Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(6): 15952–15989.
DOI: 10.3934/math.2026657
Received: 29 January 2026
Revised: 20 May 2026
Accepted: 26 May 2026
Published: 05 June 2026

Research article

Solvability and stability of mean-field stochastic differential equations driven by time-changed Lévy noise

Mahmoud Abouagwa[1,2,*] **and Maher Ibrahim Tawdrous**[3,4]

[1] College of Business, City University Ajman, Ajman 18484, United Arab Emirates
[2] Department of Mathematical Statistics, Faculty of Graduate Studies for Statistical Research, Cairo University, Giza 12613, Egypt
[3] College of Humanities, City University Ajman, Ajman 18484, United Arab Emirates
[4] Faculty of Education, Suez Canal University, Ismailia 41522, Egypt

* **Correspondence:** Email: mahmoud.aboagwa@cu.edu.eg, m.abouagwa@cu.ac.ae.

Abstract: In this article, we focused on a class of mean-field stochastic differential equations driven by time-changed Lévy noise. We first discussed the existence and uniqueness of solutions under the non-Lipschitz case with the Lipschitz condition as the special case by adopting the Carathéodory approximation. To prove our results, we established a new time-changed retarded integral inequality, which is easy to apply in practice and can be considered as a more general tool in some situations. Then, the classical Itô formula was extended to that for mean-field stochastic differential equations driven by time-changed Lévy noise. As an application of Itô's formula, we showed that the trivial solution is p-th moment asymptotically stable, stable in probability, asymptotically stable in probability, and globally asymptotically stable in probability based on the Lyapunov function. Finally, an example was presented to validate the produced results.

Keywords: mean-field stochastic differential equations; stability; time-changed Lévy noise; Itô's formula
Mathematics Subject Classification: 60B10, 60H10, 35Q83, 93E15

---

1. Introduction

In light of the innovative research of Kac [13], Mckean [26] considered the nonlinear Fokker-Plank equations using mean-field stochastic differential equations (MFSDEs), also named Mckean-Vlasov or distribution dependent SDEs in the literature, in which the drift and diffusion coefficients depend not only on the state of the unknown process but also on its probability law (distribution). These equations arise as the limiting behavior of a representative particle from a mean-field interacting particle system as the number of particles tends to infinity. For instance, we utilize the N-particles model described by SDEs

$$dY^{i,N}(t) = f(t, Y^{i,N}(t), \mathcal{L}^N(Y^{(N)}(t)))dt + g(t, Y^{i,N}(t), \mathcal{L}^N(Y^{(N)}(t)))dW_t^i, \tag{1.1}$$

where $\mathcal{L}^N(Y^{(N)}(t)) := \frac{1}{N}\sum_{\ell=1}^{N} \delta_{Y^{\ell,N}(t)}$ refers to the empirical measure of the system at time t. The mean-field limit of (1.1) can be expected as

$$dY(t) = f(t, Y(t), \mathscr{L}(Y(t)))dt + g(t, Y(t), \mathscr{L}(Y(t)))dW_t$$

as N tends to infinity and $\mathcal{L}^N(Y^{(N)}(t))$ converges weakly to $\mathscr{L}(Y(t))$ with probability 1. Hence, the coefficients depend not only on the solution pointwisely but also on its time marginal law. This was named “propagation of chaos” by Kac [13].

Due to the investigation of complex networked systems [14], mean-field games [1,2,47], stochastic control [4], and statistical physics [15], there are many papers devoted to the study of mean-field systems and the relevant theories have been well-developed (see, for example, [34–36]). So far, researchers have shown an increased interest to the study of solvability and stability of MFSDEs. For example, Bahlali et al. [4] studied the existence and uniqueness for solutions of MFSDEs driven by Brownian motion under an Osgood-type condition. Also, they studied the stability with respect to initial data, coefficients, and driving processes. Qiao and Gong [37] established the existence, uniqueness, mean-square asymptotic stability, and almost surely asymptotic stability for solutions of non-Lipschitz multivalued MFSDEs. Hammersley et al. [9] established the existence of weak solutions to mean-field SDEs under Lyapunov-type conditions. Huang [11] introduced the nonlinear Fokker-Planck equations for probability measures on path space and path-distribution dependent SDEs. Mezerdi [29] proposed the stability of mean-field SDEs under a non-Lipschitz condition. Mishura and Veretennikov [30] proved new weak and strong existence and uniqueness results for multi-dimensional mean-field SDEs driven by Brownian motion under relaxed regularity conditions. Ren and Wang [38] studied the path independence of Girsanov transformations of mean-field SDEs driven by Brownian motion. Shen et al. [41] studied the stochastic averaging principle and stability for multi-valued Mckean-Vlasov SDEs with jumps.

In recent years, the theory of time-changed semimartingales has received increased attention and became a major area of interest within the field of stochastic analysis due to its wide range of applications in modeling anomalous diffusions appearing in physics, finance, cell biology, and hydrology [22, 42]. It arises as the scaling limit of continuous time random walks [27, 28]. Since 2011, the stochastic integrals with respect to time-changed semimartingales, the time-changed Itô formula, and the duality theorem between non-time-changed stochastic differential equations (SDEs), and the associated time-changed SDEs have been investigated by Kobayashi [16]. Since then, time-changed SDEs have been used to model some real-world phenomena such as Black-Scholes models. Particularly, the stocks that are not actively traded, where their prices stay constant for some period of time can be described by time-changed Brownian motion but not by the standard Brownian motion [33]. Through the past decade, considerable research has grown up around the theory of time-changed SDEs, and we would like to mention a few here. Wu [44] studied the exponential sample-path stability, p-th moment asymptotic stability, and p-th moment exponential stability for SDEs driven by time-changed Brownian motion. Nane and Ni [31] established the stabilities in both probability and moment sense for SDEs driven by time-changed Lévy noise. For more work on SDEs driven by time-changed Brownian motion or time-changed Lévy noise, we refer the reader to [20, 32, 45, 46, 49].

Although extensive research has been carried out on the classical MFSDEs, little is known on time-changed MFSDEs. For example, Shen et al. [40] established the existence and uniqueness of strong solutions for MFSDEs driven by the time-changed Brownian motion under the one-sided Lipschitz condition. They also gave sufficient conditions for the solution to be stochastically stable, stochastically asymptotically stable, and globally stochastically asymptotically stable using the Lyapunov function. Li et al. [21] considered the existence and stability with respect to the initial data and coefficients for impulsive and non-impulsive MFSDEs driven by time-changed Brownian motion under the Lipschitz condition by means of the fixed point theorem. Wen et al. [43] provided the strong convergence on the Euler-Maruyama method for non-autonomous MFSDEs driven by time-changed Brownian motion. Finally, Li and Ren [19] considered the practical stability with respect to part of the variables for MFSDEs driven by time-changed Brownian motion.

On the other hand, in view of rapid changes in perturbation theory, the continuous sample paths of time-changed Brownian motion make it difficult to describe stochastic influences with possible jumps in financial economics, mathematical finance, stochastic filtering, and stochastic control, etc. Consequently, many real systems are very complex and may have discontinuous sample paths, which means that MFSDEs perturbed with time-changed Brownian motion fail to describe them. It is noted that MFSDEs perturbed with the time-changed Lévy noise are quite convenient to cope with such discontinuous systems [8]. Further, Lipschitz or one-sided Lipschitz assumptions are always considered as the assumptions to investigate the existence, uniqueness, and stability for solutions of MFSDEs driven by time-changed Brownian motion (see [19, 21, 40]). However, these assumptions are no longer suitable to deal with various situations in the real world and many practical models of SDEs need some weaker conditions. For instance, the one-dimensional semi-linear SDEs with Markovian switching,

$$dY(t) = \tilde{f}(r(t))Y(t)dt + \tilde{g}(r(t))\tilde{\hbar}(|Y(t)|)dW_t,$$

where W_t represents Brownian motion, $r(t)$ is a continuous-time Markov chain, and $\tilde{\hbar} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, is defined in the following manner:

$$\tilde{\hbar}(v) = \begin{cases} v\sqrt{-\ln(v)}, & 0 \le v \le e^{-1}, \\ \delta \log(\delta^{-1}) + \acute{\kappa}_2(\delta-)(v-\delta)e^{-1} + \frac{1}{2}(v - e^{-1}), & v > e^{-1}. \end{cases}$$

Such models arise in various branches of science, engineering, industry, and finance [7, 10, 18]. Then, it is necessary to consider some weaker and more generalized conditions, such as the so-called non-Lipschitz condition [25, 29, 37].

Based on the above discussion, we will make the first attempt to consider the following class of non-Lipschitz MFSDEs driven by time-changed Lévy noise:

$$\begin{aligned} dY(t) &= f(t, E_t, Y(t-), \mathscr{L}(Y(t-)))dE_t + g(t, E_t, Y(t-), \mathscr{L}(Y(t-)))dW_{E_t} \\ &\quad + \int_{|z|<c} \hbar(t, E_t, Y(t-), \mathscr{L}(Y(t-)), z)\tilde{N}(dE_t, dz), \quad t \in J = [0, T], \end{aligned} \tag{1.2}$$

on a complete probability space $(\Omega, \mathfrak{I}, \{\mathfrak{I}_t\}_{t\ge0}, \mathbb{P})$ with a right continuous and increasing filtration $\{\mathfrak{I}_t\}_{t\ge0}$, where $\mathfrak{I}_0$ contains all $\mathbb{P}$-null sets. The coefficients $f : J \times \mathbb{R}^+ \times \mathbb{R}^n \times \mathscr{M}(\mathbb{R}^n) \longmapsto \mathbb{R}^n$,

$g : J \times \mathbb{R}^+ \times \mathbb{R}^n \times \mathscr{M}(\mathbb{R}^n) \longmapsto \mathbb{R}^{n\times m}$, and $\hbar : J \times \mathbb{R}^+ \times \mathbb{R}^n \times \mathscr{M}(\mathbb{R}^n) \times (\mathbb{R}^n\backslash\{0\}) \longmapsto \mathbb{R}^n$ are Borel measurable functions, $\mathscr{L}(Y(t))$ stands for the probability law of the random variable $Y(t)$, and $\mathscr{M}(\mathbb{R}^n)$ denotes the space of all probability measures on the Borel measurable space $\mathbb{R}^n$ equiped with the weak topology. The initial value $Y(0) = Y_0 \in \mathbb{R}^n$ is an $\mathfrak{I}_0$-measurable random variable satisfying $\mathbb{E}|Y_0|^2 < \infty$. The time-change E_t is the inverse of a β-stable subordinator and the composition $(W_{E_t})_{t\in J}$ is called a time-changed Brownian motion, which is understood as a subdiffusion (see [27, 42]). $\tilde{N}(dE_t, dz)$ is the compensated Poisson random measure. The rigorous mathematical definitions are postponed to Section 2.

The main innovative points of this article are listed below:

- Our model MFSDEs (1.2) driven by time-changed Lévy noise are new to the Literature. Further, we introduce a new time-changed retarded integral inequality, which is easy to apply in practice and can be considered as a new contribution to the literature.
- The existence results in [21,40] are obtained under Lipschitz and one-sided Lipschitz conditions. However, our existence and uniqueness results for Eq (1.2) are considered under a non-Lipschitz condition by using Carathéodory approximations with more complex computations, generalizing those of [21,40].
- We extend the generalized Itô formula from classical SDEs to non-Lipschitz MFSDEs driven by time-changed Lévy noise, generalizing and containing that of [40] as a special case.
- Since the stability analysis is of great interest in SDE theory, we present sufficient conditions for the solution of Eq (1.2) to be p-th moment asymptotically stable, stable in probability, asymptotically stable in probability, and globally asymptotically stable in probability.
- Moreover, letting $\hbar \equiv 0$ in Eq (1.2), the p-th moment asymptotic stability result in Theorem 5.1 is still new for MFSDEs driven by time-changed Brownian motion and considered by [21,40].
- Finally, an example is given to justify our Conditions 3.1–3.3 and stability results.

The rest of this article is organized as follows. In Section 2, we recall some necessary notations and preliminaries for this article. In Section 3, we devote ourselves to proving the existence and uniqueness theorem of solutions for our model (1.2) based on a new integral inequality given in Lemma 3.1 and Conditions 3.1–3.3. In Section 4, we state and prove the generalized Itô formula for non-Lipschitz MFSDEs driven by time-changed Lévy noise. We present the stability results followed by an example to show the utility of the acquired theoretical results in Section 5.

2. Preliminaries

In this section, we briefly give some preliminaries that will be used in the following sections. Let $(\Omega, \mathfrak{I}, \{\mathfrak{I}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathfrak{I}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous, while $\mathfrak{I}_0$ contains all $\mathbb{P}$-null sets). Assume $W = (W_t)_{t\geq 0}$ represents an m-dimensional $\{\mathfrak{I}_t\}_{t\geq 0}$-Brownian motion and $\{\mathcal{D}(t), t \geq 0\}$ is a right continuous with a left-limit-increasing $\{\mathfrak{I}_t\}_{t\geq 0}$-Lévy process with a Lévy symbol $1 < \beta < 2$, called a β-stable subordinator, starting from $\mathcal{D}(0) = 0$ with the Laplace transform $\mathbb{E}(e^{-\lambda\mathcal{D}(\tau)}) = e^{-\tau\phi(\lambda)}$, $\lambda > 0$, where the Laplace exponent $\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda y})\alpha(dy)$ is associated with a given σ-finite measure α on $(0, \infty)$ such that $\int_0^\infty (1 \wedge y)\alpha(dy) < \infty$. Suppose that α is infinite, i.e., $\alpha(0, \infty) = \infty$. The stochastic time-change $E_t = \inf\{\tau > 0 : \mathcal{D}(\tau) > t\}$ is independent of W and can be defined as the inverse subordinator of $\mathcal{D}(t)$.

Moreover, E_t is continuous and non-decreasing, however, it is not Markovian and can be considered as the first hitting time process. Suppose that W_t is independent of $\mathcal{D}(t)$. Without loss of generality, we clarify that the filtration $\{\Im_t\}_{t\geq 0}$ is determined by

$$\Im_t = \bigcap_{\tau > t} \Big\{ \sigma_1[W_s : 0 \leq s \leq \tau] \vee \sigma_2[E_s : s \geq 0] \Big\},$$

where the notation $\sigma_1 \vee \sigma_2$ refers to the σ-algebra generated by the union of the two σ-algebras σ_1 and σ_2.

The composition $W \circ E = (W_{E_t})_{t\geq 0}$, called a time-changed Brownian motion [23], is a square integrable martingale with respect to the natural filtration $\{\mathcal{G}_t\}_{t\geq 0}$, where $\mathcal{G}_t := \Im_{E_t}$, $t \geq 0$, for the process $\{E_t\}$. Let N be the $\Im_t$-adapted Poisson random measure associated with a Lévy measure ν defined on $\mathbb{R}^n \backslash \{0\}$ and satisfying $\int_{\mathbb{R}^n \backslash \{0\}} (|y|^2 \wedge 1) \nu(dy) < \infty$ (see, e.g., [3]). $\tilde{N}(dt, dy) = N(dt, dy) - \nu(dy)dt$ is the compensated $\{\Im_t\}_{t\geq 0}$-martingale measure, where both N and $\tilde{N}$ are independent of the Brownian motion W. In Eq (1.2), dE_t describes the drift of the time-changed stochastic process and $\tilde{N}(dE_t, dz)$ is the compensated Poisson random measure that captures the jumps in addition to the time-changed Brownian motion. We define the product probability space by $(\Omega, \Im, \mathbb{P}) := (\Omega_W \times \Omega_{\mathcal{D}}, \Im^W \times \Im^{\mathcal{D}}, \mathbb{P}_W \times \mathbb{P}_{\mathcal{D}})$. Let $\mathbb{E}$ denotes the expectation under the probability measure $\mathbb{P}$, $\mathbb{E}_{\mathcal{D}}$ denotes the expectation under the probability measure $\mathbb{P}_{\mathcal{D}}$, and $\mathbb{E}_W$ refers to the expectation under the probability measure $\mathbb{P}_W$. It is shown that $\mathbb{E}(.) = \mathbb{E}_{\mathcal{D}}\mathbb{E}_W(.) = \mathbb{E}_W\mathbb{E}_{\mathcal{D}}(.)$. Assume that $f(t, E_t, 0, \delta_0) = g(t, E_t, 0, \delta_0) = \hbar(t, E_t, 0, \delta_0, z) = 0$ for all $0 \leq t, E_t < \infty$, where δ_0 is the Dirac measure at 0.

The following lemma is borrowed from [16].

Lemma 2.1. *Suppose $\alpha(t)$ and $\lambda(t)$ are integrable and $\mathcal{G}_t$-measurable. Then, with probability one,*

$$\int_0^t \alpha(\tau) dE_\tau + \int_0^t \lambda(\tau) dW_\tau = \int_0^{E_t} \alpha(\mathcal{D}(\tau-)) d\tau + \int_0^{E_t} \lambda(\mathcal{D}(\tau-)) dW_\tau, \quad t \geq 0,$$

where the $\mathcal{G}_t$-measurable time-change E_t is the general inverse of β-stable subordinator $\mathcal{D}(t)$.

For any $p \geq 2$, consider the following subspace of $\mathscr{M}(\mathbb{R}^n)$:

$$\mathscr{M}_p(\mathbb{R}^n) := \{\alpha \in \mathscr{M}(\mathbb{R}^n) : \alpha(|.|^p) := \int_{\mathbb{R}^n} |y|^p \alpha(dy) < \infty\},$$

which is a polish space under the L^p-Wasserstein distance

$$\rho_p(\alpha_1, \alpha_2) := \inf_{\pi \in \mathcal{L}(\alpha_1, \alpha_2)} \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} |y_1 - y_2|^p \pi(dy_1, dy_2) \right)^{\frac{1}{p}}, \quad \alpha_1, \alpha_2 \in \mathscr{M}_p(\mathbb{R}^n),$$

where $\mathcal{L}(\alpha_1, \alpha_2)$ is the set of probability measures on $\mathbb{R}^{n\times n}$ whose marginals are, respectively, α_1, α_2. In the case of $\alpha_1 = \mathscr{L}(Y)$, $\alpha_2 = \mathscr{L}(Z)$, the corresponding distribution of random variables Y and Z, respectively, is

$$\rho_p(\alpha_1, \alpha_2)^p \leq \mathbb{E}|Y - Z|^p.$$

Throughout the article, the letter $C > 0$ will refer to a constant whose value differs from one place to another. The dependence of a constant on the parameters will be assigned if it is necessary. Let

$C(J;\mathbb{R}^n)$ be the collection of all $\mathbb{R}^n$-valued continuous functions on J carrying the supremum norm and $\mathbb{S}^p(\Omega;C(J;\mathbb{R}^n))$ be the totality of $C(J;\mathbb{R}^n)$-valued random variables Y with $\mathbb{E}(\sup_{0\le t\le T}|Y(t)|^p)<\infty$. Hence, $\mathbb{S}^p(\Omega;C(J;\mathbb{R}^n))$ is a Banach space under the norm

$$||Y||_{\mathbb{S}^p} := (\mathbb{E}(\sup_{0\le t\le T}|Y(t)|^p))^{\frac{1}{p}}.$$

The rest of this section is devoted to recalling the concept of the L-derivative for functions on $\mathscr{M}_2(\mathbb{R}^n)$. For any $\alpha\in\mathscr{M}_2(\mathbb{R}^n)$, set

$$\mathcal{L}^2(\mathbb{R}^n,\alpha;\mathbb{R}^n) := \left\{\varphi:\mathbb{R}^n\longmapsto\mathbb{R}^n;\varphi\ \textit{is measurable with}\ \alpha(|\varphi|^2) := \int_{\mathbb{R}^n}|\varphi(y)|^2\alpha(dy)<\infty\right\}.$$

Let $|.|$ and $||.||$ refer to the vector and matrix norms on $\mathbb{R}^n$, respectively. Let $\mathcal{I}$ be the identity map on $\mathbb{R}^n$. For $\alpha\in\mathscr{M}_2(\mathbb{R}^n)$ and $\varphi\in\mathcal{L}^2(\mathbb{R}^n,\alpha;\mathbb{R}^n)$, $\alpha(\varphi) := \int_{\mathbb{R}^n}\varphi(y)\alpha(dy)$ and $\alpha\circ(\mathcal{I}+\varphi)^{-1}\in\mathscr{M}_2(\mathbb{R}^n)$. Let $\mathbb{C}^{1,1,2}(J\times\mathbb{R}^+\times\mathbb{R}^n)$ refer to the set of all functions $F(t_1,t_2,y):J\times\mathbb{R}^+\times\mathbb{R}^n\longrightarrow\mathbb{R}$, which are continuously once differentiable in t_1 and t_2 as well as continuously twice differentiable in y.

The following definitions are taken from [38, 40].

Definition 2.1. *(1) The function* $F:\mathscr{M}_2(\mathbb{R}^n)\longrightarrow\mathbb{R}$ *is called L-differentiable at* $\alpha\in\mathscr{M}_2(\mathbb{R}^n)$*, if the functional*

$$\mathcal{L}^2(\mathbb{R}^n,\alpha;\mathbb{R}^n)\ni\varphi\longrightarrow F(\alpha\circ(\mathcal{I}+\varphi)^{-1})$$

is Fréchet differentiable at $0\in\mathcal{L}^2(\mathbb{R}^n,\alpha;\mathbb{R}^n)$*; i.e., there exists a unique* $\zeta\in\mathcal{L}^2(\mathbb{R}^n,\alpha;\mathbb{R}^n)$ *such that*

$$\lim_{\alpha(|\varphi|^2)\longrightarrow 0}\frac{F(\alpha\circ(\mathcal{I}+\varphi)^{-1})-F(\alpha)-\alpha(\langle\zeta,\varphi\rangle)}{\sqrt{\alpha(|\varphi|^2)}}=0,$$

where $\langle.,.\rangle$ *is the inner product on* $\mathbb{R}^n$ *and* $\alpha(\varphi):=\int_{\mathbb{R}^n}\varphi(y)\alpha(dy)$*. Let* $\partial_\alpha F(\alpha):=\zeta$ *be the L-derivative of F at* α*.*

(2) A function $F:\mathscr{M}_2(\mathbb{R}^n)\longrightarrow\mathbb{R}$ *is said to be L-differentiable on* $\mathscr{M}_2(\mathbb{R}^n)$ *if the L-derivative* $\partial_\alpha F(\alpha)$ *exists for all* $\alpha\in\mathscr{M}_2(\mathbb{R}^n)$*.*

Definition 2.2. *A function* $F:\mathscr{M}_2(\mathbb{R}^n)\longmapsto\mathbb{R}$ *is said to be in the space* $\mathbb{C}^2(\mathscr{M}_2(\mathbb{R}^n))$*, if F is L-differentiable on* $\mathscr{M}_2(\mathbb{R}^n)$*, and its derivative* $\partial_\alpha F(x):\mathscr{M}_2(\mathbb{R}^n)\times\mathbb{R}^n\longmapsto\mathbb{R}^n$ *is continuous at every* (α,x) *for* $x\in supp(\alpha)$*.* $\partial_\alpha F(x)$ *is differentiable in x and* $\partial_x\partial_\alpha F:\mathscr{M}_2(\mathbb{R}^n)\times\mathbb{R}^n\longmapsto\mathbb{R}^{n\times n}$ *is continuous at all* (α,x) *satisfying* $x\in supp(\alpha)$*. In addition,* $\partial^2_\alpha F:\mathscr{M}_2(\mathbb{R}^n)\times\mathbb{R}^n\times\mathbb{R}^n\longmapsto\mathbb{R}^{n\times n}$ *exists and is continuous.*

Definition 2.3. *Assume that* $F:J\times\mathbb{R}^+\times\mathbb{R}^n\times\mathscr{M}_2(\mathbb{R}^n)\longmapsto\mathbb{R}$*. If* $F(.,.,.,\alpha)$ *is in* $\mathbb{C}^{1,1,2}(J\times\mathbb{R}^+\times\mathbb{R}^n)$ *for any* α*, and* $F(t,E,Y,.)\in\mathbb{C}^2(\mathbb{R}^n)$ *for every* (t,E,Y)*, and all the partial derivatives*

$$\partial_{t_1}F(t,E,Y,\alpha),\ \ \partial_{t_2}F(t,E,Y,\alpha),\ \ \partial_y F(t,E,Y,\alpha),\ \ \partial^2_y F(t,E,Y,\alpha),$$
$$\partial_\alpha F(t,E,Y,\alpha)(x),\ \ \partial_x\partial_\alpha F(t,E,Y,\alpha)(x),\ \ \partial^2_\alpha F(t,E,Y,\alpha)(x,\acute{x})$$

are continuous with respect to (t,E,Y,α)*,* (t,E,Y,α,x)*, or* $(t,E,Y,\alpha,x,\acute{x})$*, and F is said to be in the set* $\mathbb{C}^{1,1,2,(2)}(J\times\mathbb{R}^+\times\mathbb{R}^n\times\mathscr{M}_2(\mathbb{R}^n))$*.*

Definition 2.4. *A function* $F:J\times\mathbb{R}^+\times\mathbb{R}^n\times\mathscr{M}_2(\mathbb{R}^n)\longmapsto\mathbb{R}$ *is said to be in* $\wp_b(J\times\mathbb{R}^+\times\mathbb{R}^n\times\mathscr{M}_2(\mathbb{R}^n))$*, if* $F\in\mathbb{C}^{1,1,2,(2)}(J\times\mathbb{R}^+\times\mathbb{R}^n\times\mathscr{M}_2(\mathbb{R}^n))$ *and all its derivatives are uniformly bounded on* $J\times\mathbb{R}^+\times\mathbb{R}^n\times\mathscr{M}_2(\mathbb{R}^n)$*. In addition, if* $F\in\wp_b(J\times\mathbb{R}^+\times\mathbb{R}^n\times\mathscr{M}_2(\mathbb{R}^n))$ *and* $F\ge 0$*, we say that* $F\in\wp_{b,+}(J\times\mathbb{R}^+\times\mathbb{R}^n\times\mathscr{M}_2(\mathbb{R}^n))$*.*

3. Existence and uniqueness

This section is concerned with establishing the existence and uniqueness of strong solutions for Eq (1.2). In order to derive the results of this section, assume that the coefficients $f(t_1, t_2, y, \alpha)$, $g(t_1, t_2, y, \alpha)$, and $\hbar(t_1, t_2, y, \alpha, z)$ satisfy the following conditions.

Condition 3.1. There exists a positive constant $L > 0$ such that for any $t_1 \in J$, $t_2 \in \mathbb{R}^+$ and $y_1, y_2 \in \mathbb{R}^n$, $\alpha_1, \alpha_2 \in \mathscr{M}_2(\mathbb{R}^n)$,

$$
\begin{aligned}
&|f(t_1, t_2, y_1, \alpha_1) - f(t_1, t_2, y_2, \alpha_2)|^2 + \|g(t_1, t_2, y_1, \alpha_1) - g(t_1, t_2, y_2, \alpha_2)\|^2 \\
&+ \int_{|z|<c} |\hbar(t_1, t_2, y_1, \alpha_1, z) - \hbar(t_1, t_2, y_2, \alpha_2, z)|^2 \nu(dz) \le L\Gamma(|y_1 - y_2|^2 + \rho_2(\alpha_1, \alpha_2)^2),
\end{aligned}
$$

where $\Gamma(.) : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous, non-decreasing, and concave function such that $\Gamma(0) = 0$, $\Gamma(y) > 0$ for every $y > 0$ and $\int_{0^+} \frac{1}{\Gamma(y)} dy = +\infty$.

Condition 3.2. There exists a positive constant $L_1 > 0$ such that for any $t_1 \in J$, $t_2 \in \mathbb{R}^+$, $y \in \mathbb{R}^n$, and $\alpha \in \mathscr{M}_2(\mathbb{R}^n)$,

$$
\begin{aligned}
&|f(t_1, t_2, y, \alpha)|^2 + \|g(t_1, t_2, y, \alpha)\|^2 \\
&\qquad + \int_{|z|<c} |\hbar(t_1, t_2, y, \alpha, z)|^2 \nu(dz) \le L_1(1 + |y|^2 + \rho_2(\alpha, \delta_0)^2).
\end{aligned}
$$

Condition 3.3 (Technical condition). For any right-continuous $\mathcal{G}_t$-adapted stochastic process $Y(t)$ with left limits, we have

$$
f(t, E_t, Y(t), \mathscr{L}(Y(t))), g(t, E_t, Y(t), \mathscr{L}(Y(t))), \hbar(t, E_t, Y(t), \mathscr{L}(Y(t)), z) \in \mathbb{L}(\mathcal{G}_t),
$$

where $\mathbb{L}(\mathcal{G}_t)$ denotes the totality of càglàd (i.e., sample paths that are left continuous with right limits) and $\mathcal{G}_t$-adapted processes.

Remark 3.1. To demonstrate our results generality, we give some concrete examples of the function $\Gamma(.)$. Let $\acute{L} > 0$ and let $0 < \epsilon < 1$ be sufficiently small. Define

$$
\Gamma_1(y) = \acute{L}y, \qquad y \ge 0,
$$

$$
\Gamma_2(v) = \begin{cases} y \log(y^{-1}), & 0 \le y \le \epsilon, \\ \epsilon \log(\epsilon^{-1}) + \acute{\Gamma}_2(\epsilon-)(y - \epsilon), & y > \epsilon, \end{cases}
$$

$$
\Gamma_3(y) = \begin{cases} y \log(y^{-1}) \log(\log(y^{-1})), & 0 \le y \le \epsilon, \\ \epsilon \log(\epsilon^{-1}) \log(\log(\epsilon^{-1})) + \acute{\Gamma}_3(\epsilon-)(y - \epsilon), & y > \epsilon, \end{cases}
$$

where $\acute{\Gamma}_2$ and $\acute{\Gamma}_3$ denote the derivatives of functions Γ_2 and Γ_3, respectively. They are all concave and non-decreasing functions satisfying $\int_{0+} \frac{1}{\Gamma_i(y)} dy = +\infty$, $i = 1, 2, 3$.

Now, we introduce the following time-changed retarded integral inequality, which is essential in this work.

Lemma 3.1. Assume $\mathscr{D}(t)$ is a β-stable subordinator corresponding to the inverse stable subordinator E_t. Let $T > 0$ and $\Gamma : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be concave continuous and nondecreasing functions such that $\Gamma(t) > 0$

for all $t > 0$. The functions $a, d : \Omega \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are assumed to be $\mathcal{G}_t$-measurable and integrable with respect to E_t. Let $r(t) > 0$ be a monotone and nondecreasing function. If

$$a(t) \leq r(t) + \int_0^t d(s)\Gamma(a(s))dE_s, \quad t \geq 0, \tag{3.1}$$

then

$$a(t) \leq G^{-1}\Big(G(r(t)) + \int_0^t d(s)dE_s\Big)$$

holds for all $t \geq 0$ such that

$$G(r(t)) + \int_0^t d(s)dE_s \in Dom(G^{-1}),$$

where

$$G(\iota) = \int_0^\iota \frac{ds}{\Gamma(s)} \quad, \iota > 0,$$

and G^{-1} is the inverse function of G. In particular, if, moreover, $a(0) = 0$ and $\int_{0+} \frac{ds}{\Gamma(s)} = \infty$, then $a(t) = 0$, for all $t \in J$.

Proof. Let

$$e(t) := r(t) + \int_0^t d(s)\Gamma(a(s))dE_s, \quad t \geq 0. \tag{3.2}$$

Since $a(s)$ and $d(s)$ are positive and $r(t)$ is non-decreasing, the function $e(t)$ is non-decreasing. Moreover, from Eqs (3.1) and (3.2),

$$a(t) \leq e(t) \quad or \quad \Gamma(a(t)) \leq \Gamma(e(t)), \quad t \geq 0,$$

which implies that

$$e(t) \leq r(t) + \int_0^t d(s)\Gamma(e(s))dE_s, \quad t \geq 0.$$

Applying Lemma 2.1, we have

$$e(t) \leq r(t) + \int_0^{E_t} d(\mathcal{D}(s-))\Gamma(e(\mathcal{D}(s-)))ds, \quad t \geq 0. \tag{3.3}$$

Actually, for $0 \leq t \leq E_t$, $\mathcal{D}(t-)$ is defined as

$$\mathcal{D}(t-) = \inf\{s : s \in [0, T], E_s \geq t\} \wedge T,$$

which means

$$E_{\mathcal{D}(t-)} = t \quad and \quad \mathcal{D}(E_t-) \leq t. \tag{3.4}$$

Let $\tau \in [0,\infty)$, and it holds from (3.3) and (3.4) that

$$\begin{aligned} e(\mathcal{D}(\tau-)) &\leq r(\mathcal{D}(\tau-)) + \int_0^{E_{\mathcal{D}(\tau-)}} d(\mathcal{D}(s-))\Gamma(e(\mathcal{D}(s-)))ds \\ &= r(\mathcal{D}(\tau-)) + \int_0^{\tau} d(\mathcal{D}(s-))\Gamma(e(\mathcal{D}(s-)))ds. \end{aligned}$$

We apply a generalization of Bihari's inequality [5] and Lemma 3.6 in [25] path by path to yield

$$a(\mathcal{D}(\tau-)) \leq e(\mathcal{D}(\tau-)) \leq G^{-1}\Big(G(r(\mathcal{D}(\tau-))) + \int_0^{\tau} d(\mathcal{D}(s-))ds\Big).$$

For any $t \geq 0$, we can see that there is $\tau \in \mathbb{R}^+$ such that $\mathcal{D}(\tau-) = t$ provided that E is strictly increasing in some neighborhood of t. As $r(t)$ is non-decreasing, employing (3.4) and Lemma 2.1, we gain

$$\begin{aligned} a(t) &\leq e(t) \leq G^{-1}\Big(G(r(t)) + \int_0^{E_t} d(\mathcal{D}(s-))ds\Big) \\ &:= G^{-1}\Big(G(r(t)) + \int_0^{t} d(s)dE_s\Big). \end{aligned}$$

If E is a constant τ in some neighborhood $U(t;\epsilon)$ of t, then $t-\epsilon = \mathcal{D}(\tau-)$. Employing (3.4) and since $r(t)$ is non-decreasing, we have

$$a(t-\epsilon) \leq e(t-\epsilon) \leq G^{-1}\Big(G(r(t-\epsilon)) + \int_0^{E_{t-\epsilon}} d(\mathcal{D}(s-))ds\Big).$$

Then, for any $t \geq 0$, we gain by Lemma 2.1

$$a(t) \leq e(t) \leq G^{-1}\Big(G(r(t)) + \int_0^{t} d(s)dE_s\Big).$$

Hence, the proof is complete. □

Now, we construct the Carathéodory approximation as follows. For any integer $k \geq 1$, define $Y^k(t) = Y_0$ for all $-1 \leq t \leq 0$ and

$$\begin{aligned} Y^k(t) &= Y_0 + \int_0^t f(s, E_s, Y^k(s-\frac{1}{k}), \mathscr{L}(Y^k(s-\frac{1}{k})))dE_s \\ &\quad + \int_0^t g(s, E_s, Y^k(s-\frac{1}{k}), \mathscr{L}(Y^k(s-\frac{1}{k})))dW_{E_s} \\ &\quad + \int_0^t \int_{|z|<c} \hbar(s, E_s, Y^k(s-\frac{1}{k}), \mathscr{L}(Y^k(s-\frac{1}{k})), z)\tilde{N}(dE_s, dz), \quad \forall\, t \geq 0. \end{aligned} \tag{3.5}$$

Note that for $0 \leq t \leq \frac{1}{k}$, $Y^k(t)$ can be computed by

$$Y^k(t) = Y_0 + \int_0^t f(s, E_s, Y_0, \mathscr{L}(Y_0))dE_s$$

$$+\int_0^t g(s,E_s,Y_0,\mathscr{L}(Y_0))dW_{E_s}$$
$$+\int_0^t\int_{|z|<c}\hbar(s,E_s,Y_0,\mathscr{L}(Y_0),z)\tilde{N}(dE_s,dz).$$

Then for $\frac{1}{k}<t<\frac{2}{k}$,

$$\begin{aligned}Y^k(t) &= Y^k(\frac{1}{k})+\int_{\frac{1}{k}}^t f(s,E_s,Y^k(s-\frac{1}{k}),\mathscr{L}(Y^k(s-\frac{1}{k})))dE_s\\ &\quad+\int_{\frac{1}{k}}^t g(s,E_s,Y^k(s-\frac{1}{k}),\mathscr{L}(Y^k(s-\frac{1}{k})))dW_{E_s}\\ &\quad+\int_{\frac{1}{k}}^t\int_{|z|<c}\hbar(s,E_s,Y^k(s-\frac{1}{k}),\mathscr{L}(Y^k(s-\frac{1}{k})),z)\tilde{N}(dE_s,dz)\end{aligned}$$

and so on. In other words, $Y^k(t)$ can be computed step by step on the intervals $[0,\frac{1}{k}],[\frac{1}{k},\frac{2}{k}],....$

Next, we will show that the sequence of stochastic processes $\{Y^k(t)\}_{k\geq1}$ given in (3.5) is uniformly bounded.

Lemma 3.2. *Assume Conditions 3.2 and 3.3 hold. Then, for all* $k\geq1$ *and* $t\in J$, $\{Y^k(t)\}_{k\geq1}$ *is uniformly bounded with the property*

$$\mathbb{E}(\sup_{0\leq s\leq T}|Y^k(s)|^2)\leq\infty.$$

Proof. It follows from Eq (3.5) that

$$\begin{aligned}&\mathbb{E}_W(\sup_{0\leq s\leq t}|Y^k(s)|^2)\\ &\leq 4\mathbb{E}_W|Y_0|^2+4\mathbb{E}_W\Big(\sup_{0\leq s\leq t}|\int_0^s f(v,E_v,Y^k(v-\frac{1}{k}),\mathscr{L}(Y^k(v-\frac{1}{k})))dE_v|^2\Big)\\ &\quad+4\mathbb{E}_W\Big(\sup_{0\leq s\leq t}|\int_0^s g(v,E_v,Y^k(v-\frac{1}{k}),\mathscr{L}(Y^k(v-\frac{1}{k})))dW_{E_v}|^2\Big)\\ &\quad+4\mathbb{E}_W\Big(\sup_{0\leq s\leq t}|\int_0^s\int_{|z|<c}\hbar(v,E_v,Y^k(v-\frac{1}{k}),\mathscr{L}(Y^k(v-\frac{1}{k})),z)\tilde{N}(dE_v,dz)|^2\Big)\\ &:=4\mathbb{E}_W|Y_0|^2+I_1+I_2+I_3.\end{aligned}\tag{3.6}$$

For I_1, by Hölder's inequality and Condition 3.2, we gain

$$\begin{aligned}I_1 &= 4\mathbb{E}_W\Big(\sup_{0\leq s\leq t}|\int_0^s f(v,E_v,Y^k(v-\frac{1}{k}),\mathscr{L}(Y^k(v-\frac{1}{k})))dE_v|^2\Big)\\ &\leq 4E_T\mathbb{E}_W\int_0^t|f(s,E_s,Y^k(s-\frac{1}{k}),\mathscr{L}(Y^k(s-\frac{1}{k})))|^2dE_s\\ &\leq 4E_T\mathbb{E}_W\int_0^t L_1\Big(1+|Y^k(s-\frac{1}{k})|^2+\rho_2(\mathscr{L}(Y^k(s-\frac{1}{k})),\delta_0)^2\Big)dE_s\end{aligned}$$

$$
\begin{aligned}
&\le 4E_T L_1 \int_0^t \Big(1 + 2\mathbb{E}_W |Y^k(s-\frac{1}{k})|^2\Big) dE_s \\
&\le 4E_T L_1 \int_0^t (1 + 2\mathbb{E}_W(\sup_{0\le v\le s} |Y^k(v-)|^2)) dE_s. \qquad (3.7)
\end{aligned}
$$

For I_2, by the Burkholder-Davis-Gundy inequality [12] and Condition 3.2, we obtain

$$
\begin{aligned}
I_2 &= 4\mathbb{E}_W\Big(\sup_{0\le s\le t} \Big|\int_0^s g(v, E_v, Y^k(v-\frac{1}{k}), \mathscr{L}(Y^k(v-\frac{1}{k})))dW_{E_v}\Big|^2\Big) \\
&\le 4b_2 \mathbb{E}_W \int_0^t \|g(s, E_s, Y^k(s-\frac{1}{k}), \mathscr{L}(Y^k(s-\frac{1}{k})))\|^2 dE_s \\
&\le 4b_2 \mathbb{E}_W \int_0^t L_1\Big(1 + |Y^k(s-\frac{1}{k})|^2 + \rho_2(\mathscr{L}(Y^k(s-\frac{1}{k})), \delta_0)^2\Big) dE_s \\
&\le 4b_2 L_1 \int_0^t \Big(1 + 2\mathbb{E}_W |Y^k(s-\frac{1}{k})|^2\Big) dE_s \\
&\le 4b_2 L_1 \int_0^t \Big(1 + 2\mathbb{E}_W(\sup_{0\le v\le s} |Y^k(v-)|^2)\Big) dE_s, \qquad (3.8)
\end{aligned}
$$

where $b_2 > 0$ comes from [12]. For I_3, by Doob's martingale inequality, Itô's isometry (comes from [39]), and Condition 3.2, we gain

$$
\begin{aligned}
I_3 &= 4\mathbb{E}_W\Big(\sup_{0\le s\le t} \Big|\int_0^s \int_{|z|<c} \hbar(v, E_v, Y^k(v-\frac{1}{k}), \mathscr{L}(Y^k(v-\frac{1}{k})), z)\tilde{N}(dE_v, dz)\Big|^2\Big) \\
&\le 16\mathbb{E}_W \Big|\int_0^t \int_{|z|<c} \hbar(s, E_s, Y^k(s-\frac{1}{k}), \mathscr{L}(Y^k(s-\frac{1}{k})), z)\tilde{N}(dE_s, dz)\Big|^2 \\
&\le 16\mathbb{E}_W \int_0^t \int_{|z|<c} |\hbar(s, E_s, Y^k(s-\frac{1}{k}), \mathscr{L}(Y^k(s-\frac{1}{k})), z)|^2 \nu(dz) dE_s \\
&\le 16\mathbb{E}_W \int_0^t L_1\Big(1 + |Y^k(s-\frac{1}{k})|^2 + \rho_2(\mathscr{L}(Y^k(s-\frac{1}{k})), \delta_0)^2\Big) dE_s \\
&\le 16 L_1 \int_0^t \Big(1 + 2\mathbb{E}_W |Y^k(s-\frac{1}{k})|^2\Big) dE_s \\
&\le 16 L_1 \int_0^t \Big(1 + 2\mathbb{E}_W(\sup_{0\le v\le s} |Y^k(v-)|^2)\Big) dE_s. \qquad (3.9)
\end{aligned}
$$

Then, from Eqs (3.6)–(3.9), we gain

$$
\begin{aligned}
1 + 2\mathbb{E}_W(\sup_{0\le s\le t} |Y^k(s)|^2) &\le 1 + 8\mathbb{E}_W|Y_0|^2 + 8L_1(E_T + b_2 + 4) \\
&\quad \times \int_0^t \Big(1 + 2\mathbb{E}_W(\sup_{0\le v\le s} |Y^k(v-)|^2)\Big) dE_s \\
&\le 1 + 8\mathbb{E}_W|Y_0|^2 + C_{E_T} \int_0^t \Big(1 + 2\mathbb{E}_W(\sup_{0\le v\le s} |Y^k(v-)|^2)\Big) dE_s,
\end{aligned}
$$

which, with the help of the time-changed Gronwall inequality [12], gives

$$
\mathbb{E}_W(\sup_{0\le s\le t} |Y^k(s)|^2) \le (1 + 8\mathbb{E}_W|Y_0|^2) e^{C_{E_T} E_T}. \qquad (3.10)
$$

Taking $\mathbb{E}_{\mathcal{D}}$ for both sides of (3.10) and since $\alpha(0,\infty)=\infty$ guarantees that the inverse E of $\mathcal{D}$ has a finite exponential moment, we have

$$\mathbb{E}(\sup_{0\le s\le T}|Y^k(s)|^2) \le (1+8\mathbb{E}|Y_0|^2)\mathbb{E}(e^{C_{E_T}E_T})<\infty,$$

where $C_{E_T}=8L_1(E_T+b_2+4)$. □

Lemma 3.3. *Assume Conditions 3.2 and 3.3 hold. Then*

$$\mathbb{E}_W|Y^k(t)-Y^k(s)|^2\le C_{b_2,E_T}(E_t-E_s),\quad t\ge s,$$

where C_{b_2,E_T} is a positive constant depending on b_2 and E_T.

Proof. By the Eq (3.5), we have

$$\begin{aligned}
&\mathbb{E}_W|Y^k(t)-Y^k(s)|^2\\
&\le 3\mathbb{E}_W\Big|\int_s^t f(v,E_v,Y^k(v-\frac{1}{k}),\mathscr{L}(Y^k(v-\frac{1}{k})))dE_v\Big|^2\\
&\quad+3\mathbb{E}_W\Big|\int_s^t g(v,E_v,Y^k(v-\frac{1}{k}),\mathscr{L}(Y^k(v-\frac{1}{k})))dW_{E_v}\Big|^2\\
&\quad+3\mathbb{E}_W\Big|\int_s^t\int_{|z|<c}\hbar(v,E_v,Y^k(v-\frac{1}{k}),\mathscr{L}(Y^k(v-\frac{1}{k})),z)\tilde{N}(dE_v,dz)\Big|^2\\
&:=\gamma_1+\gamma_2+\gamma_3.
\end{aligned}\tag{3.11}$$

By Hölder's inequality and Condition 3.2, we get

$$\begin{aligned}
\gamma_1&\le 3(E_t-E_s)\mathbb{E}_W\int_s^t|f(v,E_v,Y^k(v-\frac{1}{k}),\mathscr{L}(Y^k(v-\frac{1}{k})))|^2dE_v\\
&\le 3(E_t-E_s)\mathbb{E}_W\int_s^t L_1\Big(1+|Y^k(v-\frac{1}{k})|^2+\rho_2(\mathscr{L}(Y^k(v-\frac{1}{k})),\delta_0)^2\Big)dE_v\\
&\le 3L_1(E_T-E_s)\int_s^t\Big(1+2\mathbb{E}_W|Y^k(v-\frac{1}{k})|^2\Big)dE_v\\
&\le 3L_1(E_T-E_s)\int_s^t(1+2\mathbb{E}_W(\sup_{0\le v_1\le v}|Y^k(v_1-)|^2))dE_v.
\end{aligned}\tag{3.12}$$

For γ_2, by the Burkholder-Davis-Gundy inequality [12] and Condition 3.2, we have

$$\begin{aligned}
\gamma_2&\le 3b_2\mathbb{E}_W\int_s^t\|g(v,E_v,Y^k(v-\frac{1}{k}),\mathscr{L}(Y^k(v-\frac{1}{k})))\|^2dE_v\\
&\le 3b_2\mathbb{E}_W\int_s^t L_1\Big(1+|Y^k(v-\frac{1}{k})|^2+\rho_2(\mathscr{L}(Y^k(v-\frac{1}{k})),\delta_0)^2\Big)dE_v\\
&\le 3b_2L_1\int_s^t\Big(1+2\mathbb{E}_W|Y^k(v-\frac{1}{k})|^2\Big)dE_v\\
&\le 3b_2L_1\int_s^t\Big(1+2\mathbb{E}_W(\sup_{0\le v_1\le v}|Y^k(v_1-)|^2)\Big)dE_v.
\end{aligned}\tag{3.13}$$

For γ_3, by Doob's martingale inequality, Itô isometry (from [39]), and Condition 3.2, we gain

$$\begin{aligned}\gamma_3 &\le 12\mathbb{E}_W\int_s^t\int_{|z|<c}|\hbar(v,E_v,Y^k(v-\frac{1}{k}),\mathscr{L}(Y^k(v-\frac{1}{k})),z)|^2\nu(dz)dE_v\\ &\le 12\mathbb{E}_W\int_s^t L_1\Big(1+|Y^k(v-\frac{1}{k})|^2+\rho_2(\mathscr{L}(Y^k(v-\frac{1}{k})),\delta_0)^2\Big)dE_v\\ &\le 12L_1\int_s^t\Big(1+2\mathbb{E}_W|Y^k(v-\frac{1}{k})|^2\Big)dE_v\\ &\le 12L_1\int_s^t\Big(1+2\mathbb{E}_W(\sup_{0\le v_1\le v}|Y^k(v_1-)|^2)\Big)dE_v. \end{aligned}\tag{3.14}$$

Thus, by Eqs (3.11)–(3.14) and Lemma 3.2, we gain

$$\begin{aligned}\mathbb{E}_W|Y^k(t)-Y^k(s)|^2 &\le 3L_1(E_T-E_s+b_2+4)\\ &\quad\times\int_s^t\Big(1+2\mathbb{E}_W(\sup_{0\le v_1\le v}|Y^k(v_1-)|^2)\Big)dE_v\\ &\le C_{b_2,E_T}(E_t-E_s),\end{aligned}$$

where $C_{b_2,E_T}=3L_1(E_T-E_s+b_2+4)(1+2\mathbb{E}_W(\sup_{0\le v_1\le v}|Y^k(v_1-)|^2))$. □

Theorem 3.1. *Assume Conditions 3.1–3.3 hold. Then, for any initial value* Y_0 *satisfying* $\mathbb{E}|Y_0|^2<\infty$, *the Eq (1.2) has a unique solution* $Y(t)$, $t\in J$.

Proof. We will split the proof into the following two parts.

First, we show the existence.

In fact, for $k>l\ge 1$, it is routine to obtain

$$\begin{aligned}&\mathbb{E}_W(\sup_{0\le s\le t}|Y^k(s)-Y^l(s)|^2)\\ &\le 3\mathbb{E}_W\Big(\sup_{0\le s\le t}|\int_0^s(f(v,E_v,Y^k(v-\frac{1}{k}),\mathscr{L}(Y^k(v-\frac{1}{k})))\\ &\quad -f(v,E_v,Y^l(v-\frac{1}{l}),\mathscr{L}(Y^l(v-\frac{1}{l}))))dE_v|^2\Big)\\ &\quad +3\mathbb{E}_W\Big(\sup_{0\le s\le t}|\int_0^s(g(v,E_v,Y^k(v-\frac{1}{k}),\mathscr{L}(Y^k(v-\frac{1}{k})))\\ &\quad -g(v,E_v,Y^l(v-\frac{1}{l}),\mathscr{L}(Y^l(v-\frac{1}{l}))))dW_{E_v}|^2\Big)\\ &\quad +3\mathbb{E}_W\Big(\sup_{0\le s\le t}|\int_0^s\int_{|z|<c}(\hbar(v,E_v,Y^k(v-\frac{1}{k}),\mathscr{L}(Y^k(v-\frac{1}{k})),z)\\ &\quad -\hbar(v,E_v,Y^l(v-\frac{1}{l}),\mathscr{L}(Y^l(v-\frac{1}{l})),z))\tilde{N}(dE_v,dz)|^2\Big)\\ &:= J_1+J_2+J_3.\end{aligned}$$

By Hölder's inequality, Condition 3.1, Jensen's inequality and Lemma 3.3, we gain

$$J_1 \le 3E_T\mathbb{E}_W\int_0^t|f(s,E_s,Y^k(s-\frac{1}{k}),\mathscr{L}(Y^k(s-\frac{1}{k})))$$

$$
\begin{aligned}
& -f(s,E_s,Y^l(s-\frac{1}{l}),\mathscr{L}(Y^l(s-\frac{1}{l})))|^2 dE_s \\
\leq\ & 6E_T\mathbb{E}_W\int_0^t |f(s,E_s,Y^k(s-\frac{1}{k}),\mathscr{L}(Y^k(s-\frac{1}{k}))) \\
& -f(s,E_s,Y^l(s-\frac{1}{k}),\mathscr{L}(Y^l(s-\frac{1}{k})))|^2 dE_s \\
& +6E_T\mathbb{E}_W\int_0^t |f(s,E_s,Y^l(s-\frac{1}{k}),\mathscr{L}(Y^l(s-\frac{1}{k}))) \\
& -f(s,E_s,Y^l(s-\frac{1}{l}),\mathscr{L}(Y^l(s-\frac{1}{l})))|^2 dE_s \\
\leq\ & 6E_T L\mathbb{E}_W\int_0^t \Gamma\Big(|Y^k(s-\frac{1}{k})-Y^l(s-\frac{1}{k})|^2 \\
& +\rho_2(\mathscr{L}(Y^k(s-\frac{1}{k})),\mathscr{L}(Y^l(s-\frac{1}{k})))^2\Big)dE_s \\
& +6E_T L\mathbb{E}_W\int_0^t \Gamma\Big(|Y^l(s-\frac{1}{k})-Y^l(s-\frac{1}{l})|^2 \\
& +\rho_2(\mathscr{L}(Y^l(s-\frac{1}{k})),\mathscr{L}(Y^l(s-\frac{1}{l})))^2\Big)dE_s \\
\leq\ & 6E_T L\int_0^t \Gamma\Big(2\mathbb{E}_W|Y^k(s-\frac{1}{k})-Y^l(s-\frac{1}{k})|^2\Big)dE_s \\
& +6E_T L\int_0^t \Gamma\Big(2\mathbb{E}_W|Y^l(s-\frac{1}{k})-Y^l(s-\frac{1}{l})|^2\Big)dE_s \\
\leq\ & 6E_T L\int_0^t \Gamma\Big(2\mathbb{E}_W|Y^k(s-\frac{1}{k})-Y^l(s-\frac{1}{k})|^2\Big)dE_s \\
& +6E_T L\int_0^t \Gamma\Big(2C_{b_2,E_T}(E_{s-\frac{1}{k}}-E_{s-\frac{1}{l}})\Big)dE_s.
\end{aligned}
$$

For J_2, by Burkholder-Davis-Gundy inequality [12], Condition 3.1, Jensen inequality and Lemma 3.3, we obtain

$$
\begin{aligned}
J_2 \leq\ & 3b_2\mathbb{E}_W\int_0^t \|g(s,E_s,Y^k(s-\frac{1}{k}),\mathscr{L}(Y^k(s-\frac{1}{k}))) \\
& -g(s,E_s,Y^l(s-\frac{1}{l}),\mathscr{L}(Y^l(s-\frac{1}{l})))\|^2 dE_s \\
\leq\ & 6b_2\mathbb{E}_W\int_0^t \|g(s,E_s,Y^k(s-\frac{1}{k}),\mathscr{L}(Y^k(s-\frac{1}{k}))) \\
& -g(s,E_s,Y^l(s-\frac{1}{k}),\mathscr{L}(Y^l(s-\frac{1}{k})))\|^2 dE_s \\
& +6b_2\mathbb{E}_W\int_0^t \|g(s,E_s,Y^l(s-\frac{1}{k}),\mathscr{L}(Y^l(s-\frac{1}{k}))) \\
& -g(s,E_s,Y^l(s-\frac{1}{l}),\mathscr{L}(Y^l(s-\frac{1}{l})))\|^2 dE_s \\
\leq\ & 6b_2 L\mathbb{E}_W\int_0^t \Gamma\Big(|Y^k(s-\frac{1}{k})-Y^l(s-\frac{1}{k})|^2
\end{aligned}
$$

$$
\begin{aligned}
&\quad +\rho_2(\mathscr{L}(Y^k(s-\frac{1}{k})),\mathscr{L}(Y^l(s-\frac{1}{k})))^2\Big)dE_s\\
&\quad +6b_2L\mathbb{E}_W\int_0^t\Gamma\Big(|Y^l(s-\frac{1}{k})-Y^l(s-\frac{1}{l})|^2\\
&\quad +\rho_2(\mathscr{L}(Y^l(s-\frac{1}{k})),\mathscr{L}(Y^l(s-\frac{1}{l})))^2\Big)dE_s\\
&\leq 6b_2L\int_0^t\Gamma\Big(2\mathbb{E}_W|Y^k(s-\frac{1}{k})-Y^l(s-\frac{1}{k})|^2\Big)dE_s\\
&\quad +6b_2L\int_0^t\Gamma\Big(2\mathbb{E}_W|Y^l(s-\frac{1}{k})-Y^l(s-\frac{1}{l})|^2\Big)dE_s\\
&\leq 6b_2L\int_0^t\Gamma\Big(2\mathbb{E}_W|Y^k(s-\frac{1}{k})-Y^l(s-\frac{1}{k})|^2\Big)dE_s\\
&\quad +6b_2L\int_0^t\Gamma\Big(2C_{b_2,E_T}(E_{s-\frac{1}{k}}-E_{s-\frac{1}{l}})\Big)dE_s.
\end{aligned}
$$

For J_3, by Doob's martingale inequality, Itô's isometry (from [39]), Condition 3.1, Jensen's inequality and Lemma 3.3, we have

$$
\begin{aligned}
J_3 &\leq 12\mathbb{E}_W\int_0^t\int_{|z|<c}|\hbar(s,E_s,Y^k(s-\frac{1}{k}),\mathscr{L}(Y^k(s-\frac{1}{k})),z)\\
&\quad -\hbar(s,E_s,Y^l(s-\frac{1}{l}),\mathscr{L}(Y^l(s-\frac{1}{l})),z)|^2\nu(dz)dE_s\\
&\leq 24\mathbb{E}_W\int_0^t\int_{|z|<c}|\hbar(s,E_s,Y^k(s-\frac{1}{k}),\mathscr{L}(Y^k(s-\frac{1}{k})),z)\\
&\quad -\hbar(s,E_s,Y^l(s-\frac{1}{k}),\mathscr{L}(Y^l(s-\frac{1}{k})),z)|^2\nu(dz)dE_s\\
&\quad +24\mathbb{E}_W\int_0^t\int_{|z|<c}|\hbar(s,E_s,Y^l(s-\frac{1}{k}),\mathscr{L}(Y^l(s-\frac{1}{k})),z)\\
&\quad -\hbar(s,E_s,Y^l(s-\frac{1}{l}),\mathscr{L}(Y^l(s-\frac{1}{l})),z)|^2\nu(dz)dE_s\\
&\leq 24L\mathbb{E}_W\int_0^t\Gamma\Big(|Y^k(s-\frac{1}{k})-Y^l(s-\frac{1}{k})|^2\\
&\quad +\rho_2(\mathscr{L}(Y^k(s-\frac{1}{k})),\mathscr{L}(Y^l(s-\frac{1}{k})))^2\Big)dE_s\\
&\quad +24L\mathbb{E}_W\int_0^t\Gamma\Big(|Y^l(s-\frac{1}{k})-Y^l(s-\frac{1}{l})|^2\\
&\quad +\rho_2(\mathscr{L}(Y^l(s-\frac{1}{k})),\mathscr{L}(Y^l(s-\frac{1}{l})))^2\Big)dE_s\\
&\leq 24L\int_0^t\Gamma\Big(2\mathbb{E}_W|Y^k(s-\frac{1}{k})-Y^l(s-\frac{1}{k})|^2\Big)dE_s\\
&\quad +24L\int_0^t\Gamma\Big(2\mathbb{E}_W|Y^l(s-\frac{1}{k})-Y^l(s-\frac{1}{l})|^2\Big)dE_s
\end{aligned}
$$

$$
\begin{aligned}
&\leq \; 24L\int_0^t \Gamma\Big(2\mathbb{E}_W|Y^k(s-\frac{1}{k})-Y^l(s-\frac{1}{k})|^2\Big)dE_s \\
&\quad +24L\int_0^t \Gamma\Big(2C_{b_2,E_T}(E_{s-\frac{1}{k}}-E_{s-\frac{1}{l}})\Big)dE_s.
\end{aligned}
$$

Hence, we gain

$$
\begin{aligned}
&\mathbb{E}_W(\sup_{0\leq s\leq t}|Y^k(s)-Y^l(s)|^2) \\
&\leq 6L(E_T+b_2+4)\int_0^t \Gamma\Big(2\mathbb{E}_W|Y^k(s-\frac{1}{k})-Y^l(s-\frac{1}{k})|^2\Big)dE_s \\
&\quad +6L(E_T+b_2+4)E_T\Gamma\Big(2C_{b_2,E_T}(E_{s-\frac{1}{k}}-E_{s-\frac{1}{l}})\Big).
\end{aligned}
$$

By the definition of E_t, and from [48] and Theorem 2 of [22], we have

$$
|E_{s-\frac{1}{k}}-E_{s-\frac{1}{l}}|\leq k_1|\frac{1}{l}-\frac{1}{k}|. \tag{3.15}
$$

Noticing (3.15), taking the limit as $k,l\longrightarrow\infty$ and using the fact that $\Gamma(0)=0$, we gain, for every $\varepsilon>0$,

$$
\begin{aligned}
Q(t) &\leq 6L(E_T+b_2+4)\int_0^t \Gamma(2Q(s))dE_s \\
&\leq \varepsilon+6L(E_T+b_2+4)\int_0^t \Gamma(2Q(s))dE_s,
\end{aligned}
$$

where $Q(t)=\lim_{k,l\longrightarrow\infty}\mathbb{E}_W(\sup_{0\leq s\leq t}|Y^k(s)-Y^l(s)|^2)$. Hence, Lemma 3.1 yields

$$
Q(t)\leq\frac{1}{2}G^{-1}\Big(G(2\varepsilon)+12L(E_T+b_2+4)E_T\Big),
$$

where $G(2\varepsilon)+12L(E_T+b_2+4)E_T\in Dom(G^{-1})$, G^{-1} is the inverse function of $G(.)$, and

$$
G(\tau)=\int_1^\tau\frac{ds}{\Gamma(s)},\quad \tau>0.
$$

By Condition 3.1, one sees that $\lim_{\varepsilon\downarrow 0}G(\varepsilon)=-\infty$ and $Dom(G^{-1})=(-\infty,G(\infty))$. Taking $\mathbb{E}_{\mathfrak{D}}$ and letting $\varepsilon\longrightarrow 0$ gives $Q(t)=0$, i.e.,

$$
\mathbb{E}(\sup_{0\leq s\leq t}|Y^k(s)-Y^l(s)|^2)\longrightarrow 0,\quad as\ k,l\longrightarrow\infty. \tag{3.16}
$$

Consequently, $\{Y^k(t)\}_{k\geq 1}$ is a Cauchy sequence in $\mathbb{S}^2(\Omega;C(J;\mathbb{R}^n))$, and then the limit is denoted by $Y(t)$. Therefore, allowing $l\longrightarrow\infty$ in (3.16), we conclude

$$
\lim_{k\longrightarrow\infty}\mathbb{E}(\sup_{0\leq s\leq T}|Y^k(s)-Y(s)|^2)=0. \tag{3.17}
$$

Now, we will show that $Y(t)$ is a solution to Eq (1.2). For all $t\in J$, we gain

$$
\mathbb{E}|Y(t)-Y^k(t-\frac{1}{k})|^2\leq 2\mathbb{E}|Y(t)-Y^k(t)|^2+2\mathbb{E}|Y^k(t)-Y^k(t-\frac{1}{k})|^2,
$$

and using Lemma 3.3 and (3.17), we obtain $\lim_{k\longrightarrow\infty}\mathbb{E}|Y(t)-Y^k(t-\frac{1}{k})|^2=0$. Hence, as $k\longrightarrow\infty$ in (3.5), we obtain

$$\begin{aligned} Y(t) &= Y_0+\int_0^t f(s,E_s,Y(s),\mathscr{L}(Y(s)))dE_s \\ &\quad +\int_0^t g(s,E_s,Y(s),\mathscr{L}(Y(s)))dW_{E_s} \\ &\quad +\int_0^t\int_{|z|<c}\hbar(s,E_s,Y(s),\mathscr{L}(Y(s)),z)\tilde{N}(dE_s,dz), \quad 0\le t\le T, \end{aligned}$$

which proves that $Y(t)$ is a solution for Eq (1.2). Therefore, the existence proof is finished.

Next, we show the uniqueness.

Let $X(t)$ and $Y(t)$ be two solutions for Eq (1.2) with $X(0)=Y(0)$. Then, using the same steps as above, we have

$$\begin{aligned} \mathbb{E}_W(\sup_{0\le s\le t}|X(s)-Y(s)|^2) &\le 3L(E_T+b_2+4) \\ &\qquad \times\int_0^t\Gamma\Big(2\mathbb{E}_W(\sup_{0\le v\le s}|X(v-)-Y(v-)|^2)\Big)dE_s, \end{aligned}$$

for all $t\in J$. Then Lemma 3.1 gives $\mathbb{E}(\sup_{0\le s\le t}|X(s)-Y(s)|^2)=0$, which implies that $X(t)=Y(t)$, $0\le t\le T$ $\mathbb{P}$-a.s. Thus, the proof of Theorem 3.1 is complete. □

Remark 3.1. Putting $\hbar\equiv 0$ in Eq (1.2), it reduces to the MFSDEs driven by time-changed Brownian motion studied by [21, 40]. Hence, Theorem 3.1 is still new for MFSDEs driven by time-changed Brownian motion. Then, our results generalize and improve those obtained in [21,40].

4. The generalized Itô formula

Nane and Ni [31] established an Itô formula for SDEs driven by time-changed Lévy noise. In this section, the Itô's formula will be extended from SDEs to MFSDEs driven by time-changed Lévy noise under Conditions 3.1–3.3.

Lemma 4.1. Assume Conditions 3.1–3.3 hold, and the function $F:\mathbb{R}^+\times\mathbb{R}^n\times\mathscr{M}_2(\mathbb{R}^n)\longmapsto\mathbb{R}$ belongs to $\wp_b(\mathbb{R}^+\times\mathbb{R}^n\times\mathscr{M}_2(\mathbb{R}^n))$. Then, for any $\mathfrak{I}_0$-measurable random variable y_0 satisfying $\mathbb{E}|y_0|^2<\infty$, the following Itô formula holds:

$$\begin{aligned} &F(E_t,Y(t),\alpha_t)-F(0,y_0,\alpha_0) \\ &\quad =\int_0^t\mathbb{L}_{f,g,\hbar}F(E_s,Y(s-),\alpha)dE_s \\ &\qquad +\int_0^t F_y(E_s,Y(s-),\alpha_0)g(s,E_s,Y(s-),\mathscr{L}(Y(s-)))dW_{E_s} \\ &\qquad +\int_0^t\int_{|z|<c}[F(E_s,Y(s-)+\hbar(s,E_s,Y(s-),\mathscr{L}(Y(s-)),z),\alpha_0) \\ &\qquad -F(E_s,Y(s-),\alpha_0)]\tilde{N}(dz,dE_s), \end{aligned} \tag{4.1}$$

with

$$\mathbb{L}_{f,g,\hbar}F(t_2,y,\alpha) = \partial_{t_2}F(t_2,y,\alpha_0)+F_y(t_2,y,\alpha_0)f(t,t_2,y,\alpha)$$

$$
\begin{aligned}
&+\frac{1}{2}tr[F_{yy}(t_2,y,\alpha_0)(gg^T)(t,t_2,y,\alpha)]\\
&+\partial_\alpha F(E_{t_1},Y(E_{t_1}),\alpha)(y)f(t,t_2,y,\alpha)\\
&+\frac{1}{2}tr[\partial_v\partial_\alpha F(E_{t_1},Y(E_{t_1}),\alpha)(y)(gg^T)(t,t_2,y,\alpha)]\\
&+\int_{|z|<c}\int_0^1\Big[\Big(\partial_\alpha F(E_{t_1},Y(E_{t_1}),\alpha)(y+\xi\hbar(t,t_2,y,\alpha,z))\\
&\quad-\partial_\alpha F(E_{t_1},Y(E_{t_1}),\alpha)(y)\Big)\hbar(t,t_2,y,\alpha,z)\Big]d\xi\nu(dz)\\
&+\int_{|z|<c}\Big[F(t_2,y+\hbar(t,t_2,y,\alpha,z),\alpha_0)-F(t_2,y,\alpha_0)\\
&\quad-F_y(t_2,y,\alpha_0)\hbar(t,t_2,y,\alpha,z)\Big]\nu(dz),
\end{aligned}
$$

where $\alpha_t := \mathscr{L}(Y(t))$, $t_1 \equiv t$, and $t_2 \equiv E_t$.

Proof: Part I. Let $f, g, \hbar$ be bounded and define $u(\alpha) = F(E, Y, \alpha)$ for fixed $(E, Y) \in \mathbb{R}^+ \times \mathbb{R}^n$. Then, we prove the Itô formula for $u(\mathscr{L}(Y(t)))$.

For any integer $N \geq 1$, define the empirical projection of u onto $\mathbb{R}^n$ as:

$$
u^N : (\mathbb{R}^n)^N \ni (Y^1(t), ..., Y^N(t)) \longmapsto u\Big(\frac{1}{N}\sum_{l=1}^{N}\eta_{Y^l(t)}\Big) = u(\bar{\alpha}_t^N), \tag{4.2}
$$

where η_y denotes the unit mass at the point $y \in \mathbb{R}^n$. Then, Proposition 3.1 in [6] concludes that u^N is $\mathbb{C}^2$ on $\mathbb{R}^{n\times N}$ and

$$
\partial_{Y^i}u^N(Y^1, ..., Y^N) = \frac{1}{N}\partial_\alpha u\Big(\frac{1}{N}\sum_{l=1}^{N}\eta_{Y^l}\Big)(Y^i) = \frac{1}{N}\partial_\alpha u(\bar{\alpha}_t^N)(Y^i), \tag{4.3}
$$

$$
\begin{aligned}
\partial^2_{Y^iY^j}u^N(Y^1, ..., Y^N) &= \frac{1}{N}\partial_v\Big[\partial_\alpha u\Big(\frac{1}{N}\sum_{l=1}^{N}\eta_{Y^l}\Big)\Big](Y^i)\delta_{i,j} + \frac{1}{N^2}\partial^2_\alpha u\Big(\frac{1}{N}\sum_{l=1}^{N}\eta_{Y^l}\Big)(Y^i, Y^j)\\
&= \frac{1}{N}\partial_v\partial_\alpha u(\bar{\alpha}_t^N)(Y^i)\delta_{i,j} + \frac{1}{N^2}\partial^2_\alpha u(\bar{\alpha}_t^N)(Y^i, Y^j),
\end{aligned} \tag{4.4}
$$

where $i, j = 1, 2, ..., N$ and $\delta_{i,j} = 1, i = j, \delta_{i,j} = 0, i \neq j$. Moreover, let $\{Y^\ell(t)\}_{t\in[0,T]}$ refer to a sequence of independent and identically distributed (i.i.d) copies of $\{Y(t)\}_{t\in[0,T]}$, i.e., for any $\ell \geq 1$,

$$
\begin{aligned}
dY^\ell(t) &= f(t, E_t, Y^\ell(t-), \mathscr{L}(Y^\ell(t-)))dE_t + g(t, E_t, Y^\ell(t-), \mathscr{L}(Y^\ell(t-)))dW^\ell_{E_t}\\
&\quad+\int_{|z|<c}\hbar(t, E_t, Y^\ell(t-), \mathscr{L}(Y^\ell(t-)), z)\tilde{N}^\ell(dE_t, dz),
\end{aligned}
$$

where $W^\ell, N^\ell, \ell = 1, 2, ..., N$ are mutually independent and have the same distributions to that of W, N, respectively. Applying Itô's formula [31] and taking the expectation, we have

$$
\mathbb{E}u^N(Y^1(t), ..., Y^N(t))
$$

$$
\begin{aligned}
&= \mathbb{E}u^N(Y^1(0), ..., Y^N(0))\\
&+ \sum_{\ell=1}^{N} \int_0^t \mathbb{E}\partial_{Y^\ell} u^N(Y^1(s), Y^2(s), ..., Y^N(s)) f(s, E_s, Y^\ell(s-), \mathscr{L}(Y^\ell(s-)))dE_s\\
&+\frac{1}{2}\sum_{\ell=1}^{N} \int_0^t \mathbb{E}tr\Big[(gg^T)(s, E_s, Y^\ell(s-), \mathscr{L}(Y^\ell(s-)))\partial^2_{Y^\ell Y^\ell} u^N(Y^1(s), Y^2(s), ..., Y^N(s))\Big]dE_s\\
&+ \int_0^t \int_{|z|<c} \mathbb{E}\Big[u^N(Y^1(s) + \hbar(s, E_s, Y^1(s-), \mathscr{L}(Y^1(s-)), z), Y^2(s), ..., Y^N(s))\\
&-u^N(Y^1(s), ..., Y^N(s)) - \partial_{Y^1} u^N(Y^1(s), ..., Y^N(s))\hbar(s, E_s, Y^1(s-), \mathscr{L}(Y^1(s-)), z)\Big]\nu(dz)dE_s\\
&+ \qquad\qquad\qquad\qquad \cdots\\
&+ \int_0^t \int_{|z|<c} \mathbb{E}\Big[u^N(Y^1(s), Y^2(s), ..., Y^N(s) + \hbar(s, E_s, Y^N(s-), \mathscr{L}(Y^N(s-)), z))\\
&-u^N(Y^1(s), ..., Y^N(s)) - \partial_{Y^N} u^N(Y^1(s), ..., Y^N(s))\hbar(s, E_s, Y^N(s-), \mathscr{L}(Y^N(s-)), z)\Big]\nu(dy)dE_s\\
&= \mathbb{E}u^N(Y^1(0), ..., Y^N(0))\\
&+N \int_0^t \mathbb{E}\partial_{Y^1} u^N(Y^1(s), Y^2(s), ..., Y^N(s)) f(s, E_s, Y^1(s-), \mathscr{L}(Y^1(s-)))dE_s\\
&+\frac{N}{2} \int_0^t \mathbb{E}tr\Big[(gg^T)(s, E_s, Y^1(s-), \mathscr{L}(Y^1(s-)))\partial^2_{Y^1 Y^1} u^N(Y^1(s), Y^2(s), ..., Y^N(s))\Big]dE_s\\
&+N \int_0^t \int_{|z|<c} \int_0^1 \mathbb{E}\Big[\Big(\partial_{Y^1} u^N(Y^1(s) + \xi\hbar(s, E_s, Y^1(s-), \mathscr{L}(Y^1(s-)), z), Y^2(s), ..., Y^N(s))\\
&-\partial_{Y^1} u^N(Y^1(s), ..., Y^N(s))\Big)\hbar(s, E_s, Y^1(s-), \mathscr{L}(Y^1(s-)), z)\Big]d\xi\nu(dz)dE_s,
\end{aligned}
$$

where we have used the fact that the processes $\{(Y^\ell(t))_{t\in[0,T]}\}_{\ell\in\{1,...,N\}}$ are i.i.d. in the second equality. Using Eqs (4.2)–(4.4), we obtain for any $t \in J$

$$
\begin{aligned}
&\mathbb{E}u^N(Y^1(t), ..., Y^N(t))\\
&\quad = \mathbb{E}u^N(Y^1(0), ..., Y^N(0))\\
&\qquad +\mathbb{E}\Big[\int_0^t \partial_\alpha u(\bar{\alpha}^N_s)(Y^1(s)) f(s, E_s, Y^1(s-), \mathscr{L}(Y^1(s-)))dE_s\Big]\\
&\qquad +\frac{1}{2}\mathbb{E}\Big[\int_0^t tr[\partial_v\partial_\alpha u(\bar{\alpha}^N_s)(Y^1(s))(gg^T)(s, E_s, Y^1(s-), \mathscr{L}(Y^1(s-)))]dE_s\Big]\\
&\qquad +\frac{1}{2N}\mathbb{E}\Big[\int_0^t tr[\partial^2_\alpha u(\bar{\alpha}^N_s)(Y^1(s), Y^1(s))(gg^T)(s, E_s, Y^1(s-), \mathscr{L}(Y^1(s-)))]dE_s\Big]\\
&\qquad + \int_0^t \int_{|z|<c} \int_0^1 \mathbb{E}\Big[\Big(\partial_\alpha u\Big(\frac{1}{N}\eta_{Y^1(s)+\xi\hbar(s,E_s,Y^1(s-),\mathscr{L}(Y^1(s-)),z)} + \frac{1}{N}\sum_{l=2}^{N}\eta_{Y^l(s)}\Big)\\
&\qquad\qquad\qquad \circ(Y^1(s) + \xi\hbar(s, E_s, Y^1(s-), \mathscr{L}(Y^1(s-)), z))\\
&\qquad -\partial_\alpha u(\bar{\alpha}^N_s)(Y^1(s))\Big)\hbar(s, E_s, Y^1(s-), \mathscr{L}(Y^1(s-)), z)\Big]d\xi\nu(dz)dE_s.
\end{aligned}
$$

Taking $N \longrightarrow \infty$ and for any $t \in J$,

$$\mathbb{P}\Big[\lim_{N\longrightarrow\infty}\rho_2(\bar{\alpha}_t^N, \alpha_t)^2 = 0\Big] = 1.$$

Then, due to the continuity and boundedness of $u, \partial_\alpha u, \partial_v\partial_\alpha u$ and the boundedness of $\partial_\alpha^2 u, f, g$, the dominated convergence theorem reads

$$\begin{aligned}
&u(\mathscr{L}(Y(t)))\\
&\quad= u(\mathscr{L}(Y(0)))\\
&\quad\quad+\mathbb{E}\Big[\int_0^t \partial_\alpha u(\mathscr{L}(Y(s)))(Y^1(s))f(s, E_s, Y^1(s-), \mathscr{L}(Y^1(s-)))dE_s\Big]\\
&\quad\quad+\frac{1}{2}\mathbb{E}\Big[\int_0^t tr[\partial_v\partial_\alpha u(\mathscr{L}(Y(s)))(Y^1(s))(gg^T)(s, E_s, Y^1(s-), \mathscr{L}(Y^1(s-)))]dE_s\Big]\\
&\quad\quad+\int_0^t\int_{|z|<c}\int_0^1 \mathbb{E}\Big[\Big(\partial_\alpha u(\mathscr{L}(Y(s)))(Y^1(s) + \xi\hbar(s, E_s, Y^1(s-), \mathscr{L}(Y^1(s-)), z))\\
&\quad\quad-\partial_\alpha u(\mathscr{L}(Y(s)))(Y^1(s))\Big)\hbar(s, E_s, Y^1(s-), \mathscr{L}(Y^1(s-)), z)\Big]d\xi\nu(dz)dE_s.
\end{aligned}\tag{4.5}$$

Part II. Assuming that Conditions 3.1–3.3 hold, the Itô formula for $u(\mathscr{L}(Y(t)))$ will be derived.

For every $n \in \mathbb{N}$, we take a smooth function $\psi_n : \mathbb{R}^n \longmapsto \mathbb{R}^n$ satisfying:

$$\begin{cases}\psi_n(y) = y, & |y| \leq n,\\ \psi_n(y) = 0, & |y| > 2n,\end{cases}$$

such that

$$|\psi_n(y)| \leq C_1, \qquad |\partial\psi_n(y)| \leq C_1, \tag{4.6}$$

where the constant $C_1 > 0$ is independent of n. Set

$$f^{(n)}(t_1, t_2, y, \alpha) := f(t_1, t_2, \psi_n(y), \alpha), \qquad g^{(n)}(t_1, t_2, y, \alpha) := g(t_1, t_2, \psi_n(y), \alpha),$$

$$\hbar^{(n)}(t_1, t_2, y, \alpha, z) := \hbar(t_1, t_2, \psi_n(y), \alpha, z).$$

Allowing $n \longrightarrow \infty$,

$$f^{(n)}(t_1, t_2, y, \alpha) \longrightarrow f(t_1, t_2, y, \alpha), \qquad g^{(n)}(t_1, t_2, y, \alpha) \longrightarrow g(t_1, t_2, y, \alpha),$$

$$\hbar^{(n)}(t_1, t_2, y, \alpha, z) \longrightarrow \hbar(t_1, t_2, y, \alpha, z).$$

Furthermore, by Condition 3.2, we notice that $f^{(n)}$, $g^{(n)}$, and $\hbar^{(n)}$ are bounded, and all $f^{(n)}$, $g^{(n)}$, $\hbar^{(n)}$ obey Conditions 3.1 and 3.3. Hence, the following equation

$$dY^{(n)}(t) \;=\; f^{(n)}(t, E_t, Y^{(n)}(t-), \mathscr{L}(Y^{(n)}(t-)))dE_t$$

$$
\begin{aligned}
&+g^{(n)}(t,E_t,Y^{(n)}(t-),\mathscr{L}(Y^{(n)}(t-)))dW_{E_t}\\
&+\int_{|z|<c}\hbar^{(n)}(t,E_t,Y^{(n)}(t-),\mathscr{L}(Y^{(n)}(t-)),z)\tilde{N}(dE_t,dz)
\end{aligned}
$$

has a unique solution $Y^{(n)}(t)$ with $Y^{(n)}(0)=Y(0)$. Then, for any $t\in J$ and according to **Part I**, we conclude that

$$
\begin{aligned}
&u(\mathscr{L}(Y^{(n)}(t)))\\
&=u(\mathscr{L}(Y^{(n)}(0)))\\
&+\mathbb{E}\Big[\int_0^t\partial_\alpha u(\mathscr{L}(Y^{(n)}(s)))(Y^{(n)}(s))f^{(n)}(s,E_s,Y^{(n)}(s-),\mathscr{L}(Y^{(n)}(s-)))dE_s\Big]\\
&+\frac{1}{2}\mathbb{E}\Big[\int_0^t tr[\partial_v\partial_\alpha u(\mathscr{L}(Y^{(n)}(s)))(Y^{(n)}(s))(g^{(n)}g^{(n)^T})(s,E_s,Y^{(n)}(s-),\mathscr{L}(Y^{(n)}(s-)))]dE_s\Big]\\
&+\int_0^t\int_{|z|<c}\int_0^1\mathbb{E}\Big[\Big(\partial_\alpha u(\mathscr{L}(Y^{(n)}(s)))(Y^{(n)}(s)+\xi\hbar^{(n)}(s,E_s,Y^{(n)}(s-),\mathscr{L}(Y^{(n)}(s-)),z))\\
&\quad-\partial_\alpha u(\mathscr{L}(Y^{(n)}(s)))(Y^{(n)}(s))\Big)\hbar^{(n)}(s,E_s,Y^{(n)}(s-),\mathscr{L}(Y^{(n)}(s-)),z)\Big]d\xi\nu(dz)dE_s. \qquad (4.7)
\end{aligned}
$$

Taking the expectation, using Hölder's inequality, the Burkholder-Davis-Gundy inequality [12], Doob's martingale inequality, and Itô's isometry (from [39]), we have

$$
\begin{aligned}
&\mathbb{E}_W|Y^{(n)}(t)-Y(t)|^2\\
&\le 3E_T\mathbb{E}_W\int_0^t|f^{(n)}(s,E_s,Y^{(n)}(s-),\mathscr{L}(Y^{(n)}(s-)))-f(s,E_s,Y(s-),\mathscr{L}(Y(s-)))|^2dE_s\\
&\quad+3b_2\mathbb{E}_W\int_0^t\|g^{(n)}(s,E_s,Y^{(n)}(s-),\mathscr{L}(Y^{(n)}(s-)))-g(s,E_s,Y(s-),\mathscr{L}(Y(s-)))\|^2dE_s\\
&\quad+12\mathbb{E}_W\int_0^t\int_{|z|<c}|\hbar^{(n)}(s,E_s,Y^{(n)}(s-),\mathscr{L}(Y^{(n)}(s-)),z)\\
&\quad-\hbar(s,E_s,Y(s-),\mathscr{L}(Y(s-)),z)|^2\nu(dz)dE_s.
\end{aligned}
$$

By Condition 3.1 and the Jensen inequality, we have

$$
\begin{aligned}
\mathbb{E}_W|Y^{(n)}(t)-Y(t)|^2 &\le 3[E_T+b_2+4]\mathbb{E}_W\int_0^t L\Gamma\Big(|\psi_n(Y^{(n)}(s-))-Y(s-)|^2\\
&\quad+\rho_2(\mathscr{L}(Y^{(n)}(s-)),\mathscr{L}(Y(s-)))^2\Big)dE_s\\
&\le 3L[E_T+b_2+4]\int_0^t\Gamma\Big(\mathbb{E}_W|\psi_n(Y^{(n)}(s-))-Y(s-)|^2\\
&\quad+\mathbb{E}_W|Y^{(n)}(s-)-Y(s-)|^2\Big)dE_s.
\end{aligned}
$$

By Eq (4.6), we have

$$
\begin{aligned}
\mathbb{E}_W|\psi_n(Y^{(n)}(s))-Y(s)|^2 &\le 2\mathbb{E}_W|\psi_n(Y^{(n)}(s))-\psi_n(Y(s))|^2+2\mathbb{E}_W|\psi_n(Y(s))-Y(s)|^2\\
&\le 2C_1\mathbb{E}_W|Y^{(n)}(s)-Y(s)|^2+2\mathbb{E}_W|\psi_n(Y(s))-Y(s)|^2,
\end{aligned}
$$

and furthermore,

$$\begin{aligned}\mathbb{E}_W|Y^{(n)}(t)-Y(t)|^2 \;\leq\; & 3L[E_T+b_2+4]\int_0^t \Gamma\Big((2C_1+1)\mathbb{E}_W|Y^{(n)}(s-)-Y(s-)|^2 \\ & +2\mathbb{E}_W|\psi_n(Y(s-))-Y(s-)|^2\Big)dE_s.\end{aligned}$$

Therefore,

$$\begin{aligned}&(2C_1+1)\mathbb{E}_W|Y^{(n)}(t)-Y(t)|^2+2\mathbb{E}_W|\psi_n(Y(t))-Y(t)|^2\\ &\quad\leq 2\mathbb{E}_W|\psi_n(Y(t))-Y(t)|^2+3L[E_T+b_2+4](2C_1+1)\\ &\qquad\times\int_0^t \Gamma\Big((2C_1+1)\mathbb{E}_W|Y^{(n)}(s-)-Y(s-)|^2+2\mathbb{E}_W|\psi_n(Y(s-))-Y(s-)|^2\Big)dE_s.\end{aligned}$$

Putting $G(t)=\int_0^t \frac{ds}{\Gamma(s)}$, it is concluded from Lemma 3.1 that

$$\begin{aligned}&(2C_1+1)\mathbb{E}_W|Y^{(n)}(t)-Y(t)|^2+2\mathbb{E}_W|\psi_n(Y(t))-Y(t)|^2\\ &\quad\leq G^{-1}\Big[G\Big(2\mathbb{E}_W|\psi_n(Y(t))-Y(t)|^2\Big)+3L[E_T+b_2+4](2C_1+1)E_T\Big].\end{aligned}$$

Using the fact that $\lim_{n\longrightarrow\infty}\psi_n(y)=y$ for all $y\in\mathbb{R}^n$, then $\lim_{n\longrightarrow\infty}\mathbb{E}_W|\psi_n(Y(t))-Y(t)|^2=0$. Referring to the hypothesis $\int_{0^+}\frac{ds}{\Gamma(s)}=\infty$, it yields that

$$G\Big(2\mathbb{E}_W|\psi_n(Y(t))-Y(t)|^2\Big)+3[E_T+b_2+4]L(2C_1+1)E_T\longrightarrow-\infty,\quad n\longrightarrow\infty.$$

On the other side, since G is a strictly increasing function, it is obtained that G has an inverse function that is strictly increasing, and $G^{-1}(-\infty)=0$, i.e.,

$$G^{-1}\Big[G\Big(2\mathbb{E}_W|\psi_n(Y(t))-Y(t)|^2\Big)+3L[E_T+b_2+4](2C_1+1)E_T\Big]\longrightarrow 0,\quad n\longrightarrow\infty.$$

We concluded that

$$\lim_{n\longrightarrow\infty}\mathbb{E}|Y^{(n)}(t)-Y(t)|^2=0.$$

Moreover, we can get

$$\lim_{n\longrightarrow\infty}\mathbb{E}|\psi_n(Y^{(n)}(t))-Y(t)|^2=0$$

and

$$\lim_{n\longrightarrow\infty}\rho_2\Big(\mathscr{L}(Y^{(n)}(t)),\mathscr{L}(Y(t))\Big)^2\leq\lim_{n\longrightarrow\infty}\mathbb{E}|Y^{(n)}(t)-Y(t)|^2=0.$$

By taking the limit on both sides of Eq (4.7), the dominated convergence theorem concludes (4.5).

Part III. We prove Itô's formula (4.1). By Itô's formula ([31, Lemma 3.1]) and Eq (4.5), it can be concluded that

$$
\begin{aligned}
&F(E_t, Y(t), \alpha_t) - F(0, y_0, \alpha_0)\\
&= F(E_t, Y(t), \alpha_t) - F(E_t, Y(t), \alpha_0) + F(E_t, Y(t), \alpha_0) - F(0, y_0, \alpha_0)\\
&= \int_0^t \partial_{t_2} F(E_s, Y(s-), \alpha_0) dE_s + \int_0^t F_y(E_s, Y(s-), \alpha_0) f(s, E_s, Y(s-), \mathscr{L}(Y(s-))) dE_s\\
&+\frac{1}{2}\int_0^t tr\Big[F_{yy}(E_s, Y(s-), \alpha_0)(gg^T)(s, E_s, Y(s-), \mathscr{L}(Y(s-)))\Big] dE_s\\
&+\int_0^t \int_{|z|<c} \Big[F(E_s, Y(s-) + \hbar(s, E_s, Y(s-), \mathscr{L}(Y(s-)), z), \alpha_0) - F(E_s, Y(s-), \alpha_0)\\
&-F_y(E_s, Y(s-), \alpha_0)\hbar(s, E_s, Y(s-), \mathscr{L}(Y(s-)), z)\Big] \nu(dz) dE_s\\
&+\int_0^t \int_{|z|<c} \Big[F(E_s, Y(s-) + \hbar(s, E_s, Y(s-), \mathscr{L}(Y(s-)), z), \alpha_0)\\
&-F(E_s, Y(s-), \alpha_0)\Big] \tilde{N}(dz, dE_s)\\
&+\int_0^t F_y(E_s, Y(s-), \alpha_0) g(s, E_s, Y(s-), \mathscr{L}(Y(s-))) dW_{E_s}\\
&+\mathbb{E}\Big[\int_0^t \partial_\alpha F(E_t, Y(t-), \alpha_s)(Y(s)) f(s, E_s, Y(s-), \mathscr{L}(Y(s-))) dE_s\Big]\\
&+\frac{1}{2}\mathbb{E}\Big[\int_0^t tr[\partial_v \partial_\alpha F(E_t, Y(t-), \alpha_s)(Y(s))(gg^T)(s, E_s, Y(s-), \mathscr{L}(Y(s-)))] dE_s\Big]\\
&+\int_0^t \int_{|z|<c} \int_0^1 \mathbb{E}\Big[\Big(\partial_\alpha F(E_t, Y(t-), \alpha_s)(Y(s) + \xi\hbar(s, E_s, Y(s-), \mathscr{L}(Y(s-)), z))\\
&-\partial_\alpha F(E_t, Y(t-), \alpha_s)(Y(s))\Big)\hbar(s, E_s, Y(s-), \mathscr{L}(Y(s-)), z)\Big] d\xi \nu(dz) dE_s.
\end{aligned}
$$

This completes the proof. □

Remark 4.1. Let F be independent of the measure α. Then $F \in \wp_b(\mathbb{R}^+ \times \mathbb{R}^n \times \mathscr{M}_2(\mathbb{R}^n))$ is simplified to $\mathbb{C}^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n)$ and the relevant Itô formula reduces to the standard Itô formula introduced by ([31, Lemma 3.1]).

5. Stability results

In this section, we study the stability of the solutions of Eq (1.2) based on the Lyapunov method. We establish some sufficient criteria to derive the p-th moment asymptotic stability, stability in probability, asymptotic stability in probability, and globally asymptotic stability in probability for the trivial solution, generalizing some results obtained in [31, 40]. For this purpose, let the initial value $Y(0) = y_0 \in \mathbb{R}^n$ be deterministic constant. Furthermore, let $\mathscr{H}$ be the set of all non-decreasing functions $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $\phi(s) > 0,\ s > 0$. Besides, for any $h > 0$, define $\mathscr{A}_h := \{y \in \mathbb{R}^n : |y| < h\}$ and $\bar{\mathscr{A}}_h := \{y \in \mathbb{R}^n : |y| \leq h\}$.

Definition 5.1. ([31,45]) The trivial solution of Eq (1.2) is called p-th moment asymptotically stable if for all $t \geq 0$, there exist a function $\upsilon(t) : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ decaying to 0 as $t \longrightarrow \infty$ and $C > 0$ such that

$$\mathbb{E}[|Y(t)|^p] < C|y_0|^p \upsilon(t),$$

where $p > 0$ and $y_0 \in \mathbb{R}^n$ is the initial value.

Theorem 5.1. Let p, a_1, a_2, a_3 be positive constants. If $F(t_2, y, \alpha) \in \wp_{b,+}(\mathbb{R}^+ \times \mathbb{R}^n \times \mathscr{M}_2(\mathbb{R}^n))$ satisfies

(1) $F(t_2, 0, \delta_0) = 0$,
(2) $a_1|y|^{2p} \leq F(t_2, y, \alpha) \leq a_2|y|^{2p}$,
(3) $\mathbb{L}_{f,g,\hbar}F(t_2, y, \alpha) + a_3F(t_2, y, \alpha_0) \leq 0$,

for all $(t_2, y, \alpha) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathscr{M}_2(\mathbb{R}^n)$, then the trivial solution of Eq (1.2) is p-th moment asymptotically stable.

Proof. Fix any $y_0 \neq 0$ in $\mathbb{R}^n$. For each $m \geq |y_0|$, define

$$\tau_m = \inf\{t \geq 0 : |Y(t)| \geq m\},$$

$$\mathcal{U}_k := k \wedge \inf\Big\{t \geq 0 : |\int_0^{\tau_m \wedge t} F_y(E_s, Y(s), \alpha_0)g(s, E_s, Y(s), \mathscr{L}(Y(s-)))dW_{E_s}| \geq k\Big\}, \tag{5.1}$$

$$\begin{aligned}\mathcal{W}_k \;\; := \;\; & k \wedge \inf\Big\{t \geq 0 : |\int_0^{\tau_m \wedge t} \int_{|z|<c} [F(E_s, Y(s) + \hbar(s, E_s, Y(s), \mathscr{L}(Y(s-)), z), \alpha_0) \\ & - F(E_s, Y(s), \alpha_0)]\tilde{N}(dz, dE_s)| \geq k\Big\},\end{aligned} \tag{5.2}$$

for $k = 1, 2, ...$ It is obvious that $\mathcal{U}_k \longrightarrow \infty$ and $\mathcal{W}_k \longrightarrow \infty$ as $k \longrightarrow \infty$. Applying Lemma 4.1 for the function $e^{a_3 t_2}F(t_2, y, \alpha)$, we obtain

$$\begin{aligned}
& e^{a_3 E_{t\wedge\tau_m\wedge\mathcal{U}_k\wedge\mathcal{W}_k}} F(E_{t\wedge\tau_m\wedge\mathcal{U}_k\wedge\mathcal{W}_k}, Y(t \wedge \tau_m \wedge \mathcal{U}_k \wedge \mathcal{W}_k), \alpha_{t\wedge\tau_m\wedge\mathcal{U}_k\wedge\mathcal{W}_k}) \\
& = F(0, y_0, \alpha_0) + \int_0^{t\wedge\tau_m\wedge\mathcal{U}_k\wedge\mathcal{W}_k} e^{a_3 E_s}[a_3F(E_s, Y(s), \alpha_0) + \partial_{t_2}F(E_s, Y(s), \alpha_0)]dE_s \\
& + \int_0^{t\wedge\tau_m\wedge\mathcal{U}_k\wedge\mathcal{W}_k} e^{a_3 E_s}F_y(E_s, Y(s), \alpha_0)f(s, E_s, Y(s), \mathscr{L}(Y(s-)))dE_s \\
& +\frac{1}{2}\int_0^{t\wedge\tau_m\wedge\mathcal{U}_k\wedge\mathcal{W}_k} e^{a_3 E_s}tr[F_{yy}(E_s, Y(s), \alpha_0)(gg^T)(s, E_s, Y(s), \mathscr{L}(Y(s-)))]dE_s \\
& + \int_0^{t\wedge\tau_m\wedge\mathcal{U}_k\wedge\mathcal{W}_k} \int_{|z|<c} e^{a_3 E_s}[F(E_s, Y(s) + \hbar(s, E_s, Y(s), \mathscr{L}(Y(s-)), z), \alpha_0) \\
& -F(E_s, Y(s), \alpha_0) - F_y(E_s, Y(s), \alpha_0)\hbar(s, E_s, Y(s), \mathscr{L}(Y(s-)), z)]\nu(dz)dE_s \\
& +\mathbb{E}\Big[\int_0^{t\wedge\tau_m\wedge\mathcal{U}_k\wedge\mathcal{W}_k} e^{a_3 E_s}\Big(\partial_\alpha F(E_{t\wedge\tau_m\wedge\mathcal{U}_k\wedge\mathcal{W}_k}, Y(t \wedge \tau_m \wedge \mathcal{U}_k \wedge \mathcal{W}_k), \alpha_s)(Y(s)) \\
& \times f(s, E_s, Y(s), \mathscr{L}(Y(s))) + \frac{1}{2}tr[\partial_v\partial_\alpha F(E_{t\wedge\tau_m\wedge\mathcal{U}_k\wedge\mathcal{W}_k}, Y(t \wedge \tau_m \wedge \mathcal{U}_k \wedge \mathcal{W}_k), \alpha_s)(Y(s))
\end{aligned}$$

$$
\begin{aligned}
&\times(gg^T)(s,E_s,Y(s),\mathscr{L}(Y(s)))]\Big)\Big]dE_s\\
&+\mathbb{E}\Big[\int_0^{t\wedge\tau_m\wedge\mathcal{U}_k\wedge\mathcal{W}_k}\int_{|z|<c}\int_0^1 e^{a_3E_s}\Big(\partial_\alpha F(E_{t\wedge\tau_m\wedge\mathcal{U}_k\wedge\mathcal{W}_k},Y(t\wedge\tau_m\wedge\mathcal{U}_k\wedge\mathcal{W}_k),\alpha_s)(Y(s)\\
&+\xi\hbar(s,E_s,Y(s),\mathscr{L}(Y(s)),z))-\partial_\alpha F(E_{t\wedge\tau_m\wedge\mathcal{U}_k\wedge\mathcal{W}_k},Y(t\wedge\tau_m\wedge\mathcal{U}_k\wedge\mathcal{W}_k),\alpha_s)(Y(s))\Big)\\
&\times\hbar(s,E_s,Y(s),\mathscr{L}(Y(s)),z)d\xi\nu(dz)\Big]dE_s\\
&+\int_0^{t\wedge\tau_m\wedge\mathcal{U}_k\wedge\mathcal{W}_k}e^{a_3E_s}F_y(E_s,Y(s),\alpha_0)g(s,E_s,Y(s),\mathscr{L}(Y(s-)))dW_{E_s}\\
&+\int_0^{t\wedge\tau_m\wedge\mathcal{U}_k\wedge\mathcal{W}_k}\int_{|z|<c}e^{a_3E_s}[F(E_s,Y(s)+\hbar(s,E_s,Y(s),\mathscr{L}(Y(s-)),z),\alpha_0)\\
&-F(E_s,Y(s),\alpha_0)]\tilde{N}(dz,dE_s)\\
&=F(0,y_0,\alpha_0)+\int_0^{t\wedge\tau_m\wedge\mathcal{U}_k\wedge\mathcal{W}_k}e^{a_3E_s}\Big[a_3F(E_s,Y(s),\alpha_0)+\mathbb{L}_{f,g,\hbar}F(E_s,Y(s),\alpha_s)\Big]dE_s\\
&+\int_0^{t\wedge\tau_m\wedge\mathcal{U}_k\wedge\mathcal{W}_k}e^{a_3E_s}F_y(E_s,Y(s),\alpha_0)g(s,E_s,Y(s),\mathscr{L}(Y(s-)))dW_{E_s}\\
&+\int_0^{t\wedge\tau_m\wedge\mathcal{U}_k\wedge\mathcal{W}_k}\int_{|z|<c}e^{a_3E_s}[F(E_s,Y(s)+\hbar(s,E_s,Y(s),\mathscr{L}(Y(s-)),z),\alpha_0)\\
&-F(E_s,Y(s),\alpha_0)]\tilde{N}(dz,dE_s).
\end{aligned}\tag{5.3}
$$

We have by Kuo [17] and Magdziarz [23] that

$$\int_0^{t\wedge\tau_m\wedge\mathcal{U}_k\wedge\mathcal{W}_k}e^{a_3E_s}F_y(E_s,Y(s),\alpha_0)g(s,E_s,Y(s),\mathscr{L}(Y(s-)))dW_{E_s}$$

and

$$
\begin{aligned}
\int_0^{t\wedge\tau_m\wedge\mathcal{U}_k\wedge\mathcal{W}_k}\int_{|z|<c}e^{a_3E_s}[F(E_s,Y(s)+\hbar(s,E_s,Y(s),\mathscr{L}(Y(s-)),z),\alpha_0)&\\
-F(E_s,Y(s),\alpha_0)]\tilde{N}(dz,dE_s)&
\end{aligned}
$$

are martingales with mean 0. Taking the expectation on both sides of Eq (5.3) and using assumption (3), we have that

$$\mathbb{E}\Big(e^{a_3E_{t\wedge\tau_m\wedge\mathcal{U}_k\wedge\mathcal{W}_k}}F(E_{t\wedge\tau_m\wedge\mathcal{U}_k\wedge\mathcal{W}_k},Y(t\wedge\tau_m\wedge\mathcal{U}_k\wedge\mathcal{W}_k),\alpha_{t\wedge\tau_m\wedge\mathcal{U}_k\wedge\mathcal{W}_k})\Big)\le F(0,y_0,\alpha_0).$$

Allowing $k,m\longrightarrow\infty$, we get

$$\mathbb{E}\Big(e^{a_3E_t}F(E_t,Y(t),\alpha_t)\Big)\le F(0,y_0,\alpha_0).$$

Then assumption (2) implies

$$a_1\mathbb{E}\Big(e^{a_3E_t}|Y(t)|^{2p}\Big)\le\mathbb{E}\Big(e^{a_3E_t}F(E_t,Y(t),\alpha_t)\Big)\le F(0,y_0,\alpha_0)\le a_2|y_0|^{2p},$$

which gives

$$\mathbb{E}(e^{a_3 E_t}|Y(t)|^{2p}) \le \frac{a_2}{a_1}|y_0|^{2p}.$$

Using the reverse Hölder inequality $\mathbb{E}|\mathcal{X}\mathcal{Y}| \ge (\mathbb{E}|\mathcal{X}|^{\frac{1}{p}})^p(\mathbb{E}|\mathcal{Y}|^{\frac{-1}{p-1}})^{-(p-1)}$ when $p = 2$, $\mathcal{X} = |Y(t)|^{2p}$, and $\mathcal{Y} = e^{a_3 E_t}$, we have

$$(\mathbb{E}|Y(t)|^p)^2(\mathbb{E}(e^{-a_3 E_t}))^{-1} \le \frac{a_2}{a_1}|y_0|^{2p}.$$

Additionally, the inverse β-stable subordinator E_t takes the Laplace transform $\mathbb{E}(e^{-a_3 E_t}) = E_\beta(-a_3 t^\beta)$, where the function $E_\beta(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(n\beta+1)}$ is the Mittag-Leffler function. Then

$$\mathbb{E}|Y(t)|^p \le \sqrt{\frac{a_2}{a_1}}|y_0|^p \sqrt{E_\beta(-a_3 t^\beta)},$$

with $E_\beta(-a_3 t^\beta) \longrightarrow 0$ as $t \longrightarrow \infty$ [24]. Therefore, the trivial solution of Eq (1.2) is p-th moment asymptotically stable and the proof is complete. □

Remark 5.1. Putting $\hbar \equiv 0$ in Eq (1.2), our results in Theorem 5.1 are new for MFSDEs driven by time-changed Brownian motion studied by [21,40].

Definition 5.2. ([31,45]) If for every pair of $\varsigma > 0$ and $0 < \varepsilon < 1$, there exists $0 < \delta = \delta(\varepsilon, \varsigma)$ such that when $|y_0| < \delta$,

$$\mathbb{P}\{|Y(t, y_0)| < \varsigma \; for \; \forall \; t \ge 0\} \ge 1 - \varepsilon,$$

then the trivial solution of Eq (1.2) is called stable in probability.

Theorem 5.2. Suppose for all $(t_2, y, \alpha) \in \mathbb{R}^+ \times \mathscr{A}_h \times \mathscr{M}_2(\mathbb{R}^n)$, there exists a function $F(t_2, y, \alpha) \in \wp_{b,+}(\mathbb{R}^+ \times \mathscr{A}_h \times \mathscr{M}_2(\mathbb{R}^n))$ with $h > 2c$ and $\phi \in \mathscr{H}$ such that

(1) $F(t_2, 0, \delta_0) = 0$,
(2) $\phi(|y|) \le F(t_2, y, \alpha)$,
(3) $\mathbb{L}_{f,g,\hbar}F(t_2, y, \alpha) \le 0$,
then the trivial solution of Eq (1.2) is stable in probability.

Proof. Select $\varepsilon \in (0, 1)$ and $\varsigma \in (0, h)$ arbitrarily. Since $F(t_2, y, \alpha)$ is continuous on $\mathbb{R}^+ \times \mathscr{A}_h \times \mathscr{M}_2(\mathbb{R}^n)$ and $F(t_2, 0, \delta_0) = 0$, $\exists$ a $\Theta = \Theta(\varepsilon, \varsigma) > 0$ such that

$$\frac{1}{\varepsilon} \sup_{y_0 \in \mathscr{A}_\Theta} F(0, y_0, \alpha_0) \le \phi(\varsigma). \tag{5.4}$$

Then by assumption (2) and (5.4), we have $\Theta < \varsigma$. Define the following stopping times:

$$\tau_\varsigma := \inf\{t \ge 0 : |Y(t, y_0)| \ge \varsigma\}, \tag{5.5}$$

$$\mathcal{U}_k := k \wedge \inf\Big\{t \ge 0 : |\int_0^{\tau_\varsigma \wedge t} F_y(E_s, Y(s), \alpha_0) g(s, E_s, Y(s), \mathscr{L}(Y(s-))) dW_{E_s}| \ge k\Big\}, \tag{5.6}$$

and

$$\begin{aligned}\mathcal{W}_k \quad := \quad & k \wedge \inf\Big\{t \geq 0 : \Big|\int_0^{\tau_\varsigma \wedge t}\int_{|z|<c}\Big[F(E_s, Y(s) + \hbar(s, E_s, Y(s), \mathscr{L}(Y(s-)), z), \alpha_0) \\ & - F(E_s, Y(s), \alpha_0)\Big]\tilde{N}(dz, dE_s)\Big| \geq k\Big\}, \end{aligned} \tag{5.7}$$

for all $k = 1, 2, ...$, where the initial value $y_0 \in \mathscr{A}_\Theta$ is fixed. It is clear that $\mathcal{U}_k \longrightarrow \infty$ and $\mathcal{W}_k \longrightarrow \infty$ as $k \longrightarrow \infty$. Applying Lemma 4.1 to Eq (1.2), we have

$$\begin{aligned} & F(E_{\tau_\varsigma \wedge \mathcal{U}_k \wedge \mathcal{W}_k}, Y(\tau_\varsigma \wedge \mathcal{U}_k \wedge \mathcal{W}_k), \alpha_{\tau_\varsigma \wedge \mathcal{U}_k \wedge \mathcal{W}_k}) \\ & = F(0, y_0, \alpha_0) + \int_0^{\tau_\varsigma \wedge \mathcal{U}_k \wedge \mathcal{W}_k} \mathbb{L}_{f,g,\hbar} F(E_s, Y(s), \alpha_s) dE_s \\ & \quad + \int_0^{\tau_\varsigma \wedge \mathcal{U}_k \wedge \mathcal{W}_k} F_y(E_s, Y(s), \alpha_0) g(s, E_s, Y(s), \mathscr{L}(Y(s-))) dW_{E_s} \\ & \quad + \int_0^{\tau_\varsigma \wedge \mathcal{U}_k \wedge \mathcal{W}_k} \int_{|z|<c} [F(E_s, Y(s) + \hbar(s, E_s, Y(s), \mathscr{L}(Y(s-)), z), \alpha_0) \\ & \quad - F(E_s, Y(s), \alpha_0)]\tilde{N}(dz, dE_s). \end{aligned} \tag{5.8}$$

Owing to Kuo [17] and Magdziarz [23], the following:

$$\int_0^{\tau_\varsigma \wedge \mathcal{U}_k \wedge \mathcal{W}_k} F_y(E_s, Y(s), \alpha_0) g(s, E_s, Y(s), \mathscr{L}(Y(s-))) dW_{E_s}$$

and

$$\begin{aligned} \int_0^{\tau_\varsigma \wedge \mathcal{U}_k \wedge \mathcal{W}_k} \int_{|z|<c} & [F(E_s, Y(s) + \hbar(s, E_s, Y(s), \mathscr{L}(Y(s-)), z), \alpha_0) \\ & -F(E_s, Y(s), \alpha_0)]\tilde{N}(dz, dE_s) \end{aligned}$$

are square integrable martingales with respect to $\mathcal{G}_t = \Im_{E_t}$ having mean 0. Using assumption (3) and taking the expectation on all sides of (5.8), we get

$$\mathbb{E}F(E_{\tau_\varsigma \wedge \mathcal{U}_k \wedge \mathcal{W}_k}, Y(\tau_\varsigma \wedge \mathcal{U}_k \wedge \mathcal{W}_k), \alpha_{\tau_\varsigma \wedge \mathcal{U}_k \wedge \mathcal{W}_k}) \leq F(0, y_0, \alpha_0).$$

Then by letting $k \longrightarrow \infty$,

$$\mathbb{E}F(E_{\tau_\varsigma}, Y(\tau_\varsigma), \alpha_{\tau_\varsigma}) \leq F(0, y_0, \alpha_0).$$

As $F(t_2, y, \alpha) > 0$,

$$\mathbb{E}\{F(E_{\tau_\varsigma}, Y(\tau_\varsigma), \alpha_{\tau_\varsigma}) 1_{\{\tau_\varsigma < \infty\}}\} \leq \mathbb{E}F(E_{\tau_\varsigma}, Y(\tau_\varsigma), \alpha_{\tau_\varsigma}) \leq F(0, y_0, \alpha_0). \tag{5.9}$$

Assumption (2) reads

$$\mathbb{E}\{\phi(|Y(\tau_\varsigma)|) 1_{\{\tau_\varsigma < \infty\}}\} \leq \mathbb{E}\{F(E_{\tau_\varsigma}, Y(\tau_\varsigma), \alpha_{\tau_\varsigma}) 1_{\{\tau_\varsigma < \infty\}}\}. \tag{5.10}$$

Using Eqs (5.5), and (5.10) and as the function ϕ is non-decreasing, we obtain

$$|Y(\tau_\varsigma)| \geq \varsigma$$

and

$$\phi(\varsigma)\mathbb{P}\{\tau_\varsigma < \infty\} \leq \mathbb{E}\{F(E_{\tau_\varsigma}, Y(\tau_\varsigma), \alpha_{\tau_\varsigma})1_{\{\tau_\varsigma<\infty\}}\}. \tag{5.11}$$

Combining (5.4), (5.9), and (5.11), we get $\mathbb{P}\{\tau_\varsigma < \infty\} \leq \varepsilon$, which indicates $\mathbb{P}\{\tau_\varsigma = \infty\} \geq 1 - \varepsilon$, i.e.,

$$\mathbb{P}\{|Y(t, y_0)| < \varsigma \;\; for\ all\ t \geq 0\} \geq 1 - \varepsilon.$$

Hence $Y(t, y_0)$ is stable in probability and the proof is finished. □

Definition 5.3. ([31,45]) If for every $\varepsilon \in (0, 1)$, there exists $0 < \delta_0 = \delta_0(\varepsilon)$ such that when $|y_0| < \delta_0$,

$$\mathbb{P}\{\lim_{t\longrightarrow\infty} Y(t, y_0) = 0\} \geq 1 - \varepsilon,$$

then the trivial solution of (1.2) is called asymptotically stable in probability provided that it is stable in probability.

Theorem 5.3. Suppose for all $(t_2, y, \alpha) \in \mathbb{R}^+ \times \mathscr{A}_h \times \mathscr{M}_2(\mathbb{R}^n)$ there exists a function $F(t_2, y, \alpha) \in \wp_{b,+}(\mathbb{R}^+ \times \mathscr{A}_h \times \mathscr{M}_2(\mathbb{R}^n))$ with $h > 2c$ and $\phi \in \mathscr{H}$ such that

(1) $F(t_2, 0, \delta_0) = 0$,
(2) $\phi(|y|) \leq F(t_2, y, \alpha)$,
(3) for any $\mu \in (0, h)$, $y \in \mathscr{A}_h - \bar{\mathscr{A}_\mu}$, $\mathbb{L}_{f,g,\hbar}F(t_2, y, \alpha) \leq -\gamma(\mu)$ a.s.,

where $\gamma(\mu) \geq 0$ and $\gamma(\mu) \neq 0$ when $\mu \neq 0$. Then the trivial solution of (1.2) is asymptotically stable in probability.

Proof. Depending on Theorem 5.2 and for any $0 < \varepsilon < 1$, there exists a positive $\Theta = \Theta(\varepsilon)$ such that

$$\mathbb{P}\{|Y(t, y_0)| < h\ for\ \forall\ t \geq 0\} \geq 1 - \frac{\varepsilon}{5}, \tag{5.12}$$

where $y_0 \in \mathscr{A}_\Theta$. Let $0 < \mu < \varpi < |y_0|$ be arbitrary and define

$$\tau_h := \inf\{t \geq 0 : |Y(t, y_0)| \geq h\}, \quad \tau_\mu := \inf\{t \geq 0 : |Y(t, y_0)| \leq \mu\},$$

$$\begin{aligned}\mathcal{U}_k &:= k \wedge \inf\Big\{t \geq 0 : |\int_0^{\tau_h\wedge\tau_\mu\wedge t} F_y(E_s, Y(s), \alpha_0)g(s, E_s, Y(s), \mathscr{L}(Y(s-)))dW_{E_s}| \geq k\Big\},\\ \mathcal{W}_k &:= k \wedge \inf\Big\{t \geq 0 : |\int_0^{\tau_h\wedge\tau_\mu\wedge t}\int_{|z|<c} [F(E_s, Y(s) + \hbar(s, E_s, Y(s), \mathscr{L}(Y(s-)), z), \alpha_0)\\ &\qquad - F(E_s, Y(s), \alpha_0)]\tilde{N}(dz, dE_s)| \geq k\Big\},\end{aligned} \tag{5.13}$$

for $k = 1, 2, ...$, where $y_0 \in \mathscr{A}_\Theta$ is fixed. Clearly, $\mathcal{U}_k, \mathcal{W}_k \longrightarrow \infty$ as $k \longrightarrow \infty$. Applying Lemma 4.1 and assumption (3) to (1.2), we have

$$\begin{array}{rcl} 0 & \leq & \mathbb{E}F(E_{t\wedge\tau_h\wedge\tau_\mu\wedge\mathcal{U}_k\wedge\mathcal{W}_k}, Y(t \wedge \tau_h \wedge \tau_\mu \wedge \mathcal{U}_k \wedge \mathcal{W}_k), \alpha_{t\wedge\tau_h\wedge\tau_\mu\wedge\mathcal{U}_k\wedge\mathcal{W}_k}) \\ & \leq & F(0, y_0, \alpha_0) - \gamma(\mu)\mathbb{E}E_{t\wedge\tau_h\wedge\tau_\mu\wedge\mathcal{U}_k\wedge\mathcal{W}_k}. \end{array}$$

Putting $t, k \longrightarrow \infty$, we get

$$\gamma(\mu)\mathbb{E}E_{\tau_h\wedge\tau_\mu} \leq F(0, y_0, \alpha_0).$$

Because $E_t \longrightarrow \infty$ as $t \longrightarrow \infty$ and $\gamma(\mu) \neq 0$,

$$\mathbb{P}\{\tau_h \wedge \tau_\mu < \infty\} = 1. \tag{5.14}$$

From (5.12) and (5.13), it follows that $\mathbb{P}(\tau_h < \infty) \leq \frac{\varepsilon}{5}$. Therefore,

$$1 = \mathbb{P}\{\tau_h \wedge \tau_\mu < \infty\} \leq \mathbb{P}\{\tau_h < \infty\} + \mathbb{P}\{\tau_\mu < \infty\} \leq \mathbb{P}\{\tau_\mu < \infty\} + \frac{\varepsilon}{5}.$$

Hence, $\mathbb{P}\{\tau_\mu < \infty\} \geq 1 - \frac{\varepsilon}{5}$. Also, there exists a positive constant $\bar{\Theta} = \bar{\Theta}(\mu)$ such that

$$\mathbb{P}\{\tau_\mu < \bar{\Theta}\} \geq 1 - \frac{2\varepsilon}{5}.$$

Additionally,

$$\begin{array}{rcl} \mathbb{P}\{\tau_\mu < \tau_h \wedge \bar{\Theta}\} & \geq & \mathbb{P}(\{\tau_\mu < \bar{\Theta}\} \cap \{\tau_h = \infty\}) \\ & = & \mathbb{P}(\tau_\mu < \bar{\Theta}) - \mathbb{P}(\{\tau_\mu < \bar{\Theta}\} \cap \{\tau_h < \infty\}) \\ & \geq & \mathbb{P}(\tau_\mu < \bar{\Theta}) - \mathbb{P}(\tau_h < \infty) \geq 1 - \dfrac{2\varepsilon}{5} - \dfrac{\varepsilon}{5} = 1 - \dfrac{3\varepsilon}{5}. \end{array} \tag{5.15}$$

We define the stopping times

$$\varrho := \begin{cases} \tau_\mu, & if \ \ \tau_\mu < \tau_h \wedge \bar{\Theta}, \\ \infty, & otherwise, \end{cases} \tag{5.16}$$

$$\begin{aligned} \tau_\varpi &:= \inf\{t \geq \varrho : |Y(t, y_0)| \geq \varpi\}, \\ M_\ell &:= \inf\Big\{t \geq \varrho : \Big|\int_\varrho^{\tau_\varpi\wedge t} F_y(E_s, Y(s), \alpha_0) g(s, E_s, Y(s), \mathscr{L}(Y(s-))) dW_{E_s}\Big| \geq \ell\Big\}, \\ R_\ell &:= \inf\Big\{t \geq \varrho : \Big|\int_\varrho^{\tau_\varpi\wedge t} \int_{|z|<c} [F(E_s, Y(s) + \hbar(s, E_s, Y(s), \mathscr{L}(Y(s-)), z), \alpha_0) \\ &\qquad - F(E_s, Y(s), \alpha_0)]\tilde{N}(dz, dE_s)\Big| \geq \ell\Big\}, \end{aligned} \tag{5.17}$$

for $\ell = 1, 2, ...$, where $M_\ell, R_\ell \longrightarrow \infty$ as $\ell \longrightarrow \infty$. Again, using Lemma 4.1 and assumption (3), we have

$$\begin{aligned} &\mathbb{E}F(E_{t\wedge\tau_h\wedge\tau_\varpi\wedge M_\ell\wedge R_\ell}, Y(t \wedge \tau_h \wedge \tau_\varpi \wedge M_\ell \wedge R_\ell), \alpha_{t\wedge\tau_h\wedge\tau_\varpi\wedge M_\ell\wedge R_\ell}) \\ &\leq \mathbb{E}F(E_{t\wedge\tau_h\wedge\varrho}, Y(t \wedge \tau_h \wedge \varrho), \alpha_{t\wedge\tau_h\wedge\varrho}) \end{aligned}$$

for all $\ell = 1, 2, ...$ and $t \in \mathbb{R}^+$. Letting $\ell \longrightarrow \infty$ followed by $t \longrightarrow \infty$,

$$\mathbb{E}F(E_{\tau_\varpi \wedge \tau_h}, Y(\tau_\varpi \wedge \tau_h), \alpha_{\tau_\varpi \wedge \tau_h}) \leq \mathbb{E}F(E_{\varrho \wedge \tau_h}, Y(\varrho \wedge \tau_h), \alpha_{\varrho \wedge \tau_h}). \tag{5.18}$$

Combining Eqs (5.16), (5.17), and (5.18),

$$\mathbb{E}\{F(E_{\tau_\varpi \wedge \tau_h}, Y(\tau_\varpi \wedge \tau_h), \alpha_{\tau_\varpi \wedge \tau_h})1_{\{\tau_\varpi < \infty\}}\} \leq \mathbb{E}\{F(E_{\varrho \wedge \tau_h}, Y(\varrho \wedge \tau_h), \alpha_{\varrho \wedge \tau_h})1_{\{\varrho < \infty\}}\}, \tag{5.19}$$

which implies, from (5.15), that

$$\mathbb{E}\{F(E_{\tau_\varpi}, Y(\tau_\varpi), \alpha_{\tau_\varpi})1_{\{\tau_\varpi \wedge < \infty\} \cap \{\tau_h = \infty\}}\} \leq \mathbb{E}\{F(E_{\tau_\mu}, Y(\tau_\mu), \alpha_{\tau_\mu})1_{\{\tau_\mu < \tau_h \wedge \bar{\Theta}\}}\}. \tag{5.20}$$

Furthermore, we define

$$A_\mu := \sup\{F(t_2, y, \alpha) : (t_2, y, \alpha) \in \mathbb{R}^+ \times \bar{\mathscr{A}}_\mu \times \mathscr{M}_2(\mathbb{R}^n)\}, \tag{5.21}$$

with $\lim_{\mu \longrightarrow 0} A_\mu = 0$ according to assumption (1). So, there exists a small μ such that

$$\frac{A_\mu}{\phi(\varpi)} < \frac{\varepsilon}{5}. \tag{5.22}$$

From (5.20)–(5.22),

$$\mathbb{P}\Big\{\{\tau_\varpi < \infty\} \cap \{\tau_h = \infty\}\Big\} \leq \frac{A_\mu}{\phi(\varpi)} < \frac{\varepsilon}{5}. \tag{5.23}$$

Similar to (5.15), it is easy to derive

$$\mathbb{P}\Big\{\{\tau_\varpi < \infty\} \cap \{\tau_h = \infty\}\Big\} \geq \mathbb{P}\{\tau_\varpi < \infty\} - \mathbb{P}\{\tau_h < \infty\} \geq \mathbb{P}\{\tau_\varpi < \infty\} - \frac{\varepsilon}{5}. \tag{5.24}$$

From (5.23) and (5.24), we obtain

$$\mathbb{P}\{\tau_\varpi < \infty\} < \frac{2\varepsilon}{5}. \tag{5.25}$$

Combining (5.15) and (5.25), we obtain

$$\mathbb{P}\{\varrho < \infty \textit{ and } \tau_\varpi = \infty\} \geq \mathbb{P}\{\varrho < \infty\} - \mathbb{P}\{\tau_\varpi < \infty\} > 1 - \varepsilon,$$

which indicates

$$\mathbb{P}\{\omega : \limsup_{t \longrightarrow \infty} |Y(t, y_0)| \leq \varpi\} > 1 - \varepsilon.$$

Lastly, since ϖ is arbitrary, we put $\varpi \longrightarrow 0$ to arrive at

$$\mathbb{P}\{\omega : \lim_{t \longrightarrow \infty} |Y(t, y_0)| = 0\} \geq 1 - \varepsilon.$$

The proof is then complete. □

Definition 5.4. ([31, 45]) Assume for all $y_0 \in \mathbb{R}^n$ the trivial solution of (1.2) is stable in probability such that

$$\mathbb{P}\{\lim_{t\longrightarrow\infty} |Y(t, y_0)| = 0\} = 1.$$

Then it is called globally asymptotically stable in probability or stochastically asymptotically stable in large.

Theorem 5.4. Suppose for all $(t_2, y, \alpha) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathscr{M}_2(\mathbb{R}^n)$, there exists a function $F(t_2, y, \alpha) \in \wp_{b,+}(\mathbb{R}^+ \times \mathbb{R}^n \times \mathscr{M}_2(\mathbb{R}^n))$ with $h > 2c$ and $\phi \in \mathscr{H}$ such that

(1) $F(t_2, 0, \delta_0) = 0$,
(2) $\phi(|y|) \le F(t_2, y, \alpha)$,
(3) $\lim_{|y|\longrightarrow\infty} \inf_{t_2\ge0,\alpha\in\mathscr{M}_2(\mathbb{R}^n)} F(t_2, y, \alpha) = \infty$,
(4) $\mathbb{L}_{f,g,\hbar}F(t_2, y, \alpha) \le -\gamma(y)$ a.s.,

where $\gamma(y) \ge 0$ and $\gamma(y) \ne 0$ when $y \ne 0$. Then the trivial solution of (1.2) is globally asymptotically stable in probability.

Proof. The proof uses a similar idea as Theorem 4 in [40], so we omit it here. □

We close this section by giving an example to enhance the theoretical results.

Example 5.1. Consider the following one-dimensional mean-field stochastic differential equation driven by time-changed Lévy noise:

$$\begin{aligned} dY(t) &= -b(t, E_t)\sin\Big(\int_{\mathbb{R}} x\alpha_t(dx)\Big)dE_t + \sigma(t, E_t)\sqrt{\Big|\int_{\mathbb{R}} x\alpha_t(dx)\Big|}dW_{E_t} \\ &\quad + \int_{|z|<c} \sigma_1(t, E_t, z)Y(t)\tilde{N}(dE_t, dz), \end{aligned} \tag{5.26}$$

where

$$\begin{aligned} f(t_1, t_2, y, \alpha) &= -b(t_1, t_2)\sin\Big(\int_{\mathbb{R}} x\alpha(dx)\Big), \\ g(t_1, t_2, y, \alpha) &= \sigma(t_1, t_2)\sqrt{\Big|\int_{\mathbb{R}} x\alpha(dx)\Big|}, \\ \hbar(t_1, t_2, y, \alpha, z) &= \sigma_1(t_1, t_2, z)y, \end{aligned}$$

with initial value $Y(0) = y_0$, where $b(t_1, t_2)$, $\sigma(t_1, t_2)$, and $\sigma(t_1, t_2, z)$ are real-valued and $\mathcal{G}_t$-measurable functions satisfying

$$0 < b^2(t_1, t_2), \sigma^2(t_1, t_2), \sigma_1^2(t_1, t_2, z) \le K,$$

for all $t_1, t_2 \in \mathbb{R}^+ \times \mathbb{R}^+$ and $K > 0$. So, one can justify that for $(t_1, t_2, y, \alpha) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \times \mathscr{M}_2(\mathbb{R})$ and using the fact that $\sin(l) \le l$, we have

$$|f(t_1, t_2, y, \alpha)|^2 + \|g(t_1, t_2, y, \alpha)\|^2 + \int_{|z|<c} |\hbar(t_1, t_2, y, \alpha, z)|^2\nu(dz)$$

$$
\begin{aligned}
&\le K\Big\{\Big|\sin\Big(\int_{\mathbb{R}} x\alpha(dx)\Big)\Big|^2 + \Big|\int_{\mathbb{R}} x\alpha(dx)\Big| + |y|^2\Big\}\\
&\le K\Big\{\Big|\sin\Big(\int_{\mathbb{R}} x\alpha(dx)\Big)\Big| + \Big|\int_{\mathbb{R}} x\alpha(dx)\Big| + |y|^2\Big\}\\
&\le K\Big\{|y|^2 + 2\Big|\int_{\mathbb{R}} x\alpha(dx)\Big|\Big\}\\
&\le K\Big\{1 + |y|^2 + \Big|\int_{\mathbb{R}} x\alpha(dx)\Big|^2\Big\}\\
&\le K\{1 + |y|^2 + \rho_2^2(\alpha, \delta_0)\}.
\end{aligned}
$$

Moreover, for $(y_1, \alpha_1), (y_2, \alpha_2) \in \mathbb{R} \times \mathscr{M}_2(\mathbb{R})$,

$$
\begin{aligned}
&|f(t_1, t_2, y_1, \alpha_1) - f(t_1, t_2, y_2, \alpha_2)|^2 + \|g(t_1, t_2, y_1, \alpha_1) - g(t_1, t_2, y_2, \alpha_2)\|^2\\
&\quad + \int_{|z|<c} |\hbar(t_1, t_2, y_1, \alpha_1, z) - \hbar(t_1, t_2, y_2, \alpha_2, z)|^2 \nu(dz)\\
&\le K\Big\{|y_1 - y_2|^2 + \Big|\sqrt{\Big|\int_{\mathbb{R}} x\alpha_1(dx)\Big|} - \sqrt{\Big|\int_{\mathbb{R}} x\alpha_2(dx)\Big|}\Big|^2\\
&\quad + \Big|\sin\Big(\int_{\mathbb{R}} x\alpha_1(dx)\Big) - \sin\Big(\int_{\mathbb{R}} x\alpha_2(dx)\Big)\Big|^2\Big\}\\
&\le K\Big\{|y_1 - y_2|^2 + \Big|\sqrt{\Big|\int_{\mathbb{R}} x\alpha_1(dx) - \int_{\mathbb{R}} x\alpha_2(dx)\Big|}\Big|^2\\
&\quad + 4\Big|\sin\frac{\int_{\mathbb{R}} x\alpha_1(dx) - \int_{\mathbb{R}} x\alpha_2(dx)}{2}\Big|^2\Big\}\\
&\le K\Big\{|y_1 - y_2|^2 + \Big|\sqrt{\Big|\int_{\mathbb{R}} x\alpha_1(dx) - \int_{\mathbb{R}} x\alpha_2(dx)\Big|}\Big|^2\\
&\quad + \Big|\int_{\mathbb{R}} x\alpha_1(dx) - \int_{\mathbb{R}} x\alpha_2(dx)\Big|^2\Big\}\\
&\le K\Big\{|y_1 - y_2|^2 + \Big|\int_{\mathbb{R}} x\alpha_1(dx) - \int_{\mathbb{R}} x\alpha_2(dx)\Big|^2\\
&\quad + \kappa^2\Big(\Big|\int_{\mathbb{R}} x\alpha_1(dx) - \int_{\mathbb{R}} x\alpha_2(dx)\Big|\Big)\Big\},
\end{aligned}
$$

where we have used the fact that $\kappa(u) = \sqrt{u} \in [0, \infty)$, which is increasing with $\kappa(0) = 0$ and satisfying $\int_{0^+} \frac{1}{\kappa(u)} du = \infty$. Now, taking $\hat{\kappa}(u) = u + \kappa^2(u^{\frac{1}{2}})$, we obtain

$$
\begin{aligned}
&|f(t_1, t_2, y_1, \alpha_1) - f(t_1, t_2, y_2, \alpha_2)|^2 + \|g(t_1, t_2, y_1, \alpha_1) - g(t_1, t_2, y_2, \alpha_2)\|^2\\
&\quad + \int_{|z|<c} |\hbar(t_1, t_2, y_1, \alpha_1, z) - \hbar(t_1, t_2, y_2, \alpha_2, z)|^2 \nu(dz)\\
&\le K\Big\{|y_1 - y_2|^2 + \Big|\int_{\mathbb{R}} x\alpha_1(dx) - \int_{\mathbb{R}} x\alpha_2(dx)\Big|^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \kappa^2\bigg(\bigg(\bigg|\int_{\mathbb{R}} x\alpha_1(dx) - \int_{\mathbb{R}} x\alpha_2(dx)\bigg|^2\bigg)^{\frac{1}{2}}\bigg)\bigg\} \\
&\le K\bigg\{|y_1 - y_2|^2 + \hat{\kappa}\bigg(\bigg|\int_{\mathbb{R}} x\alpha_1(dx) - \int_{\mathbb{R}} x\alpha_2(dx)\bigg|^2\bigg)\bigg\} \\
&\le K\{|y_1 - y_2|^2 + \hat{\kappa}(\rho_2^2(\alpha_1, \alpha_2))\},
\end{aligned}
$$

where $\hat{\kappa}(u)$ is a non-decreasing function on $\mathbb{R}^+$ and $\hat{\kappa}(0) = 0$. Moreover, we can obtain

$$
\int_{0^+} \frac{ds}{\hat{\kappa}(s)} = \int_{0^+} \frac{1}{\kappa^2(s^{\frac{1}{2}}) + s} ds = \int_{0^+} \frac{u}{\kappa^2(u) + u^2} du = \infty.
$$

Thus, the coefficients $f, g, \hbar$ satisfy Conditions 3.1–3.3, which yields that Eq (5.26) has a strong solution.

For some $0 < \beta < 1$, we define

$$
F(t_2, y, \alpha) = \bigg(1 + \cos\bigg(\int_{\mathbb{R}} x\alpha(dx)\bigg)\bigg)|y|^{\beta},
$$

and it is easily seen that

$$
\begin{aligned}
\partial_{t_2} F(t_2, y, \alpha) = 0, \qquad & F_y(t_2, y, \alpha) = \beta\bigg(1 + \cos\bigg(\int_{\mathbb{R}} x\alpha(dx)\bigg)\bigg)|y|^{\beta-1}, \\
F_{yy}(t_2, y, \alpha) &= \beta(\beta - 1)\bigg(1 + \cos\bigg(\int_{\mathbb{R}} x\alpha(dx)\bigg)\bigg)|y|^{\beta-2}, \\
\partial_{\alpha} F(t_2, y, \alpha)(v) = -\sin\bigg(\int_{\mathbb{R}} x\alpha(dx)\bigg)|y|^{\beta}, \qquad & \partial_v \partial_{\alpha} F(t_2, y, \alpha)(v) = 0.
\end{aligned}
$$

Hence, it holds that

$$
\begin{aligned}
&\mathbb{L}_{f,g,\hbar} F(t_2, y, \alpha) \\
&\quad = -\beta \sin\bigg(\int_{\mathbb{R}} x\alpha(dx)\bigg)\bigg(1 + \cos\bigg(\int_{\mathbb{R}} x\alpha_0(dx)\bigg)\bigg) b(t_1, t_2)|y|^{\beta-1} \\
&\qquad + \frac{\beta(\beta - 1)}{2}\bigg(1 + \cos\bigg(\int_{\mathbb{R}} x\alpha_0(dx)\bigg)\bigg)\sigma^2(t_1, t_2)\bigg|\int_{\mathbb{R}} x\alpha(dx)\bigg||y|^{\beta-2} \\
&\qquad + \sin^2\bigg(\int_{\mathbb{R}} x\alpha(dx)\bigg) b(t_1, t_2)\bigg(\int_{\mathbb{R}} x\alpha(dx)\bigg)|y|^{\beta} \\
&\qquad + \int_{|z|<c} \int_0^1 \bigg[\bigg(-\sin\bigg(\int_{\mathbb{R}} x\alpha(dx)\bigg)|y|^{\beta}(y + \xi\sigma_1(t_1, t_2, z)y) \\
&\qquad\quad + \sin\bigg(\int_{\mathbb{R}} x\alpha(dx)\bigg)|y|^{\beta}\bigg(\int_{\mathbb{R}} x\alpha(dx)\bigg)\bigg)\sigma_1(t_1, t_2, z)y\bigg] d\xi \nu(dz) \\
&\qquad + \int_{|z|<c} \bigg[\bigg(1 + \cos\bigg(\int_{\mathbb{R}} x\alpha_0(dx)\bigg)\bigg)|y + \sigma_1(t_1, t_2, z)y|^{\beta} \\
&\qquad\quad - \bigg(1 + \cos\bigg(\int_{\mathbb{R}} x\alpha_0(dx)\bigg)\bigg)|y|^{\beta} - \beta\bigg(1 + \cos\bigg(\int_{\mathbb{R}} x\alpha_0(dx)\bigg)\bigg) \\
&\qquad\quad \times \sigma_1(t_1, t_2, z)|y|^{\beta}\bigg]\nu(dz).
\end{aligned}
$$

Therefore, if

$$\begin{cases} b(t_1,t_2)\Big(\int_{\mathbb{R}} x\alpha(dx)\Big)\sin^2\Big(\int_{\mathbb{R}} x\alpha(dx)\Big)+\Big(1+\cos\Big(\int_{\mathbb{R}} x\alpha_0(dx)\Big)\Big) \\ \qquad \times\int_{|z|<c}[|1+\sigma_1(t_1,t_2,z)|^{\beta}-1-\beta\sigma_1(t_1,t_2,z)]\nu(dz)\le 0, \\ \int_{|z|<c}\sin\Big(\int_{\mathbb{R}} x\alpha(dx)\Big)\Big(\int_{\mathbb{R}} x\alpha(dx)\Big)\sigma_1(t_1,t_2,z)\nu(dz)\le 0, \\ \int_{|z|<c}\int_0^1\sin\Big(\int_{\mathbb{R}} x\alpha(dx)\Big)(1+\xi\sigma_1(t_1,t_2,z))\sigma_1(t_1,t_2,z)d\xi\nu(dz)\ge 0, \\ \beta b(t_1,t_2)\sin\Big(\int_{\mathbb{R}} x\alpha(dx)\Big)\Big(1+\cos\Big(\int_{\mathbb{R}} x\alpha_0(dx)\Big)\Big)\ge 0, \\ \frac{\beta(\beta-1)}{2}\sigma^2(t_1,t_2)\Big|\int_{\mathbb{R}} x\alpha(dx)\Big|\Big(1+\cos\Big(\int_{\mathbb{R}} x\alpha_0(dx)\Big)\Big)\le 0 \end{cases} \tag{5.27}$$

hold a.s. for all $(t_1,t_2)\in(J,\mathbb{R}^+)$, then from Theorem 5.2, it is known that the trivial solution of Eq (5.26) is stable in probability. If inequalities (5.27) hold strictly, Theorem 5.4 guarantees that the trivial solution of Eq (5.26) is globally asymptotically stable in probability.

6. Conclusions

In this work, we have investigated a new class of MFSDEs perturbed by time-changed Lévy noise. To show the existence and uniqueness of solutions, we have derived a new time-changed retarded integral inequality and used the Carathéodory approximations and non-Lipschitz condition, generalizing some existing results. Second, the classical Itô formula is extended from SDEs to MFSDEs perturbed by time-changed Lévy noise. Then, to show the applicability of Itô formula, we have studied the p-th moment asymptotic stability, stability in probability, asymptotic stability in probability, and globally asymptotic stability in probability for the trivial solution depending on the Lyapunov function. Finally, the theoretical results are validated through an example. For the future investigations, we may study the practical stability with respect to part of the variables of MFSDEs perturbed by time-changed Lévy noise.

Author contributions

M. Abouagwa: Validation, writing–original draft, supervision, conceptualization, writing–review and editing; M. I. Tawdrous: Validation, writing–review and editing. All authors have read and agreed to the revised version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors sincerely thank the anonymous reviewers and academic editor who have helped to improve the paper.

Conflict of interest

The authors declare no conflicts.

References

1. M. Abouagwa, A. D. Khalaf, N. Gul, S. Alyobi, A. S. Mohamed, Multivalued impulsive SDEs driven by G-Brownian noise: Periodic averaging result, *Complexity*, 2022, 1–16. https://doi.org/10.1155/2022/5619693
2. M. Abouagwa, An averaging limit theorem for impulsive delay stochastic fractional differential equations, *Contemp. Math.*, **6** (2025), 6671–6688. https://doi.org/10.37256/cm.6520257180
3. D. Applebaum, *Lévy processes and stochastic calculus*, Cambridge University Press, **116** (2009). https://doi.org/10.1017/CBO9780511809781
4. K. Bahlali, M. A. Mezerdi, B. Mezerdi, Stability of McKean-Vlasov stochastic differential equations and applications, *Stoch. Dyn.*, **20** (2020), 2050007. https://doi.org/10.1142/S0219493720500070
5. I. Bihari, A generalization of a lemma of Bellman and its application to uniqueness problem of differential equations, *Acta Math. Hung.*, **7** (1956), 81–94. https://doi.org/10.1007/bf02022967
6. J. F. Chassagneux, D. Crisan, F. Delarue, A Probabilistic approach to classical solutions of the master equation for large population equilibria, *Mem. Amer. Math. Soc.*, **280** (2022). https://doi.org/10.1090/memo/1379
7. J. C. Cox, J. E. Ingersoll, S. A. Ross, A theory of the term structure of interest rates, *Econometrica*, **53** (1985), 385–407.
8. M. Gao, J. Zhao, W. Sun, Stochastic H_2/H_∞ control for discrete-time mean-field systems with Poisson Jump, *J. Frankli. Inst.*, **358** (2021), 2933–2947. https://doi.org/10.1016/j.jfranklin.2021.02.005
9. W. R. P. Hammersley, D. Šiška, L. Szpruch, McKean-Vlasov SDEs under measure dependent Lyapunov conditions, *Ann. Inst. H. Poincaré Probab. Statist.*, **57** (2021), 1032–1057. https://doi.org/10.1214/20-AIHP1106
10. S. L. Heston, A closed-form solution for options with stochastic volatiltiy with applications to bond and currency options, *Rev. Fin. Stud.*, **6** (1993), 327–343. https://doi.org/10.1093/rfs/6.2.327
11. X. Huang, M. Röckner, F. Y. Wang, Nonlinear Fokker-Planck equations for probability measures on path space and path-distribution dependent SDEs, *Discrt. Cont. Dyn. Syst. B*, **39** (2019), 3017–3035. https://doi.org/10.3934/dcds.2019125

12. S. Jin, K. Kobayashi, Strong approximation of stochastic differential equations driven by a time-changed Brownian motion with time-space-dependent coefficients, *J. Math. Anal. Appl.*, **476** (2019), 619–636. https://doi.org/10.1016/j.jmaa.2019.04.001
13. M. Kac, *Foundations of kinetic theory*, In: Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, Vol. III, 1956, 171–197. Available from: `https://api.semanticscholar.org/CorpusID:67833762`.
14. A. D. Khalaf, A. Zeb, T. Saeed, M. Abouagwa, S. Djilali, H. M. Alshehri, A special study of the mixed weighted fractional Brownian motion, *Fractal Fract.*, **5** (2021), 1–10. https://doi.org/10.3390/fractalfract5040192
15. A. D. Khalaf, T. Saeed, R. Abu-Shanab, W. Almutiry, M. Abouagwa, Estimating drift parameters in a sub-fractional Vasicek-type process, *Fractal Fract.*, **24** (2022), 1–19. https://doi.org/10.3390/e24050594
16. K. Kobayashi, Stochastic calculus for a time-changed semimartingales and the associated stochastic differential equations, *J. Theor. Probab.*, **24** (2011), 789–820. https://doi.org/10.1007/s10959-010-0320-9
17. H. H. Kuo, *Introduction to stochastic integration*, Springer New York, 2005. https://doi.org/10.1007/0-387-31057-6
18. Y. K. Kwow, Pricing multi-asset options with an external barrier, *Int. J. Theor. Appl. Fin.*, **1** (1998), 523–541. https://doi.org/10.1142/S021902499800028X
19. X. Li, Y. Ren, On the practical stability with regard to a part of the variables for distribution-dependent SDEs driven by time-changed Brownian motion, *Int. J. Control*, **96** (2023), 2911–2916. https://doi.org/10.1080/00207179.2022.2117088
20. Z. Li, L. P. Xu, W. Ma, Global attracting sets and exponential stability of stochastic functional differential equations driven by the time-changed Brownian motion, *Syst. Contr. Lett.*, **160** (2022), 105103. https://doi.org/10.1016/j.sysconle.2021.105103
21. Z. Li, L. Xu, L. Yan, Mckean-Vlasov stochastic differential equations driven by time-changed Brownian motion, *J. Math. Anal. Appl.*, **527** (2023), 127336. https://doi.org/10.1016/j.jmaa.2023.127336
22. M. Magdziarz, Stochastic representation of subdiffusion processes with time-dependent drift, *Stoch. Proc. Appl.*, **119** (2009), 3238–3252. https://doi.org/10.1016/j.spa.2009.05.006
23. M. Magdziarz, Path properties of subdiffusion-A martingale approach, *Stoch. Models*, **26** (2010), 256–271. https://doi.org/10.1080/15326341003756379
24. F. Mainardi, On some properties of the Mittag-Leffler function $E_\alpha(-t^\alpha)$, completely monotone for $t > 0$ with $0 < \alpha < 1$, *Discrt. Cont. Dyn. Syst. B*, **19** (2014), 2267–2278. https://doi.org/10.3934/dcdsb.2014.19.2267
25. X. Mao, Adapted solutions of backward stochastic differential equations with non-Lipschitz coefficents, *Stoch. Process. Appl.*, **58** (1995), 281–292. https://doi.org/10.1016/0304-4149(95)00024-2
26. H. P. McKean, A class of Markov processes associated with nonlinear parabolic equations, *P. Natl. Acad. Sci. USA*, **56** (1966), 1907–1911. https://doi.org/10.1073/pnas.56.6.1907

27. M. M. Meerscheart, H. P. Scheffler, Limit theorems for continuous time random walks with infinite mean waiting times, *J. Appl. Prob.*, **41** (2004), 623–638. Available from: `https://www.jstor.org/stable/4141342`.
28. M. M. Meerscheart, H. P. Scheffler, Triangular array limits for continuous time random walks, *Stoch. Process. Appl.*, **118** (2008), 1606–1633. https://doi.org/10.1016/j.spa.2007.10.005
29. M. A. Mezerdi, N. Khelfallah, Stability and prevalence of McKean-Vlasov stochastic differential equations with non-Lipschitz coefficients, *Random Oper. Stoch. Equ.*, **29** (2021), 67–78. https://doi.org/10.1515/rose-2021-2053
30. Y. Mishura, A. Veretennikov, Existence and uniqueness theorems for solutions of McKean-Vlasov stochastic equations, *Theor. Probab. Math. Stat.*, **103** (2020), 59–101. https://doi.org/10.1090/tpms/1135
31. E. Nane, Y. Ni, Stability of the solution of stochastic differential equation driven by time-changed Lévy noise, *Proc. Am. Math. Soc.*, **145** (2017), 3085–3104. http://dx.doi.org/10.1090/proc/13447
32. E. Nane, Y. Ni, Path stability of stochastic differential equations driven by time-changed Lévy noises, *ALEA Lat. Am. J. Probab. Math. Stat.*, **15** (2018), 479–507. http://dx.doi.org/10.30757/ALEA.v15-20
33. E. Nane, Y. Ni, A time-changed stochastic control problem and its maximum principle theory, *Probab. Math. Stat.*, **41** (2021), 193–215. http://dx.doi.org/10.37190/0208-4147.41.2.1
34. C. Peng, W. Zhang, L. Ma, Infinite horizon multiobjective optimal control of stochastic cooperative linear-quadratic dynamic difference games, *J. Frankl. Inst.*, **358** (2021), 8288–8307. https://doi.org/10.1016/j.jfranklin.2021.08.022
35. C. Peng, W. Zhang, Pareto optimality in infinite horizon mean-field stochastic cooperative linear-quadratic difference games, *IEEE T. Automat. Contr.*, **68** (2023), 4113–4126. https://doi.org/10.1109/TAC.2022.3202824
36. H. Pham, X. Wei, Dynamic programming for optimal control of stochastic Mckean-Vlasov dynamics, *SIAM J. Control Optim.*, **55** (2017), 1069–1101. https://doi.org/10.1137/16M1071390
37. H. Qiao, J. Gong, The stability for multivalued McKean-Vlasov SDEs with non-Lipschitz coefficients, *Front. Math.*, **20** (2025), 905–932. https://doi.org/10.1007/s11464-022-0273-1
38. P. Ren, F. Y. Wang, Space-Distribution PDEs for path independent additive functionals of McKean–Vlasov SDEs, *Infin. Dimens. Anal. Qu.*, **23** (2020), 2050018. https://doi.org/10.1142/S0219025720500186
39. G. Shen, W. Xu, J. L. Wu, An averagig principle for stochastic differential delay equations driven by time-changed Lévy noise, *Acta Math. Sci.*, **42** (2022), 540–550. https://doi.org/10.1007/s10473-022-0208-7
40. G. Shen, T. Zhang, J. Song, J. L. Wu, On a class of distribution dependent stochastic differential equations driven by time-changed Brownian motions, *Appl. Math. Optim.*, **88** (2023), 1–31. https://doi.org/10.1007/s00245-023-10007-3
41. G. Shen, J. Xiang, J. L. Wu, Stochastic averaging principle and stability for multi-valued McKean-Vlasov stochastic differential equations with jumps, *arXiv preprint*, 2023. https://doi.org/10.48550/arXiv.2308.02195

42. S. Umarov, M. Hahn, K. Kobayashi, *Beyond the triangle: Brownian motion, Itô calculus, and Fokker-Planck equation-fractional generalizations*, World Scientific, 2018.
43. X. Q. Wen, Z. Li, L. Xu, Strong approximation of non-autonomous time-changed McKean-Vlasov stochastic differential equations, *Commun. Nonlinear Sci.*, **119** (2023), 107122. https://doi.org/10.1016/j.cnsns.2023.107122
44. Q. Wu, Stability analysis for a class of nonlinear time-changed systems, *Cogent Math.*, **3** (2016), 1228273. https://doi.org/10.1080/23311835.2016.1228273
45. X. W. Yin, W. T. Xu, G. J. Shen, Stability of stochastic differential equations driven by the time-changed Lévy process with impulsive effects, *Int. J. Syst. Sci.*, **52** (2021), 2338–2357. https://doi.org/10.1080/00207721.2021.1885763
46. X. Zhang, C. Yuan, Razumikhin-type theorem on time-changed stochastic functional differential equations with Markovian switching, *Open Math.*, **17** (2019), 689–699. https://doi.org/10.1515/math-2019-0055
47. W. Zhang, C. Peng, Indefinite mean-field stochastic cooperative linear-quadratic dynamic difference game with its application to the network security model, *IEEE Trans. Cybern.*, **52** (2022), 11805–11818. https://doi.org/10.1109/TCYB.2021.3070352
48. S. Zhang, Z. Q. Chen, Stochastic maximum principle for subdiffusions and its applications, *SIAM J. Control Optim.*, **62** (2024), 953–981. https://doi.org/10.1137/23M157168X
49. M. Zhu, J. P. Li, D.Z. Liu, Exponential stability for time-changed stochastic differential equations, *Acta Math. Appl. Sin.-E.*, **37** (2021), 617–627. https://doi.org/10.1007/s10255-021-1031-y

AIMS Press