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Research article

LP-Kenmotsu manifolds admitting almost $*$-Ricci-Bourguignon solitons and gradient almost $*$-Ricci-Bourguignon solitons

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Abstract: In this work, a solitonic study of LP-Kenmotsu n-manifolds (in brief $(LPK)_n$) was performed. Key properties of almost $*$-RBSs are established, thus generalizing some of the known results. Additionally, we examine gradient almost $*$-RBSs, thereby deriving conditions for the realization of the soliton structure. These results provide new insights into the theory of Ricci-Bourguignon solitons and geometric flows in semi-Riemannian manifolds. It is proven that an $(LPK)_n$ that admits an almost $*$-RBSs or a gradient almost $*$-RBSs is a generalized ω-Einstein spacetime. Moreover, under certain assumptions, an $(LPK)_n$ that admits an almost $*$-RBSs or a gradient almost $*$-RBSs is a perfect fluid spacetime.

Keywords: almost $*$-Ricci-Bourguignon solitons; gradient almost $*$-Ricci-Bourguignon solitons; LP-Kenmotsu manifolds; screened Poisson equation
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1. Introduction

The idea of the Ricci-Bourguignon flow, proposed in [1] as a generalization of the Ricci flow [2], is an evolutionary equation for the metrics on a Riemannian (or semi-Riemannian) manifold (M, g) given by the following:

$$g(0) = g_0, \qquad \frac{\partial g}{\partial t} = 2(\rho r g - S),$$

where r is the scalar curvature, S is the Rcci tensor, and $\rho(\neq 0) \in \mathbb{R}$ (set of real numbers). It is worth noting that for certain values of ρ, the Ricci-Bourguignon flow turns to the following [3]:
(i) the Einstein flow for $\rho = \frac{1}{2}$;
(ii) the Schouten flow for $\rho = \frac{1}{2(n-1)}$;
(iii) the Ricci flow for $\rho = 0$.

Quasi-Einstein metrics and Ricci solitons appear as particular solutions of the Ricci flow, which motivates the study of a broader class of solitons through the lens of the almost Ricci-Bourguignon flow. An (M, g) of dimension $n(> 2)$ is named an almost Ricci-Bourguignon soliton (or almost RBS) if [4]

$$2(\rho r + \Lambda)g = 2S + \pounds_V g, \tag{1.1}$$

where $\Lambda \in C^\infty(M)$, namely the set of smooth functions. It is called an RBS if $\Lambda \in \mathbb{R}$. If V is Killing, then an almost RBS reduces to an RBS, since the condition forces $\Lambda \in \mathbb{R}$. Recently, Prakasha et al. studied almost RBSs on Kenmotsu manifolds [5].

If $V = \nabla f$ for $f \in C^\infty(M)$, then M is said to admit a gradient almost RBS. Hence, (1.1) becomes the following:

$$Hessf + S = (\rho r + \Lambda)g, \tag{1.2}$$

where $Hessf$ indicates the Hessian of f, which is defined by the following:

$$Hessf(\mathfrak{A}_1, \mathfrak{A}_2) = \nabla^2 f(\mathfrak{A}_1, \mathfrak{A}_2) = g(\nabla_{\mathfrak{A}_1} Df, \mathfrak{A}_2),$$

where $\mathfrak{A}_1, \mathfrak{A}_2 \in \chi(M)$, namely the set of all smooth vector fields on M, and Df is the gradient of f with respect to the metric g. Several authors have contributed to the theory of RBSs (or ρ-Einstein solitons). In [6], Huang investigated integral pinched gradient shrinking ρ-Einstein solitons and obtained several rigidity results. In 2022, characterizations of ρ-Einstein solitons on Sasakian manifolds were established by Patra [7]. More recently, Haseeb et al. [8] investigated soliton structures on Riemannian manifolds endowed with a semi-symmetric metric ξ-connection. Additional developments in the theory of RBSs can be found in the works of Shaikh and coauthors [9–11].

In 2010, Blair [12] introduced the idea of the $*$-Ricci tensor S^* on a contact metric manifold M, which is defined by the following:

$$S^*(\mathfrak{A}_1, \mathfrak{A}_2) = g(Q^*\mathfrak{A}_1, \mathfrak{A}_2) = \text{Trace}\{\varphi \circ \mathcal{R}(\mathfrak{A}_1, \varphi\mathfrak{A}_2)\},$$

where $\mathfrak{A}_1, \mathfrak{A}_2 \in \chi(M)$. Here $\mathcal{R}$, Q^*, and φ represent the curvature tensor, the $*$-Ricci operator, and a $(1, 1)$ tensor field on M, respectively. The concept of the $*$-Ricci tensor on complex manifolds was investigated by Tachibana [13] and Hamada [14]. In [15], Kaimakamis and Panagiotidou studied the $*$-Ricci tensor of real hypersurfaces in non-flat complex space forms. The properties of a $*$-Ricci tensor on a Riemannian (or semi-Riemannian) manifold have been extensively investigated by several researchers [16–18].

An (M, g) of dimension $n(> 2)$ admits an almost $*$-Ricci-Bourguignon soliton (almost $*$-RBS) if

$$\pounds_V g + 2S^* - 2(\rho r^* + \Lambda)g = 0, \tag{1.3}$$

where $\Lambda \in C^{\infty}(M)$, and r^{*} denotes the $*$-scalar curvature of g. An almost $*$-RBS is said to be a $*$-RBS if $\Lambda \in \mathbb{R}$ [19].

If $V = \nabla f$, then (M, g) is named a gradient almost $*$-Ricci-Bourguignon soliton (or gradient almost $*$-RBS). Hence, (1.3) turns into

$$Hessf + S^{*} = (\rho r^{*} + \Lambda)g. \tag{1.4}$$

The solitons defined in (1.1)–(1.4) are classified as expanding, steady, or shrinking if $\Lambda :< 0, = 0$, or > 0, respectively.

The study of geometric flows is of high importance in mathematics, physics, engineering etc. In [20], Bourguignon and Lawson studied stability and isolation phenomena for Yang–Mills fields. Barros and Ribeiro obtained several characterizations of compact almost Ricci solitons in [21], while Barbosa and Ribeiro investigated conformal solutions to the Yamabe flow in [22]. For further information on related work within the framework of semi-Riemannian geometry, the readers are referred to [23–25] and the references therein.

In this study, we undertake an investigation of almost $*$-RBSs and gradient almost $*$-RBSs in $(LPK)_n$. The article is outlined as follows: in Section 2 (Preliminaries), we collect basic information about $(LPK)_n$ manifolds, along with some well-known definitions and useful lemmas; Section 3 focuses on the study of almost $*$-RBSs in $(LPK)_n$ manifolds, where several important results are established; and in Section 4, we discuss gradient almost $*$-RBSs on $(LPK)_n$ manifolds.

2. Preliminaries and basic results on $(LPK)_n$

A smooth manifold M ($\dim M = n$) with a quadruple $(\varphi, \zeta, \omega, g)$ is called a Lorentzian almost para-contact metric manifold if φ is a $(1,1)$-tensor field, ζ is a vector field, ω is a 1-form, and g is a Lorentzian metric that fulfills the following relations [26]:

$$\omega \otimes \zeta + I = \varphi^{2}, \quad 1 + \omega(\zeta) = 0, \tag{2.1}$$

which give

$$\omega \circ \varphi = 0, \quad \varphi\zeta = 0. \tag{2.2}$$

Let the 1-form ω of M be associated with ζ by $\omega(\cdot) = g(\cdot, \zeta)$ and

$$g(\cdot, \cdot) + \omega(\cdot)\omega(\cdot) = g(\varphi\cdot, \varphi\cdot).$$

Definition 2.1. *A Lorentzian almost para-contact manifold M is named a Lorentzian para-Kenmotsu (LP-Kenmotsu) manifold if*

$$(\nabla_{\mathfrak{A}_1}\varphi)\mathfrak{A}_2 = -g(\varphi\mathfrak{A}_1, \mathfrak{A}_2)\zeta - \omega(\mathfrak{A}_2)\varphi\mathfrak{A}_1, \tag{2.3}$$

for $\mathfrak{A}_1, \mathfrak{A}_2 \in \chi(M)$ *[27, 28].*

For an $(LPK)_n$, we have the following:

$$\nabla_{\mathfrak{A}_1}\zeta = -\mathfrak{A}_1 - \omega(\mathfrak{A}_1)\zeta, \tag{2.4}$$

$$(\nabla_{\mathfrak{A}_1}\omega)\mathfrak{A}_2 = -g(\mathfrak{A}_1, \mathfrak{A}_2) - \omega(\mathfrak{A}_1)\omega(\mathfrak{A}_2), \tag{2.5}$$

where ∇ indicates the Levi-Civita connection that corresponds to g.

Moreover, for an $(LPK)_n$, the following relations hold [27]:

$$g(\mathcal{R}(\mathfrak{A}_1, \mathfrak{A}_2)\mathfrak{A}_3, \zeta) = \omega(\mathcal{R}(\mathfrak{A}_1, \mathfrak{A}_2)\mathfrak{A}_3) = g(\mathfrak{A}_2, \mathfrak{A}_3)\omega(\mathfrak{A}_1) - g(\mathfrak{A}_1, \mathfrak{A}_3)\omega(\mathfrak{A}_2),$$

$$\mathcal{R}(\zeta, \mathfrak{A}_1)\mathfrak{A}_2 = -\mathcal{R}(\mathfrak{A}_1, \zeta)\mathfrak{A}_2 = g(\mathfrak{A}_1, \mathfrak{A}_2)\zeta - \omega(\mathfrak{A}_2)\mathfrak{A}_1, \tag{2.6}$$

$$\mathcal{R}(\mathfrak{A}_1, \mathfrak{A}_2)\zeta = \omega(\mathfrak{A}_2)\mathfrak{A}_1 - \omega(\mathfrak{A}_1)\mathfrak{A}_2,$$

$$\mathcal{R}(\zeta, \mathfrak{A}_1)\zeta = \mathfrak{A}_1 + \omega(\mathfrak{A}_1)\zeta,$$

$$S(\mathfrak{A}_1, \zeta) = (n-1)\omega(\mathfrak{A}_1), \quad S(\zeta, \zeta) + (n-1) = 0, \tag{2.7}$$

$$Q\zeta = (n-1)\zeta, \tag{2.8}$$

for any $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3 \in \chi(LPK)_n$. In this context, the Ricci tensor S is given by the following:

$$S(\mathfrak{A}_1, \mathfrak{A}_2) = \sum_{i=1}^{n} \delta_i g(\mathcal{R}(e_i, \mathfrak{A}_1)\mathfrak{A}_2, e_i) = g(Q\mathfrak{A}_1, \mathfrak{A}_2).$$

Here, $\delta_i = g(e_i, e_i) = +1$ or -1, Q is the Ricci operator, and $\{e_i\}_{i=1}^n$ is an orthonormal frame of vector fields on $(LPK)_n$.

Definition 2.2. *An $(LPK)_n$ is said to be a generalized $*$-ω-Einstein manifold if its Ricci tensor $S(\neq 0)$ is of the form*

$$S^*(\mathfrak{A}_1, \mathfrak{A}_2) = \ell_1 g(\mathfrak{A}_1, \mathfrak{A}_2) + \ell_2\omega(\mathfrak{A}_1)\omega(\mathfrak{A}_1) + \ell_3\Phi(\mathfrak{A}_1, \mathfrak{A}_2), \tag{2.9}$$

where ℓ_1, ℓ_2, and ℓ_3 are scalar functions on $(LPK)_n$ and $\Phi(\mathfrak{A}_1, \mathfrak{A}_2) = g(\varphi\mathfrak{A}_1, \mathfrak{A}_2)$. If $\ell_3 = 0$, then the manifold reduces to a $*$-ω-Einstein manifold.

We begin by recalling the following well-known results of $(LPK)_n$, which may be useful in establishing our results in the subsequent sections.

Lemma 2.1. *[29] For an $(LPK)_n$, one obtains*

$$(\nabla_{\mathfrak{A}_1}Q)\zeta = Q\mathfrak{A}_1 - (n-1)\mathfrak{A}_1, \tag{2.10}$$

$$(\nabla_{\zeta}Q)\mathfrak{A}_1 = 2Q\mathfrak{A}_1 - 2(n-1)\mathfrak{A}_1, \tag{2.11}$$

for any $\mathfrak{A}_1 \in \chi(LPK)_n$.

By contracting (2.11), we obtain the following lemma.

Lemma 2.2. *[30] For an $(LPK)_n$, we have the following:*

$$\zeta(r) + 2(n(n-1) - r) = 0. \tag{2.12}$$

If $r \in \mathbb{R}$, then it follows that $r = n(n-1)$.

Lemma 2.3. *For an $(LPK)_n$, the following statements are valid:*

$$\mathfrak{A}_1(r) - 2(n(n-1) - r)\omega(\mathfrak{A}_1) = 0, \tag{2.13}$$

$$\omega(\nabla_\zeta Dr) + 4(n(n-1) - r) = 0 \tag{2.14}$$

for any $\mathfrak{A}_1 \in \chi(LPK)_n$ [30]. Here, D is the gradient operator of g such that $\mathfrak{A}_1(r) = g(\mathfrak{A}_1, Dr)$.

Lemma 2.4. *[31] For an $(LPK)_n$, the following relations are satisfied:*

$$S^*(\mathfrak{A}_1, \mathfrak{A}_2) = S(\mathfrak{A}_1, \mathfrak{A}_2) - ng(\mathfrak{A}_1, \mathfrak{A}_2) + ag(\mathfrak{A}_1, \varphi\mathfrak{A}_2) - \omega(\mathfrak{A}_1)\omega(\mathfrak{A}_2), \tag{2.15}$$

$$r^* = r - n^2 + 1 + a^2 \tag{2.16}$$

for any $\mathfrak{A}_1, \mathfrak{A}_2 \in \chi(LPK)_n$, and a(= trace of φ) is a smooth function.

Example: Consider a manifold $M = \left\{(v_1, v_2, v_3, v_4) \in \mathbb{R}^4 : v_4 > 0\right\}$ (dim $M = 4$), where (v_1, v_2, v_3, v_4) are the standard coordinates in R^4. Let ϵ_1, ϵ_2, ϵ_3, and ϵ_4 be the vector fields on M given by [30]

$$\epsilon_1 = v_4\frac{\partial}{\partial v_1}, \;\; \epsilon_2 = v_4\frac{\partial}{\partial v_2}, \;\; \epsilon_3 = v_4\frac{\partial}{\partial v_3}, \;\; \epsilon_4 = v_4\frac{\partial}{\partial v_4} = \zeta, \; respectively,$$

which are linearly independent at each point of M. Let the metric g be defined by the following:

$$g(\epsilon_i, \epsilon_i) = 1, \quad \text{for} \quad 1 \le i \le 3 \quad \text{and} \quad g(\epsilon_4, \epsilon_4) = -1,$$

$$g(\epsilon_i, \epsilon_j) = 0, \quad \text{for} \quad i \neq j, \quad 1 \le i, j \le 4.$$

Let the 1-form ω be defined by $\omega(\mathfrak{A}_1) = g(\mathfrak{A}_1, \epsilon_4) = g(\mathfrak{A}_1, \zeta)$, and let the $(1,1)$-tensor field φ be defined by the following:

$$\varphi\epsilon_1 = -\epsilon_1, \;\; \varphi\epsilon_2 = -\epsilon_2, \;\; \varphi\epsilon_3 = -\epsilon_3, \;\; \varphi\epsilon_4 = 0.$$

By applying the linearity of g and φ, we have

$$\omega(\zeta) = -1, \;\; \varphi^2\mathfrak{A}_1 = \mathfrak{A}_1 + \omega(\mathfrak{A}_1)\zeta \;\; \text{and} \;\; g(\varphi\mathfrak{A}_1, \varphi\mathfrak{A}_2) = g(\mathfrak{A}_1, \mathfrak{A}_2) + \omega(\mathfrak{A}_1)\omega(\mathfrak{A}_2)$$

for all $\mathfrak{A}_1, \mathfrak{A}_2 \in \chi(LPK)_n$. Thus, for $\epsilon_4 = \zeta$, the structure $(\varphi, \zeta, \omega, g)$ defines a Lorentzian almost paracontact metric structure on M.

Then, we have the following:

$$[\epsilon_i, \epsilon_4] = -\epsilon_i, \quad \text{for} \quad 1 \le i \le 3,$$

$$[\epsilon_i, \epsilon_j] = 0 \quad \text{otherwise}.$$

By using Koszul's formula, we can easily find the following:

$$\nabla_{\epsilon_1}\epsilon_1 = -\epsilon_4, \;\; \nabla_{\epsilon_1}\epsilon_2 = 0, \;\; \nabla_{\epsilon_1}\epsilon_3 = 0, \;\; \nabla_{\epsilon_1}\epsilon_4 = -\epsilon_1,$$

$$\nabla_{\epsilon_2}\epsilon_1 = 0, \;\; \nabla_{\epsilon_2}\epsilon_2 = -\epsilon_4, \;\; \nabla_{\epsilon_2}\epsilon_3 = 0, \;\; \nabla_{\epsilon_2}\epsilon_4 = -\epsilon_2,$$

$$\nabla_{\epsilon_3}\epsilon_1 = 0, \;\; \nabla_{\epsilon_3}\epsilon_2 = 0, \;\; \nabla_{\epsilon_3}\epsilon_3 = -\epsilon_4, \;\; \nabla_{\epsilon_3}\epsilon_4 = -\epsilon_3,$$

$$\nabla_{\epsilon_4}\epsilon_1 = 0, \;\; \nabla_{\epsilon_4}\epsilon_2 = 0, \;\; \nabla_{\epsilon_4}\epsilon_3 = 0, \;\; \nabla_{\epsilon_4}\epsilon_4 = 0.$$

Additionally, one can easily verify that

$$\nabla_{\mathfrak{A}_1}\zeta + \mathfrak{A}_1 + \omega(\mathfrak{A}_1)\zeta = 0 \quad \text{and} \quad (\nabla_{\mathfrak{A}_1}\varphi)\mathfrak{A}_2 + g(\varphi\mathfrak{A}_1, \mathfrak{A}_2)\zeta + \omega(\mathfrak{A}_2)\varphi\mathfrak{A}_1 = 0.$$

Therefore, M is an LP-Kenmotsu manifold.

The components of $\mathcal{R}$ are obtained as follows:

$$\begin{cases} \mathcal{R}(\epsilon_1, \epsilon_2)\epsilon_1 = -\epsilon_2, \ \mathcal{R}(\epsilon_1, \epsilon_2)\epsilon_2 = \epsilon_1, \ \mathcal{R}(\epsilon_1, \epsilon_3)\epsilon_1 = -\epsilon_3, \\ \mathcal{R}(\epsilon_1, \epsilon_3)\epsilon_3 = \epsilon_1, \ \mathcal{R}(\epsilon_1, \epsilon_4)\epsilon_1 = -\epsilon_4, \ \mathcal{R}(\epsilon_1, \epsilon_4)\epsilon_4 = -\epsilon_1, \\ \mathcal{R}(\epsilon_2, \epsilon_3)\epsilon_2 = -\epsilon_3, \ \mathcal{R}(\epsilon_2, \epsilon_3)\epsilon_3 = \epsilon_2, \ \mathcal{R}(\epsilon_2, \epsilon_4)\epsilon_2 = -\epsilon_4, \\ \mathcal{R}(\epsilon_2, \epsilon_4)\epsilon_4 = -\epsilon_2, \ \mathcal{R}(\epsilon_3, \epsilon_4)\epsilon_3 = -\epsilon_4, \ \mathcal{R}(\epsilon_3, \epsilon_4)\epsilon_4 = -\epsilon_3. \end{cases} \tag{2.17}$$

Now, using the components of $\mathcal{R}$ given in (2.17), it follows that

$$\mathcal{R}(\mathfrak{A}_1, \mathfrak{A}_2)\mathfrak{A}_3 = g(\mathfrak{A}_2, \mathfrak{A}_3)\mathfrak{A}_1 - g(\mathfrak{A}_1, \mathfrak{A}_3)\mathfrak{A}_2. \tag{2.18}$$

Thus, M is of a constant curvature.

Taking the inner product of (2.18) with $\mathfrak{A}_4$, we have the following:

$$'\mathcal{R}(\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, \mathfrak{A}_4) = g(\mathfrak{A}_2, \mathfrak{A}_3)\omega(\mathfrak{A}_1) - g(\mathfrak{A}_1, \mathfrak{A}_3)\omega(\mathfrak{A}_2), \tag{2.19}$$

where $'\mathcal{R}(\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, \mathfrak{A}_4) = g(\mathcal{R}(\mathfrak{A}_1, \mathfrak{A}_2)\mathfrak{A}_3, \mathfrak{A}_4)$. By contracting (2.19) over $\mathfrak{A}_1$, and $\mathfrak{A}_4$, we find $S(\mathfrak{A}_2, \mathfrak{A}_3) = 3g(\mathfrak{A}_2, \mathfrak{A}_3)$. This gives $r = 12$. Moreover, $r \in \mathbb{R} \implies \zeta(r) = 0$. Therefore, from (2.12), it follows that $r = 12$, where $n = 4$. This verifies Lemma 2.2.

3. Almost $*$-RBSs on $(LPK)_n$

Let the metric of $(LPK)_n$ be an almost $*$-RBS, and let the soliton vector field V be parallel to ζ (i.e., $V = \sigma\zeta$, where, $\sigma(\neq 0) \in C^\infty(M)$). Then, from (1.3), we have

$$(\pounds_{\sigma\zeta} g)(\mathfrak{A}_1, \mathfrak{A}_2) + 2S^*(\mathfrak{A}_1, \mathfrak{A}_2) = 2(\Lambda + \rho r^*)g(\mathfrak{A}_1, \mathfrak{A}_2),$$

which is equivalent to

$$\begin{aligned} \sigma g(\nabla_{\mathfrak{A}_1}\zeta, \mathfrak{A}_2) + (\mathfrak{A}_1\sigma)\omega(\mathfrak{A}_2) + g(\mathfrak{A}_1, \nabla_{\mathfrak{A}_2}\zeta) + (\mathfrak{A}_2\sigma)\omega(\mathfrak{A}_1) \\ +2S^*(\mathfrak{A}_1, \mathfrak{A}_2) = 2(\Lambda + \rho r^*)g(\mathfrak{A}_1, \mathfrak{A}_2). \end{aligned} \tag{3.1}$$

Using (2.4) in (3.1), we have the following:

$$\begin{aligned} &2S^*(\mathfrak{A}_1, \mathfrak{A}_2) + \mathfrak{A}_1(\sigma)\omega(\mathfrak{A}_2) + \mathfrak{A}_2(\sigma)\omega(\mathfrak{A}_1) \\ &= 2(\Lambda + \rho r^* + \sigma)g(\mathfrak{A}_1, \mathfrak{A}_2) + 2\sigma\omega(\mathfrak{A}_1)\omega(\mathfrak{A}_2). \end{aligned} \tag{3.2}$$

Taking $\mathfrak{A}_2 = \zeta$ in (3.2), and then using (2.1) and (2.15), we obtain the following:

$$\mathfrak{A}_1(\sigma) = [-2(\Lambda + \rho r^*) + \zeta(\sigma)]\omega(\mathfrak{A}_1). \tag{3.3}$$

Again, taking $\mathfrak{A}_1 = \zeta$ in (3.3) and using (2.1), it follows that

$$\zeta(\sigma) = \Lambda + \rho r^*. \tag{3.4}$$

From (3.3) and (3.4), we find the following:

$$\mathfrak{A}_1(\sigma) = -(\Lambda + \rho r^*)\omega(\mathfrak{A}_1) = -\zeta(\sigma)\omega(\mathfrak{A}_1). \tag{3.5}$$

By virtue of (3.4) and (3.5), (3.2) gives the following:

$$S^*(\mathfrak{A}_1, \mathfrak{A}_2) = [\sigma + \zeta(\sigma)]g(\mathfrak{A}_1, \mathfrak{A}_2) + [\sigma + \zeta(\sigma)]\omega(\mathfrak{A}_1)\omega(\mathfrak{A}_2). \tag{3.6}$$

By contracting (3.6) over $\mathfrak{A}_1$ and $\mathfrak{A}_2$, we have the following:

$$r^* = (n-1)[\sigma + \zeta(\sigma)]. \tag{3.7}$$

In view of (3.7), (3.6) takes the following form:

$$S^*(\mathfrak{A}_1, \mathfrak{A}_2) = \frac{r^*}{n-1}g(\mathfrak{A}_1, \mathfrak{A}_2) + \frac{r^*}{n-1}\omega(\mathfrak{A}_1)\omega(\mathfrak{A}_2). \tag{3.8}$$

From (3.8), it follows that

$$Q^*\mathfrak{A}_1 = \frac{r^*}{n-1}\mathfrak{A}_1 + \frac{r^*}{n-1}\omega(\mathfrak{A}_1)\zeta. \tag{3.9}$$

Taking the covariant derivative of (3.9) along $\mathfrak{A}_2$ and using (2.4) and (2.5), we have the following:

$$\begin{aligned}(\nabla_{\mathfrak{A}_2} Q^*)\mathfrak{A}_1 &= \frac{\mathfrak{A}_2(r^*)}{n-1}\mathfrak{A}_1 + \frac{\mathfrak{A}_2(r^*)}{n-1}\omega(\mathfrak{A}_1)\zeta \\ &- \frac{r^*}{n-1}(g(\mathfrak{A}_1, \mathfrak{A}_2)\zeta + \omega(\mathfrak{A}_1)\mathfrak{A}_2 + 2\omega(\mathfrak{A}_1)\omega(\mathfrak{A}_2)\zeta).\end{aligned}$$

On contracting the foregoing equation along $\mathfrak{A}_2$, we have

$$(n-3)\mathfrak{A}_1(r^*) + 2(n-1)r^*\omega(\mathfrak{A}_1) = 2\zeta(r^*)\omega(\mathfrak{A}_1), \tag{3.10}$$

which, by taking $\mathfrak{A}_1 = \zeta$, gives $\zeta(r^*) = 2r^*$. Thus, (3.10) takes the following form:

$$\mathfrak{A}_1(r^*) = -\zeta(r^*)\omega(\mathfrak{A}_1). \tag{3.11}$$

From (2.12), (2.16), (3.11), and the relation $\zeta(r^*) = 2r^*$, we deduce that

$$r^* = r - n(n-1) + \frac{1}{2}\zeta(a^2). \tag{3.12}$$

Using (3.12) in (3.8), it follows that

$$\begin{aligned}S^*(\mathfrak{A}_1, \mathfrak{A}_2) &= (\frac{r}{n-1} + \frac{\zeta(a^2)}{2(n-1)} - n)g(\mathfrak{A}_1, \mathfrak{A}_2) \\ &+(\frac{r}{n-1} + \frac{\zeta(a^2)}{2(n-1)} - n)\omega(\mathfrak{A}_1)\omega(\mathfrak{A}_2).\end{aligned} \tag{3.13}$$

Thus, we obtain the following result.

Theorem 3.1. *If the metric of an $(LPK)_n$ represents an almost $*$-RBS such that $V = \sigma\zeta$, for some smooth function σ, then $(LPK)_n$ is $*$-ω-Einstein manifold.*

By equating (3.7) and (3.12), it follows that

$$\sigma + \zeta(\sigma) = \frac{r}{n-1} + \frac{\zeta(a^2)}{2(n-1)} - n. \tag{3.14}$$

If r and a are constants, then from (3.13) and (3.14), we respectively obtain $S^* = 0$ and $\zeta(\sigma) = -\sigma \implies D\sigma = \sigma\zeta$, where (2.13) is used. Thus, we have the following corollary.

Corollary 3.1. *Let an $(LPK)_n$ with constant values of r and a admit an almost $*$-RBS such that $V = \sigma\zeta$ for some smooth function σ. Then, $(LPK)_n$ is $*$-Ricci flat, and the gradient of σ is parallel to the Reeb vector field ζ.*

Further, if $\sigma(\neq 0) \in \mathbb{R}$, then from (3.4), we have $\Lambda = -\rho r^*$, which, using (3.12), leads to the following:

$$\Lambda = \frac{\rho}{2}[h - 2(r - n(n-1))], \text{ where } \zeta(a^2) = -h(\in \mathbb{R}). \tag{3.15}$$

If r is constant, then (3.15) reduces to the following:

$$\Lambda = \frac{\rho h}{2}.$$

Thus, we have the following corollary.

Corollary 3.2. *Assume that the metric of an $(LPK)_n$ with the constant scalar curvature defines an almost $*$-RBS such that $V = \sigma\zeta$, where σ is a constant function. Then, we have the following (see Table 1):*

Table 1. Nature of almost $*$-RBS on an $(LPK)_n$ with constant scalar curvature.

Values of ρ	Values of h	Values of Λ	Nature of the $*$-RBS
	(i) >0	(i) >0	shrinking
>0	(ii) $=0$	(ii) $=0$	steady
	(iii) <0	(iii) <0	expanding
$=0$	>0, $=0$ or <0	$=0$	steady
	(i) >0	(i) <0	expanding
<0	(ii) $=0$	(ii) $=0$	steady
	(iii) <0	(iii) >0	shrinking

Next, we assume that the metric of an $(LPK)_n$ is an almost $*$-RBS. Then, (1.3) holds. By substituting (2.15) into (1.3), we obtain

$$\begin{aligned}(\pounds_V g)(\mathfrak{A}_1, \mathfrak{A}_2) &= -2S(\mathfrak{A}_1, \mathfrak{A}_2) + 2(\Lambda + n + \rho r^*)g(\mathfrak{A}_1, \mathfrak{A}_2) \\ &\quad +2\omega(\mathfrak{A}_1)\omega(\mathfrak{A}_2) - 2ag(\mathfrak{A}_1, \varphi\mathfrak{A}_2),\end{aligned} \tag{3.16}$$

for any $\mathfrak{A}_1, \mathfrak{A}_2 \in \chi(LPK)_n$.

Taking the covariant derivative of (3.16) along $\mathfrak{A}_3$, and using (2.3), (2.5), and (3.16), we find the following:

$$\begin{aligned}(\nabla_{\mathfrak{A}_3}\pounds_V g)(\mathfrak{A}_1, \mathfrak{A}_2) &= -2(\nabla_{\mathfrak{A}_3}S)(\mathfrak{A}_1, \mathfrak{A}_2) + 2\mathfrak{A}_3(\Lambda + \rho r^*)g(\mathfrak{A}_1, \mathfrak{A}_2)\\ &\quad -2\{g(\mathfrak{A}_1, \mathfrak{A}_3)\omega(\mathfrak{A}_2) + g(\mathfrak{A}_2, \mathfrak{A}_3)\omega(\mathfrak{A}_1)\}\\ &\quad -2(\mathfrak{A}_3 a)g(\mathfrak{A}_1, \phi\mathfrak{A}_2) - 4\omega(\mathfrak{A}_1)\omega(\mathfrak{A}_2)\omega(\mathfrak{A}_3)\\ &\quad +2a\{g(\varphi\mathfrak{A}_3, \mathfrak{A}_2)\omega(\mathfrak{A}_1) + g(\varphi\mathfrak{A}_3, \mathfrak{A}_1)\omega(\mathfrak{A}_2)\}. \end{aligned} \tag{3.17}$$

As g is parallel w.r.t. (with respect to) ∇ (*i.e.*, $\nabla g = 0$), we can apply it into the following identity (see [32]):

$$(\pounds_V\nabla_{\mathfrak{A}_1}g - \nabla_{\mathfrak{A}_1}\pounds_V g - \nabla_{[V,\mathfrak{A}_1]}g)(\mathfrak{A}_2, \mathfrak{A}_3) = -g((\pounds_V\nabla)(\mathfrak{A}_1, \mathfrak{A}_2), \mathfrak{A}_3) - g((\pounds_V\nabla)(\mathfrak{A}_1, \mathfrak{A}_3), \mathfrak{A}_2),$$

which turns into

$$(\nabla_{\mathfrak{A}_1}\pounds_V g)(\mathfrak{A}_2, \mathfrak{A}_3) = g((\pounds_V\nabla)(\mathfrak{A}_1, \mathfrak{A}_2), \mathfrak{A}_3) + g((\pounds_V\nabla)(\mathfrak{A}_1, \mathfrak{A}_3), \mathfrak{A}_2).$$

Owing to the symmetry of $\pounds_V\nabla$, we obtain the following:

$$2g((\pounds_V\nabla)(\mathfrak{A}_1, \mathfrak{A}_2), \mathfrak{A}_3) = (\nabla_{\mathfrak{A}_1}\pounds_V g)(\mathfrak{A}_2, \mathfrak{A}_3) + (\nabla_{\mathfrak{A}_2}\pounds_V g)(\mathfrak{A}_1, \mathfrak{A}_3) - (\nabla_{\mathfrak{A}_3}\pounds_V g)(\mathfrak{A}_1, \mathfrak{A}_2),$$

which, using (3.17), takes the form

$$\begin{aligned}g((\pounds_V\nabla)(\mathfrak{A}_1, \mathfrak{A}_2), \mathfrak{A}_3) &= (\nabla_{\mathfrak{A}_3}S)(\mathfrak{A}_1, \mathfrak{A}_2) - (\nabla_{\mathfrak{A}_1}S)(\mathfrak{A}_2, \mathfrak{A}_3) - (\nabla_{\mathfrak{A}_2}S)(\mathfrak{A}_1, \mathfrak{A}_3)\\ &\quad +\mathfrak{A}_1(\Lambda + \rho r^*)g(\mathfrak{A}_2, \mathfrak{A}_3) + \mathfrak{A}_2(\Lambda + \rho r^*)g(\mathfrak{A}_1, \mathfrak{A}_3)\\ &\quad -\mathfrak{A}_3(\Lambda + \rho r^*)g(\mathfrak{A}_1, \mathfrak{A}_2) - 2\{g(\phi\mathfrak{A}_1, \varphi\mathfrak{A}_2)\}\omega(\mathfrak{A}_3)\\ &\quad -(\mathfrak{A}_1 a)g(\mathfrak{A}_2, \varphi\mathfrak{A}_3) - (\mathfrak{A}_2 a)g(\mathfrak{A}_1, \varphi\mathfrak{A}_3)\\ &\quad +(\mathfrak{A}_3 a)g(\mathfrak{A}_1, \varphi\mathfrak{A}_2) + 2ag(\varphi\mathfrak{A}_1, \mathfrak{A}_2)\omega(\mathfrak{A}_3). \end{aligned} \tag{3.18}$$

By putting $\mathfrak{A}_2 = \zeta$ in (3.18) and then using (2.1), (2.2), and Lemma 2.1, we find the following:

$$\begin{aligned}g((\pounds_V\nabla)(\mathfrak{A}_1, \zeta), \mathfrak{A}_3) &= -2S(\mathfrak{A}_1, \mathfrak{A}_3) + 2(n-1)g(\mathfrak{A}_1, \mathfrak{A}_3) + \mathfrak{A}_1(\Lambda + \rho r^*)\omega(\mathfrak{A}_3)\\ &\quad +\zeta(\Lambda + \rho r^*)g(\mathfrak{A}_1, \mathfrak{A}_3) - \mathfrak{A}_3(\Lambda + \rho r^*)\omega(\mathfrak{A}_1) - (\zeta a)g(\mathfrak{A}_1, \varphi\mathfrak{A}_3).\end{aligned}$$

This gives the following:

$$\begin{aligned}(\pounds_V\nabla)(\mathfrak{A}_1, \zeta) &= -2Q\mathfrak{A}_1 + 2(n-1)\mathfrak{A}_1 + \mathfrak{A}_1(\Lambda + \rho r^*)\zeta + \zeta(\Lambda + \rho r^*)\mathfrak{A}_1\\ &\quad -D(\Lambda + \rho r^*)\omega(\mathfrak{A}_1) - (\zeta a)\varphi\mathfrak{A}_1.\end{aligned} \tag{3.19}$$

Now, assuming that $\Lambda + \rho r^*$ is invariant under ζ (i.e., $\zeta(\Lambda + \rho r^*) = 0$), then (3.19) can be written as follows:

$$\begin{aligned}(\pounds_V\nabla)(\mathfrak{A}_2, \zeta) &= -2Q\mathfrak{A}_2 + 2(n-1)\mathfrak{A}_2 + g(D\Lambda, \mathfrak{A}_2)\zeta - (D\Lambda)\omega(\mathfrak{A}_2)\\ &\quad -\rho\{2(r - n(n-1))\omega(\mathfrak{A}_2)\zeta + (Dr)\omega(\mathfrak{A}_2)\}\end{aligned} \tag{3.20}$$

$$+\rho\{g(Da^2, \mathfrak{A}_2)\zeta - (Da^2)\omega(\mathfrak{A}_2)\} - (\zeta a)\varphi\mathfrak{A}_2,$$

where (2.16) is used.

The covariant differentiation of (3.20) w.r.t. $\mathfrak{A}_1$, and using (2.3)–(2.5), (2.13), and (3.20), we are led to the following expression:

$$\begin{aligned}(\nabla_{\mathfrak{A}_1}\pounds_V\nabla)(\mathfrak{A}_2,\zeta) &= (\pounds_V\nabla)(\mathfrak{A}_1,\mathfrak{A}_2) - 2\{Q\mathfrak{A}_2 - (n-1)\mathfrak{A}_2\}\omega(\mathfrak{A}_1)\\ &\quad -(\zeta a)\omega(\mathfrak{A}_1)\varphi\mathfrak{A}_2 - 2(\nabla_{\mathfrak{A}_1}Q)\mathfrak{A}_2 + g(\nabla_{\mathfrak{A}_1}D\Lambda, \mathfrak{A}_2)\zeta\\ &\quad -g(D\Lambda,\mathfrak{A}_2)\mathfrak{A}_1 - (\nabla_{\mathfrak{A}_1}D\Lambda)\omega(\mathfrak{A}_2)\\ &\quad +(D\Lambda)g(\mathfrak{A}_1,\mathfrak{A}_2) + \rho\{6(r-n(n-1))\omega(\mathfrak{A}_1)\omega(\mathfrak{A}_2)\zeta\\ &\quad +2(r-n(n-1))g(\mathfrak{A}_1,\mathfrak{A}_2)\zeta + 2(r-n(n-1))\omega(\mathfrak{A}_2)\mathfrak{A}_1\\ &\quad -(\nabla_{\mathfrak{A}_1}Dr)\omega(\mathfrak{A}_2) + (Dr)g(\mathfrak{A}_1,\mathfrak{A}_2)\} + \rho\{g(\nabla_{\mathfrak{A}_1}Da^2,\mathfrak{A}_2)\zeta\\ &\quad -g(Da^2,\mathfrak{A}_2)\mathfrak{A}_1 - (\nabla_{\mathfrak{A}_1}Da^2)\omega(\mathfrak{A}_2) + (Da^2)g(\mathfrak{A}_1,\mathfrak{A}_2)\}\\ &\quad -(\nabla_{\mathfrak{A}_1}(\zeta a))\varphi\mathfrak{A}_2 + (\zeta a)g(\varphi\mathfrak{A}_1,\mathfrak{A}_2)\zeta + (\zeta a)\omega(\mathfrak{A}_2)\varphi\mathfrak{A}_1.\end{aligned} \tag{3.21}$$

Again, from [32], we have

$$(\pounds_V\mathcal{R})(\mathfrak{A}_1,\mathfrak{A}_2)\mathfrak{A}_3 - (\nabla_{\mathfrak{A}_1}\pounds_V\nabla)(\mathfrak{A}_2,\mathfrak{A}_3) + (\nabla_{\mathfrak{A}_2}\pounds_V\nabla)(\mathfrak{A}_1,\mathfrak{A}_3) = 0,$$

which, by setting $\mathfrak{A}_3 = \zeta$ and using (3.21), turns into

$$\begin{aligned}(\pounds_V\mathcal{R})(\mathfrak{A}_1,\mathfrak{A}_2)\zeta &= 2\{\omega(\mathfrak{A}_2)Q\mathfrak{A}_1 - \omega(\mathfrak{A}_1)Q\mathfrak{A}_2 + (n-1)(\omega(\mathfrak{A}_1)\mathfrak{A}_2 - \omega(\mathfrak{A}_2)\mathfrak{A}_1)\}\\ &\quad +2\{(\nabla_{\mathfrak{A}_2}Q)\mathfrak{A}_1 - (\nabla_{\mathfrak{A}_1}Q)\mathfrak{A}_2\} + g(\nabla_{\mathfrak{A}_1}D\Lambda,\mathfrak{A}_2)\zeta\\ &\quad -g(\nabla_{\mathfrak{A}_2}D\Lambda,\mathfrak{A}_1)\zeta + g(D\Lambda,\mathfrak{A}_1)\mathfrak{A}_2 - g(D\Lambda,\mathfrak{A}_2)\mathfrak{A}_1 + (\nabla_{\mathfrak{A}_2}D\Lambda)\omega(\mathfrak{A}_1)\\ &\quad -(\nabla_{\mathfrak{A}_1}D\Lambda)\omega(\mathfrak{A}_2) + \rho\{2(r-n(n-1))(\omega(\mathfrak{A}_2)\mathfrak{A}_1 - \omega(\mathfrak{A}_1)\mathfrak{A}_2)\\ &\quad +(\nabla_{\mathfrak{A}_2}Dr)\omega(\mathfrak{A}_1) - (\nabla_{\mathfrak{A}_1}Dr)\omega(\mathfrak{A}_2) + g(\nabla_{\mathfrak{A}_1}Da^2,\mathfrak{A}_2)\zeta\\ &\quad -g(\nabla_{\mathfrak{A}_2}Da^2,\mathfrak{A}_1)\zeta + g(Da^2,\mathfrak{A}_1)\mathfrak{A}_2 - g(Da^2,\mathfrak{A}_2)\mathfrak{A}_1\\ &\quad +(\nabla_{\mathfrak{A}_2}Da^2)\omega(\mathfrak{A}_1) - (\nabla_{\mathfrak{A}_1}Da^2)\omega(\mathfrak{A}_2)\} + (\nabla_{\mathfrak{A}_2}(\zeta a))\varphi\mathfrak{A}_1\\ &\quad -(\nabla_{\mathfrak{A}_1}(\zeta a))\varphi\mathfrak{A}_2 + 2(\zeta a)\omega(\mathfrak{A}_2)\varphi\mathfrak{A}_1 - 2(\zeta a)\omega(\mathfrak{A}_1)\varphi\mathfrak{A}_2.\end{aligned} \tag{3.22}$$

Now, putting $\mathfrak{A}_2 = \zeta$ in (3.22) then using (2.1), (2.2), (2.4), (2.8), and Lemma 2.1, we have the following:

$$\begin{aligned}(\pounds_V\mathcal{R})(\mathfrak{A}_1,\zeta)\zeta &= \omega(\nabla_{\mathfrak{A}_1}D\Lambda)\zeta - g(\nabla_\zeta D\Lambda,\mathfrak{A}_1)\zeta + g(D\Lambda,\mathfrak{A}_1)\zeta - \omega(D\Lambda)\mathfrak{A}_1\\ &\quad +(\nabla_\zeta D\Lambda)\omega(\mathfrak{A}_1) + (\nabla_{\mathfrak{A}_1}D\Lambda) + \rho\{-2(r-n(n-1))\mathfrak{A}_1\\ &\quad -2(r-n(n-1))\omega(\mathfrak{A}_1)\zeta + (\nabla_\zeta Dr)\omega(\mathfrak{A}_1) + (\nabla_{\mathfrak{A}_1}Dr)\\ &\quad +\omega(\nabla_{\mathfrak{A}_1}Da^2)\zeta - g(\nabla_\zeta Da^2,\mathfrak{A}_1)\zeta + g(Da^2,\mathfrak{A}_1)\zeta - \omega(Da^2)\mathfrak{A}_1\\ &\quad +(\nabla_\zeta Da^2)\omega(\mathfrak{A}_1) + (\nabla_{\mathfrak{A}_1}Da^2)\} + \zeta(\nabla_\zeta a)\varphi\mathfrak{A}_1 - 2(\zeta a)\varphi\mathfrak{A}_1.\end{aligned} \tag{3.23}$$

On contracting (3.23) over $\mathfrak{A}_1$, we obtain the following:

$$
\begin{aligned}
(\pounds_V S)(\zeta,\zeta) \;=\; & \triangle\Lambda-(n-1)\omega(D\Lambda)+\omega(\nabla_\zeta D\Lambda)++\zeta(\nabla_\zeta a)a-2(\zeta a)a \\
& +\rho\{\triangle r+\triangle a^2+\omega(\nabla_\zeta Da^2)+\omega(\nabla_\zeta Dr) \\
& -(n-1)\omega(Da^2)+2(n-1)(n(n-1)-r)\},
\end{aligned} \tag{3.24}
$$

where Δ symbolizes the Laplacian operator of g.

The Lie derivative of (2.7) and $1+g(\zeta,\zeta)=0$ along V infers

$$
(\pounds_V S)(\zeta,\zeta)=-2(n-1)\omega(\pounds_V\zeta), \tag{3.25}
$$

and

$$
(\pounds_V g)(\zeta,\zeta)=-2\omega(\pounds_V\zeta), \tag{3.26}
$$

respectively.

By setting $\mathfrak{A}_1=\mathfrak{A}_2=\zeta$ in (3.16) then utilizing (2.1), (2.2), and (2.7), we have the following:

$$
(\pounds_V g)(\zeta,\zeta)=-2(\Lambda+\rho r^*). \tag{3.27}
$$

Now, by combining (3.24)–(3.27), we deduce the following:

$$
\begin{aligned}
\triangle r=\;&\frac{1}{\rho}\{(n-1)\omega(D\Lambda)-\triangle\Lambda-2(n-1)(\Lambda+\rho r^*) \\
&-\omega(\nabla_\zeta D\Lambda)-\zeta(\zeta a)a+2(\zeta a)a\} \\
&-\triangle a^2-\omega(\nabla_\zeta Da^2)-2(n-3)(n(n-1)-r)+(n-1)\omega(Da^2),
\end{aligned} \tag{3.28}
$$

where (2.14) is used. Thus, we obtain the following result.

Theorem 3.2. *An $(LPK)_n$ that admits an almost $*$-RBS satisfies the partial differential equation* (3.28)*, provided ζ leaves $\Lambda+\rho r^*$ invariant.*

If we assume $Da^2=h\zeta$ and $D\Lambda=l\zeta$, where $h,l\in\mathbb{R}$, then we deduce the following values:

$$
\begin{cases}
(i)\;\; \nabla_{\mathfrak{A}_1}(Da^2)=-h(\mathfrak{A}_1+\omega(\mathfrak{A}_1)\zeta), \\
(ii)\;\; \mathfrak{A}_1(a^2)=h\omega(\mathfrak{A}_1), \\
(iii)\;\; \nabla_\zeta(D\Lambda)=0, \\
(iv)\;\; \omega(D\Lambda)=-l, \\
(v)\;\; \triangle r=-\frac{1}{\rho}\{2(n-1)(\Lambda+\rho r^*)+\zeta(\zeta a)a+h\} \\
\qquad\qquad -2(n-3)(n(n-1)-r).
\end{cases} \tag{3.29}
$$

The screened Poisson equation (also called the Yukawa equation or modified Helmholtz equation) is a partial differential equation (PDE) of the following form:

$$
(\triangle-\mathfrak{F}^2)\mathfrak{m}=\mathfrak{d}, \tag{3.30}
$$

where $\mathfrak{F} > 0$ is a constant called the screening parameter, $\mathfrak{m}$ is a source term, and $\mathfrak{d}$ is the unknown function. Additionally, (3.30) appears in plasma screening, such as in limits of the Thomas-Fermi theory [33] or the Debye-Hückel theory [34]. Moreover, (3.30) has applicability in granular fluid flow [35] and emerges in the Klein-Gordon equation. This is a rapidly growing subfield at the intersection of differential geometry, differential equations, and mathematical physics.

Thus, from (3.29)(v), we have the following:

$$(\triangle - 2(n-3))r = \mathfrak{D},$$

where $\mathfrak{D} = -\frac{1}{\rho}\{2(n-1)(\Lambda + \rho r^*) + \zeta(\zeta a)a + h\} - 2n(n-3)(n-1)$ for $dimM > 3$. Thus, we summarize our outcome as follows.

Corollary 3.3. *An $(LPK)_n$ that admits an almost $*$-RBS and satisfies $Da^2 = h\zeta$ and $D\Lambda = l\zeta$ also satisfies the screened Poisson equation $(\triangle - 2(n-3))r = \mathfrak{D}$.*

4. Gradient almost $*$-RBSs on $(LPK)_n$

This section characterizes $(LPK)_n$ which admits a gradient almost $*$-RBS metric.

Let an $(LPK)_n$ admit a gradient almost $*$-RBS metric. Then, from (1.4), it follows that

$$\nabla_{\mathfrak{A}_1} Df + Q^*\mathfrak{A}_1 = (\Lambda + \rho r^*)\,\mathfrak{A}_1, \tag{4.1}$$

for any $\mathfrak{A}_1 \in \chi(LPK)_n$.

The covariant differentiation of (4.1) w.r.t. $\mathfrak{A}_1$ leads to the following:

$$\begin{aligned} \nabla_{\mathfrak{A}_2}\nabla_{\mathfrak{A}_1} Df &= -\{(\nabla_{\mathfrak{A}_2} Q^*)\mathfrak{A}_1 + Q^*(\nabla_{\mathfrak{A}_2}\mathfrak{A}_1\} + (\Lambda + \rho r^*)\,\nabla_{\mathfrak{A}_2}\mathfrak{A}_1 \\ &\quad +\{(\mathfrak{A}_2\Lambda) + \rho(\mathfrak{A}_2 r^*)\}\mathfrak{A}_1. \end{aligned} \tag{4.2}$$

By interchanging $\mathfrak{A}_1$ and $\mathfrak{A}_2$ in (4.2), we obtain the following:

$$\begin{aligned} \nabla_{\mathfrak{A}_1}\nabla_{\mathfrak{A}_2} Df &= -\{(\nabla_{\mathfrak{A}_1} Q^*)\mathfrak{A}_2 + Q^*(\nabla_{\mathfrak{A}_1}\mathfrak{A}_2\} + (\Lambda + \rho r^*)\,\nabla_{\mathfrak{A}_1}\mathfrak{A}_2 \\ &\quad +\{(\mathfrak{A}_1\Lambda) + \rho(\mathfrak{A}_1 r^*)\}\mathfrak{A}_2. \end{aligned} \tag{4.3}$$

By using (4.1)–(4.3) in the curvature identity

$$\mathcal{R}(\mathfrak{A}_1, \mathfrak{A}_2) = [\nabla_{\mathfrak{A}_1}, \nabla_{\mathfrak{A}_2}] - \nabla_{[\mathfrak{A}_1,\mathfrak{A}_2]},$$

we get

$$\begin{aligned} \mathcal{R}(\mathfrak{A}_1, \mathfrak{A}_2)Df &= -(\nabla_{\mathfrak{A}_1} Q^*)\mathfrak{A}_2 + (\nabla_{\mathfrak{A}_2} Q^*)\mathfrak{A}_1 \\ &\quad +\{(\mathfrak{A}_1\Lambda) + \rho(\mathfrak{A}_1 r^*)\}\mathfrak{A}_2 - \{(\mathfrak{A}_2\Lambda) + \rho(\mathfrak{A}_2 r^*)\}\mathfrak{A}_1. \end{aligned} \tag{4.4}$$

Here, $[\cdot\,,\,\cdot]$ denotes the Lie bracket, and it is defined as

$$[\mathfrak{A}_1, \mathfrak{A}_2]f = \mathfrak{A}_1(\mathfrak{A}_2 f) - \mathfrak{A}_2(\mathfrak{A}_1 f)$$

for a smooth function f on M.

By contracting (4.4) along $\mathfrak{A}_1$, we find the following:

$$S(\mathfrak{A}_2, Df) = \{\frac{1}{2} - (n-1)\rho\}(\mathfrak{A}_2 r^*) - (n-1)(\mathfrak{A}_2 \Lambda). \tag{4.5}$$

From (2.7), we have the following:

$$S(\zeta, Df) = (n-1)g(\zeta, Df). \tag{4.6}$$

Thus, from (4.5) and (4.6), it follows that

$$(\zeta f) = \{\frac{1}{2(n-1)} - \rho\}(\zeta r^*) - (\zeta \Lambda). \tag{4.7}$$

Now, we consider two cases in our study as follows.

Case I. Let $\rho = \frac{1}{2(n-1)}$. Then, (4.7) turns into

$$(\zeta f) = -(\zeta \Lambda). \tag{4.8}$$

The covariant differentiation of (4.8) along $\mathfrak{A}_1$ gives the following:

$$(\Lambda + \rho r^*)\omega(\mathfrak{A}_1) - (\mathfrak{A}_1 f) = -g(\nabla_{\mathfrak{A}_1} D\Lambda, \zeta) + (\mathfrak{A}_1 \Lambda), \tag{4.9}$$

where (2.2), (2.7), (2.15), (4.1), and (4.8) are used.

By setting $\mathfrak{A}_1 = \zeta$ in (4.9) and utilizing (2.1) and (4.8), we obtain the following:

$$g(\nabla_\zeta D\Lambda, \zeta) = \Lambda + \rho r^*. \tag{4.10}$$

By using (3.29) (*iii*), (4.10) transforms into the following:

$$\Lambda + \rho r^* = 0. \tag{4.11}$$

Now, from (4.9)–(4.11), we conclude that

$$\mathfrak{A}_1 f = -\mathfrak{A}_1 \Lambda \Longleftrightarrow Df = -D\Lambda = -\beta\zeta. \tag{4.12}$$

Thus, the gradient of $f(\in C^\infty(M))$ is a constant multiple of the time-like vector field ζ.

In view of (2.15) and (4.11), (4.1) gives the following:

$$Hessf + S - ng - \omega \otimes \omega = -ag(\varphi\cdot, \cdot). \tag{4.13}$$

Thus, we summarize our result as follows.

Theorem 4.1. *Let an $(LPK)_n$ admit a gradient almost $*$-Schouten soliton. If $\Lambda + \rho r^* = 0$, then the gradient of the potential function of the gradient almost $*$-Schouten soliton is proportional to the unit time-like vector field ζ and $Hessf + S - ng - \omega \otimes \omega = -ag(\varphi\cdot, \cdot)$.*

Suppose $a = 0$ in (4.13). Then, by Theorem 4.1, the gradient almost $*$-Schouten soliton reduces to a shrinking gradient ω-Ricci soliton. Based on the above discussion, we summarize our finding as follows.

Corollary 4.1. *Let an $(LPK)_n$ admit a gradient almost $*$-Schouten soliton; if the gradient of its potential function is proportional to ζ and $a = 0$, then the soliton transforms to a shrinking gradient ω-Ricci soliton.*

Corollary 4.2. *If an $(LPK)_n$ admits a gradient almost $*$-Schouten soliton whose potential function has gradient proportional to the unit time-like vector field ζ, then the soliton will be expanding, steady, or shrinking if $r^* > 0$, $= 0$ or $r^* < 0$, respectively.*

The covariant derivative of (4.12) along $\mathfrak{A}_1$ gives

$$\nabla_{\mathfrak{A}_1} Df = \beta(\mathfrak{A}_1 + \omega(\mathfrak{A}_1)\zeta),$$

which, with the help of (2.15) and (4.1), gives

$$Q\mathfrak{A}_1 = (n-\beta)\mathfrak{A}_1 - (\beta-1)\omega(\mathfrak{A}_1)\zeta - a\varphi\mathfrak{A}_1.$$

It is well known that an (M, g) is called a generalized u-Einstein manifold if its Ricci tensor $S(\neq 0)$ is expressed as follows:

$$S = \mathfrak{b}_1 g + \mathfrak{b}_2 u \otimes u + \mathfrak{b}_3 g(\varphi\cdot, \cdot), \tag{4.14}$$

where $\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_3 \in C^\infty(M)$, and u is a 1-form associated with the vector field $\mathfrak{I}$ such that $u(\cdot) = g(\cdot, \mathfrak{I})$. If we set $\mathfrak{b}_3 = 0$, $u = \omega$, and $\mathfrak{I} = \zeta$ in (4.14), then the generalized ω-Einstein manifold reduces to a perfect fluid spacetime. In the study of perfect fluid spacetimes, Lorentzian para-Sasakian structures and related geometric frameworks have been investigated by several authors in [36–38], while their interaction with soliton structures has been further explored in different contexts, including Yamabe solitons by De et al. [39] and hyperbolic Ricci solitons by Azami et al. [40].

Based on the above discussion, we can state the following results.

Corollary 4.3. *An $(LPK)_n$ that admits a gradient almost $*$-Schouten soliton whose potential function has a gradient proportional to the vector field ζ is a generalized ω-Einstein manifold.*

Corollary 4.4. *If an $(LPK)_n$ admits a gradient almost $*$-Schouten soliton whose potential function has a gradient proportional to the vector field ζ, and if $a = 0$, then the manifold is a perfect fluid spacetime.*

Case II. We suppose that $\rho \neq \frac{1}{2(n-1)}$. By replacing $\mathfrak{A}_1 = \zeta$ in (4.4) and then using Lemma 2.4, we have the following:

$$\begin{aligned}\mathcal{R}(\zeta, \mathfrak{A}_2)Df &= -Q\mathfrak{A}_2 + (n-2)\mathfrak{A}_2 - \omega(\mathfrak{A}_2)\zeta + a\varphi\mathfrak{A}_2 - \zeta(a)\varphi\mathfrak{A}_2 \\ &\quad + \{(\zeta\Lambda) + \rho(\zeta r^*)\}\mathfrak{A}_2 - \{(\mathfrak{A}_2\Lambda) + \rho(\mathfrak{A}_2 r^*)\}\zeta.\end{aligned}$$

Additionally, from (2.6), we have the following:

$$\mathcal{R}(\zeta, \mathfrak{A}_2)Df = (\mathfrak{A}_2 f)\zeta - (\zeta f)\mathfrak{A}_2.$$

Setting the last two expressions equal to each other, we find the following:

$$\begin{aligned}(\mathfrak{A}_2 f)\zeta - (\zeta f)\mathfrak{A}_2 &= -Q\mathfrak{A}_2 + (n-2)\mathfrak{A}_2 - \omega(\mathfrak{A}_2)\zeta + a\varphi\mathfrak{A}_2 - \zeta(a)\varphi\mathfrak{A}_2 \\ &\quad + \{(\zeta\Lambda) + \rho(\zeta r^*)\}\mathfrak{A}_2 - \{(\mathfrak{A}_2\Lambda) + \rho(\mathfrak{A}_2 r^*)\}\zeta.\end{aligned} \tag{4.15}$$

By contracting (4.15) along $\mathfrak{A}_2$, we have the following:

$$(\zeta f) = -\frac{1}{n-1}\{a(a-\zeta(a)) + n(n-2) + 1 - r)\} - \{(\zeta\Lambda) + \rho(\zeta r^*)\}. \tag{4.16}$$

Now, from (2.16) and (2.13), we can easily find the following:

$$(\mathfrak{A}_2 r^*) = -2(r - n(n-1))\omega(\mathfrak{A}_2) + (\mathfrak{A}_2 a^2). \tag{4.17}$$

This gives

$$(\zeta r^*) = 2(r - n(n-1)) - h, \tag{4.18}$$

since $h = -g(\zeta, Da^2)$.

By substituting (4.18) into (4.16), we obtain the following:

$$\begin{aligned}(\zeta f) \;\; = \;\; & -\frac{1}{n-1}\{a(a-\zeta(a)) + n(n-2) + 1 - r)\} \\ & -\{(\zeta\Lambda) - \rho h - 2\rho((n-1)n - r)\}.\end{aligned} \tag{4.19}$$

From (4.15), (4.17), and (4.19), it follows that

$$\begin{aligned}(\mathfrak{A}_2 f)\zeta \;\; = \;\; & -\frac{1}{n-1}\{a(a-\zeta(a)) + n(n-2) + 1 - r)\}\mathfrak{A}_2 \\ & -Q\mathfrak{A}_2 + (n-2)\mathfrak{A}_2 - \omega(\mathfrak{A}_2)\zeta + (a - \zeta(a))\varphi\mathfrak{A}_2 \\ & -\{(\mathfrak{A}_2\Lambda) - 2\rho(r - n(n-1))\omega(\mathfrak{A}_2) + \rho h\omega(\mathfrak{A}_2)\}\zeta.\end{aligned} \tag{4.20}$$

Applying the inner product with ζ to (4.20) then using (2.1), (2.2), and (2.7) results in the following:

$$\begin{aligned}(\mathfrak{A}_2 f) \;\; = \;\; & \frac{1}{n-1}\{a(a-\zeta(a)) + n(n-2) + 1 - r)\}\omega(\mathfrak{A}_2) \\ & -\{(\mathfrak{A}_2\Lambda) + 2\rho(n(n-1) - r)\omega(\mathfrak{A}_2) + \rho h\omega(\mathfrak{A}_2)\}.\end{aligned} \tag{4.21}$$

It follows from (4.21) that

$$\begin{aligned}(Df) \;\; = \;\; & \frac{1}{n-1}\{a(a-\zeta(a)) + n(n-2) + 1 - r)\}\zeta + D\Lambda \\ & +\rho\{h - 2(r - n(n-1))\}\zeta.\end{aligned} \tag{4.22}$$

The inner product of (4.20) with $\mathfrak{A}_1$ gives

$$\begin{aligned}(\mathfrak{A}_2 f)\omega(\mathfrak{A}_1) = & -\frac{1}{n-1}\{a(a-\zeta(a)) + n(n-2) + 1 - r)\}g(\mathfrak{A}_2, \mathfrak{A}_1) \\ & -S(\mathfrak{A}_1, \mathfrak{A}_2) + (n-2)g(\mathfrak{A}_2, \mathfrak{A}_1) - \omega(\mathfrak{A}_2)\omega(\mathfrak{A}_1) + (a - \zeta(a))g(\varphi\mathfrak{A}_2, \mathfrak{A}_1) \\ & -\{(\mathfrak{A}_2\Lambda) - 2\rho(r - n(n-1))\omega(\mathfrak{A}_2) + \rho h\omega(\mathfrak{A}_2)\}\omega(\mathfrak{A}_1),\end{aligned}$$

which, in view of (4.21), reduces to the following form:

$$\begin{aligned}S(\mathfrak{A}_1, \mathfrak{A}_2) = & \frac{1}{n-1}\{r - (n-1) - a(a-\zeta(a))\}g(\mathfrak{A}_2, \mathfrak{A}_1) \\ & +\frac{1}{n-1}\{r - n(n-1) - a(a-\zeta(a))\}\omega(\mathfrak{A}_2)\omega(\mathfrak{A}_1) + (a - \zeta(a))g(\varphi\mathfrak{A}_2, \mathfrak{A}_1),\end{aligned} \tag{4.23}$$

which, by (4.14), infers that $(LPK)_n$ is a generalized ω-Einstein manifold, and the potential function f is determined by (4.22). Thus, we summarize our result as follows.

Theorem 4.2. *An $(LPK)_n$ admitting a gradient almost $*$-RBS is a generalized ω-Einstein manifold, provided $\rho \neq \frac{1}{2(n-1)}$. Additionally, the potential function f of the gradient $*$-RBS is evaluated by* (4.22).

Particularly, we suppose that a satisfies the PDE $\zeta(a) = a$. Then, (4.23) can be reduced to the following:

$$S(\mathfrak{A}_1, \mathfrak{A}_2) = \frac{1}{1-n}\{(n-1) - r\}g(\mathfrak{A}_2, \mathfrak{A}_1) + \frac{1}{1-n}\{n(n-1) - r\}\omega(\mathfrak{A}_2)\omega(\mathfrak{A}_1),$$

which shows that an $(LPK)_n$ is a perfect fluid spacetime. Thus, we have the following result.

Corollary 4.5. *Let an $(LPK)_n$ admit a gradient $*$-RBS and $\rho \neq \frac{1}{2(n-1)}$. If $\zeta(a) = a$, then $(LPK)_n$ is a perfect fluid spacetime.*

Let an $(LPK)_n$ admit a gradient $*$-RBS and $\rho \neq \frac{1}{2(n-1)}$, and it satisfies the Einstein field equations without a cosmological constant. Then, we have the following:

$$S - \frac{1}{2}rg = \kappa T, \tag{4.24}$$

where κ and T denote the gravitational constant and the energy-momentum tensor, respectively. In consequence of (4.23) and (4.24), we get the following expression of T:

$$\begin{aligned} T(\mathfrak{A}_1, \mathfrak{A}_2) &= -\frac{1}{\kappa(n-1)}\{a(a-\zeta(a)) + n(n-1) - r\}\omega(\mathfrak{A}_2)\omega(\mathfrak{A}_1) \\ &\quad -\frac{1}{\kappa(n-1)}\{r\left(\frac{n-3}{2(n-1)}\right) - (n-1) - a(a-\zeta(a))\}g(\mathfrak{A}_2, \mathfrak{A}_1) \\ &\quad +\frac{1}{\kappa}(a-\zeta(a))g(\varphi\mathfrak{A}_2, \mathfrak{A}_1). \end{aligned} \tag{4.25}$$

Thus, we can state the following corollary.

Corollary 4.6. *Let an $(LPK)_n$ admit a gradient $*$-RBS with $\rho \neq \frac{1}{2(n-1)}$, and suppose it satisfies the Einstein field equations without a cosmological constant; then, T is given by* (4.25).

If the matter content of the fluid is filled with a perfect fluid, then T is defined by the following:

$$T(\mathfrak{A}_1, \mathfrak{A}_2) = pg(\mathfrak{A}_1, \mathfrak{A}_2) + (p+\mu)\omega(\mathfrak{A}_1)\omega(\mathfrak{A}_2),$$

where p and μ refer to the isotropic pressure and the energy density of the fluid, respectively.

Together, the last equations give

$$S(\mathfrak{A}_1, \mathfrak{A}_2) - \frac{1}{2}rg(\mathfrak{A}_1, \mathfrak{A}_2) = \kappa\{pg(\mathfrak{A}_1, \mathfrak{A}_2) + (p+\mu)\omega(\mathfrak{A}_1)\omega(\mathfrak{A}_2)\} \tag{4.26}$$

for arbitrary vector fields $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Putting $\mathfrak{A}_2 = \zeta$ in (4.26), we find the following:

$$\kappa\mu = \frac{1}{2}\{r - 2(n-1)\}. \tag{4.27}$$

Let us consider a set of orthonormal vector fields, and by contracting (4.26) over $\mathfrak{A}_1$ and $\mathfrak{A}_2$, we obtain

$$-\kappa p = \frac{r}{2}\left(\frac{n-3}{n-1}\right) + 1, \tag{4.28}$$

since (4.27) is used. Together, (4.27) and (4.28) give the following:

$$\frac{p}{\mu} = -\frac{2 + r\left(\frac{n-3}{n-1}\right)}{r - 2(n-1)}, \tag{4.29}$$

which is the equation of state (EOS).

For $n = 4$, (4.29) assumes the form $p + \mu = 0$. The EOS $\theta = \frac{p}{\mu} = -1$ represents the cosmological constant, or vacuum energy, responsible for the dark energy which drives the current accelerated expansion of the universe [41–43]. Thus, we summarize our result as follows.

Corollary 4.7. *Let an $(LPK)_n$ admit a gradient $*$-RBS with $\rho \neq \frac{1}{2(n-1)}$, and suppose it satisfies the Einstein field equations without a cosmological constant. Then, the EOS parameter represents the cosmological constant, or vacuum energy, which is responsible for the dark energy which drives the current accelerated expansion of the universe.*

5. Conclusions

In the present work, we studied an $(LPK)_n$ that admits almost $*$-RBSs and gradient almost $*$-RBSs, and obtained certain key results. First, we proved that an $(LPK)_n$ that admits almost $*$-RBSs (where the soliton vector field is parallel to the timelike vector field ζ) is a $*$-ω-Einstein manifold. Moreover, we proved that an $(LPK)_n$ which admits almost $*$-RBSs satisfies a screened Poisson equation. Furthermore, under certain assumptions, we characterized an $(LPK)_n$ which admits gradient almost $*$-RBSs in two cases: (I) $\rho = \frac{1}{2(n-1)}$, where in this case, we showed that in an $(LPK)_n$ which admits a gradient almost $*$-Schouten soliton, the gradient of the potential function f is proportional to the unit time-like vector filed ζ, and $(LPK)_n$ represents a generalized ω-Einstein spacetime; and (II) $\rho \neq \frac{1}{2(n-1)}$, where in this case, we proved that an $(LPK)_n$ which admits a gradient almost $*$-RBS is a generalized ω-Einstein spacetime, and in a particular case, it is a perfect fluid spacetime.

Author contributions

Abdul Haseeb: Conceptualization, investigation, methodology, writing-original draft; Sudhakar Kumar Chaubey: Conceptualization, methodology, writing-review & editing; Fatemah Mofarreh: Investigation, methodology, writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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