function* $\phi(t)$ *evolve according to the system (1.2) over the interval* $t \in [0, T]$. *Suppose* u *is a positive solution to Eq (1.1) defined on the parabolic cylinder* $Q_{2R,T}$ *of radius* $R > 0$, *and suppose that* u *satisfies the upper constraint*

$$\frac{(p-1)k}{p} \geq u^{p-1}$$

for some fixed real constant k. *Assume constants* $l_1, l_2, l_3, l_4 > 0$ *exist such that the weighted Bakry–Émery curvature tensor and the evolving tensor* h *satisfy*

$$\mathrm{Ric}_{\phi}^{m-n} \geq -(m-1)l_1 g, \quad -l_2 g \leq h \leq l_3 g, \quad \textit{and} \quad |\nabla h| \leq l_4.$$

Then for any $p > 1$ *and* $\alpha > 1$, *there are positive constants* c_0, c_1, *and* c_2 *ensuring that on* $Q_{2R,T}$ *the following gradient estimate holds:*

$$\frac{|\nabla\vartheta|^2}{\vartheta} - \alpha\frac{\vartheta_t}{\vartheta} - C\vartheta^q \leq \frac{m(p-1)\alpha^2}{1+\alpha m(p-1)}\frac{1}{t} + \frac{m(p-1)\alpha^2}{1+\alpha m(p-1)}N, \tag{3.1}$$

where $\vartheta = \frac{p}{p-1}u^{p-1}$, $q = s - 1$, *and*

$$\begin{aligned}N &= l_2c_2 + k\left(-\frac{c_0}{R}(1-m)\left(\sqrt{l_1} + \frac{2}{R}\right) + \frac{c_1}{R^2}\right) + 2(p-1)k\frac{c_1}{R^2}\\ &\quad + \frac{m(p-1)\alpha^2}{2(\alpha-1)}\frac{c_1}{R^2}k - 2k_2 + \left\{\frac{1+\alpha m(p-1)}{m(p-1)\alpha^2}\right\}^{1/2}\\ &\quad \times \Bigg[\alpha^2(p-1)n\max\{l_2^2, l_3^2\} + \frac{9}{8}n(p-1)\alpha^2 l_4 + \frac{1}{2}l_2\alpha(p-1)k\theta_1^2\\ &\quad + \frac{1}{4}\alpha(p-1)k\theta_2^2 - k_1 + \frac{1}{4}\frac{m(p-1)\alpha^2}{(\alpha-1)^2 + \alpha(\alpha-1)m(p-1)}\\ &\quad \times \left[2(l_2 + \alpha l_3) + (2l_1k + 2l_4k + 2l_2\alpha + \alpha)(p-1)\right]^2 - 2k_3\Bigg]^{1/2}.\end{aligned}$$

For two points (y_1, s_1) and (y_2, s_2) in $M \times (0, T]$ with $s_1 < s_2$, define the action functional

$$\mathcal{J}(y_1, s_1, y_2, s_2) := \inf_{\zeta} \int_{s_1}^{s_2} |\zeta'(t)|^2_{g(t)}\, dt,$$

where the infimum is calculated across all smooth paths $\zeta\colon [s_1, s_2] \to M$ that connect y_1 to y_2.

As a direct and significant result of Theorem 3.1, we derive the following Harnack inequality:

Corollary 3.1. *Under the assumptions stated above, for any* $(y_1, s_1), (y_2, s_2) \in M \times (0, T]$ *with* $s_1 < s_2$, *it holds that*

$$\vartheta(y_1, s_1) \le \vartheta(y_2, s_2)\left(\frac{s_2}{s_1}\right)^{\frac{m(p-1)\alpha}{1+\alpha m(p-1)}} \times \exp\left\{\frac{\alpha}{4\tilde{k}}\mathcal{J}(y_1, s_1, y_2, s_2) + (s_2 - s_1)\frac{m(p-1)\alpha}{1+\alpha m(p-1)}N_2\right\},$$

where $\tilde{k} := \inf_{M\times(0,T]}\vartheta$.

4. Proof of the main results

In the following, we present the detailed proof of Theorem 3.1 along with the derivation of the corresponding Harnack inequality. Starting with the WPME (1.1), and setting $\vartheta = \frac{p}{p-1}u^{p-1}$, we reformulate (1.1) as

$$\vartheta_t = (p-1)\vartheta\Delta_\phi\vartheta + |\nabla\vartheta|^2 + B\vartheta^s, \tag{4.1}$$

where the constants B and s are defined by

$$B = pC\left(\frac{p-1}{p}\right)^{\frac{p-2+q}{p-1}}, \quad s = \frac{p-2+q}{p-1}.$$

We begin by aiming to derive a localized space-time gradient bound of Li–Yau type for the coupled systems (1.1) and (1.2), under the assumption that Ric_ϕ^{m-n} admits a uniform lower bound.

To facilitate the proof, we recall essential lemmas, beginning with the following result from [16]:

Lemma 4.1. *Assume that the metric changes as described in* (1.2)*. For any differentiable function* ω*, the subsequent identities in the term of partial differential equations are valid:*

$$\frac{\partial}{\partial t}|\nabla\omega|^2 = 2\langle\nabla\omega, \nabla\omega_t\rangle - 2h(\nabla\omega, \nabla\omega)$$

and

$$\begin{aligned}(\Delta_\phi\omega)_t &= \Delta_\phi\omega_t - 2\langle h, \mathrm{Hess}\omega\rangle - 2\langle -\tfrac{1}{2}\nabla(\mathrm{tr}_g h) + \mathrm{div}h, \nabla\omega\rangle \\ &\quad + 2h(\nabla\phi, \nabla\omega) - \langle\nabla\omega, \nabla\phi_t\rangle \\ &= \Delta_\phi\omega_t - 2\langle h, \mathrm{Hess}\omega\rangle - 2\langle -\tfrac{1}{2}\nabla(\mathrm{tr}_g h) + \mathrm{div}h, \nabla\omega\rangle \\ &\quad - \langle\nabla\omega, \nabla\Delta\phi\rangle + 2h(\nabla\phi, \nabla\omega),\end{aligned}$$

where, div *denotes the divergence operator.*

Let $w := |\nabla\vartheta|^2 > 0$ and

$$\mathcal{L} = \partial_t - (p-1)\vartheta\Delta_\phi. \tag{4.2}$$

We now present the following lemma:

Lemma 4.2. *Suppose* $(M^n, g, e^{-\phi}dv)$ *is a weighted Riemannian manifold with the metric* $g(t)$ *evolving as in Eq* (1.2)*. Suppose there exist positive constants* l_1*–*l_4 *such that on* $Q_{2R,T}$ *the following bounds hold:*

$$\mathrm{Ric}_\phi^{m-n} \geq -(m-1)l_1 g, \quad -l_2 g \leq h \leq l_3 g, \quad |\nabla h| \leq l_4, \tag{4.3}$$

and define

$$\theta_1 := \sup_{Q_{2R,T}} |\nabla\phi|, \quad \theta_2 := \sup_{Q_{2R,T}} |\nabla\Delta\phi|. \tag{4.4}$$

Set

$$\mathcal{H} = -\alpha\frac{\vartheta_t}{\vartheta} - C\vartheta^{s-1} + \frac{|\nabla\vartheta|^2}{\vartheta}.$$

Then the following statements hold:

(i) Provided that $p > 1$*, the following inequality holds*

$$\begin{aligned}\mathcal{L}(\mathcal{H}) \leq\, & 2p\langle\nabla\vartheta, \nabla\mathcal{H}\rangle - \left[1 - \frac{1}{m(1-p)}\right]\left(-(1-p)\Delta_\phi\vartheta\right)^2 - (\alpha-1)\left(\frac{\vartheta_t}{\vartheta}\right)^2 - \alpha'\frac{\vartheta_t}{\vartheta} \\ & - 2(1-p)l_1|\nabla\vartheta|^2 - 2\frac{|\nabla\vartheta|^2}{\vartheta}(l_2 + \alpha l_3) - \alpha^2(1-p)n\max\{l_3^2, l_2^2\} \\ & - 3\alpha(1-p)\sqrt{n}l_4|\nabla\vartheta| - 2\alpha(1-p)l_2\langle\nabla\vartheta, \nabla\phi\rangle - \alpha(1-p)\langle\nabla\Delta\phi, \nabla\vartheta\rangle \\ & + (1-p)\Delta_\phi\vartheta\left(B\vartheta^{s-1}[(2s-1) + 2\vartheta(1-p)]\right) - C(1-p)(s-1)(s-4)\vartheta^{s-2}|\nabla\vartheta|^2.\end{aligned} \tag{4.5}$$

(ii) If $p < 1$*, then*

$$\begin{aligned}\mathcal{L}(\mathcal{H}) \geq\, & 2p\langle\nabla\mathcal{H}, \nabla\vartheta\rangle - \left[1 - \frac{1}{m(1-p)}\right]\left((1-p)\Delta_\phi\vartheta\right)^2 - (\alpha-1)\left(\frac{\vartheta_t}{\vartheta}\right)^2 - \alpha'\frac{\vartheta_t}{\vartheta} \\ & - 2(1-p)l_1|\nabla\vartheta|^2 - 2(l_3 + \alpha l_2)\frac{|\nabla\vartheta|^2}{\vartheta} - \alpha^2(1-p)n\max\{l_3^2, l_2^2\} \\ & - 3\alpha(1-p)\sqrt{n}l_4|\nabla\vartheta| - 2\alpha(1-p)l_2\langle\nabla\vartheta, \nabla\phi\rangle - \alpha(1-p)\langle\nabla\Delta\phi, \nabla\vartheta\rangle \\ & + (1-p)\Delta_\phi\vartheta\left(B\vartheta^{s-1}[(2s-1) - 2\vartheta(1-p)]\right) - C(1-p)(s-1)(s-4)\vartheta^{s-2}|\nabla\vartheta|^2.\end{aligned} \tag{4.6}$$

Proof. A straightforward calculation using the linearized operator $\mathcal{L}$ applied to functions $\sigma, \gamma \in C^\infty(M)$ yields

$$\mathcal{L}\left(\frac{\sigma}{\gamma}\right) = -\frac{\sigma}{\gamma^2}\mathcal{L}(\gamma) + \frac{1}{\gamma}\mathcal{L}(\sigma) - 2(1-p)\vartheta\left\langle \nabla \log\gamma, \nabla\left(\frac{\sigma}{\gamma}\right)\right\rangle. \tag{4.7}$$

By the definition of $\mathcal{L}$, we compute

$$\begin{aligned}\mathcal{L}(\vartheta_t) &= \partial_t(\vartheta_t) + (1-p)\vartheta\Delta_\phi(\vartheta_t)\\ &= \left[B\vartheta^s + |\nabla\vartheta|^2 - (1-p)\vartheta\Delta_\phi\vartheta\right]_t + (1-p)\vartheta\Delta_\phi(\vartheta_t)\\ &= Bs\vartheta^{s-1}\vartheta_t + 2\langle\nabla\vartheta_t, \nabla\vartheta\rangle - 2h(\nabla\vartheta,\nabla\vartheta) - (1-p)\vartheta(\Delta_\phi\vartheta)_t\\ &\quad - (1-p)\vartheta_t\Delta_\phi\vartheta + (1-p)\vartheta\Delta_\phi(\vartheta_t).\end{aligned} \tag{4.8}$$

Using the time derivative of the weighted Laplacian and the evolving metric properties, the right-hand side expands further as

$$\begin{aligned}\mathcal{L}(\vartheta_t) = {}& 2(1-p)\vartheta\left\langle \operatorname{div} h - \tfrac{1}{2}\nabla(\operatorname{tr}_g h), \nabla\vartheta\right\rangle + 2(1-p)\vartheta\langle h, \operatorname{Hess}\vartheta\rangle - (1-p)\vartheta_t\Delta_\phi\vartheta\\ &+ (1-p)\vartheta\langle\nabla\Delta\phi, \nabla\vartheta\rangle - 2(1-p)\vartheta h(\nabla\phi,\nabla\vartheta) - 2h(\nabla\vartheta,\nabla\vartheta)\\ &+ Bs\vartheta^{s-1}\vartheta_t + 2\langle\nabla\vartheta,\nabla\vartheta_t\rangle.\end{aligned}$$

Let us recall the Bochner identity adapted to the weighted setting, valid for any smooth scalar function σ:

$$\frac{1}{2}\Delta_\phi|\nabla\sigma|^2 = \operatorname{Ric}_\phi(\nabla\sigma,\nabla\sigma) + \langle\nabla\sigma, \nabla\Delta_\phi\sigma\rangle + |\operatorname{Hess}\sigma|^2. \tag{4.9}$$

Applying this to ϑ, we find

$$\begin{aligned}\mathcal{L}(|\nabla\vartheta|^2) = {}& 2\langle\nabla\vartheta_t, \nabla\vartheta\rangle - 2h(\nabla\vartheta,\nabla\vartheta)\\ &+ 2(1-p)\vartheta\left(\operatorname{Ric}_\phi(\nabla\vartheta,\nabla\vartheta) + \langle\nabla\vartheta,\nabla\Delta_\phi\vartheta\rangle + |\operatorname{Hess}\vartheta|^2\right).\end{aligned}$$

Substituting Eq (4.1) into the above, we obtain

$$\begin{aligned}\mathcal{L}(|\nabla\vartheta|^2) = {}& -2\left(h - (1-p)\vartheta\operatorname{Ric}_\phi\right)(\nabla\vartheta,\nabla\vartheta) - 2(1-p)\Delta_\phi\vartheta\,|\nabla\vartheta|^2\\ &+ 2(1-p)\vartheta\,|\operatorname{Hess}\vartheta|^2 + 2Bs\vartheta^{s-1}|\nabla\vartheta|^2 + 2\langle\nabla|\nabla\vartheta|^2, \nabla\vartheta\rangle.\end{aligned} \tag{4.10}$$

Combining Eqs (4.8) and (4.10) into (4.7), we derive

$$\begin{aligned}\mathcal{L}\left(\frac{\vartheta_t}{\vartheta}\right) &= -2(1-p)\vartheta\left\langle\nabla\log\vartheta, \nabla\left(\frac{\vartheta_t}{\vartheta}\right)\right\rangle - \frac{\vartheta_t}{\vartheta^2}\mathcal{L}(\vartheta) + \frac{1}{\vartheta}\mathcal{L}(\vartheta_t)\\ &= -\frac{\vartheta_t}{\vartheta}\left(B\vartheta^{s-1} + \frac{|\nabla\vartheta|^2}{\vartheta}\right) + Bs\vartheta^{s-2}\vartheta_t + \frac{2}{\vartheta}\langle\nabla\vartheta,\nabla\vartheta_t\rangle - \frac{2}{\vartheta}h(\nabla\vartheta,\nabla\vartheta)\\ &\quad + (1-p)\langle\nabla\Delta\phi,\nabla\vartheta\rangle - 2(1-p)h(\nabla\vartheta,\nabla\phi) + 2(1-p)\left\langle\nabla\vartheta, \operatorname{div} h - \frac{1}{2}\nabla(\operatorname{tr}_g h)\right\rangle\\ &\quad + 2(1-p)\langle\operatorname{Hess}\vartheta, h\rangle - (1-p)\frac{\vartheta_t}{\vartheta}\Delta_\phi\vartheta.\end{aligned}$$

Similarly, substituting (4.7) and (4.10) into (4.7) for the term $\frac{|\nabla\vartheta|^2}{\vartheta}$, we have

$$\begin{aligned}
\mathcal{L}\left(\frac{|\nabla\vartheta|^2}{\vartheta}\right) &= -2(1-p)\vartheta\left\langle\nabla\log\vartheta,\nabla\left(\frac{|\nabla\vartheta|^2}{\vartheta}\right)\right\rangle - \frac{|\nabla\vartheta|^2}{\vartheta^2}\mathcal{L}(\vartheta) + \frac{1}{\vartheta}\mathcal{L}(|\nabla\vartheta|^2) \\
&= -\frac{|\nabla\vartheta|^2}{\vartheta^2}B\vartheta^s - \frac{|\nabla\vartheta|^4}{\vartheta^2} + 2Bs\vartheta^{s-2}|\nabla\vartheta|^2 + 2(1-p)|\mathrm{Hess}\,\vartheta|^2 + \frac{2}{\vartheta}\langle\nabla|\nabla\vartheta|^2,\nabla\vartheta\rangle \\
&\quad - 2(1-p)\frac{|\nabla\vartheta|^2}{\vartheta}\Delta_\phi\vartheta - 2\left(\frac{1}{\vartheta}h-(1-p)\mathrm{Ric}_\phi\right)(\nabla\vartheta,\nabla\vartheta) \\
&\quad - 2(1-p)\vartheta\left\langle\nabla\log\vartheta,\nabla\left(\frac{|\nabla\vartheta|^2}{\vartheta}\right)\right\rangle.
\end{aligned}$$

Therefore, combining these results, we conclude that

$$\begin{aligned}
\mathcal{L}(\mathcal{H}) =& -C\mathcal{L}(\vartheta^{s-1}) - \alpha'\frac{\vartheta_t}{\vartheta} - \alpha\mathcal{L}\left(\frac{\vartheta_t}{\vartheta}\right) + \mathcal{L}\left(\frac{|\nabla\vartheta|^2}{\vartheta}\right) \\
=& \frac{2}{\vartheta}\langle\nabla|\nabla\vartheta|^2,\nabla\vartheta\rangle + 2(p-1)\Delta_\phi\vartheta\frac{|\nabla\vartheta|^2}{\vartheta} - 2\left((p-1)\mathrm{Ric}_\phi + \frac{1}{\vartheta}h\right)(\nabla\vartheta,\nabla\vartheta) \\
&+ 2Bs\vartheta^{s-2}|\nabla\vartheta|^2 + 2(p-1)\vartheta\left\langle\nabla\log\vartheta,\nabla\left(\frac{|\nabla\vartheta|^2}{\vartheta}\right)\right\rangle + 2(1-p)|\mathrm{Hess}\,\vartheta|^2 \\
&+ 2\alpha(p-1)\langle\mathrm{Hess}\,\vartheta,h\rangle - \alpha(p-1)\frac{\vartheta_t}{\vartheta}\Delta_\phi\vartheta - \frac{|\nabla\vartheta|^4}{\vartheta^2} - B\vartheta^s\frac{|\nabla\vartheta|^2}{\vartheta^2} \\
&- \alpha(1-p)\langle\nabla\Delta\phi,\nabla\vartheta\rangle + 2\alpha(1-p)\langle\nabla\vartheta,h(\nabla\phi,\cdot)\rangle \\
&- 2\alpha(1-p)\left\langle\nabla\vartheta,\mathrm{div}\,h-\frac{1}{2}\nabla(\mathrm{tr}_g h)\right\rangle - \alpha\frac{2}{\vartheta}\langle\nabla\vartheta_t,\nabla\vartheta\rangle + \alpha\frac{2}{\vartheta}h(\nabla\vartheta,\nabla\vartheta) \\
&+ 2\alpha(1-p)\vartheta\left\langle\nabla\log\vartheta,\nabla\left(\frac{\vartheta_t}{\vartheta}\right)\right\rangle + \alpha\frac{\vartheta_t}{\vartheta}\frac{|\nabla\vartheta|^2}{\vartheta} \\
&+ \alpha B\vartheta^{s-2}\vartheta_t - \alpha Bs\vartheta^{s-2}\vartheta_t - C(s-1)\vartheta^{s-2}\vartheta_t - \alpha'\frac{\vartheta_t}{\vartheta} \\
&- C(1-p)(s-1)(s-2)\vartheta^{s-2}|\nabla\vartheta|^2 + C(p-1)(s-1)\vartheta^{s-1}\Delta_\phi\vartheta.
\end{aligned} \tag{4.11}$$

It is straightforward to derive the identity

$$\begin{aligned}
&-2\alpha(p-1)\vartheta\left\langle\nabla\log\vartheta,\nabla\left(\frac{\vartheta_t}{\vartheta}\right)\right\rangle - 2(1-p)\vartheta\left\langle\nabla\log\vartheta,\nabla\left(\frac{|\nabla\vartheta|^2}{\vartheta}\right)\right\rangle \\
&= 2(1-p)C(s-1)\vartheta^{s-2}|\nabla\vartheta|^2 - 2(1-p)\langle\nabla F,\nabla\vartheta\rangle.
\end{aligned}$$

and also,

$$\begin{aligned}
&-\alpha\frac{2}{\vartheta}\langle\nabla\vartheta,\nabla\vartheta_t\rangle + \frac{2}{\vartheta}\left\langle\nabla\vartheta,\nabla|\nabla\vartheta|^2\right\rangle \\
&= 2F\cdot\frac{|\nabla\vartheta|^2}{\vartheta} + 2C(s-1)\vartheta^{s-1}\cdot\frac{|\nabla\vartheta|^2}{\vartheta} + 2\langle\nabla\vartheta,\nabla F\rangle + 2C\vartheta^{s-1}\cdot\frac{|\nabla\vartheta|^2}{\vartheta} \\
&= 2\langle\nabla\vartheta,\nabla F\rangle + 2F\cdot\frac{|\nabla\vartheta|^2}{\vartheta} + \left[2C(s-1)\vartheta^{s-2} + 2C\vartheta^{s-1}\right]\cdot\frac{|\nabla\vartheta|^2}{\vartheta}.
\end{aligned}$$

Utilizing identity (1.1), we deduce the relation:

$$\begin{aligned} &-2(1-p)\frac{|\nabla\vartheta|^2}{\vartheta}\Delta_\phi\vartheta - \frac{|\nabla\vartheta|^4}{\vartheta^2} + \alpha(1-p)\frac{\vartheta_t}{\vartheta}\Delta_\phi\vartheta + \alpha\frac{\vartheta_t}{\vartheta}\cdot\frac{|\nabla\vartheta|^2}{\vartheta}\\ &=(2\alpha+2)\cdot\frac{\vartheta_t}{\vartheta}\cdot\frac{|\nabla\vartheta|^2}{\vartheta} - 3\cdot\frac{|\nabla\vartheta|^4}{\vartheta^2} - \alpha\left(\frac{\vartheta_t}{\vartheta}\right)^2. \end{aligned}\tag{4.12}$$

Consequently, combining all the above expressions yields:

$$\begin{aligned} &-2\alpha(1-p)\vartheta\left\langle\nabla\log\vartheta,\nabla\left(\frac{\vartheta_t}{\vartheta}\right)\right\rangle + 2(1-p)\vartheta\left\langle\nabla\log\vartheta,\nabla\left(\frac{|\nabla\vartheta|^2}{\vartheta}\right)\right\rangle - \alpha\cdot\frac{2}{\vartheta}\langle\nabla\vartheta_t,\nabla\vartheta\rangle\\ &+\frac{2}{\vartheta}\langle\nabla|\nabla\vartheta|^2,\nabla\vartheta\rangle - \frac{|\nabla\vartheta|^4}{\vartheta^2} - 2(1-p)\frac{|\nabla\vartheta|^2}{\vartheta}\Delta_\phi\vartheta + \alpha(1-p)\frac{\vartheta_t}{\vartheta}\Delta_\phi\vartheta + \alpha\frac{\vartheta_t}{\vartheta}\cdot\frac{|\nabla\vartheta|^2}{\vartheta}\\ &=2p\langle\mathcal{H},\nabla\vartheta\rangle - \left(\frac{|\nabla\vartheta|^2}{\vartheta} - \frac{\vartheta_t}{\vartheta}\right)^2 - (\alpha-1)\left(\frac{\vartheta_t}{\vartheta}\right)^2\\ &-2(p-2)C(s-1)\vartheta^{s-2}|\nabla\vartheta|^2. \end{aligned}\tag{4.13}$$

By substituting this result into Eq (4.11), we eventually arrive at the inequality

$$\begin{aligned} \mathcal{L}(\mathcal{H}) \le &-(\alpha-1)\left(\frac{\vartheta_t}{\vartheta}\right)^2 - \alpha'\cdot\frac{\vartheta_t}{\vartheta} + 2p\langle\nabla\vartheta,\nabla\mathcal{H}\rangle - \left(\frac{|\nabla\vartheta|^2}{\vartheta} - \frac{\vartheta_t}{\vartheta}\right)^2\\ &-2\left(\frac{1}{\vartheta}h - (1-p)\operatorname{Ric}_\phi\right)(\nabla\vartheta,\nabla\vartheta) - 2\alpha(1-p)\langle\operatorname{div}h - \tfrac{1}{2}\nabla(\operatorname{tr}_g h),\nabla\vartheta\rangle\\ &+2(1-p)|\operatorname{Hess}\vartheta|^2 - 2\alpha(1-p)\langle h,\operatorname{Hess}\vartheta\rangle + \frac{2\alpha}{\vartheta}h(\nabla\vartheta,\nabla\vartheta)\\ &-\alpha(1-p)\langle\nabla\vartheta,\nabla\Delta\phi\rangle + 2\alpha(1-p)h(\nabla\phi,\nabla\vartheta)\\ &-C(s-1)(1-p)(2-s)\vartheta^{s-2}|\nabla\vartheta|^2 + B(2s-1)\vartheta^{s-1}\left(\frac{|\nabla\vartheta|^2}{\vartheta} - \frac{\vartheta_t}{\vartheta}\right)\\ &-BC(s-1)\vartheta^{2s-2}. \end{aligned}\tag{4.14}$$

From the conditions imposed in the aforementioned Lemma, we arrive at the inequality

$$|h|^2 \le n\max\{l_2^2, l_3^2\}.$$

Applying Young's inequality subsequently yields

$$\begin{aligned} \langle h,\operatorname{Hess}\vartheta\rangle &\le \frac{1}{2\alpha}|\operatorname{Hess}\vartheta|^2 + \frac{\alpha}{2}|h|^2\\ &\le \frac{1}{2\alpha}|\operatorname{Hess}\vartheta|^2 + \frac{n\alpha}{2}\max\{l_2^2, l_3^2\}. \end{aligned}\tag{4.15}$$

In addition, the following estimate holds:

$$\left|\operatorname{div}h - \frac{1}{2}\nabla(\operatorname{tr}_g h)\right| = \left|g^{ij}\nabla_i h_{jl} - \frac{1}{2}g^{ij}\nabla_l h_{ij}\right| \le \frac{3}{2}|g|\,|\nabla h| \le \frac{3}{2}\sqrt{n}l_4.\tag{4.16}$$

Now, for any $m > n$, consider the following expression:

$$
\begin{aligned}
0 &\leq \left(\sqrt{\frac{n}{m(m-n)}}\,\langle \nabla\phi, \nabla\vartheta \rangle + \sqrt{\frac{1}{mn}(m-n)}\,\Delta\vartheta \right)^2 \\
&= -\left(\frac{1}{m} - \frac{1}{n}\right)(\Delta\vartheta)^2 + \frac{2}{m}\langle \nabla\vartheta, \nabla\phi \rangle \Delta\vartheta - \left(\frac{1}{m} - \frac{1}{m-n}\right)\langle \nabla\vartheta, \nabla\phi \rangle^2 \\
&\leq -\frac{1}{m}\left[\langle \nabla\phi, \nabla\vartheta \rangle^2 - 2\Delta\vartheta\langle \nabla\phi, \nabla\vartheta \rangle + (\Delta\vartheta)^2\right] + |\mathrm{Hess}\,\vartheta|^2 + \frac{1}{m-n}\langle \nabla\phi, \nabla\vartheta \rangle^2 \\
&= -\frac{1}{m}(\Delta_\phi\vartheta)^2 + \frac{1}{m-n}\langle \nabla\phi, \nabla\vartheta \rangle^2 + |\mathrm{Hess}\,\vartheta|^2.
\end{aligned}
$$

Consequently, the following lower bound is obtained:

$$
\frac{1}{m-n}\langle \nabla\phi, \nabla\vartheta \rangle^2 - \frac{(\Delta_\phi\vartheta)^2}{m} + |\mathrm{Hess}\,\vartheta|^2 \geq 0. \tag{4.17}
$$

Substituting Eqs (4.15), (4.17), and (4.18) into (4.14), we derive the following differential inequality for the case $p > 1$:

$$
\begin{aligned}
\mathcal{L}(\mathcal{H}) \leq{}& -\alpha' \cdot \frac{\vartheta_t}{\vartheta} - (\alpha - 1)\left(\frac{\vartheta_t}{\vartheta}\right)^2 - (1-p)^2(\Delta_\phi\vartheta)^2 + 2p\langle \nabla\mathcal{H}, \nabla\vartheta \rangle + (1-p)|\mathrm{Hess}\,\vartheta|^2 \\
&+ \frac{2l_2}{\vartheta}|\nabla\vartheta|^2 + 2(1-p)\mathrm{Ric}_\phi(\nabla\vartheta, \nabla\vartheta) - 3\alpha(1-p)\sqrt{n}l_4|\nabla\vartheta| - \alpha^2(1-p)n\max\{l_3^2, l_2^2\} \\
&- \alpha(1-p)\langle \nabla\Delta\phi, \nabla\vartheta \rangle - 2\alpha(1-p)\langle \nabla\phi, \nabla\vartheta \rangle + (1-p)\cdot\Delta_\phi\vartheta\cdot B\vartheta^{s-1}(2v(p-1) \\
&+ (2s-1)) + \frac{2\alpha l_3}{\vartheta}|\nabla\vartheta|^2 - C(1-p)(s-1)(s-4)\vartheta^{s-2}|\nabla\vartheta|^2,
\end{aligned}
$$

which can alternatively be rearranged into the more compact form:

$$
\begin{aligned}
\mathcal{L}(\mathcal{H}) \leq{}& -(\alpha - 1)\left(\frac{\vartheta_t}{\vartheta}\right)^2 - \left[1 - \frac{1}{m(1-p)}\right](1-p)^2(\Delta_\phi\vartheta)^2 + 2p\langle \nabla\mathcal{H}, \nabla\vartheta \rangle + \frac{2(l_2 + \alpha l_3)}{\vartheta}|\nabla\vartheta|^2 \\
&- 2(1-p)l_1|\nabla\vartheta|^2 - \alpha' \cdot \frac{\vartheta_t}{\vartheta} - 3\alpha(1-p)\sqrt{n}l_4|\nabla\vartheta| - \alpha^2(1-p)n\max\{l_3^2, l_2^2\} \\
&- \alpha(1-p)\langle \nabla\Delta\phi, \nabla\vartheta \rangle - 2\alpha l_2(1-p)\langle \nabla\phi, \nabla\vartheta \rangle - C(1-p)(s-1)(s-4)\vartheta^{s-2}|\nabla\vartheta|^2 \\
&+ (1-p)\Delta_\phi\vartheta \cdot B\vartheta^{s-1}(2v(p-1) + (2s-1)).
\end{aligned}
$$

For the opposite case where $p < 1$, a corresponding lower bound is established as:

$$
\begin{aligned}
\mathcal{L}(\mathcal{H}) \geq{}& -(\alpha - 1)\left(\frac{\vartheta_t}{\vartheta}\right)^2 - \left[1 - \frac{1}{m(1-p)}\right](1-p)^2(\Delta_\phi\vartheta)^2 + 2p\langle \nabla\mathcal{H}, \nabla\vartheta \rangle + \frac{(2l_3 + 2\alpha l_2)}{\vartheta}|\nabla\vartheta|^2 \\
&- 2(1-p)l_1|\nabla\vartheta|^2 - \alpha' \cdot \frac{\vartheta_t}{\vartheta} - 3\alpha(1-p)\sqrt{n}l_4|\nabla\vartheta| - \alpha^2(1-p)n\max\{l_3^2, l_2^2\} \\
&- \alpha(1-p)\langle \nabla\Delta\phi, \nabla\vartheta \rangle - 2\alpha l_2(1-p)\langle \nabla\phi, \nabla\vartheta \rangle + (1-p)\Delta_\phi\vartheta \cdot B\vartheta^{s-1}(2v(p-1) + (2s-1)) \\
&- C(1-p)(s-1)(s-4)\vartheta^{s-2}|\nabla\vartheta|^2.
\end{aligned} \tag{4.18}
$$

This concludes the proof of the lemma. □

Now we proceed to derive a local space-time Li–Yau-type gradient bound for the solution of Eqs (1.1) and (1.2), assuming a lower bound on the (m,n)-Bakry–Émery Ricci curvature.

Proof of Theorem 3.1. Since the Ricci curvature is bounded and the Riemannian metric evolves smoothly over time, it follows from [2, Corollary 6.11] that the metric $g(t)$ remains uniformly comparable to the initial metric $g(0)$, namely

$$e^{-2l_2T}g(0) \leq g(t) \leq e^{2l_3T}g(0).$$

It follows that the Riemannian manifold $(M, g(t))$ remains complete throughout the entire time interval $t \in [0,T]$.

Let $A\colon [0,+\infty) \to \mathbb{R}$ be a twice continuously differentiable cut-off function defined by

$$A(s) = \begin{cases} 1, & \text{for } 0 \leq s \leq 1, \\ 0, & \text{for } s \geq 2, \end{cases}$$

with the following properties:

$$0 \leq A(s) \leq 1, \quad -c_0 \leq A'(s) \leq 0, \quad A''(s) \geq -c_1, \quad \frac{|A''(s)|^2}{A(s)} \leq c_1,$$

where c_0 and c_1 are constants. For any $R \geq 1$, define the spatial cut-off function as

$$\psi(z,t) = A\left(\frac{r(z,t)}{R}\right),$$

where $r(z,t) = d(x,x_0,t)$ denotes the distance with respect to the metric $g(t)$. The function $A(s)$ is Lipschitz continuous, and following the arguments in [5, 17], we may apply the maximum principle and use Calabi's trick, ensuring the regularity of ψ. By invoking the generalized Laplacian comparison theorem [18–20] under the curvature assumption $\mathrm{Ric}_\phi^{m-n} \geq -(m-1)l_1$, we obtain the following estimate:

$$\Delta_\phi r(x) \leq (m-1)\sqrt{l_1}\coth(\sqrt{l_1}r(x)).$$

As a consequence, we estimate the weighted Laplacian of ψ by

$$\begin{aligned} \Delta_\phi \psi &= A'\frac{\Delta_\phi r}{R} + A''\frac{|\nabla r|^2}{R^2} \\ &\geq -\frac{c_0}{R}(m-1)\sqrt{l_1}\coth(\sqrt{l_1}r(x)) - \frac{c_1}{R^2} \\ &\geq -\frac{c_0}{R}(m-1)\left(\sqrt{l_1} + \frac{2}{R}\right) - \frac{c_1}{R^2}. \end{aligned} \tag{4.19}$$

Additionally, we estimate

$$\frac{|\nabla\psi|^2}{\psi} = \frac{|A'|^2|\nabla r|^2}{R^2A} \leq \frac{c_1}{R^2}. \tag{4.20}$$

Define the auxiliary function $W = t\psi F$ and fix an arbitrary $T_1 \in (0,T]$. Suppose that $\max_{(z,t)\in M\times(0,T]} W > 0$, and let $(z_0,t_0) \in Q_{2R,T_1}$ be the point where W attains its maximum. At this maximum point, we have:

$$\nabla W = 0, \qquad \mathcal{L}W \geq 0.$$

This yields

$$\nabla\mathcal{H} = -\frac{\mathcal{H}}{\psi}\nabla\psi \tag{4.21}$$

and

$$0 \leq \mathcal{L}W = \psi\mathcal{H} + t_0\mathcal{H}\mathcal{L}\psi + t_0\psi\mathcal{L}\mathcal{H} - 2t_0(p-1)\vartheta\langle\nabla\psi, \nabla\mathcal{H}\rangle. \tag{4.22}$$

Using the approach from [21, Theorem 1], one can find a constant c_2 such that

$$-F\psi_t \geq -c_2 l_2 F.$$

Inserting the estimates from (4.19)–(4.21) into (4.22), we obtain:

$$0 \leq \frac{W}{t_0} + \left[c_2 l_2 + k\left(\frac{c_0}{R}(m-1)(\sqrt{l_1} + \frac{2}{R}) + \frac{c_1}{R^2}\right)\right] t_0\mathcal{H} + t_0\psi\mathcal{L}\mathcal{H} - 2t_0(p-1)\vartheta\langle\nabla\psi, \nabla\mathcal{H}\rangle. \tag{4.23}$$

Now, substituting the expression of $\mathcal{L}(\mathcal{H})$ from Lemma 4.2 into the above inequality and evaluating at (z_0, t_0), we find:

$$\begin{aligned}
0 \leq & \frac{W}{t_0} + \left[c_2 l_2 + k\left(\frac{c_0}{R}(m-1)(\sqrt{l_1} + \frac{2}{R}) + \frac{c_1}{R^2}\right)\right] t_0\mathcal{H} \\
& - 2t_0(p-1)\vartheta\langle\nabla\psi, \nabla\mathcal{H}\rangle + 2pt_0\psi\langle\nabla\vartheta, \nabla\mathcal{H}\rangle \\
& - t_0\psi\left[\frac{1}{-m(1-p)} + 1\right]\left((p-1)\Delta_\phi\vartheta\right)^2 - (\alpha-1)t_0\psi\left(\frac{\vartheta_t}{\vartheta}\right)^2 \\
& + 2(p-1)t_0\psi l_1|\nabla\vartheta|^2 + (2l_2 + 2\alpha l_3)t_0\psi\frac{|\nabla\vartheta|^2}{\vartheta} \\
& - \alpha^2(1-p)nt_0\psi\max\{l_2^2, l_3^2\} - 3\alpha(1-p)\sqrt{n}l_4 t_0\psi|\nabla\vartheta| \\
& - 2l_2\alpha(1-p)t_0\psi\langle\nabla\vartheta, \nabla\phi\rangle - \alpha(1-p)t_0\psi\langle\nabla\vartheta, \nabla\Delta\phi\rangle \\
& + t_0\psi(1-p)\Delta_\phi\vartheta\left(B\vartheta^{s-1}\left[-2v(1-p) + (2s-1)\right]\right) \\
& - t_0\psi C(1-p)(s-1)(s-4)\vartheta^{s-2}|\nabla\vartheta|^2.
\end{aligned} \tag{4.24}$$

By applying Eq (4.1) along with the definition of the quantity F, the following identity is derived:

$$(p-1)\Delta_\phi\vartheta = -\frac{1}{\alpha}\mathcal{H} + \frac{1-\alpha}{\alpha}\cdot\frac{|\nabla\vartheta|^2}{\vartheta} - \left(\frac{C}{\alpha} + B\right)\vartheta^{s-1}. \tag{4.25}$$

From this, squaring both sides yields:

$$\begin{aligned}
\left[(p-1)\Delta_\phi\vartheta\right]^2 = & \left(B + \frac{C}{\alpha}\right)^2\vartheta^{2s-2} + 2\vartheta^{s-1}\left(\frac{B}{\alpha} + \frac{C}{\alpha^2}\right)\mathcal{H} \\
& + 2\vartheta^{s-1}\frac{|\nabla\vartheta|^2}{\vartheta}\left(\frac{\alpha+1}{\alpha}\right)\left(B + \frac{C}{\alpha}\right) \\
& + \frac{(\alpha-1)^2}{\alpha^2}\frac{|\nabla\vartheta|^4}{\vartheta^2} + \frac{2(\alpha-1)}{\alpha^2}\frac{|\nabla\vartheta|^2}{\vartheta}\mathcal{H} + \frac{1}{\alpha^2}\mathcal{H}^2.
\end{aligned} \tag{4.26}$$

On the other hand, a straightforward calculation yields:

$$\left(\frac{\vartheta_t}{\vartheta}\right)^2 = \frac{1}{\alpha^2}\mathcal{H}^2 + \frac{1}{\alpha^2}\cdot\frac{|\nabla\vartheta|^4}{\vartheta^2} + \frac{C^2}{\alpha^2}\vartheta^{2s-2} - \frac{2}{\alpha^2}\mathcal{H}\cdot\frac{|\nabla\vartheta|^2}{\vartheta} + \frac{2C}{\alpha^2}\mathcal{H}v^{s-1} - \frac{2C}{\alpha^2}\cdot\frac{|\nabla\vartheta|^2}{\vartheta}\vartheta^{s-1}. \tag{4.27}$$

Utilizing the Cauchy–Schwarz inequality, the following estimates hold:

$$\psi\langle\nabla\vartheta,\nabla\mathcal{H}\rangle = -\mathcal{H}\langle\nabla\vartheta,\nabla\psi\rangle \leq \frac{\sqrt{c_1}}{R}\psi^{1/2}k^{1/2}\mathcal{H}\cdot\frac{|\nabla\vartheta|}{\vartheta^{1/2}}, \tag{4.28}$$

$$\langle\nabla\vartheta,\nabla\phi\rangle \leq \theta_1 k^{1/2}\cdot\frac{|\nabla\vartheta|}{\vartheta^{1/2}} \leq \frac{|\nabla\vartheta|^2}{\vartheta} + \frac{1}{4}k\theta_1^2, \tag{4.29}$$

$$\langle\nabla\vartheta,\nabla\Delta\phi\rangle \leq \theta_2 k^{1/2}\cdot\frac{|\nabla\vartheta|}{\vartheta^{1/2}} \leq \frac{|\nabla\vartheta|^2}{\vartheta} + \frac{1}{4}k\theta_2^2. \tag{4.30}$$

Moreover, Young's inequality gives:

$$3\alpha\sqrt{n}l_4|\nabla\vartheta| \leq 2l_4k\cdot\frac{|\nabla\vartheta|^2}{\vartheta} + \frac{9}{8}n\alpha^2 l_4. \tag{4.31}$$

Additionally, from Eq (4.20), one obtains:

$$-\langle\nabla\psi,\nabla\mathcal{H}\rangle = \mathcal{H}\cdot\frac{|\nabla\psi|^2}{\psi} \leq \frac{c_1}{R^2}\mathcal{H}. \tag{4.32}$$

Substituting the estimates (4.25)–(4.32) into (4.24), the inequality becomes:

$$\begin{aligned}
0 \leq & \frac{W}{t_0} + \left[c_2l_2 + k\left(\frac{c_0}{R}(m-1)\left(\frac{2}{R} + \sqrt{l_1}\right) + \frac{c_1}{R^2}\right) - 2(1-p)k\cdot\frac{c_1}{R^2}\right]t_0\mathcal{H} \\
& + 2pt_0\cdot\frac{\sqrt{c_1}}{R}\psi^{1/2}k^{1/2}\mathcal{H}\cdot\frac{|\nabla\vartheta|}{\vartheta^{1/2}} - t_0\psi\left(\frac{1}{-m(1-p)} + \alpha\right)\frac{\mathcal{H}^2}{\alpha^2} \\
& - t_0\psi\left[\alpha^2 - \alpha + \frac{(\alpha-1)^2}{-m(1-p)}\right]\cdot\frac{1}{\alpha^2}\cdot\frac{|\nabla\vartheta|^4}{\vartheta^2} + 2t_0\psi\cdot\frac{\alpha-1}{-m(1-p)\alpha^2}\mathcal{H}\cdot\frac{|\nabla\vartheta|^2}{\vartheta} \\
& - 2(1-p)t_0\psi l_1k\cdot\frac{|\nabla\vartheta|^2}{\vartheta} + 2(l_2+\alpha l_3)t_0\psi\cdot\frac{|\nabla\vartheta|^2}{\vartheta} - \frac{9}{8}n(1-p)t_0\psi\alpha^2l_4 \\
& - \alpha^2n(1-p)t_0\psi\cdot\max\{l_2^2, l_3^2\} - 2l_4k(1-p)t_0\psi\cdot\frac{|\nabla\vartheta|^2}{\vartheta} \\
& - 2l_2\alpha(1-p)t_0\psi\cdot\frac{|\nabla\vartheta|^2}{\vartheta} - \frac{1}{2}l_2\alpha(1-p)t_0\psi k\theta_1^2 - \alpha(1-p)t_0\psi\cdot\frac{|\nabla\vartheta|^2}{\vartheta} \\
& - \frac{1}{4}\alpha(1-p)t_0\psi k\theta_2^2 - t_0\psi k_1 - 2t_0\psi k_2\mathcal{H} - 2t_0\psi k_3\cdot\frac{|\nabla\vartheta|^2}{\vartheta}.
\end{aligned}$$

Here, the constants k_1, k_2, and k_3 are explicitly defined by:

$$\begin{aligned}
k_1 &= \left[\left(\frac{1}{-m(1-p)} + 1\right)\left(B + \frac{C}{\alpha}\right)^2 + (\alpha-1)\frac{C^2}{\alpha^2} - B[-2k(1-p) + (2s-1)]\left(B + \frac{C}{\alpha}\right)\right]k^{2s-2}, \\
k_2 &= \left[\left(\frac{1}{-m(1-p)} + 1\right)\left(\frac{B}{\alpha} + \frac{C}{\alpha^2}\right) + (\alpha-1)\frac{C}{\alpha^2} - \frac{B}{2\alpha}[-2k(1-p) + (2s-1)]\right]k^{s-1}, \\
k_3 &= \left[\left(\frac{1}{-m(1-p)} + 1\right)\left(\frac{\alpha-1}{\alpha}\right)\left(B + \frac{C}{\alpha}\right) - (\alpha-1)\frac{C}{\alpha^2}\right]k^{s-1} \\
&\quad - \left[\left(\frac{\alpha-1}{\alpha}\right)B[-2k(1-p) + (2s-1)] + C(p-1)(s-1)(s-4)\right]k^{s-1}.
\end{aligned}$$

Consequently, we derive the inequality,

$$\begin{aligned}0 \leq &\frac{W}{t_0} + t_0\mathcal{H}\Big[c_2l_2 + k\Big(\frac{c_0}{R}(m-1)(\sqrt{l_1} + \frac{2}{R}) + \frac{c_1}{R^2}\Big) + 2(p-1)k\frac{c_1}{R^2}\Big] \\ &+ 2p\frac{\sqrt{c_1}}{R}\psi^{-\frac{1}{2}}k^{\frac{1}{2}}W\frac{|\nabla\vartheta|}{\vartheta^{\frac{1}{2}}} - t_0\psi\Big(\frac{1}{-m(1-p)} + \alpha\Big)\frac{\mathcal{H}^2}{\alpha^2} \\ &- t_0\psi\Big(\frac{(\alpha-1)^2}{-m(1-p)} + \alpha^2 - \alpha\Big)\frac{|\nabla\vartheta|^4}{\alpha^2\vartheta^2} - \frac{2(\alpha-1)}{-m(1-p)\alpha^2}W\frac{|\nabla\vartheta|^2}{\vartheta} \\ &+ t_0\psi\Big[(p-1)(2l_1k + 2l_4k + 2l_2\alpha + \alpha) + 2(l_2 + \alpha l_3) - 2k_3\Big]\frac{|\nabla\vartheta|^2}{\vartheta} \\ &+ \alpha^2(p-1)nt_0\psi\max\{l_2^2, l_3^2\} + \frac{9}{8}n(p-1)t_0\psi\alpha^2 l_4 \\ &+ \frac{1}{2}l_2\alpha(p-1)t_0\psi k\theta_1^2 + \frac{1}{4}\alpha(p-1)t_0\psi k\theta_2^2 - t_0\psi k_1 - 2k_2W. \end{aligned} \tag{4.33}$$

Applying the classical inequality $-Ax^2 + Bx \leq \frac{B^2}{4A}$ (valid for any real number B and positive constant A), we obtain

$$\begin{aligned}&- t_0\psi\Big(\frac{(1-\alpha)^2}{-m(1-p)} - \alpha + \alpha^2\Big)\frac{|\nabla\vartheta|^4}{\alpha^2\vartheta^2} \\ &+ t_0\psi\Big[2(l_2 + \alpha l_3) - (2l_1k + 2l_4k + 2l_2\alpha + \alpha)(1-p) - 2k_3\Big]\frac{|\nabla\vartheta|^2}{\vartheta} \\ &\leq t_0\psi C_1, \end{aligned} \tag{4.34}$$

where the upper bound constant C_1 is explicitly given by

$$C_1 = \frac{m(p-1)\alpha^2}{4(\alpha-1)((\alpha-1) + \alpha m(p-1))}\Big[2(l_2 + \alpha l_3) + (2l_1k + 2l_4k + 2l_2\alpha + \alpha)(p-1) - 2k_3\Big]^2.$$

Using a similar argument, we can estimate the combined gradient terms as

$$-\frac{2(\alpha-1)}{m(p-1)\alpha^2}W\frac{|\nabla\vartheta|^2}{\vartheta} + 2p\frac{\sqrt{c_1}}{R}\psi^{-\frac{1}{2}}k^{\frac{1}{2}}W\frac{|\nabla\vartheta|}{\vartheta^{1/2}} \leq \frac{m(p-1)\alpha^2p^2}{2(\alpha-1)}\cdot\frac{c_1}{R^2}\cdot\psi^{-1}kW. \tag{4.35}$$

Substituting the inequalities (4.34) and (4.35) into (4.33), we obtain the bound

$$\begin{aligned}0 \leq &\frac{W}{t_0} + t_0\mathcal{H}\Big[c_2l_2 + k\Big(\frac{c_0}{R}(m-1)(\sqrt{l_1} + \frac{2}{R}) + \frac{c_1}{R^2}\Big) + 2(p-1)k\frac{c_1}{R^2}\Big] \\ &+ \frac{m(p-1)\alpha^2p^2}{2(\alpha-1)}\cdot\frac{c_1}{R^2}\cdot\psi^{-1}kW - t_0\psi\Big(\frac{1}{m(p-1)} + \alpha\Big)\frac{\mathcal{H}^2}{\alpha^2} \\ &+ \alpha^2(p-1)nt_0\psi\max\{l_2^2, l_3^2\} + \frac{9}{8}n(p-1)t_0\psi\alpha^2 l_4 \\ &+ \frac{1}{2}l_2\alpha(p-1)t_0\psi k\theta_1^2 + \frac{1}{4}\alpha(p-1)t_0\psi k\theta_2^2 \\ &+ t_0\psi(C_1 - k_1) - 2k_2W. \end{aligned} \tag{4.36}$$

Multiplying the entire inequality (4.36) by $t_0\eta$, we reach the final form:

$$\begin{aligned}0 \leq &-\Big(\alpha - \frac{1}{m(1-p)}\Big)\frac{W^2}{\alpha^2} + \Big\{\psi + t_0\Big[c_2 l_2 + k\Big(\frac{c_0}{R}(m-1)(\sqrt{l_1} + \frac{2}{R}) + \frac{c_1}{R^2}\Big)\\ &- 2(1-p)k\frac{c_1}{R^2}\Big] + \frac{m(p-1)\alpha^2 p^2}{2(\alpha-1)} \cdot \frac{c_1}{R^2}t_0 k - 2t_0\psi k_2\Big\}W\\ &+ \alpha^2(p-1)nt_0^2\psi^2 \max\{l_2^2, l_3^2\} + \frac{9}{8}n(p-1)t_0^2\psi^2\alpha^2 l_4\\ &+ \frac{1}{2}l_2\alpha(p-1)t_0^2\psi^2 k\theta_1^2 + \frac{1}{4}\alpha(p-1)t_0^2\psi^2 k\theta_2^2 + t_0^2\psi^2(C_1 - k_1). \end{aligned} \tag{4.37}$$

Assume $\tilde{a} > 0$, $\tilde{b} \in \mathbb{R}$, and $\tilde{c} \geq 0$. Then, the inequality $\tilde{c} + \tilde{b}x - \tilde{a}x^2 \geq 0$.holds only if

$$x \leq \frac{1}{2\tilde{a}}\left(\sqrt{\tilde{b}^2 + 4\tilde{a}\tilde{c}} + \tilde{b}\right).$$

Utilizing this result, we get:

$$\begin{aligned}W &\leq \frac{1}{2} \cdot \frac{-m(1-p)\alpha^2}{1-\alpha m(1-p)}\left(C_2 + \sqrt{C_2^2 + 4t_0^2\psi^2 C_3 \cdot \frac{1-\alpha m(1-p)}{-m(1-p)\alpha^2}}\right)\\ &\leq \frac{-m(1-p)\alpha^2}{1-\alpha m(1-p)}\left(C_2 + t_0\psi\sqrt{C_3 \cdot \frac{1-\alpha m(1-p)}{-m(1-p)\alpha^2}}\right). \end{aligned} \tag{4.38}$$

Here, the constants C_2 and C_3 are given by:

$$\begin{aligned}C_2 =&\psi + t_0\left[c_2 l_2 + k\left(\frac{c_0}{R}(m-1)\left(\sqrt{l_1} + \frac{2}{R}\right) + \frac{c_1}{R^2}\right) + 2(p-1)k\frac{c_1}{R^2}\right]\\ &+ \frac{m(p-1)\alpha^2 p^2}{2(\alpha-1)} \cdot \frac{c_1}{R^2}t_0 k - 2t_0\psi k_2,\\ C_3 =&\alpha^2(p-1)n\max\{l_2^2, l_3^2\} + \frac{9}{8}n(p-1)\alpha^2 l_4 + \frac{1}{2}l_2\alpha(p-1)k\theta_1^2\\ &+ \frac{1}{4}\alpha(p-1)k\theta_2^2 + C_1 - k_1. \end{aligned}$$

Suppose x lies within a suitable neighborhood in M such that $\psi(z, T_1) = 1$ for all $d(z, z_0, T_1) < 2R$. Since the maximum of W over the domain $Q_{2R,T}$ occurs at (z_0, t_0), then for any $T_1 \in (0, T]$ satisfying $d(z, z_0, T_1) < R$, it follows that

$$\mathcal{H}(z, T_1) = \frac{W(z, T_1)}{T_1} \leq \frac{W(z_0, t_0)}{T_1}, \quad \forall z \in M.$$

Substituting the bound from (4.38) yields

$$\mathcal{H}(z, T_1) \leq \frac{-m(1-p)\alpha^2}{1-\alpha m(1-p)}\left(C_4 + \sqrt{C_3 \cdot \frac{1-\alpha m(1-p)}{-m(1-p)\alpha^2}}\right),$$

where

$$C_4 = \frac{1}{T_1} + \left[c_2 l_2 + k\left(\frac{c_0}{R}(m-1)\left(\sqrt{l_1} + \frac{2}{R}\right) + \frac{c_1}{R^2}\right) + 2(p-1)k\frac{c_1}{R^2}\right] + \frac{m(p-1)\alpha^2 p^2}{2(\alpha-1)} \cdot \frac{c_1}{R^2}k - 2k_2.$$

This concludes the argument, since T_1 was taken arbitrarily. □

Remark 4.1. *Given that the metric $g(t)$ remains uniformly equivalent to the initial configuration $g(0)$, the manifold $(M, g(t))$ preserves its completeness and noncompactness for all times $t \in [0, T]$. Taking the limit as $R \to +\infty$ in inequality* (3.1) *leads to the estimate*

$$-\alpha \cdot \frac{\vartheta_t}{\vartheta} + \frac{|\nabla\vartheta|^2}{\vartheta} \le \frac{-m(1-p)\alpha^2}{1-\alpha m(1-p)}\left(\frac{1}{t} + N_1\right), \tag{4.39}$$

where the constant N_1 is explicitly given by

$$\begin{aligned} N_1 = c_2 l_2 + \sqrt{\frac{1-\alpha m(1-p)}{-m(1-p)\alpha^2}}\Bigg[&-\alpha^2(1-p)n\max\{l_2^2, l_3^2\} - \frac{9}{8}n(1-p)\alpha^2 l_4 \\ &- \frac{1}{2}l_2\alpha(1-p)k\Theta_1^2 - \frac{1}{4}\alpha(1-p)k\Theta_2^2 \\ &+ \frac{1}{4}\frac{m(p-1)\alpha^2}{(\alpha-1)^2 + \alpha(\alpha-1)m(p-1)}\Big[2(l_2 + \alpha l_3) \\ &+ (2l_1 k + 2l_4 k + 2l_2\alpha + \alpha)(p-1) - 2k_3\Big]^2\Bigg]^{\frac{1}{2}}. \end{aligned}$$

It is important to remark that the quantities Θ_1 and Θ_2 are defined as the uniform suprema over the spacetime domain:

$$\Theta_1 := \sup_{M\times[0,T]} |\nabla\phi|, \quad \Theta_2 := \sup_{M\times[0,T]} |\nabla\Delta\phi|. \tag{4.40}$$

Proof of Corollary 3.1. Consider the geodesic path $\zeta(t)$ parametrized such that it connects points y_1 and y_2 with $\zeta(s_1) = y_1$ and $\zeta(s_2) = y_2$. For any point in spacetime $(\zeta(t), t) \in M \times (0, T]$, invoking Theorem 3.1 alongside the preceding remark, we deduce the differential inequality

$$-(\ln\vartheta)_t \le -\frac{1}{\alpha}\vartheta|\nabla\ln\vartheta|^2 + \frac{m(p-1)\alpha}{1+\alpha m(p-1)}\frac{1}{t} + \frac{m(p-1)\alpha}{1+\alpha m(p-1)}N_2. \tag{4.41}$$

Integrating along the geodesic ζ, we express

$$\begin{aligned} \ln\frac{\vartheta(y_1, s_1)}{\vartheta(y_2, s_2)} &= -\int_{s_1}^{s_2} \frac{d}{dt}\ln\vartheta(\zeta(t), t)\,dt \\ &= -\int_{s_1}^{s_2} \Big[(\ln\vartheta)_t + \langle\dot\zeta(t), \nabla\ln\vartheta\rangle\Big]dt \\ &\le \int_{s_1}^{s_2}\left[-\frac{\vartheta}{\alpha}|\nabla\ln\vartheta|^2 + \frac{m(p-1)\alpha}{1-\alpha m(1-p)}\frac{1}{t} + \frac{-m(1-p)\alpha}{1+\alpha m(p-1)}N_2 - \langle\nabla\ln\vartheta, \dot\zeta(t)\rangle\right]dt. \end{aligned}$$

Utilizing the inequality $-\tilde{a}x^2 - \tilde{b}x \le \frac{\tilde{b}^2}{4\tilde{a}}$ for positive $\tilde{a}$, the integrand is bounded above by

$$\frac{\alpha}{4\tilde{k}}|\dot\zeta(t)|^2 + \frac{-m(1-p)\alpha}{1-\alpha m(1-p)}\frac{1}{t} - \frac{m(1-p)\alpha}{1-\alpha m(1-p)}N_2.$$

Consequently,

$$\ln \frac{\vartheta(y_1, s_1)}{\vartheta(y_2, s_2)} \leq \frac{\alpha}{4\tilde{k}} \int_{s_1}^{s_2} |\dot{\zeta}(t)|^2 dt + (s_2 - s_1) \frac{-m(1-p)\alpha}{1-\alpha m(1-p)} N_2 - \frac{m(1-p)\alpha}{1-\alpha m(1-p)} \ln \frac{s_2}{s_1}.$$

Exponentiating both sides, we obtain the Harnack-type inequality

$$\vartheta(y_1, s_1) \leq \vartheta(y_2, s_2) \left(\frac{s_2}{s_1} \right)^{\frac{m(p-1)\alpha}{1+\alpha m(p-1)}} \exp \left\{ \frac{\alpha}{4\tilde{k}} \int_{s_1}^{s_2} |\dot{\zeta}(t)|^2 dt + (s_2 - s_1) \frac{-m(1-p)\alpha}{1-\alpha m(1-p)} N_2 \right\}.$$

This concludes the proof. □

5. Conclusions

In this paper, we established new Li–Yau-type gradient estimates for positive solutions of weighted nonlinear diffusion equations with a potential term on weighted Riemannian manifolds evolving under geometric flows. Under suitable assumptions on the weighted Bakry–Émery curvature and the evolving tensor field, we derived localized space-time gradient bounds and obtained a corresponding Harnack-type inequality.

Our results extend several existing gradient estimates by incorporating weighted geometry, nonlinear diffusion, and evolving metrics within a unified framework. Since the setting includes important geometric flows such as the Ricci, Yamabe, Ricci–Bourguignon, and extended Ricci flows, the obtained estimates may prove useful in the study of qualitative properties of nonlinear diffusion equations. Future work may focus on more general nonlinear operators and weaker geometric assumptions.

Author contributions

Majid Ali Chaudhary: conceptualization, methodology, investigation, writing—original draft, writing—review and editing; Foued Aloui: conceptualization, methodology, investigation, writing—original draft, writing—review and editing; Maged Z. Youssef: conceptualization, methodology, investigation, writing—original draft, writing—review and editing; Mohammad Nazrul Islam Khan: conceptualization, methodology, investigation, writing—original draft, writing—review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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