Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(6): 15912–15925.
DOI: 10.3934/math.2026655
Received: 24 February 2026
Revised: 27 May 2026
Accepted: 01 June 2026
Published: 05 June 2026

Research article

On primes of the form $\lfloor \alpha p^c + \beta \rfloor$ for almost all α

Yanbo Song*

Department of Mathematics, School of Information Engineering, Xi'an University, Xi'an 710065, China

* **Correspondence:** Email: syb888202106@163.com; Tel: +86-18291806025.

Abstract: In this paper, we prove that for almost all irrational numbers $\alpha > 0$ (in the sense of Lebesgue measure), there exist infinitely many primes p in a Beatty sequence such that $\lfloor \alpha p^c + \beta \rfloor$ is also a prime, where $c \in (1, \frac{150}{119})$.

Keywords: prime number; Goldbach conjecture; twin prime conjecture; Selberg's sieve method; generalized Piatetski-Shapiro sequence; Beatty sequence
Mathematics Subject Classification: 11P32, 11L07

1. Introduction

In 2009, motivated by the Goldbach conjecture and the twin prime conjecture, Li and Pan [8] proposed the following conjecture.

Conjecture 1. *Let $\alpha > 0$ be an irrational number, and let β be a real number. Then, there exist infinitely many primes p such that $\lfloor \alpha p + \beta \rfloor$ is also a prime.*

Concerned with the above conjecture, Li and Pan [8] proved a weaker theorem (Theorem 1 in [8]) as follows.

Theorem 1.1. *Let β be a real number. Then, for almost all irrational numbers $\alpha > 0$ (in the sense of Lebesgue measure),*

$$\limsup_{x\to\infty} \frac{\pi^*_{\alpha,\beta}(x)(\log^2 x)}{x} \geq 1,$$

where

$$\pi^*_{\alpha,\beta}(x) = |\{p \leq x : \text{both } p \text{ and } \lfloor \alpha p + \beta \rfloor \text{ are primes}\}|.$$

On the other hand, the Piatetski-Shapiro sequence is a fundamental object in analytic number theory, named after the Soviet mathematician Israel Moiseevich Piatetski-Shapiro. It refers to sequences of the form $\{\lfloor n^c \rfloor\}_{n=1}^{\infty}$, where $c > 1$ is a real number. The theory of Piatetski-Shapiro primes plays an important role in analytic number theory. In 1953, Piatetski-Shapiro [12] gave an asymptotic formula for the counting function of Piatetski-Shapiro prime for $c \in (1, 12/11)$. It is worth emphasizing that the range $c \in (1, 12/11)$ was improved subsequently by Kolesnik [6], Heath-Brown [4], Kolesnik [7], and Liu and Rivat [9]. The best known result in this direction is due to Rivat and Sargos [13], which gave an asymptotic formula for the counting function of Piatetski-Shapiro prime for $c \in (1, 2817/2426)$. We remark that Rivat and Wu [14] gave the lower bound for the counting function for $c \in (1, 243/205)$. This result is the best regarding the infinitude of the Piatetski-Shapiro primes. In [10], Li, Qi and Zhang studied prime of the general form $\lfloor \alpha n^c + \beta \rfloor$, and they called it generalized Piatetski-Shapiro prime. In [3], Guo studied prime numbers in the intersection of Piatetski-Shapiro sequences and Beatty sequences. Motivated by these results, in this paper we prove the following result.

Theorem 1.2. *For almost all irrational numbers $\alpha > 0$ (in the sense of Lebesgue measure), let $c \in (1, 150/119)$, let β, β_1 be real numbers, and let $\alpha_1 > 1$ be an irrational number with finite type (for the definition of the type of irrational numbers see Definition 2.1), and let $P = \{p = \lfloor \alpha_1 m + \beta_1 \rfloor : m \in \mathbf{N}\}$. Then, we have*

$$\limsup_{x \to \infty} \frac{\pi^{c*}_{\alpha,\beta}(x)(c\alpha_1 \log x)}{\pi(x)} \geq 1,$$

where

$$\pi^{c*}_{\alpha,\beta}(x) = |\{p \leq x : \ p \text{ is a prime}, p \in P, \lfloor \alpha p^c + \beta \rfloor \text{ is a prime}\}|.$$

2. Some notations

Let p always denote a prime. Let $mes(A)$ denote the Lebesgue measure of A. We shall frequently use ε with a slight abuse of notation to mean a small positive number, possibly a different one each time. Given a real number t, we write $e(t) = e^{2\pi i t}$, $\{\alpha\}$ for the fractional part of α, and $\lfloor \alpha \rfloor$ for the largest integer not exceeding α. We recall that for functions F and real nonnegative G the notations $F \ll G$ and $F = O(G)$ are equivalent to the statement that the inequality $|F| \leq CG$ holds for some constant $C > 0$. We also write $F \sim G$ to indicate that $F \ll G$ and $G \ll F$.

Next, we give the definition of the type of the irrational number.

Definition 2.1. *For any irrational number α, we define its type $\tau = \tau(\alpha)$ by the following formula:*

$$\tau := \sup\{t \in \mathbb{R} : \liminf_{n \to \infty} n^t \parallel \alpha n \parallel = 0\}.$$

Remark 1. By Dirichlet's approximation theorem, we have $\tau \geq 1$ for every irrational number α. By the work of Khinchin [5], it is known that $\tau = 1$ for almost all real numbers, in the sense of the Lebesgue measure. Moreover, if α is an irrational number of type $\tau < \infty$, then so are α^{-1} and $n\alpha^{-1}$ for all integers $n \geq 1$.

3. Some lemmas

Before proving the main theorem, we state the proof strategy of the theorem.

Remark 2. We remark that the method in this paper simplifies the proof of the main theorems in [8, 15] due to the use of the so-called division trick. First, the division trick yields a characteristic function for the condition $d|\lfloor\alpha p^c+\beta\rfloor$. We then combine Fourier analysis with estimation of exponential sums to establish Lemma 3.12. Finally, the theorem is proved by applying a standard sieve method.

For readers' convenience, we list some lemmas before proving Theorem 1.2. First, we need the following well-known approximation of Vaaler [16].

Lemma 3.1. *Let x be a real number, and let $\psi(x) = x - [x] - 1/2$. Then, for any $H \geq 1$, there are numbers a_h, b_h such that*

$$\left|\psi(x) - \sum_{0<|h|\leq H} a_h e(xh)\right| \leq \sum_{|h|\leq H} b_h e(xh),$$

where

$$a_h \ll \frac{1}{|h|}, b_h \ll \frac{1}{H}.$$

Lemma 3.2. *Let α and β be real numbers. An integer m can be written in the form $\lfloor\alpha n+\beta\rfloor$ where $n \in \mathbb{N}$ if and only if $F_{\alpha,\beta}(m) := \lfloor-\alpha^{-1}(m-\beta)\rfloor - \lfloor-\alpha^{-1}(m-\beta+1)\rfloor = 1$.*

Proof. The proof of this lemma is easy, so we leave it to the reader. □

We need the following lemma to detect the condition $d|\lfloor\alpha p^c+\beta\rfloor$, which we refer to as the division trick.

Lemma 3.3. *Let d be a natural number, let α be an irrational number, and let $t > 0$ and β be real numbers. Then, $d|\lfloor\alpha t+\beta\rfloor$ if and only if $E_{\alpha,\beta,d}(t) := \left\lfloor\frac{\alpha t+\beta}{d}\right\rfloor - \left\lfloor\frac{\alpha t+\beta-1}{d}\right\rfloor = 1$.*

Proof. First, we suppose that $d|\lfloor\alpha t+\beta\rfloor$. Then, we have $\lfloor\alpha t+\beta\rfloor = dk$ for some integer k, and then

$$\begin{aligned}
&\alpha t+\beta-1 < dk \leq \alpha t+\beta,\\
&\Rightarrow \frac{\alpha t+\beta-1}{d} < k \leq \frac{\alpha t+\beta}{d},\\
&\Rightarrow \left\lfloor\frac{\alpha t+\beta-1}{d}\right\rfloor < k \leq \left\lfloor\frac{\alpha t+\beta}{d}\right\rfloor.
\end{aligned}$$

Because

$$0 \leq \left\lfloor\frac{\alpha t+\beta}{d}\right\rfloor - \left\lfloor\frac{\alpha t+\beta-1}{d}\right\rfloor \leq 1,$$

if $d|\lfloor\alpha t+\beta\rfloor$, we have

$$\left\lfloor\frac{\alpha t+\beta}{d}\right\rfloor - \left\lfloor\frac{\alpha t+\beta-1}{d}\right\rfloor = 1.$$

Next, we suppose $\left\lfloor\frac{\alpha t+\beta}{d}\right\rfloor - \left\lfloor\frac{\alpha t+\beta-1}{d}\right\rfloor = 1$. Letting $\lfloor\alpha t+\beta\rfloor = dk+l$, $0 \leq l \leq d-1$, we have

$$\left\lfloor\frac{\alpha t+\beta}{d}\right\rfloor - \left\lfloor\frac{\alpha t+\beta-1}{d}\right\rfloor$$

$$= \left\lfloor \frac{l + \{\alpha t + \beta\}}{d} \right\rfloor - \left\lfloor \frac{l + \{\alpha t + \beta\} - 1}{d} \right\rfloor.$$

If $l \geq 1$, we have

$$\left\lfloor \frac{\alpha t + \beta}{d} \right\rfloor - \left\lfloor \frac{\alpha t + \beta - 1}{d} \right\rfloor = 0,$$

so $l = 0$. This completes the proof of this lemma. □

Lemma 3.4. *Let α be an irrational number. Suppose that*

$$\left| \alpha - \frac{a}{q} \right| \leq \frac{1}{q^2},$$

where a and q are two integers with $(a, q) = 1$ and $1 \leq a < q$. Then, there holds

$$\sum_{p \leq x} e(\alpha p) \ll x^{\epsilon}(xq^{-1/2} + x^{4/5} + x^{1/2}q^{1/2}).$$

Proof. See Chapter 25 of [1]. □

Lemma 3.5. *Suppose that a is a fixed irrational number of finite type $\tau < \infty$. Let $h \geq 1$ and M be positive integers. Then, we have*

$$\sum_{m \leq M} \Lambda(m)e(ahm) \ll h^{1/2}M^{1-3/(10\tau)+\varepsilon} + M^{4/5+\varepsilon}.$$

Proof. Since a is of type τ, when n is sufficiently large, say $n > c$, we have

$$||an|| \gg n^{-\tau-\varepsilon}.$$

When $n \leq c$, we have only finite integers n, so there exists a constant c' such that

$$||an|| \geq c'n^{-\tau-\varepsilon}.$$

So, we have

$$||an|| \gg n^{-\tau-\varepsilon}. \tag{3.1}$$

By Dirichlet's approximation theorem, there exist natural numbers $d \leq M^{3/5}$ and $b < d$ with $(b, d) = 1$, such that

$$\left| ah - \frac{b}{d} \right| \leq \frac{1}{dM^{3/5}} \leq \frac{1}{d^2}, \tag{3.2}$$

which combined with (3.1) yields

$$M^{-3/5} \geq |ahd - b| \geq ||ahd|| \gg (hd)^{-\tau-\varepsilon}.$$

So, we have

$$d \gg h^{-1}M^{\frac{3}{5(\tau+\varepsilon)}}. \tag{3.3}$$

Combining $d \leq M^{3/5}$ and (3.3) with Lemma 3.4, by partial summation, we have

$$\begin{aligned}\sum_{m\leq M}\Lambda(m)e(ahm) =& \sum_{p\leq M}\log pe(ahp) + O(M^{1/2+\varepsilon})\\ &\ll M^{\varepsilon}(Md^{-1/2} + M^{4/5} + M^{1/2}d^{1/2})\\ &\ll h^{1/2}M^{1-3/(10\tau)+\varepsilon} + M^{4/5+\varepsilon}.\end{aligned}$$

□

The next lemma is the famous Vaughan decomposition for the arithmetic function $\Lambda(n)$.

Lemma 3.6. *Let* $3 < U < V < W < X$ *be reals with*

$$\{W\} = \frac{1}{2}, \quad X \geq 64W^2U, \quad W \geq 4U^2, \quad V^3 \geq 32X.$$

We further assume that $F(n)$ *is a complex-valued function such that* $|F(n)| = 1$. *Then, the sum*

$$\sum_{X<n\leq 2X}\Lambda(n)F(n)$$

may be decomposed into $O(\log^{10} X)$ *sums, each of which is either of type I:*

$$\sum_{M<m\leq 2M} a(m) \sum_{N<n\leq 2N} F(mn)$$

with $N > W$*, where* $a(m) \ll m^{\epsilon}$*,* $X \ll MN \ll X$*, or of type II:*

$$\sum_{M<m\leq 2M} a(m) \sum_{N<n\leq 2N} b(n)F(mn)$$

with $U \ll M \ll V$*, where* $a(m) \ll m^{\epsilon}$*,* $b(n) \ll n^{\epsilon}$*,* $X \ll MN \ll X$*.*

Proof. This is Lemma 3 in [4]. □

Lemma 3.7. *Let* M *and* M' *be two integers, and let* f *be a real function with three continuous derivatives on* (M, M')*. Suppose that* $M < M' \leq 2M$*. If* $|f'''| \sim \lambda$*, then*

$$\sum_{M<n<M'} e(f(n)) \ll M\lambda^{1/6} + M^{3/4} + M^{1/4}\lambda^{-1/4}.$$

Proof. This is Theorem 2.6 in [2]. □

Lemma 3.8. *Let* $z(n)$ *be any complex numbers. Then, for* $1 \leqslant Q \leqslant N$*, there holds*

$$\left|\sum_{N<n\leqslant CN} z(n)\right|^2 \ll \frac{N}{Q}\sum_{0\leqslant|q|\leqslant Q}\left(1-\frac{|q|}{Q}\right)\Re\sum_{N<n\leqslant CN-q} z(n)\overline{z(n+q)}.$$

Proof. The proof of this lemma can be found in [2]. □

Lemma 3.9. *Let a, h, and $1 < c < \frac{150}{119}$ be reals. Then, for sufficiently large reals x and any $d < \min(x^{\frac{150-119c}{833}}, x^{\frac{9}{476}-\varepsilon})$, $N \geq \lfloor x^{50/119} \rfloor + 1/2$, and $MN \sim x$, we have*

$$S_I := \sum_{n\sim N}\sum_{m\sim M} a(m)e\left(amn + h(mn)^c\right) \ll \frac{x^{1-\varepsilon}}{d},$$

where $a(m) \ll m^{\varepsilon}$ and $1/d \ll h \ll dx^{\varepsilon}$.

Proof. We have

$$S_I = \sum_{m\sim M} a(m) \sum_{n\sim N} e\left(amn + h(mn)^c\right)$$

$$\ll \sum_{m\sim M}\left|\sum_{n\sim N} e\left(amn + h(mn)^c\right)\right|.$$

Applying Lemma 3.7 to the innermost sum, we get

$$S_I \ll d^{1/6}x^{94/119+c/6+\varepsilon} + x^{213/238+\varepsilon} + d^{1/4}x^{1-c/4+\varepsilon} \ll \frac{x^{1-\varepsilon}}{d}.$$

This completes the proof of this lemma. □

Lemma 3.10. *Let a, h, and $1 < c < \frac{150}{119}$ be reals. Then, for sufficiently large reals x and any $d < \min(x^{\frac{150-119c}{833}}, x^{\frac{9}{476}-\varepsilon})$, $x^{18/119} \ll M \ll x^{47/119}$, and $MN \sim x$, we have*

$$S_{II} := \sum_{n\sim N}\sum_{m\sim M} a(m)b(n)e\left(amn + h(mn)^c\right) \ll \frac{x^{1-\varepsilon}}{d},$$

where $a(m) \ll m^{\varepsilon}$, $b(n) \ll n^{\varepsilon}$ and $1/d \ll h \ll dx^{\varepsilon}$.

Proof. Applying Lemma 3.8 with $Q = x^{\varepsilon}d^2$, we get

$$S_{II}^2 \ll \frac{x^2}{Q} + \frac{x}{Q}\sum_{q\leq Q}\sum_{n\sim N}\left|\sum_{m\sim M} e\left(aqm + h((n+q)^c - n^c)m^c\right)\right|.$$

Let

$$S' = \sum_{q\leq Q}\sum_{n\sim N}\left|\sum_{m\sim M} e\left(aqm + h((n+q)^c - n^c)m^c\right)\right|.$$

By Lemma 3.7, we have

$$S' \ll d^{5/2}x^{459/714+c/6+\varepsilon} + d^2x^{229/238+\varepsilon} + d^{7/4}x^{5/4-9/238-c/4+\varepsilon}.$$

So, we have

$$S_{II} \ll \frac{x^{1-\varepsilon}}{d}.$$

This completes the proof of this lemma. □

Lemma 3.11. *Let a, h, and* $1 < c < \frac{150}{119}$ *be real numbers, and let d be a positive integer such that* $d < \min(x^{\frac{150-119c}{833}}, x^{\frac{9}{476}-\varepsilon})$. *Then, we have*

$$\left|\sum_{n\sim x}\Lambda(n)e(an+hn^c)\right| \ll \frac{x^{1-\varepsilon}}{d}, \tag{3.4}$$

where $1/d \ll h \ll dx^{\varepsilon}$.

Proof. Let $W = \lfloor x^{50/119}\rfloor + 1/2$, $V = x^{47/119}$ and $U = x^{18/119}$, then by Lemma 3.6, we have that the above sum can be decomposed into $O(\log^{10} X)$ sums, each of which either of type I,

$$\sum_{n\sim N}\sum_{m\sim M} a(m)e\left(amn + h(mn)^c\right)$$

with $N > W$, where $a(m) \ll m^{\epsilon}$, $x \ll MN \ll x$, or of type II,

$$\sum_{n\sim N}\sum_{m\sim M} a(m)b(n)e\left(amn + h(mn)^c\right)$$

with $U \ll M \ll V$, where $a(m) \ll m^{\epsilon}$, $b(n) \ll n^{\epsilon}$, and $x \ll MN \ll x$.
By Lemma 3.9, the type I sums are

$$\ll \frac{x^{1-\varepsilon}}{d}.$$

By Lemma 3.10, the type II sums are

$$\ll \frac{x^{1-\varepsilon}}{d}.$$

So we have

$$\left|\sum_{n\sim x}\Lambda(n)e(an+hn^c)\right| \ll \frac{x^{1-\varepsilon}}{d}.$$

This completes the proof of this lemma. □

Lemma 3.12. *Suppose that* $b_2 > b_1 > 0$, $1 < c < \frac{150}{119}$, β, *and* β_1 *are arbitrary real numbers. Suppose* $\alpha_1 > 1$ *is an irrational number with finite type* τ. *Let P denote the set* $\{p : p = \lfloor\alpha_1 m + \beta_1\rfloor\}$. *Then, for any square-free* $d < \min(x^{\frac{150-119c}{833}}, x^{\frac{9}{476}-\varepsilon}, x^{\frac{1}{5\tau}-\varepsilon})$ *and any* $\alpha \in (b_1, b_2)$, *we have*

$$|\{1 \le p \le x, p \in P : d|\lfloor\alpha p^c + \beta\rfloor\}| = \frac{\pi(x)}{\alpha_1 d} + O\left(\frac{x^{1-\varepsilon}}{d}\right).$$

Proof. Let

$$S := \sum_{\substack{p\in P\\ d|\lfloor\alpha p^c+\beta\rfloor}} 1.$$

By Lemmas 3.2 and 3.3, we have

$$F_{\alpha_1,\beta_1}(p) = \alpha_1^{-1} + \psi(-\alpha_1^{-1}(p-\beta_1+1)) - \psi(-\alpha_1^{-1}(p-\beta_1)), \tag{3.5}$$

$$E_{\alpha,\beta,d}(p^c) = \frac{1}{d} + \psi(\frac{\alpha p^c+\beta-1}{d}) - \psi(\frac{\alpha p^c+\beta}{d}). \tag{3.6}$$

So, by (3.5) and (3.6), we obtain

$$\begin{aligned} S =& \sum_{p\le x} F_{\alpha_1,\beta_1}(p)E_{\alpha,\beta,d}(p^c) \\ =& \sum_{p\le x}(\alpha_1^{-1} + \psi(-\alpha_1^{-1}(p-\beta+1)) - \psi(-\alpha_1^{-1}(p-\beta))) \\ & \times (\frac{1}{d} + \psi(\frac{\alpha p^c+\beta-1}{d}) - \psi(\frac{\alpha p^c+\beta}{d})) \\ =& S_1 + S_2 + S_3 + S_4, \end{aligned} \tag{3.7}$$

where

$$S_1 = \sum_{p\le x}\frac{1}{\alpha_1 d}, \tag{3.8}$$

$$S_2 = \sum_{p\le x}\alpha_1^{-1}(\psi(\frac{\alpha p^c+\beta-1}{d}) - \psi(\frac{\alpha p^c+\beta}{d})), \tag{3.9}$$

$$S_3 = \sum_{p\le x}\frac{1}{d}(\psi(-\alpha_1^{-1}(p-\beta_1+1)) - \psi(-\alpha_1^{-1}(p-\beta_1))), \tag{3.10}$$

$$\begin{aligned} S_4 =& \sum_{p\le x}(\psi(-\alpha_1^{-1}(p-\beta_1+1)) - \psi(-\alpha_1^{-1}(p-\beta_1))) \\ & \times (\psi(\frac{\alpha p^c+\beta-1}{d}) - \psi(\frac{\alpha p^c+\beta}{d})). \end{aligned} \tag{3.11}$$

For S_1, we have

$$S_1 = \frac{\pi(x)}{d\alpha_1}. \tag{3.12}$$

For S_i $(i = 2, 3, 4)$, it is sufficient to show that

$$S_i \ll \frac{x^{1-\varepsilon}}{d} \quad (2 \le i \le 4).$$

By Lemma 3.1 with $H_1 = H_2 = dx^{\varepsilon}$, we have

$$\begin{aligned} & \psi(-\alpha_1^{-1}(p-\beta+1)) - \psi(-\alpha_1^{-1}(p-\beta)) \\ =& \sum_{0<|h_1|\le H_1} a_{h_1}(e(h_1\alpha_1^{-1}(p-\beta+1)) - e(h_1\alpha_1^{-1}(p-\beta)) \\ & + O(\sum_{|h_1|\le H_1} b_{h_1}(e(h_1\alpha_1^{-1}(p-\beta+1)) - e(h_1\alpha_1^{-1}(p-\beta)))), \\ & \psi(\frac{\alpha p^c+\beta-1}{d}) - \psi(\frac{\alpha p^c+\beta}{d}) \\ =& \sum_{0<|h_2|\le H_2} a_{h_2}(e(h_2\frac{\alpha p^c+\beta-1}{d}) - e(h_2\frac{\alpha p^c+\beta}{d})) \\ & + O(\sum_{|h_2|\le H_2} b_{h_2}(e(h_2\frac{\alpha p^c+\beta-1}{d}) - e(h_2\frac{\alpha p^c+\beta}{d}))). \end{aligned} \tag{3.13}$$

For S_3, we have

$$S_3 = \frac{1}{d}\sum_{p\le x}\sum_{0<|h_1|\le H_1} a_{h_1}(e(h_1\alpha_1^{-1}(p-\beta_1+1)) - e(h_1\alpha_1^{-1}(p-\beta_1)))$$
$$+\frac{1}{d}\sum_{p\le x}\sum_{|h_1|\le H_1} b_{h_1}(e(h_1\alpha_1^{-1}(p-\beta_1+1)) + e(h_1\alpha_1^{-1}(p-\beta_1))). \tag{3.14}$$

By partial summation, we obtain

$$S_3 \ll \frac{1}{d}\sum_{0<|h_1|\le H_1}|h_1|^{-1}\left|\sum_{n\le x}\Lambda(n)e((h_1\alpha_1^{-1})n)\right|. \tag{3.15}$$

Then by Lemma 3.5, we conclude that

$$S_3 \ll \frac{x^{1-\varepsilon}}{d}. \tag{3.16}$$

For S_4, we write

$$S_4 = O(S_{41}+S_{42}+S_{43}+S_{44}) + O(\frac{x^{1-\varepsilon}}{d}),$$

where

$$S_{41} = \sum_{p\le x}\sum_{0<|h_1|\le H_1} a_{h_1}(e(h_1\alpha_1^{-1}(p-\beta_1+1))) - e(h_1\alpha_1^{-1}(p-\beta_1))$$
$$\times\sum_{0<|h_2|\le H_2} a_{h_2}(e(h_2\frac{\alpha p^c+\beta-1}{d}) - e(h_2\frac{\alpha p^c+\beta}{d})), \tag{3.17}$$

$$S_{42} = \sum_{p\le x}\sum_{0<|h_1|\le H_1} a_{h_1}(e(h_1\alpha_1^{-1}(p-\beta_1+1))) - e(h_1\alpha_1^{-1}(p-\beta_1))$$
$$\times\sum_{0<|h_2|\le H_2} b_{h_2}(e(h_2\frac{\alpha p^c+\beta-1}{d}) + e(h_2\frac{\alpha p^c+\beta}{d})), \tag{3.18}$$

$$S_{43} = \sum_{p\le x}\sum_{0<|h_1|\le H_1} b_{h_1}(e(h_1\alpha_1^{-1}(p-\beta_1+1)) + e(h_1\alpha_1^{-1}(p-\beta_1)))$$
$$\times\sum_{0<|h_2|\le H_2} a_{h_2}(e(h_2\frac{\alpha p^c+\beta-1}{d}) - e(h_2\frac{\alpha p^c+\beta}{d})), \tag{3.19}$$

$$S_{44} = \sum_{p\le x}\sum_{0<|h_1|\le H_1} b_{h_1}(e(h_1\alpha_1^{-1}(p-\beta_1+1)) + e(h_1\alpha_1^{-1}(p-\beta_1)))$$
$$\times\sum_{0<|h_2|\le H_2} b_{h_2}(e(h_2\frac{\alpha p^c+\beta-1}{d}) + e(h_2\frac{\alpha p^c+\beta}{d})). \tag{3.20}$$

For S_{41}, we find that

$$S_{41} \ll \sum_{0<|h_1|\le H_1}|h_1|^{-1}\sum_{0<|h_2|\le H_2}|h_2|^{-1}\left|\sum_{n\le x}\Lambda(n)e(h_1\alpha_1^{-1}n + h_2\alpha d^{-1}n^c)\right|.$$

By Lemma 3.11, we conclude that

$$S_{41} \ll \frac{x^{1-\varepsilon}}{d}. \tag{3.21}$$

S_{42}, S_{43}, and S_{44} can be estimated similarly, so we have

$$S_4 \ll \frac{x^{1-\varepsilon}}{d}. \tag{3.22}$$

For S_2, similarly we have

$$S_2 \ll \frac{x^{1-\varepsilon}}{d}. \tag{3.23}$$

By (3.7), (3.12), (3.16), (3.22), and (3.23), we derive

$$S = \frac{\pi(x)}{\alpha_1 d} + O\left(\frac{x^{1-\varepsilon}}{d}\right). \tag{3.24}$$

This completes the proof of this lemma. □

Lemma 3.13. *Suppose that $b_2 > b_1 > 0$, $1 < c < \frac{150}{119}$, β, and β_1 are arbitrary real numbers. Suppose $\alpha_1 > 1$ is an irrational number with finite type. Let P denote the set $\{p : p = \lfloor \alpha_1 m + \beta_1 \rfloor\}$. Then, for all sufficiently large real numbers x and any irrational $\alpha \in (b_1, b_2)$, we have*

$$|\{1 \le p \le x, p \in P : \lfloor \alpha p^c + \beta \rfloor \text{ is a prime}\}| \ll \frac{\pi(x)}{\alpha_1 \log x}.$$

Proof. Put

$$z = \min(x^{\frac{150-119c}{1666}}, x^{\frac{9}{952}-\varepsilon}, x^{\frac{1}{10\tau}-\varepsilon}),$$
$$P(z) = \prod_{p<z, p \text{ prime}} p,$$
$$S(A, z) = \{a \in A : (a, P(z)) = 1\}.$$

Let $A = \{\lfloor \alpha p^c + \beta \rfloor : p \le x, \ \ p \in P\}$. We find

$$\{\lfloor \alpha p^c + \beta \rfloor : \alpha^{-1/c}(z - \beta + 1)^{1/c} < p < x, p \in P, \lfloor \alpha p^c + \beta \rfloor \text{ is prime}\}$$

a subset of $S(A, z)$. Then, by Lemma 3.12, we know that for any square-free $d < \min(x^{\frac{150-119c}{833}}, x^{\frac{9}{476}-\varepsilon}, x^{\frac{1}{5\tau}-\varepsilon})$ and any irrational $\alpha \in (b_1, b_2)$,

$$|A_d| = \frac{\pi(x)}{\alpha_1 d} + O\left(\frac{x^{1-\varepsilon}}{d}\right),$$

where $A_d = \{a \in A : d|a\}$. We define a completely multiplicative function such that $g(p) = \frac{1}{p}$ for each prime p. And, let $G(z) = \sum_{m<z, m|P(z)} g(m)$. By Selberg's sieve method, we have

$$|S(A, z)| \le \frac{|A|}{G(z)} + O\left(\sum_{d<z^2} 3^{\omega(d)} x^{1-\varepsilon}/d\right), \tag{3.25}$$

where $\omega(d)$ denotes the number of distinct prime divisors of d. So, we have $3^{\omega(d)} \ll d^{\varepsilon}$. So, we have

$$\sum_{d<z^2} 3^{\omega(d)} x^{1-\varepsilon}/d \ll x^{1-\varepsilon} \sum_{d<z^2} d^{\varepsilon-1} \ll x^{1-\varepsilon}.$$

So, the error term in (3.25) is negligible.
By the prime number theorem and Theorem 7.14 of [11], we have

$$G(z) \gg \prod_{p<z} \left(1 - \frac{1}{p}\right)^{-1} \gg \log z.$$

This completes the proof of this lemma. □

4. Proof of Theorem 1.2

The proof of Theorem 1.2 is similar to the proof of Theorem 1 in [8]. The only difference is the formula (4) in [8]. For completeness, we give the full proof of Theorem 1.2.
Suppose that $b_2 > b_1 > 0$. Let

$$\mathcal{F} = \{\alpha \in (b_1, b_2) : \limsup_{x\to\infty} \pi^{c*}_{\alpha,\beta}(x) c\alpha_1 \log x/\pi(x) < 1\},$$

and

$$\mathcal{F}_n = \{\alpha \in (b_1, b_2) : \limsup_{x\to\infty} \pi^{c*}_{\alpha,\beta}(x) c\alpha_1 \log x/\pi(x) < 1 - 1/n\}.$$

Obviously $\mathcal{F} = \bigcup_{n>1} \mathcal{F}_n$. So, it suffices to show $mes(\mathcal{F}_n) = 0$ for every $n > 1$. Assume on the contrary that there exists $n > 1$ such that $mes(\mathcal{F}_n) > 0$. Letting $I := (c_1, c_2)$ be an arbitrary sub-interval of (b_1, b_2), we have

$$\begin{aligned}
\int_{c_1}^{c_2} \pi^{c*}_{\alpha,\beta}(x) d\alpha &= \int_{c_1}^{c_2} \left(\sum_{\substack{p\le x\\ p\in P}} \sum_{\substack{\alpha p^c+\beta-1<q\le\alpha p^c+\beta\\ q \text{ is a prime}}} 1 \right) d\alpha \\
\ge & \sum_{\substack{p\le x\\ p\in P}} \sum_{\substack{c_1 p^c+\beta-1<q\le c_2 p^c+\beta\\ q \text{ is a prime}}} mes([(q-\beta)/p^c, (q+1-\beta)/p^c) \cap [c_1, c_2]) \\
\ge & \sum_{\substack{p\le x\\ p\in P}} \sum_{\substack{c_1 p^c+\beta-1<q\le c_2 p^c+\beta\\ q \text{ is a prime}}} \frac{1}{p^c} \\
= & (c_2 - c_1) \sum_{\substack{p\le x\\ p\in P}} \frac{1}{c \log p} \left(1 + O\left(\frac{1}{\log(c_1 p)}\right)\right) \\
\ge & (c_2 - c_1) \frac{\pi(x)}{c\alpha_1 \log x} \left(1 + O\left(\frac{1}{\log x}\right)\right). \qquad (4.1)
\end{aligned}$$

Suppose that $C' > 1$ is the implied constant in Lemma 3.13. Let $C = C'c$, and let $\mathcal{L}_I = \mathcal{F}_n \cap I$ and

$$\mathcal{L}_{I,\delta}(x) = \{\alpha \in I : \pi^{c*}_{\alpha,\beta}(x) c\alpha_1 \log x/\pi(x) < 1 - \delta\}.$$

For any two primes p, q,

$$J_{p,q} := \{\alpha \in I : \lfloor \alpha p^c + \beta \rfloor = q\}$$

is an interval or empty set. Let

$$\mathcal{H}_{I,\delta}(x) = \left(\bigcup_{\substack{k>(1-\delta)\pi(x)/(c\alpha_1 \log x) \\ p_1,\cdots,p_k \le x \text{ are distinct and } p_i \in P \\ q_1,\cdots,q_k \text{ are primes}}} \bigcap_{j=1}^{k} J_{p_j,q_j} \right).$$

So, we have that

$$\mathcal{L}_{I,\delta}(x) = I \setminus \mathcal{H}_{I,\delta}(x)$$

is measurable.
By Lemma 3.13,

$$\int_{c_1}^{c_2} \pi_{\alpha,\beta}^{c*}(x) d\alpha \le mes(\mathcal{L}_{I,\delta}(x)) \frac{(1-\delta)\pi(x)}{c\alpha_1 \log x} + (c_2 - c_1 - mes(\mathcal{L}_{I,\delta}(x))) \frac{C\pi(x)}{c\alpha_1 \log x}.$$

Combining the above result with (4.1), we have

$$mes(\mathcal{L}_{I,\delta}(x)) \le \frac{C-1}{C-1+\delta} mes(I). \tag{4.2}$$

We claim that

$$\mathcal{L}_I = \bigcap_{m>n} \bigcup_{\substack{y \ge 1 \\ y \in \mathbf{Z}}} \bigcap_{\substack{x \ge y \\ x \in \mathbf{Z}}} \mathcal{L}_{I,1/n-1/m}(x). \tag{4.3}$$

In fact, for any $m > n$, if

$$\limsup_{x \to \infty} \pi_{\alpha,\beta}^{c*}(x) c\alpha_1 \log x / \pi(x) < 1 - 1/n + 1/m,$$

then there exists y_0 such that for any $x \ge y_0$,

$$\pi_{\alpha,\beta}^{c*}(x) \le (1 - 1/n + 1/m)\pi(x)/(c\alpha_1 \log x).$$

On the other hand, if $\alpha \in \cup_y \cap_{x \ge y} \mathcal{L}_{I,1/n-1/m}(x)$, clearly we have

$$\limsup_{x \to \infty} \pi_{\alpha,\beta}^{c*}(x) c\alpha_1 \log x / \pi(x) < 1 - 1/n + 1/m.$$

By (4.2) and (4.3), we get

$$mes(\mathcal{L}_I) \le \limsup_{x \to \infty} mes(\mathcal{L}_{I,1/3n}(x)) \le \frac{C-1}{C-1+1/3n} mes(I). \tag{4.4}$$

Since $mes(\mathcal{F}_n) > 0$, there exist open intervals $I_1, I_2, \cdots \subseteq (b_1, b_2)$ such that

$$\mathcal{F}_n \subseteq \bigcup_{k=1}^{\infty} I_k$$

and

$$\sum_{k=1}^{\infty} mes(I_k) \leq \frac{C - 1 + 1/4n}{C - 1} mes(\mathcal{F}_n). \tag{4.5}$$

But, by (4.4) and (4.5),

$$mes(\mathcal{F}_n) \leq \sum_{k=1}^{\infty} mes(\mathcal{L}_{I_k}) \leq \frac{C - 1}{C - 1 + 1/3n} \sum_{k=1}^{\infty} mes(I_k) \leq \frac{C - 1 + 1/4n}{C - 1 + 1/3n} mes(\mathcal{F}_n).$$

This leads to a contradiction.

5. Discussion and conclusions

Let $P = \{p = \lfloor \alpha_1 m + \beta_1 \rfloor : m \in \mathbf{N}\}$. In this paper, we proved for that almost all irrational numbers $\alpha > 0$ (in the sense of Lebesgue measure) and every real β,

$$\limsup_{x \to \infty} \frac{\pi_{\alpha,\beta}^{c*}(x)(c\alpha_1 \log x)}{\pi(x)} \geq 1,$$

where

$$\pi_{\alpha,\beta}^{c*}(x) = |\{p \leq x : \ p \ is \ a \ prime, p \in P, \lfloor \alpha p^c + \beta \rfloor \ is \ a \ prime\}|.$$

Clearly, studies concerning the distribution of primes play an important role in number theory.

We give an open problem related to this article. Let c_i $(i = 1, 2, \cdots, k)$ be non-integers, then whether for almost all irrational numbers $\alpha_i > 0$ (in the sense of Lebesgue measure) and every irrational β there exist infinitely many primes $p \in P$ such that $\lfloor \alpha_1 p^{c_1} + \alpha_2 p^{c_2} + \cdots + \alpha_k p^{c_k} + \beta \rfloor$ is also a prime. According to the method we used in the current paper, we should estimate the exponential sum with a non-integer polynomial, which is the difficulty of this problem.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author would like to thank the editors and reviewers for their valuable comments, which have greatly improved this article.

Conflict of interest

The author declares no conflict of interest.

References

1. H. Davenport, *Multiplicative number theory*, New York: Springer, 1980. https://doi.org/10.1007/978-1-4757-5927-3
2. S. W. Graham, G. Kolesnik, *Van der Corput's method of exponential sums*, Cambridge: Cambridge University Press, 1991. https://doi.org/10.1017/CBO9780511661976
3. V. Z. Guo, Piatetski-Shapiro primes in a Beatty sequence, *J. Number Theory*, **156** (2015), 317–330. https://doi.org/10.1016/j.jnt.2015.04.010
4. D. R. Heath-Brown, The Pjateckij-Sapiro prime number theorem, *J. Number Theory*, **16** (1983), 242–266. https://doi.org/10.1016/0022-314X(83)90044-6
5. A. Khintchine, Zur metrischen Theorie der diophantischen Approximationen, *Math. Z.*, **24** (1926), 706–714. https://doi.org/10.1007/BF01216806
6. G. Kolesnik, Distribution of primes in sequences of the form $\lfloor n^c \rfloor$, *Mat. Zametki.*, **2** (1967), 553–560. https://doi.org/10.1007/BF01094244
7. G. Kolesnik, Primes of the form $\lfloor n^c \rfloor$, *Pacific J. Math.*, **118** (1985), 437–447. https://doi.org/10.2140/pjm.1985.118.437
8. H. Li, H. Pan, Primes of the form $\lfloor \alpha p + \beta \rfloor$, *J. Number Theory*, **129** (2009), 2328–2334. https://doi.org/10.1016/j.jnt.2009.03.009
9. H. Q. Liu, J. Rivat, On the Pjateckij-Sapiro prime number theorem, *Bull. London Math. Soc.*, **24** (1992), 143–147. https://doi.org/10.1112/blms/24.2.143
10. J. Li, J. Qi, M. Zhang, A generalization of Piatetski-Shapiro sequences (II), *Indian J. Pure Ap. Mat.*, **56** (2025), 1293–1303. https://doi.org/10.1007/s13226-024-00532-4
11. C. D. Pan, C. B. Pan, *Goldbach conjecture*, Beijing: Science Press, 1992.
12. I. I. Piatetski-Shapiro, On the distribution of prime numbers in the sequence of the form $[f(n)]$, *Mat. Sb.*, **33** (1953), 559–566.
13. J. Rivat, P. Sargos, Nombres premiers de la forme $\lfloor n^c \rfloor$, *Canad. J. Math.*, **53** (2001), 414–433. https://doi.org/10.4153/CJM-2001-017-0
14. J. Rivat, J. Wu, Prime numbers of the form $\lfloor n^c \rfloor$, *Glasg. Math. J.*, **43** (2001), 237–254. https://doi.org/10.1017/S0017089501000204
15. Y. Song, A note on primes of the form $\lfloor \alpha p + \beta \rfloor$, *J. Number Theory*, **225** (2021), 1–17. https://doi.org/10.1016/j.jnt.2021.01.005
16. J. D. Vaaler, Some extremal functions in Fourier analysis, *Bull. Amer. Math. Soc.* **12** (1985), 183-216. https://doi.org/10.1090/s0273-0979-1985-15349-2