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Research article

Mirror Hom-Lie algebras in semi-Euclidean spaces

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Abstract: In this paper, first we introduce the notion of a mirror Hom-Lie algebra and give some examples. Then, we endow the space of general linear mappings with a mirror Hom-Lie algebraic structure $(\mathfrak{gl}(V), [\cdot,\cdot]_\alpha, Ad_\alpha)$. Next, we study representations of mirror Hom-Lie algebras and that the pseudo-adjoint representation $ad^* : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$, which is defined by $ad^*_x(y) = -[x, y]$, is a morphism from the mirror Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \beta)$ to the mirror Hom-Lie algebra $(\mathfrak{gl}(\mathfrak{g}), [\cdot,\cdot]_\beta, Ad_\beta)$. Then, we provide the coboundary operator of mirror Hom-Lie algebras. As an application, there exists a mirror Hom-Lie algebra $(R^4_2, [\cdot,\cdot]_\theta, P)$ in semi-Euclidean spaces. For the null space of semi-Euclidean spaces, there is a subset V^* of the null space, and V^* is invariant under the actions of $[\cdot,\cdot]_\theta$ and P.

Keywords: deformations of Lie algebras; Hom-Lie algebras; representations; semi-Euclidean spaces; null space
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1. Introduction

The notion of Hom-Lie algebras was introduced by Hartwig, Larsson, and Silvestrov in [1] as part of a study of deformations of the Witt and Virasoro algebras. In a Hom-Lie algebra, the Jacobi identity is twisted by a linear map, called the Hom-Jacobi identity. Some q-deformations of the Witt and Virasoro algebras have the structure of a Hom-Lie algebra [1, 2]. Because of its close relationship to discrete and deformed vector fields and differential calculus [1, 3, 4], researchers have increasingly paid attention to this algebraic structure [5–9]. If $(\mathfrak{a}, [\cdot,\cdot], \gamma)$ is a Hom-Lie algebra, k is a real number, if $k \neq 1$, then $(\mathfrak{a}, [\cdot,\cdot], k\gamma)$ is not a Hom-Lie algebra. Naturally, what algebraic structure could it be? On the other hand, the semi-Euclidean space is a vector space with a pseudoscalar product that is different from Euclidean space. The study of semi-Euclidean spaces has produced fruitful results [10–14]. It is well known that there exist spacelike submanifolds, timelike submanifolds, and null submanifolds in semi-Euclidean space. Null submanifolds appear in many physics papers. For example, null submanifolds are of interest because they provide models of different horizon types such as event horizons of Kerr black holes, Cauchy horizons, isolated horizons, Kruskal horizons, and Killing horizons [15–20]. Null submanifolds are also studied in the theory of electromagnetism.

In this paper, we discuss the Hom-Lie algebra structure over R^3. We obtain a kind of deformations of Lie algebras; these deformations are associated with Hom-Lie algebras. We have: if $(\mathfrak{a}, [\cdot,\cdot], \gamma)$ is a Hom-Lie algebra, then $(\mathfrak{a}, [\cdot,\cdot], -\gamma)$ is a mirror Hom-Lie algebra. Then, we give some properties of these deformations and study their representations. Next, we give the coboundary operator of these deformations. Finally, we find out these deformations in null spaces of semi-Euclidean spaces.

The paper is organized as follows: In Section 2, we introduce the notion of mirror Hom-Lie algebras, study their representations, and give some properties. In Section 3, we give their coboundary operator. In Section 4, we construct a series of mirror Hom-Lie algebras $(R^4_2, [\cdot,\cdot]_\theta, P)$ in semi-Euclidean 4-spaces; then, we prove that for the null space of semi-Euclidean 4-spaces , there exists a subset V^* of the null space, and V^* is invariant under the actions of $[\cdot,\cdot]_\theta$ and P.

2. Mirror Hom-Lie algebras and their representations

The notion of a Hom-Lie algebra was introduced in [1]; see also [21, 22] for more information.

Definition 2.1. (i) *A Hom-Lie algebra is a triple* $(\mathfrak{a}, [\cdot,\cdot], \gamma)$ *consisting of a vector space* $\mathfrak{a}$*, a skew symmetric bilinear map (bracket)* $[\cdot,\cdot] : \wedge^2\mathfrak{a} \longrightarrow \mathfrak{a}$ *and a linear transformation* $\gamma : \mathfrak{a} \longrightarrow \mathfrak{a}$ *satisfying* $\gamma[x, y] = [\gamma(x), \gamma(y)]$*, and the following Hom-Jacobi identity:*

$$[[y,z], \gamma(x)] + [[z, x], \gamma(y)] + [[x, y], \gamma(z)] = 0, \quad \forall x, y, z \in \mathfrak{g}.$$

A Hom-Lie algebra is called a regular Hom-Lie algebra if γ *is a linear automorphism.*

(ii) *A subspace* $\mathfrak{b} \subset \mathfrak{a}$ *is a Hom-Lie sub-algebra of* $(\mathfrak{a}, [\cdot,\cdot], \gamma)$ *if* $\gamma(\mathfrak{b}) \subset \mathfrak{b}$ *and* $\mathfrak{b}$ *is closed under the bracket operation* $[\cdot,\cdot]$*, i.e., for all* $x, y \in \mathfrak{b}$*,* $[x, y] \in \mathfrak{b}$*.*

(iii) *A morphism from the Hom-Lie algebra* $(\mathfrak{a}, [\cdot,\cdot]_\mathfrak{a}, \gamma)$ *to the Hom-Lie algebra* $(\mathfrak{b}, [\cdot,\cdot]_\mathfrak{b}, \delta)$ *is a linear map* $\psi : \mathfrak{a} \longrightarrow \mathfrak{b}$ *such that* $\psi([x, y]_\mathfrak{a}) = [\psi(x), \psi(y)]_\mathfrak{b}$ *and* $\psi \circ \gamma = \delta \circ \psi$*.*

When $(\mathfrak{a}, [\cdot,\cdot], \gamma)$ is a Hom-Lie algebra, then $\gamma^k([x, y]) = [\gamma^k(x), \gamma^k(y)]$, $k = 1, 2, 3, \cdots$. Now, we give the notion of mirror Hom-Lie algebras.

Definition 2.2. (1) *A mirror Hom-Lie algebra is a triple* $(\mathfrak{g}, [\cdot,\cdot], \beta)$ *consisting of a vector space* $\mathfrak{g}$*, a skew symmetric bilinear map (bracket)* $[\cdot,\cdot] : \wedge^2\mathfrak{g} \longrightarrow \mathfrak{g}$ *and a linear transformation* $\beta : \mathfrak{g} \longrightarrow \mathfrak{g}$ *satisfying* $\beta([x, y]) = -[\beta(x), \beta(y)]$*, and the following Hom-Jacobi identity:*

$$[[y,z], \beta(x)] + [[z, x], \beta(y)] + [[x, y], \beta(z)] = 0, \quad \forall x, y, z \in V. \tag{2.1}$$

A mirror Hom-Lie algebra is called a regular mirror Hom-Lie algebra if β *is a linear automorphism.*

(2) *A subspace* $\mathfrak{h} \subset \mathfrak{g}$ *is a mirror Hom-Lie sub-algebra of* $(\mathfrak{g}, [\cdot,\cdot], \beta)$ *if* $\beta(\mathfrak{h}) \subset \mathfrak{h}$ *and* $\mathfrak{h}$ *is closed under the bracket operation* $[\cdot,\cdot]$*, i.e., for all* $x, y \in \mathfrak{h}$*,* $[x, y] \in \mathfrak{h}$*.*

(3) *A morphism from the mirror Hom-Lie algebra* $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \beta)$ *to the mirror Hom-Lie algebra* $(\mathfrak{h}, [\cdot,\cdot]_\mathfrak{h}, \lambda)$ *is a linear map* $f : \mathfrak{g} \longrightarrow \mathfrak{h}$ *such that* $f([x, y]_\mathfrak{g}) = -[f(x), f(y)]_\mathfrak{h}$ *and* $f \circ \beta = \lambda \circ f$*.*

Remark 2.3. (1) *Let* $(\mathfrak{a}, [\cdot,\cdot], \gamma)$ *be a Hom-Lie algebra, when* γ *is an identity, then* $(\mathfrak{a}, [\cdot,\cdot], \gamma)$ *is a Lie algebra; but when* $(\mathfrak{g}, [\cdot,\cdot], \beta)$ *is a mirror Hom-Lie algebra, map* β *can not be an identity; so a Lie algebra is a Hom-Lie algebra, but it is not a mirror Hom-Lie algebra;*

(2) $(\mathfrak{a}, [\cdot,\cdot], \gamma)$ *is a Hom-Lie algebra, then* $(\mathfrak{a}, [\cdot,\cdot], (-1)\gamma)$ *is a mirror Hom-Lie algebra;*
(3) *when* $(\mathfrak{g}, [\cdot,\cdot], \beta)$ *is a mirror Hom-Lie algebra, we have:* $\beta^{2k}([x,y]) = [\beta^{2k}(x), \beta^{2k}(y)]$*, and* $\beta^{2k-1}([x,y]) = -[\beta^{2k-1}(x), \beta^{2k-1}(y)]$*,* $k = 1, 2, 3, \cdots$ *.*
(4) *If* f *is a morphism from the mirror Hom-Lie algebra* $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \beta)$ *to the mirror Hom-Lie algebra* $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, \lambda)$*, and* g *is a morphism from the mirror Hom-Lie algebra* $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, \lambda)$ *to the mirror Hom-Lie algebra* $(\mathfrak{k}, [\cdot,\cdot]_{\mathfrak{k}}, \chi)$*, then* $-g \circ f$ *is a morphism from the mirror Hom-Lie algebra* $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \beta)$ *to the mirror Hom-Lie algebra* $(\mathfrak{k}, [\cdot,\cdot]_{\mathfrak{k}}, \chi)$*.*

Example 2.4. *Let* R^3 *be a 3-dimensional vector space. For* $x, y \in R^3$*,* $x = (x_1, x_2, x_3)^T$*, ,* $y = (y_1, y_2, y_3)^T$*, we define a skew symmetric bilinear map (bracket)*

$$[x,y] = x \wedge y = \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix},$$

where $\{e_1, e_2, e_3\}$ *is the canonical basis of* R^3*. We also have the scalar product* $< x, y >_1 = x^T y$*. Let* A *be a* 3×3*-matrix,* A *acts on* R^3 *by* $A(x) = Ax$*; then,* A *is a linear map. When* $AA^T = \mathrm{id}$*, for* $x, y, z \in R^3$*, we have:*

$$< A([x,y]), z >_1 = < x \wedge y, A^T z >_1 = \begin{vmatrix} x^T \\ y^T \\ (A^T z)^T \end{vmatrix}.$$

On the other hand, we have:

$$< [Ax, Ay], z >_1 = < Ax \wedge Ay, z >_1 = \begin{vmatrix} (Ax)^T \\ (Ay)^T \\ z^T \end{vmatrix} = |A| \begin{vmatrix} x^T \\ y^T \\ (A^T z)^T \end{vmatrix} = |A| < A([x,y]), z >_1 .$$

So, there is

$$|A| = 1, \quad A([x,y]) = [A(x), A(y)];$$
$$|A| = -1, \quad A([x,y]) = -[A(x), A(y)].$$

Let

$$[x,y]_1 = A([x,y]),$$

we have:

(i) *when* $|A| = 1$*,* $(R^3, [\cdot,\cdot]_1, A)$ *is a Hom-Lie algebra;*
(ii) *when* $|A| = -1$*,* $(R^3, [\cdot,\cdot]_1, A)$ *is a mirror Hom-Lie algebra.*

Proposition 2.5. *Let* $\alpha \in \mathfrak{gl}(V)$ *and* $\alpha^2 = -\mathrm{id}$*, we define a linear map*

$$Ad_\alpha : \mathfrak{gl}(V) \longrightarrow \mathfrak{gl}(V)$$

by $Ad_\alpha(B) = \alpha B \alpha$*, and a bilinear map (bracket)*

$$[A, B]_\alpha = \alpha A \alpha B \alpha - \alpha B \alpha A \alpha,$$

then $(\mathfrak{gl}(V), [\cdot,\cdot]_\alpha, Ad_\alpha)$ *is a mirror Hom-Lie algebra.*

Proof. Obviously, $[\cdot,\cdot]$ is a skew bilinear map. Also, we have:

$$\begin{aligned} Ad_\alpha([A,B]_\alpha) &= \alpha^2 A\alpha B\alpha^2 - \alpha^2 B\alpha A\alpha^2 \\ &= A\alpha B - B\alpha A. \end{aligned}$$

On the other hand,

$$\begin{aligned} [Ad_\alpha(A), Ad_\alpha(B)]_\alpha &= \alpha^2 A\alpha^3 B\alpha^2 - \alpha^2 B\alpha^3 A\alpha^2 \\ &= -A\alpha B + B\alpha A. \end{aligned}$$

So, $Ad_\alpha([A,B]_\alpha) = -[Ad_\alpha(A), Ad_\alpha(B)]_\alpha$.

For all $A,B,C \in \mathfrak{gl}(V)$, we have

$$\begin{aligned} & [[A,B]_\alpha, Ad_\alpha(C)]_\alpha + \circlearrowleft_{A,B,C} \\ = \ & [\alpha A\alpha B\alpha, \alpha C\alpha]_\alpha - [\alpha B\alpha A\alpha, \alpha C\alpha]_\alpha + \circlearrowleft_{A,B,C} \\ = \ & -A\alpha B\alpha C + C\alpha A\alpha B + B\alpha A\alpha C - C\alpha B\alpha A + \circlearrowleft_{A,B,C} \\ = \ & 0, \end{aligned}$$

where $\circlearrowleft_{A,B,C}$ denotes summation over the cyclic permutation on A,B,C. Thus, $(\mathfrak{gl}(V), [\cdot,\cdot]_\alpha, Ad_\alpha)$ is a mirror Hom-Lie algebra. □

Remark 2.6. $(\mathfrak{gl}(V), [\cdot,\cdot]_\alpha, Ad_\alpha^2)$ *is not a Hom-Lie algebra. By straightforward computations,*

$$\begin{aligned} & [[A,B]_\alpha, Ad_\alpha^2(C)]_\alpha + \circlearrowleft_{A,B,C} \\ = \ & [\alpha A\alpha B\alpha, C]_\alpha - [\alpha B\alpha A\alpha, C]_\alpha + \circlearrowleft_{A,B,C} \\ = \ & A\alpha BC\alpha - \alpha CA\alpha B - B\alpha AC\alpha + \alpha CB\alpha A + \circlearrowleft_{A,B,C} \\ \neq \ & 0. \end{aligned}$$

Definition 2.7. *A representation of the mirror Hom-Lie algebra* $(\mathfrak{g}, [\cdot,\cdot], \beta)$ *on a vector space* V *with respect to* $\phi \in \mathfrak{gl}(V)$ *is a linear map* $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$*, such that for all* $x, y \in \mathfrak{g}$*, the following equalities are satisfied:*

$$\rho(\beta(x)) \circ \phi = -\phi \circ \rho(x); \tag{2.2}$$

$$\rho([x,y]) \circ \phi = \rho(\beta(x)) \circ \rho(y) - \rho(\beta(y)) \circ \rho(x). \tag{2.3}$$

Theorem 2.8. *Let* $(\mathfrak{g}, [\cdot,\cdot], \beta)$ *be a mirror Hom-Lie algebra,* V *is a vector space,* $\alpha^2 = -\mathrm{id}$*. Then,* $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ *is a representation of* $(\mathfrak{g}, [\cdot,\cdot], \beta)$ *on* V *with respect to* α *if and only if* $\rho : (\mathfrak{g}, [\cdot,\cdot], \beta) \longrightarrow (\mathfrak{gl}(V), [\cdot,\cdot]_\alpha, Ad_\alpha)$ *is a morphism of mirror Hom-Lie algebras.*

Proof. If $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is a representation of $(\mathfrak{g}, [\cdot,\cdot], \beta)$ on V with respect to α, we have

$$\rho(\beta(x)) \circ \alpha = -\alpha \circ \rho(x), \tag{2.4}$$

$$\rho([x,y]) \circ \alpha = \rho(\beta(x))\rho(y) - \rho(\beta(y))\rho(x). \tag{2.5}$$

By (2.4), we deduce that

$$\rho \circ \beta = Ad_\alpha \circ \rho.$$

Furthermore, by (2.4) and (2.5), we have

$$\begin{aligned}\rho([x,y]) &= -\rho(\beta(x))\circ\rho(y)\alpha+\rho(\beta(y))\circ\rho(x)\alpha\\ &= -\alpha\rho(x)\alpha\rho(y)\alpha+\alpha\rho(y)\alpha\rho(x)\alpha\\ &= -[\rho(x),\rho(y)]_{\alpha}.\end{aligned}$$

Thus, ρ is a morphism of mirror Hom-Lie algebras. The converse part is easy to be checked. The proof is completed. □

Definition 2.9. *Let* $(\mathfrak{g},[\cdot,\cdot],\beta)$ *be a mirror Hom-Lie algebra, the pseudo-adjoint representation* $ad^*:\mathfrak{g}\longrightarrow\mathfrak{gl}(\mathfrak{g})$*, which is defined by* $ad^*_x y=-[x,y]$.

Remark 2.10. *Because of* $\beta\neq\mathrm{id}$*, a Lie algebra is not a mirror Hom-Lie algebra, for* $x,y,z\in\mathfrak{g}$

$$\begin{aligned}ad^*_{[x,y]}\circ\beta(z) &= -[[x,y],\beta(z)]\\ &= [\beta(x),-[y,z]]+[\beta(y),[x,z]]\\ &= -ad^*_{\beta(x)}\circ ad^*_y(z)+ad^*_{\beta(y)}\circ ad^*_x(z).\end{aligned}$$

Proposition 2.11. *Let* $(\mathfrak{g},[\cdot,\cdot],\beta)$ *be a mirror Hom-Lie algebra and* $\beta^2=-\mathrm{id}$*. Then, the pseudo-adjoint representation* $ad^*:\mathfrak{g}\longrightarrow\mathfrak{gl}(\mathfrak{g})$*, which is defined by* $ad^*_x y=-[x,y]$*, is a morphism from* $(\mathfrak{g},[\cdot,\cdot],\beta)$ *to* $(\mathfrak{gl}(\mathfrak{g}),[\cdot,\cdot]_\beta,Ad_\beta)$.

Proof. For $x,y,z\in\mathfrak{g}$, we have: $Ad_\beta\circ ad^*_x(y)=-[\beta(x),y]=ad^*_{\beta(x)}(y)$, and

$$\begin{aligned}ad^*_{[x,y]}(z) &= [[x,y],\beta^2(z)]\\ &= -[[y,\beta(z)],\beta(x)]-[[\beta(z),x],\beta(y)]\\ &= [\beta(x),[y,\beta(z)]]-[\beta(y),[x,\beta(z)]].\end{aligned}$$

By straightforward computations,

$$\begin{aligned}\beta\circ ad^*_x\circ\beta\circ ad^*_y\circ\beta(z) &= \beta\circ ad^*_x\circ\beta(-[y,\beta(z)])\\ &= \beta(-[x,[\beta(y),-z]])=[\beta(x),\beta([\beta(y),-z])]\\ &= -[\beta(x),[y,\beta(z)]].\end{aligned}$$

So, we have: $ad^*_{[x,y]}=-[ad^*_x,ad^*_y]_\beta$. The proof is completed. □

3. Coboundary operators of mirror Hom-Lie algebras

Let $(\mathfrak{g},[\cdot,\cdot],\beta)$ be a mirror Hom-Lie algebra, V be a vector space, $\rho:\mathfrak{g}\longrightarrow\mathfrak{gl}(V)$ be a representation of $(\mathfrak{g},[\cdot,\cdot],\beta)$ on the vector space V with respect to $\phi\in GL(V)$, where ϕ is invertible.

The set of k-cochains on $\mathfrak{g}$ with values in V, which we denote by $C^k(\mathfrak{g};V)$, is the set of skew symmetric k-linear maps from $\mathfrak{g}\times\cdots\times\mathfrak{g}$($k$-times) to V:

$$C^k(\mathfrak{g};V):=\{\eta:\wedge^k\mathfrak{g}\longrightarrow V\text{ is a linear map}\}.$$

For $s = 0, 1, 2, \ldots$, define $d^s : C^k(\mathfrak{g}; V) \longrightarrow C^{k+1}(\mathfrak{g}; V)$ by

$$\begin{aligned} d^s\eta(x_1, \cdots, x_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i+1}\phi^{k+1+s}\rho(x_i)\phi^{-k-2-s}\eta(\beta(x_1), \cdots, \hat{x}_i, \cdots, \beta(x_{k+1})) \\ &\quad + \sum_{i<j}(-1)^{i+j}\eta([x_i, x_j], \beta(x_1), \cdots, \widehat{x_{i,j}}, \cdots, \beta(x_{k+1})), \end{aligned}$$

where ϕ^{-1} is the inverse of ϕ, $\eta \in C^k(\mathfrak{g}; V)$.

Proposition 3.1. *With the above notations, the map d^s is a coboundary operator, i.e., $d^s \circ d^s = 0$.*

Proof. For any $\eta \in C^k(\mathfrak{g}; V)$, by straightforward computations, we have

$$\begin{aligned} d^s \circ d^s\eta(x_1, \cdots, x_{k+2}) &= \sum_{i=1}^{k+2}(-1)^{i+1}\phi^{k+2+s}\rho(x_i)\phi^{-k-3-s}d^s\eta(\beta(x_1), \cdots, \hat{x}_i, \cdots, \beta(x_{k+2})) \\ &\quad + \sum_{i<j}(-1)^{i+j}d^s\eta([x_i, x_j], \beta(x_1), \cdots, \widehat{x_{i,j}}, \cdots, \beta(x_{k+2})). \end{aligned}$$

And

$$\begin{aligned} d^s\eta(\beta(x_1), \cdots, \hat{x}_i, \cdots, \beta(x_{k+2})) &= \sum_{l<i}(-1)^{l+1}\phi^{k+1+s}\rho(\beta(x_l))\phi^{-k-2-s}\eta(\beta^2(x_1), \cdots, \widehat{x_{l,i}}, \cdots, \beta^2(x_{k+2})) \\ &\quad + \sum_{l>i}(-1)^{l}\phi^{k+1+s}\rho(\beta(x_l))\phi^{-k-2-s}\eta(\beta^2(x_1), \cdots, \widehat{x_{i,l}}, \cdots, \beta^2(x_{k+2})) \\ &\quad + \sum_{m<n<i}(-1)^{m+n}\eta(\beta([x_i, x_j]), \beta^2(x_1), \cdots, \widehat{x_{m,n,i}}, \cdots, \beta^2(x_{k+2})) \\ &\quad + \sum_{m<i<n}(-1)^{m+n-1}\eta(\beta([x_i, x_j]), \beta^2(x_1), \cdots, \widehat{x_{m,i,n}}, \cdots, \beta^2(x_{k+2})) \\ &\quad + \sum_{i<m<n}(-1)^{m+n}\eta(\beta([x_i, x_j]), \beta^2(x_1), \cdots, \widehat{x_{i,m,n}}, \cdots, \beta^2(x_{k+2})). \end{aligned}$$

At the same time, we have

$$\begin{aligned} & d^s\eta([x_i, x_j], \beta(x_1), \cdots, \widehat{x_{i,j}}, \cdots, \beta(x_{k+2})) \\ = \; & \phi^{k+1+s}\rho([x_i, x_j])\phi^{-k-2-s}\eta(\beta^2(x_1), \cdots, \widehat{x_{i,j}}, \cdots, \beta^2(x_{k+2})) \\ & + \sum_{p<i<j}(-1)^{p}\phi^{k+1+s}\rho(\beta(x_p))\phi^{-k-2-s}\eta(\beta([x_i, x_j]), \beta^2(x_1), \cdots, \widehat{x_{p,i,j}}, \cdots, \beta^2(x_{k+2})) \\ & + \sum_{i<p<j}(-1)^{p+1}\phi^{k+1+s}\rho(\beta(x_p))\phi^{-k-2-s}\eta(\beta([x_i, x_j]), \beta^2(x_1), \cdots, \widehat{x_{i,p,j}}, \cdots, \beta^2(x_{k+2})) \\ & + \sum_{i<j<p}(-1)^{p}\phi^{k+1+s}\rho(\beta(x_p))\phi^{-k-2-s}\eta(\beta([x_i, x_j]), \beta^2(x_1), \cdots, \widehat{x_{i,j,p}}, \cdots, \beta^2(x_{k+2})) \\ & + \sum_{q<i<j}(-1)^{1+q}\eta([[x_i, x_j], \beta(x_q)], \beta^2(x_1), \cdots, \widehat{x_{q,i,j}}, \cdots, \beta^2(x_{k+2})) \qquad (3.1) \\ & + \sum_{i<q<j}(-1)^{q}\eta([[x_i, x_j], \beta(x_q)], \beta^2(x_1), \cdots, \widehat{x_{i,q,j}}, \cdots, \beta^2(x_{k+2})) \qquad (3.2) \end{aligned}$$

$$+\sum_{i<j<q}(-1)^{1+q}\eta([[x_i,x_j],\beta(x_q)],\beta^2(x_1),\cdots,\widehat{x_{i,j,q}},\cdots,\beta^2(x_{k+2})) \quad (3.3)$$

$$+\sum_{m<n<i<j}(-1)^{m+n}\eta([\beta(x_m),\beta(x_n)],\beta([x_i,x_j]),\beta^2(x_1),\cdots,\widehat{x_{m,n,i,j}},\cdots,\beta^2(x_{k+2})) \quad (3.4)$$

$$+\cdots$$

By Hom-Jacobi identity,

$$(3.1)+(3.2)+(3.3)=0,$$

and we have: $(3.4)+\cdots=0$.

By $\rho(\beta(x))\phi=-\phi\rho(x)$ and $\rho(\beta(x))=-\phi\rho(x)\phi^{-1}$, $\beta([x_i,x_j])=-[\beta(x_i),\beta(x_j)]$, we have

$$\begin{aligned}
& d^s\circ d^s\eta(x_1,\cdots,x_{k+2})\\
= & \sum_{l<i}(-1)^{l+i}\phi^{k-1+s}\rho(\beta^3(x_i))\rho(\beta^2(x_l))\phi^{-k-1-s}\eta(\beta^2(x_1),\cdots,\widehat{x_{l,i}},\cdots,\beta^2(x_{k+2})) \quad (3.5)\\
& +\sum_{l>i}(-1)^{l+i+1}\phi^{k-1+s}\rho(\beta^3(x_i))\rho(\beta^2(x_l))\phi^{-k-1-s}\eta(\beta^2(x_1),\cdots,\widehat{x_{i,l}},\cdots,\beta^2(x_{k+2})) \quad (3.6)\\
& +\sum_{m<n<i}(-1)^{m+n+i}\phi^{k+2+s}\rho(x_i)\phi^{-k-3-s}\eta([\beta(x_i),\beta(x_j)],\beta^2(x_1),\cdots,\widehat{x_{m,n,i}},\cdots,\beta^2(x_{k+2}))\\
& +\sum_{m<i<n}(-1)^{m+n+i+1}\phi^{k+2+s}\rho(x_i)\phi^{-k-3-s}\eta([\beta(x_i),\beta(x_j)],\beta^2(x_1),\cdots,\widehat{x_{m,i,n}},\cdots,\beta^2(x_{k+2}))\\
& +\sum_{i<m<n}(-1)^{m+n+i}\beta^{k+2+s}\rho(x_i)\beta^{-k-3-s}\eta([\beta(x_i),\beta(x_j)],\beta^2(x_1),\cdots,\widehat{x_{i,m,n}},\cdots,\beta^2(x_{k+2}))\\
& +\sum_{i<j}(-1)^{i+j}\phi^{k-1+s}\rho([\beta^2(x_i),\beta^2(x_j)])\phi\phi^{-k-1-s}\eta(\beta^2(x_1),\cdots,\widehat{x_{i,j}},\cdots,\beta^2(x_{k+2})) \quad (3.7)\\
& +\sum_{p<i<j}(-1)^{p+i+j+1}\phi^{k+2+s}\rho(x_p)\phi^{-k-3-s}\eta([\beta(x_i),\beta(x_j)],\beta^2(x_1),\cdots,\widehat{x_{p,i,j}},\cdots,\beta^2(x_{k+2}))\\
& +\sum_{i<p<j}(-1)^{p+i+j}\phi^{k+2+s}\rho(x_p)\phi^{-k-3-s}\eta([\beta(x_i),\beta(x_j)],\beta^2(x_1),\cdots,\widehat{x_{i,p,j}},\cdots,\beta^2(x_{k+2}))\\
& +\sum_{i<j<p}(-1)^{p+i+j+1}\phi^{k+2+s}\rho(x_p)\phi^{-k-3-s}\eta([\beta(x_i),\beta(x_j)],\beta^2(x_1),\cdots,\widehat{x_{i,j,p}},\cdots,\beta^2(x_{k+2})).
\end{aligned}$$

By $\rho([x,y])\phi=\rho(\beta(x))\rho(y)-\rho(\beta(y))\rho(x)$, we have

$$(3.5)+(3.6)+(3.7)=0.$$

About the above equations, the sum of the remaining six equations is zero. So, we prove that $d^s\circ d^s=0$. □

4. Mirror Hom-Lie algebras in semi-Euclidean 4-spaces

Let R^4 be a 4-dimensional vector space. For any two vectors $x = (x_1, x_2, x_3, x_4)^T$ and $y = (y_1, y_2, y_3, y_4)^T$ in R^4, their pseudoscalar product is defined by

$$< x, y >= x^T \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} y = -x_1y_1 - x_2y_2 + x_3y_3 + x_4y_4.$$

The space $(R^4, < \cdot, \cdot >)$ is called semi-Euclidean 4-space with index 2 and denoted by R^4_2. A non-zero vector $x \in R^4_2$ is called spacelike, null, or timelike if $< x, x >> 0$, $< x, x >= 0$, or $< x, x >< 0$, respectively. The null space of semi-Euclidean 4-spaces is $V_0 = \{x \in R^4_2 | < x, x >= 0\}$. The null space V_0 is not a vector space.

$(\mathfrak{gl}(V), [\cdot, \cdot]_\alpha, Ad_\alpha)$ is the mirror Hom-Lie algebra in Section 2. When V is the vector space R^2, let

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$Ad_\alpha(e_{11}, e_{12}, e_{21}, e_{22}) = (e_{11}, e_{12}, e_{21}, e_{22})P,$$

for $B \in \mathfrak{gl}(R^2)$, we have: $Ad_\alpha \circ Ad_\alpha(B) = B$. Thus, we have:

$$P^2 = \text{id}.$$

For the map Ad_α, we named P as the corresponding map.

Let $\alpha = \begin{pmatrix} -\theta & \sqrt{1+\theta^2} \\ -\sqrt{1+\theta^2} & \theta \end{pmatrix}$, $\theta \in \mathbb{R}$, then $\alpha^2 = -\text{id}$ and

$$\alpha \circ \alpha^T = \begin{pmatrix} 1 + 2\theta^2 & 2\theta\sqrt{1+\theta^2} \\ 2\theta\sqrt{1+\theta^2} & 1 + 2\theta^2 \end{pmatrix},$$

when $\theta \neq 0$, α is not an orthogonal matrix.

By straightforward computations, we have

$$\begin{aligned} Ad_\alpha(e_{11}) &= \begin{pmatrix} -\theta & \sqrt{1+\theta^2} \\ -\sqrt{1+\theta^2} & \theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\theta & \sqrt{1+\theta^2} \\ -\sqrt{1+\theta^2} & \theta \end{pmatrix} \\ &= \begin{pmatrix} -\theta & 0 \\ -\sqrt{1+\theta^2} & 0 \end{pmatrix} \begin{pmatrix} -\theta & \sqrt{1+\theta^2} \\ -\sqrt{1+\theta^2} & \theta \end{pmatrix} \\ &= \begin{pmatrix} \theta^2 & -\theta\sqrt{1+\theta^2} \\ \theta\sqrt{1+\theta^2} & -1-\theta^2 \end{pmatrix} \\ &= (e_{11}, e_{12}, e_{21}, e_{22}) \begin{pmatrix} \theta^2 \\ -\theta\sqrt{1+\theta^2} \\ \theta\sqrt{1+\theta^2} \\ -1-\theta^2 \end{pmatrix}. \end{aligned}$$

$$\begin{aligned}
Ad_\alpha(e_{12}) &= \begin{pmatrix} -\theta & \sqrt{1+\theta^2} \\ -\sqrt{1+\theta^2} & \theta \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -\theta & \sqrt{1+\theta^2} \\ -\sqrt{1+\theta^2} & \theta \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\theta \\ 0 & -\sqrt{1+\theta^2} \end{pmatrix}\begin{pmatrix} -\theta & \sqrt{1+\theta^2} \\ -\sqrt{1+\theta^2} & \theta \end{pmatrix} \\
&= \begin{pmatrix} \theta\sqrt{1+\theta^2} & -\theta^2 \\ 1+\theta^2 & -\theta\sqrt{1+\theta^2} \end{pmatrix} \\
&= (e_{11}, e_{12}, e_{21}, e_{22})\begin{pmatrix} \theta\sqrt{1+\theta^2} \\ -\theta^2 \\ 1+\theta^2 \\ -\theta\sqrt{1+\theta^2} \end{pmatrix}.
\end{aligned}$$

$$\begin{aligned}
Ad_\alpha(e_{21}) &= \begin{pmatrix} -\theta & \sqrt{1+\theta^2} \\ -\sqrt{1+\theta^2} & \theta \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -\theta & \sqrt{1+\theta^2} \\ -\sqrt{1+\theta^2} & \theta \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{1+\theta^2} & 0 \\ \theta & 0 \end{pmatrix}\begin{pmatrix} -\theta & \sqrt{1+\theta^2} \\ -\sqrt{1+\theta^2} & \theta \end{pmatrix} \\
&= \begin{pmatrix} -\theta\sqrt{1+\theta^2} & 1+\theta^2 \\ -\theta^2 & \theta\sqrt{1+\theta^2} \end{pmatrix} \\
&= (e_{11}, e_{12}, e_{21}, e_{22})\begin{pmatrix} -\theta\sqrt{1+\theta^2} \\ 1+\theta^2 \\ -\theta^2 \\ \theta\sqrt{1+\theta^2} \end{pmatrix}.
\end{aligned}$$

$$\begin{aligned}
Ad_\alpha(e_{22}) &= \begin{pmatrix} -\theta & \sqrt{1+\theta^2} \\ -\sqrt{1+\theta^2} & \theta \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\theta & \sqrt{1+\theta^2} \\ -\sqrt{1+\theta^2} & \theta \end{pmatrix} \\
&= \begin{pmatrix} 0 & \sqrt{1+\theta^2} \\ 0 & \theta \end{pmatrix}\begin{pmatrix} -\theta & \sqrt{1+\theta^2} \\ -\sqrt{1+\theta^2} & \theta \end{pmatrix} \\
&= \begin{pmatrix} -1-\theta^2 & \theta\sqrt{1+\theta^2} \\ -\theta\sqrt{1+\theta^2} & \theta^2 \end{pmatrix} \\
&= (e_{11}, e_{12}, e_{21}, e_{22})\begin{pmatrix} -1-\theta^2 \\ \theta\sqrt{1+\theta^2} \\ -\theta\sqrt{1+\theta^2} \\ \theta^2 \end{pmatrix}.
\end{aligned}$$

Then, the corresponding map P is

$$\begin{pmatrix} \theta^2 & \theta\sqrt{1+\theta^2} & -\theta\sqrt{1+\theta^2} & -1-\theta^2 \\ -\theta\sqrt{1+\theta^2} & -\theta^2 & 1+\theta^2 & \theta\sqrt{1+\theta^2} \\ \theta\sqrt{1+\theta^2} & 1+\theta^2 & -\theta^2 & -\theta\sqrt{1+\theta^2} \\ -1-\theta^2 & -\theta\sqrt{1+\theta^2} & \theta\sqrt{1+\theta^2} & \theta^2 \end{pmatrix}.$$

When $\theta \neq 0$, P is not an orthogonal matrix. Let $r = (-\theta, \sqrt{1+\theta^2}, -\sqrt{1+\theta^2}, \theta)^T$, by straightforward computations, we have $Pr = -r$. For $x = (x_1, x_2, x_3, x_4)^T$, $y = (y_1, y_2, y_3, y_4)^T$, we define $[\cdot,\cdot]_\theta$ by

$$[x, y]_\theta = Px \wedge r \wedge y - Py \wedge r \wedge x,$$

where

$$Px \wedge r \wedge y = \begin{vmatrix} e_1 & -e_2 & e_3 & e_4 \\ & & (Px)^T & \\ & & r^T & \\ & & y^T & \end{vmatrix},$$

and $\{e_1, e_2, e_3, e_4\}$ is the canonical basis of R^4. By straightforward computations,

$$\begin{aligned} & Px \wedge r \wedge y \\ = \; & (-\sqrt{1+\theta^2}[x_1(y_2+y_3) + (x_2+x_3)y_4] + \theta(x_3y_2 - x_2y_3), \\ & \theta(x_1y_3 + x_3y_1 + x_3y_4 + x_4y_3) + \sqrt{1+\theta^2}(x_4y_4 - x_1y_1), \\ & \theta(x_1y_2 + x_2y_1 + x_2y_4 + x_4y_2) + \sqrt{1+\theta^2}(x_1y_1 - x_4y_4), \\ & \sqrt{1+\theta^2}[(x_2+x_3)y_1 + (y_2+y_3)x_4] + \theta(x_3y_2 - x_2y_3))^T \end{aligned}$$

$$\begin{aligned} & Py \wedge r \wedge x \\ = \; & (-\sqrt{1+\theta^2}[(x_2+x_3)y_1 + (y_2+y_3)x_4] + \theta(x_2y_3 - x_3y_2), \\ & \theta(x_1y_3 + x_3y_1 + x_3y_4 + x_4y_3) + \sqrt{1+\theta^2}(x_4y_4 - x_1y_1), \\ & \theta(x_1y_2 + x_2y_1 + x_2y_4 + x_4y_2) + \sqrt{1+\theta^2}(x_1y_1 - x_4y_4), \\ & \sqrt{1+\theta^2}[(y_2+y_3)x_1 + (x_2+x_3)y_4] + \theta(x_2y_3 - x_3y_2))^T \end{aligned}$$

then $[x, y]_\theta = (a, 0, 0, a)^T$, where

$$a = \sqrt{1+\theta^2}[(x_4 - x_1)(y_2+y_3) + (x_2+x_3)(y_1 - y_4)] + 2\theta(x_3y_2 - x_2y_3).$$

Because of $P^2 = \mathrm{id}$,

$$[Px, Py]_\theta = x \wedge r \wedge Py - y \wedge r \wedge Px = Px \wedge r \wedge y - Py \wedge r \wedge x = [x, y]_\theta,$$

and

$$P\begin{pmatrix} a \\ 0 \\ 0 \\ a \end{pmatrix} = \begin{pmatrix} \theta^2 & \theta\sqrt{1+\theta^2} & -\theta\sqrt{1+\theta^2} & -1-\theta^2 \\ -\theta\sqrt{1+\theta^2} & -\theta^2 & 1+\theta^2 & \theta\sqrt{1+\theta^2} \\ \theta\sqrt{1+\theta^2} & 1+\theta^2 & -\theta^2 & -\theta\sqrt{1+\theta^2} \\ -1-\theta^2 & -\theta\sqrt{1+\theta^2} & \theta\sqrt{1+\theta^2} & \theta^2 \end{pmatrix}\begin{pmatrix} a \\ 0 \\ 0 \\ a \end{pmatrix} = -\begin{pmatrix} a \\ 0 \\ 0 \\ a \end{pmatrix},$$

then we have

$$P([x, y]_\theta) = P(a, 0, 0, a)^T = -(a, 0, 0, a)^T = -[x, y]_\theta = -[Px, Py]_\theta.$$

For $z \in R^4$,

$$[[x, y]_\theta, Pz]_\theta$$

$$
\begin{aligned}
&= P([x,y]_\theta)\wedge r\wedge Pz - PPz\wedge r\wedge [x,y]_\theta\\
&= \begin{pmatrix}-a\\0\\0\\-a\end{pmatrix}\wedge r\wedge Pz - z\wedge r\wedge \begin{pmatrix}a\\0\\0\\a\end{pmatrix}\\
&= (Pz-z)\wedge r\wedge \begin{pmatrix}a\\0\\0\\a\end{pmatrix}\\
&= \begin{vmatrix} e_1 & -e_2 & e_3 & e_4\\ b_1 & b_2 & b_3 & b_4\\ -\theta & \sqrt{1+\theta^2} & -\sqrt{1+\theta^2} & \theta\\ a & 0 & 0 & a\end{vmatrix}\\
&= \begin{pmatrix} -a\sqrt{1+\theta^2}(b_2+b_3)\\ a(2\theta b_3+\sqrt{1+\theta^2}(b_4-b_1))\\ a(2\theta b_2+\sqrt{1+\theta^2}(b_1-b_4))\\ a\sqrt{1+\theta^2}(b_2+b_3)\end{pmatrix}\\
&= (0,0,0,0)^T,
\end{aligned}
$$

where

$$
\begin{aligned}
b_1 &= ((\theta^2-1)z_1+\theta\sqrt{1+\theta^2}z_2)-(\theta\sqrt{1+\theta^2}z_3+(1+\theta^2)z_4),\\
b_2 &= -(\theta\sqrt{1+\theta^2}z_1+(\theta^2+1)z_2)+((1+\theta^2)z_3+\theta\sqrt{1+\theta^2}z_4),\\
b_3 &= (\theta\sqrt{1+\theta^2}z_1+(1+\theta^2)z_2)-((\theta^2+1)z_3+\theta\sqrt{1+\theta^2}z_4),\\
b_4 &= -((1+\theta^2)z_1+\theta\sqrt{1+\theta^2}z_2)+(\theta\sqrt{1+\theta^2}z_3+(\theta^2-1)z_4).
\end{aligned}
$$

Similarly, we have $[[y,z]_\theta, Px]_\theta = 0$ and $[[z,x]_\theta, Py]_\theta = 0$, so the Hom-Jacobi identity holds. Thus, we have the following:

Proposition 4.1. *With the above notations, $(R^4_2, [\cdot,\cdot]_\theta, P)$ is a mirror Hom-Lie algebra.*

Theorem 4.2. *The null space of semi-Euclidean 4-spaces is $V_0 = \{x \in R^4_2 | < x, x >= 0\}$, let*

$$V^* = \{x = (x_1,x_2,x_3,x_4)^T \in V_0 | x_1x_2 = x_3x_4\},$$

there exists a mirror Hom-Lie algebra $(R^4_2, [\cdot,\cdot]_\theta, P)$, such that

(a1) $r = (-\theta, \sqrt{1+\theta^2}, -\sqrt{1+\theta^2}, \theta)^T \in V^*$*, and* $Pr \in V^*$*;*
(a2) *for all* $x, y \in R^4_2, \quad [x,y]_\theta \in V^*$*;*
(a3) *for all* $z \in V^*, \quad Pz \in V^*$.

Proof. We just need to prove (a3), for $z = (z_1,z_2,z_3,z_4)^T \in V^*$,

$$(Pz)^T\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}Pz = z^TP^T\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}Pz$$

$$
\begin{aligned}
&= z^T \begin{pmatrix} 1+2\theta^2 & 2\theta\sqrt{1+\theta^2} & 0 & 0 \\ 2\theta\sqrt{1+\theta^2} & 1+2\theta^2 & 0 & 0 \\ 0 & 0 & -1-2\theta^2 & -2\theta\sqrt{1+\theta^2} \\ 0 & 0 & -2\theta\sqrt{1+\theta^2} & -1-2\theta^2 \end{pmatrix} z \\
&= (1+2\theta^2)(z_1^2+z_2^2-z_3^2-z_4^2) + 4\theta\sqrt{1+\theta^2}(z_1z_2-z_3z_4) \\
&= 0.
\end{aligned}
$$

So, $Pz \in V_0$, and by straightforward computations,

$$
\begin{aligned}
Pz &= \begin{pmatrix} \theta^2 & \theta\sqrt{1+\theta^2} & -\theta\sqrt{1+\theta^2} & -1-\theta^2 \\ -\theta\sqrt{1+\theta^2} & -\theta^2 & 1+\theta^2 & \theta\sqrt{1+\theta^2} \\ \theta\sqrt{1+\theta^2} & 1+\theta^2 & -\theta^2 & -\theta\sqrt{1+\theta^2} \\ -1-\theta^2 & -\theta\sqrt{1+\theta^2} & \theta\sqrt{1+\theta^2} & \theta^2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \\
&= \begin{pmatrix} (\theta^2 z_1 + \theta\sqrt{1+\theta^2}z_2) - (\theta\sqrt{1+\theta^2}z_3 + (1+\theta^2)z_4) \\ -(\theta\sqrt{1+\theta^2}z_1 + \theta^2 z_2) + ((1+\theta^2)z_3 + \theta\sqrt{1+\theta^2}z_4) \\ (\theta\sqrt{1+\theta^2}z_1 + (1+\theta^2)z_2) - (\theta^2 z_3 + \theta\sqrt{1+\theta^2}z_4) \\ -((1+\theta^2)z_1 + \theta\sqrt{1+\theta^2}z_2) + (\theta\sqrt{1+\theta^2}z_3 + \theta^2 z_4) \end{pmatrix}.
\end{aligned}
$$

By $z_1^2 + z_2^2 = z_3^2 + z_4^2$ and $z_1z_2 = z_3z_4$, we have: $Pz \in V^*$. □

5. Conclusions

In this study, we discuss the Hom-Lie algebra structure over R^3. We obtain a kind of deformations of Lie algebras; these deformations are associated with Hom-Lie algebras. We have: if $(\mathfrak{a}, [\cdot,\cdot], \gamma)$ is a Hom-Lie algebra, then $(\mathfrak{a}, [\cdot,\cdot], -\gamma)$ is a mirror Hom-Lie algebra. We give some properties of these deformations and study their representations, and we give the coboundary operator of these deformations. More, We find out these deformations in null spaces of semi-Euclidean spaces.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declare no conflicts of interest in this paper.

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