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Research article

Analysis of Newton-type inequalities for differentiable hyperbolic p-convex functions via RL-integrals

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Abstract: In this paper, new generalizations of Newton-type inequalities for the class of hyperbolic p-convex functions by utilizing Riemann-Liouville fractional integrals are established. By means of an auxiliary identity connected with the Riemann-Liouville fractional integrals, new estimates are obtained for functions that are hyperbolic p-convex. The established inequalities are further improved by the effective use of Hölder's inequality and the power-mean inequality. Examples along with graphs are provided to demonstrate the validity of the newly established inequalities and comparisons with existing results. The results of this paper will open up new avenues of research and may be generalized to other types of fractional operators and generalized convex functions.

Keywords: convex functions; Riemann-Liouville fractional integrals; Simpson inequality; mathematical operators; hyperbolic p-convex function
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1. Introduction

The classical concept of convex functions has been generalized to a great extent over the past few decades, leading to the introduction of various new classes of extended functions with wider applications. Some of the early developments include s-convex functions [1], m-convex functions [2], and the unifying h-convexity [3]. In 2021, Kadakal et al. [4] proposed trigonometrically convex functions using sine and cosine weights. A more recent development is the class of hyperbolic p-convex functions introduced by Dragomir and Torebek [5]. These functions replace the linear weights by hyperbolic sine weights as follows

$$F(\psi\kappa_1 + (1-\psi)\kappa_2) \le \frac{\sinh[p(1-\psi)(\kappa_2-\kappa_1)]}{\sinh[p(\kappa_2-\kappa_1)]}F(\kappa_1) + \frac{\sinh[p\psi(\kappa_2-\kappa_1)]}{\sinh[p(\kappa_2-\kappa_1)]}F(\kappa_2).$$

For $p > 0$, these weights satisfy $\frac{\sinh(p(1-\psi)\Delta)}{\sinh(p\Delta)} \geq 1-\psi$ and $\frac{\sinh(p\psi\Delta)}{\sinh(p\Delta)} \geq \psi$, so they are larger than the linear weights. Consequently, when the derivative values at the endpoints significantly differ, the resulting bounds can be sharper than the classical convex ones. As $p \to 0$, the hyperbolic weights reduce to the linear ones, thus recovering classical convexity. Subsequently, a modified class of hyperbolic p-convex functions was studied by Wang et al. [6].

In the theory of inequalities, convexity plays a central role. Classic examples include the Jensen inequality [7] and Giaccardi-type inequalities [8]. Fractional calculus has become an indispensable tool to model memory and hereditary effects [9]. Fractional versions of Hermite-Hadamard, Simpson, Ostrowski, and Newton-type inequalities have been extensively studied. Sarikaya et al. [10] pioneered the use of fractional integrals in Hermite-Hadamard inequalities. Set [11] extended Ostrowski's inequality within the fractional calculus framework. Işcan and Wu [12] derived Hermite-Hadamard-type inequalities using harmonic convexity. In particular, Sitthiwirattham et al. [13] proved a fundamental integral identity for Riemann-Liouville (RL) fractional integrals and used it to establish fractional Newton-type inequalities for differentiable convex functions.

Recent years have seen substantial progress in fractional Newton-type inequalities. For instance, Budak et al. [14] obtained Newton-type inequalities via conformable integrals, and Hezenci and Budak [15] derived error bounds for Newton's formula using tempered fractional integrals. Moreover, new classes of convexity, such as exponential-type, harmonic-type, and φ-convexity, have been actively investigated. For example, a variant of the Jensen-type inequality for harmonic convex functions was investigated by Baloch et al. [16]. The stability and controllability of fractional delay systems were studied by Hammad and De la Sen [17]. Neumann problems for p-harmonic functions were addressed by Capogna et al. [18]. The gradient regularity for (s, p)-harmonic functions was established by Bögelein et al. [19]. These contributions highlight the ongoing interest in fractional calculus and convex analyses.

Moreover, the interplay between fractional calculus, convex analyses, and engineering applications has been demonstrated in numerous recent studies. For example, a fault diagnosis using neural networks was explored by Zhao et al. [20], robust control for uncertain systems was explored by Lu et al. [21], zeroing neural networks for dynamic equations was explored by Li et al. [22], real-time parametric curve interpolations were explored by Song et al. [23], vibration control via fractional-order models was explored by Zhang et al. [24], and finite-time control for jump-diffusion systems was explored by Yan et al. [25]. Nondestructive defect detection was developed by Liu et al. [26], thermo-elastic parameter identification was developed by Qiang et al. [27], antioxidant screening was developed by Jia et al. [28], link prediction using Kolmogorov-Arnold networks was developed by Zhang et al. [29], knowledge graph link prediction was developed by Lu et al. [30], and membrane technology for oil separation was developed by Amini et al. [31]. These developments further motivate the need for refined integral inequalities in fractional calculus. However, to the best of our knowledge, fractional Newton-type inequalities for hyperbolic p-convex functions have not yet been investigated.

Comparison with existing results

Now, we explicitly compare our contributions with the most relevant existing works. Sitthiwirattham et al. [13] proved fractional Newton-type inequalities for differentiable convex functions using RL integrals. Their bounds are based on linear weights $1 - \psi$ and ψ. In contrast, we replace these by hyperbolic weights $\frac{\sinh(p(1-\psi)\Delta)}{\sinh(p\Delta)}$ and $\frac{\sinh(p\psi\Delta)}{\sinh(p\Delta)}$, which are larger for $p > 0$. This enlargement can lead to tighter estimates when the derivative at one endpoint dominates (see Tables 1–3). As $p \to 0$, our results exactly reduce to those of [13], thereby generalizing them.

Wang et al. [6] introduced a modified class of hyperbolic p-convex functions and established Hermite-Hadamard and Simpson-type inequalities, but they did not consider Newton-type inequalities. Our work is the first to apply hyperbolic p-convexity to fractional Newton-type inequalities. Compared to recent fractional Newton-type results for Caputo and other operators (e.g., [14, 15]), our approach is distinct because it incorporates hyperbolic sine weights and provides explicit upper bounds that involve integrals of $|\psi^{\alpha} - c|$ multiplied by those weights.

The specific mathematical properties that make hyperbolic p-convexity suitable for Newton-type inequalities are the facts that its weights are larger than the linear ones (for $p > 0$) and that they tend to the linear weights as $p \to 0$. This allows the bounds to adapt to the behavior of $|F'|$ at the endpoints: when $|F'(\kappa_1)|$ is much larger than $|F'(\kappa_2)|$, the enlarged weights on $|F'(\kappa_1)|$ can increase the bound; however, for many functions, the opposite holds, thus leading to sharper estimates than the classical convex case. We chose RL integrals because the fundamental identity [13] (Lemma 2.4) is specific to this operator. Extensions to Caputo, Hadamard, or Atangana-Baleanu operators would require analogous integral identities; such extensions are left as future research directions.

In this paper, we prove three families of Newton-type inequalities (Theorems 3.1–3.7) for functions whose derivative is hyperbolic p-convex using RL fractional integrals. The proofs explicitly show how hyperbolic p-convexity is applied to each integral term, and how the power-mean and Hölder inequalities are employed step by step. The structure of the coefficients A_i (Theorem 3.1), C_i (Theorem 3.4) and D_i (Theorem 3.7) directly reflects the hyperbolic weights. The choice of RL integrals is dictated by the fundamental identity [13] (Lemma 2.4), which is the cornerstone of our proofs.

The paper is organized as follows: Section 2 recalls the necessary definitions, including the RL fractional integral and hyperbolic p-convexity, and states the fundamental integral identity; Section 3 presents the main Newton-type inequalities for differentiable hyperbolic p-convex functions using the power-mean and Hölder inequalities, along with numerical examples, graphical illustrations, and comparisons with existing results; and Section 4 concludes the paper with some suggestions for future research.

2. Preliminaries

In this section, we recall some fundamental definitions and results that will be used throughout the paper.

Definition 2.1 (Riemann-Liouville Fractional Integral [9])**.** *Let $F \in L_1[\kappa_1, \kappa_2]$ and $\alpha > 0$. The Riemann-Liouville fractional integrals of order α are defined by*

$$\Im^{\alpha}_{\kappa_1^+} F(x) = \frac{1}{\Gamma(\alpha)} \int_{\kappa_1}^{x} (x - \psi)^{\alpha-1} F(\psi)\, d\psi, \quad x > \kappa_1, \tag{2.1}$$

and

$$\Im^{\alpha}_{\kappa_2^-} F(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\kappa_2} (\psi - x)^{\alpha-1} F(\psi)\, d\psi, \quad x < \kappa_2. \tag{2.2}$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2 (Hyperbolic p-convex function [5]). *Let $\Im \subset \mathbb{R}$ be an interval and $F : \Im \to \mathbb{R}$ be a function. Then, F is called hyperbolic p-convex on $\Im$ if for all $\kappa_1, \kappa_2 \in \Im$ with $\kappa_1 < \kappa_2$, for any $\psi \in [0, 1]$, and for $p > 0$, the following inequality holds:*

$$F(\psi\kappa_1 + (1-\psi)\kappa_2) \leq \frac{\sinh[p(1-\psi)(\kappa_2-\kappa_1)]}{\sinh[p(\kappa_2-\kappa_1)]}F(\kappa_1) + \frac{\sinh[p\psi(\kappa_2-\kappa_1)]}{\sinh[p(\kappa_2-\kappa_1)]}F(\kappa_2). \tag{2.3}$$

(The condition $\kappa_1 < \kappa_2$ ensures that $\sinh(p(\kappa_2 - \kappa_1)) \neq 0$ and that the weights are well-defined positive numbers.) If the inequality holds with " $\geq$ ", then F is called hyperbolic p-concave on $\Im$.

Remark 2.3. *For $\varrho = (1-\psi)\kappa_1 + \psi\kappa_2$, Condition (2.3) can be equivalently written as*

$$F((1-\psi)\kappa_1 + \psi\kappa_2) \leq \frac{\sinh[p(1-\psi)(\kappa_2-\kappa_1)]}{\sinh[p(\kappa_2-\kappa_1)]}F(\kappa_1) + \frac{\sinh[p\psi(\kappa_2-\kappa_1)]}{\sinh[p(\kappa_2-\kappa_1)]}F(\kappa_2), \tag{2.4}$$

for all $\psi \in [0, 1]$.

The following integral identity, established by Sitthiwirattham et al. [13], is fundamental to derive fractional Newton-type inequalities.

Lemma 2.4 ([13], Lemma 1). *Let $F : \Im \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\Im^\circ$ (the interior of $\Im$) such that $F' \in L_1[\kappa_1, \kappa_2]$ with $\kappa_1, \kappa_2 \in \Im$, $\kappa_1 < \kappa_2$. Then, the following identity holds:*

$$\begin{aligned} &\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha}\left[\Im^\alpha_{\kappa_1^+}F\left(\frac{2\kappa_1+\kappa_2}{3}\right) + \Im^\alpha_{\frac{2\kappa_1+\kappa_2}{3}^+}F\left(\frac{\kappa_1+2\kappa_2}{3}\right) + \Im^\alpha_{\frac{\kappa_1+2\kappa_2}{3}^+}F(\kappa_2)\right] \\ &- \frac{1}{8}\left[F(\kappa_1) + 3F\left(\frac{2\kappa_1+\kappa_2}{3}\right) + 3F\left(\frac{\kappa_1+2\kappa_2}{3}\right) + F(\kappa_2)\right] \\ &= \frac{\kappa_2-\kappa_1}{9}\left[I_1 + I_2 + I_3\right], \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} I_1 &= \int_0^1 \left(\psi^\alpha - \frac{5}{8}\right)F'\left(\psi\kappa_1 + (1-\psi)\frac{2\kappa_1+\kappa_2}{3}\right)d\psi, \\ I_2 &= \int_0^1 \left(\psi^\alpha - \frac{1}{2}\right)F'\left(\psi\frac{2\kappa_1+\kappa_2}{3} + (1-\psi)\frac{\kappa_1+2\kappa_2}{3}\right)d\psi, \\ I_3 &= \int_0^1 \left(\psi^\alpha - \frac{3}{8}\right)F'\left(\psi\frac{\kappa_1+2\kappa_2}{3} + (1-\psi)\kappa_2\right)d\psi. \end{aligned}$$

(Since $F' \in L_1[\kappa_1, \kappa_2]$ and the kernels $\psi^\alpha - \frac{5}{8}$, $\psi^\alpha - \frac{1}{2}$, $\psi^\alpha - \frac{3}{8}$ are bounded on $[0, 1]$, each I_1, I_2, I_3 exists as a finite number.)

The following fundamental inequalities are helpful to establish the main findings of this work.

Theorem 2.5 (Hölder's Integral Inequality [32]). *Let $l, v > 1$ with $\frac{1}{l} + \frac{1}{v} = 1$. If F and g are real functions defined on $[\kappa_1, \kappa_2]$ such that $|F|^l$ and $|G|^v$ are integrable on $[\kappa_1, \kappa_2]$, then*

$$\int_{\kappa_1}^{\kappa_2} |F(\psi)G(\psi)|d\psi \leq \left(\int_{\kappa_1}^{\kappa_2} |F(\psi)|^l d\psi\right)^{1/l}\left(\int_{\kappa_1}^{\kappa_2} |G(\psi)|^v d\psi\right)^{1/v}. \tag{2.6}$$

Theorem 2.6 (Power-Mean Integral Inequality [32]). *Let $v \geq 1$. If F and G are real functions defined on $[\kappa_1, \kappa_2]$ such that $|F|$ and $|F||G|^v$ are integrable on $[\kappa_1, \kappa_2]$, then*

$$\int_{\kappa_1}^{\kappa_2} |F(\psi)G(\psi)|d\psi \leq \left(\int_{\kappa_1}^{\kappa_2} |F(\psi)|d\psi\right)^{1-1/v}\left(\int_{\kappa_1}^{\kappa_2} |F(\psi)||G(\psi)|^v d\psi\right)^{1/v}. \tag{2.7}$$

3. Main results

Theorem 3.1. *Assume that the conditions of Lemma 2.4 hold. If* $|F'|$ *is a hyperbolic p-convex function on* $[\kappa_1, \kappa_2]$*, then the following Newton-type inequality holds:*

$$\begin{aligned}&\left|\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[\Im^{\alpha}_{\kappa_1^+}F\left(\frac{2\kappa_1+\kappa_2}{3}\right)+\Im^{\alpha}_{\frac{2\kappa_1+\kappa_2}{3}^+}F\left(\frac{\kappa_1+2\kappa_2}{3}\right)+\Im^{\alpha}_{\frac{\kappa_1+2\kappa_2}{3}^+}F(\kappa_2)\right]\right.\\ &\left.-\frac{1}{8}\left[F(\kappa_1)+3F\left(\frac{2\kappa_1+\kappa_2}{3}\right)+3F\left(\frac{\kappa_1+2\kappa_2}{3}\right)+F(\kappa_2)\right]\right| \\ &\leq \frac{\kappa_2-\kappa_1}{9}\left[|F'(\kappa_2)|(A_1+A_2+A_3)+|F'(\kappa_1)|(A_4+A_5+A_6)\right],\end{aligned}\tag{3.1}$$

where

$$\begin{aligned}A_1 &= \int_0^1\left|\psi^{\alpha}-\frac{5}{8}\right|\left(\frac{\sinh\left(p\left(\frac{1-\psi}{3}\right)(\kappa_2-\kappa_1)\right)}{\sinh(p(\kappa_2-\kappa_1))}\right)d\psi,\\ A_2 &= \int_0^1\left|\psi^{\alpha}-\frac{1}{2}\right|\left(\frac{\sinh\left(p\left(\frac{2-\psi}{3}\right)(\kappa_2-\kappa_1)\right)}{\sinh(p(\kappa_2-\kappa_1))}\right)d\psi,\\ A_3 &= \int_0^1\left|\psi^{\alpha}-\frac{3}{8}\right|\left(\frac{\sinh\left(p\left(\frac{3-\psi}{3}\right)(\kappa_2-\kappa_1)\right)}{\sinh(p(\kappa_2-\kappa_1))}\right)d\psi,\\ A_4 &= \int_0^1\left|\psi^{\alpha}-\frac{5}{8}\right|\left(\frac{\sinh\left(p\left(\frac{2+\psi}{3}\right)(\kappa_2-\kappa_1)\right)}{\sinh(p(\kappa_2-\kappa_1))}\right)d\psi,\\ A_5 &= \int_0^1\left|\psi^{\alpha}-\frac{1}{2}\right|\left(\frac{\sinh\left(p\left(\frac{1+\psi}{3}\right)(\kappa_2-\kappa_1)\right)}{\sinh(p(\kappa_2-\kappa_1))}\right)d\psi,\\ A_6 &= \int_0^1\left|\psi^{\alpha}-\frac{3}{8}\right|\left(\frac{\sinh\left(p\left(\frac{\psi}{3}\right)(\kappa_2-\kappa_1)\right)}{\sinh(p(\kappa_2-\kappa_1))}\right)d\psi.\end{aligned}$$

Proof. By taking modulus in Lemma 2.4 and applying the triangle inequality, we obtain the following:

$$\begin{aligned}&\left|\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[\Im^{\alpha}_{\kappa_1^+}F\left(\frac{2\kappa_1+\kappa_2}{3}\right)+\Im^{\alpha}_{\frac{2\kappa_1+\kappa_2}{3}^+}F\left(\frac{\kappa_1+2\kappa_2}{3}\right)+\Im^{\alpha}_{\frac{\kappa_1+2\kappa_2}{3}^+}F(\kappa_2)\right]\right.\\ &\left.-\frac{1}{8}\left[F(\kappa_1)+3F\left(\frac{2\kappa_1+\kappa_2}{3}\right)+3F\left(\frac{\kappa_1+2\kappa_2}{3}\right)+F(\kappa_2)\right]\right| \\ &\leq \frac{\kappa_2-\kappa_1}{9}\left[\int_0^1\left|\psi^{\alpha}-\frac{5}{8}\right|\left|F'\left(\psi\kappa_1+(1-\psi)\frac{2\kappa_1+\kappa_2}{3}\right)\right|d\psi\right.\\ &+\int_0^1\left|\psi^{\alpha}-\frac{1}{2}\right|\left|F'\left(\psi\frac{2\kappa_1+\kappa_2}{3}+(1-\psi)\frac{\kappa_1+2\kappa_2}{3}\right)\right|d\psi\\ &\left.+\int_0^1\left|\psi^{\alpha}-\frac{3}{8}\right|\left|F'\left(\psi\frac{\kappa_1+2\kappa_2}{3}+(1-\psi)\kappa_2\right)\right|d\psi\right].\end{aligned}$$

Now, we rewrite the arguments as convex combinations of κ_1 and κ_2 as follows

$$\begin{aligned} \psi\kappa_1+(1-\psi)\frac{2\kappa_1+\kappa_2}{3} &= \frac{1-\psi}{3}\kappa_2+\frac{2+\psi}{3}\kappa_1, \\ \psi\frac{2\kappa_1+\kappa_2}{3}+(1-\psi)\frac{\kappa_1+2\kappa_2}{3} &= \frac{2-\psi}{3}\kappa_2+\frac{1+\psi}{3}\kappa_1, \\ \psi\frac{\kappa_1+2\kappa_2}{3}+(1-\psi)\kappa_2 &= \frac{3-\psi}{3}\kappa_2+\frac{\psi}{3}\kappa_1. \end{aligned}$$

Thus,

$$\begin{aligned} &\le \frac{\kappa_2-\kappa_1}{9}\left[\int_0^1\left|\psi^\alpha-\frac{5}{8}\right|\left|F'\left(\frac{1-\psi}{3}\kappa_2+\frac{2+\psi}{3}\kappa_1\right)\right|d\psi\right. \\ &+\int_0^1\left|\psi^\alpha-\frac{1}{2}\right|\left|F'\left(\frac{2-\psi}{3}\kappa_2+\frac{1+\psi}{3}\kappa_1\right)\right|d\psi \\ &\left.+\int_0^1\left|\psi^\alpha-\frac{3}{8}\right|\left|F'\left(\frac{3-\psi}{3}\kappa_2+\frac{\psi}{3}\kappa_1\right)\right|d\psi\right]. \end{aligned}$$

Since $|F'|$ is hyperbolic p-convex, we have the following bounds:

$$\begin{aligned} \left|F'\left(\frac{1-\psi}{3}\kappa_2+\frac{2+\psi}{3}\kappa_1\right)\right| &\le \frac{\sinh\left(p\frac{1-\psi}{3}(\kappa_2-\kappa_1)\right)}{\sinh(p(\kappa_2-\kappa_1))}|F'(\kappa_2)|+\frac{\sinh\left(p\frac{2+\psi}{3}(\kappa_2-\kappa_1)\right)}{\sinh(p(\kappa_2-\kappa_1))}|F'(\kappa_1)|, \\ \left|F'\left(\frac{2-\psi}{3}\kappa_2+\frac{1+\psi}{3}\kappa_1\right)\right| &\le \frac{\sinh\left(p\frac{2-\psi}{3}(\kappa_2-\kappa_1)\right)}{\sinh(p(\kappa_2-\kappa_1))}|F'(\kappa_2)|+\frac{\sinh\left(p\frac{1+\psi}{3}(\kappa_2-\kappa_1)\right)}{\sinh(p(\kappa_2-\kappa_1))}|F'(\kappa_1)|, \\ \left|F'\left(\frac{3-\psi}{3}\kappa_2+\frac{\psi}{3}\kappa_1\right)\right| &\le \frac{\sinh\left(p\frac{3-\psi}{3}(\kappa_2-\kappa_1)\right)}{\sinh(p(\kappa_2-\kappa_1))}|F'(\kappa_2)|+\frac{\sinh\left(p\frac{\psi}{3}(\kappa_2-\kappa_1)\right)}{\sinh(p(\kappa_2-\kappa_1))}|F'(\kappa_1)|. \end{aligned}$$

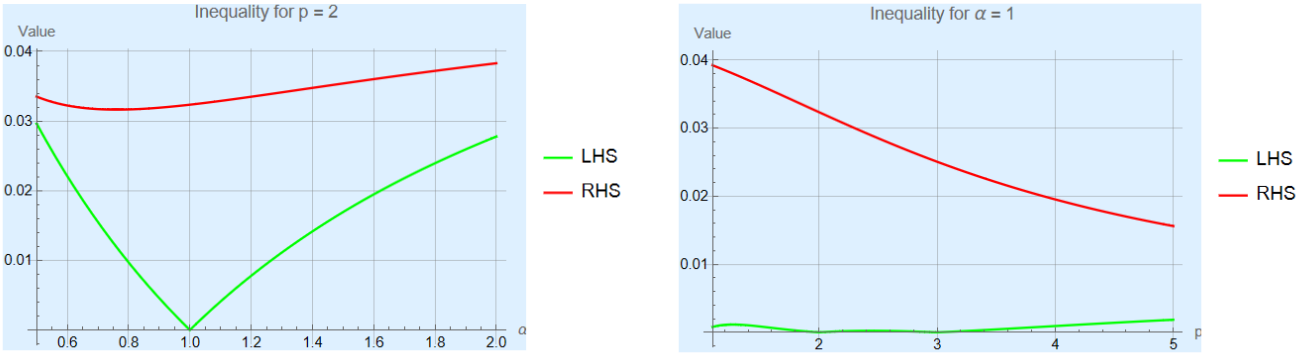
Substituting these estimates and collecting the coefficients of $|F'(\kappa_2)|$ and $|F'(\kappa_1)|$ exactly yields the integrals A_1 to A_6. Hence,

$$\begin{aligned} &\left|\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha}\left[\Im^\alpha_{\kappa_1^+}F\left(\frac{2\kappa_1+\kappa_2}{3}\right)+\Im^\alpha_{\frac{2\kappa_1+\kappa_2}{3}^+}F\left(\frac{\kappa_1+2\kappa_2}{3}\right)+\Im^\alpha_{\frac{\kappa_1+2\kappa_2}{3}^+}F(\kappa_2)\right]\right. \\ &\left.-\frac{1}{8}\left[F(\kappa_1)+3F\left(\frac{2\kappa_1+\kappa_2}{3}\right)+3F\left(\frac{\kappa_1+2\kappa_2}{3}\right)+F(\kappa_2)\right]\right| \\ &\le \frac{\kappa_2-\kappa_1}{9}\left[|F'(\kappa_2)|(A_1+A_2+A_3)+|F'(\kappa_1)|(A_4+A_5+A_6)\right]. \end{aligned}$$

This completes the proof. □

Remark 3.2. *Let* $\Delta=\kappa_2-\kappa_1$. *As* $p\to 0$, *we have* $\frac{\sinh(p(1-\psi)\Delta)}{\sinh(p\Delta)}\to 1-\psi$ *and* $\frac{\sinh(p\psi\Delta)}{\sinh(p\Delta)}\to\psi$. *Hence, Theorem 3.1 reduces to Theorem* 4 *of [13].*

Example 3.3. *Let* $[\kappa_1,\kappa_2]=[0,1]$, *and define the function* $F:[0,1]\to\mathbb{R}$ *by* $F(\psi)=\dfrac{\psi^p}{p}$ *such that* $F'(\psi)=\psi^{p-1}$ *and* $|F'|$ *is hyperbolic p-convex on* $[0,1]$. *By taking* $\kappa_2=1,\kappa_1=0,p=2$, *and* $\alpha=0.5$, *under these assumptions, Inequality* (3.1) *reduces to* $0.0296296\le 0.0335073$. *The validity of the* *inequality* $lefthandside(LHS) \leq righthandside(RHS)$ *is illustrated in Figures 1 and 2 for different values of* $\alpha > 0$ *and* $p > 1$.



(a) LHS and RHS vs α for fixed $p = 2$. (b) LHS and RHS vs p for fixed $\alpha = 1$.

Figure 1. Graphical verification of Theorem 3.1 for Example 3.3 (function $F(\psi) = \psi^p/p$ on $[0, 1]$). (a) and (b) are 2D slices showing that RHS always lies above LHS.

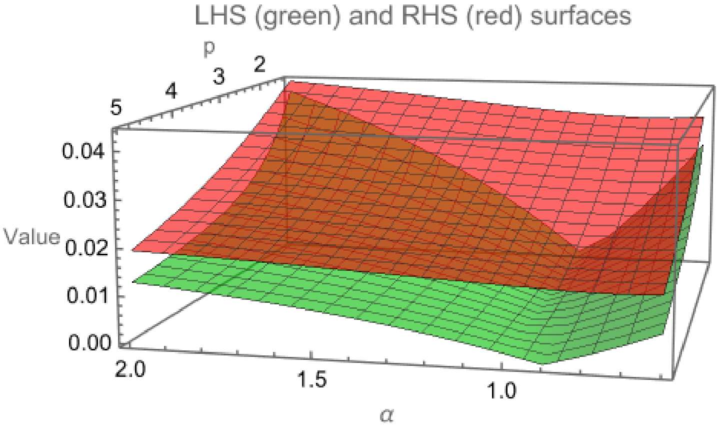


Figure 2. 3D surfaces of the LHS (green) and RHS (red) for the same example.

3.1. Comparison 1

We shall prove Theorem 3.1 by means of Lemma 2.4, and Thanin Sitthiwirattham *et al.* employed the same lemma to prove Theorem 4 in their paper [13]. Now, let us compare the two theorems.

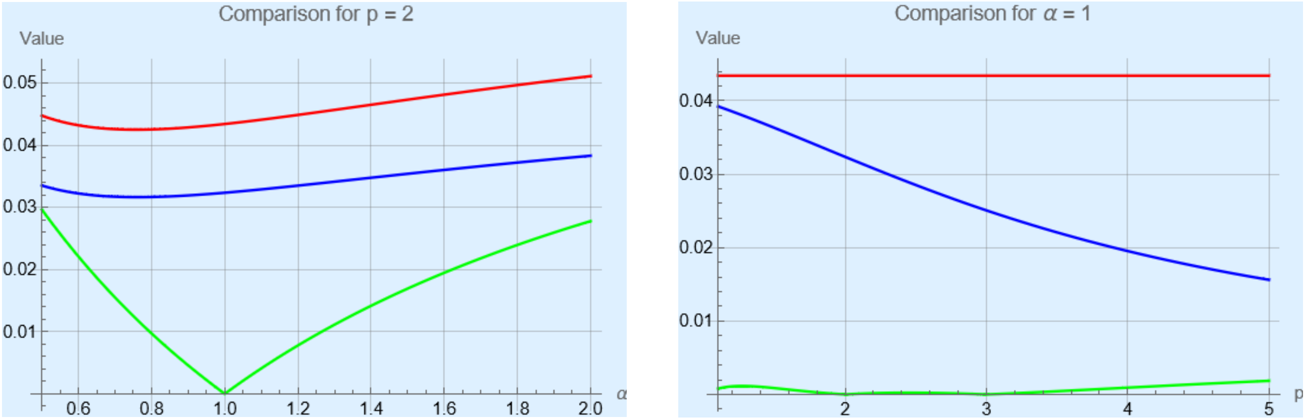
Consider the function $F(\psi) = \dfrac{\psi^p}{p}$ on the interval $[0, 1]$ (so $\kappa_1 = 0$, $\kappa_2 = 1$) with the derivative $F'(\psi) = \psi^{p-1}$. Taking $p = 2$ and $\alpha = 0.6$, Inequality (11) becomes $0.0220798 \leq 0.0431942$, while Inequality (3.1) becomes $0.0220798 \leq 0.0322282$. The bound of difference for Theorem 3.1 is given by 0.0101484, which is smaller than the corresponding bound of difference for Theorem 4 in [13], (i.e., 0.0211144). Therefore, we can conclude that Theorem 3.1 gives a tighter bound than Theorem 4.

Table 1 shows that Theorem 3.1 yields better bounds than Theorem 4 of [13].

Table 1. Comparison between Theorems 3.1 and 4 of [13] for the function $F(\psi) = \psi^p/p$ on $[0, 1]$. The data show that our bound of (3.1) is always smaller than the paper's bound of (11) of [13] and both satisfy the inequality.

α	p	*LHS*	*RHS* of (3.1)	*RHS* of (11) of [13]	*Bounds* of (3.1)	*Bounds* of (11) of [13]
0.6	1.5	0.028120	0.036158	0.043194	0.008039	0.015074
0.6	2.0	0.022080	0.032228	0.043194	0.010148	0.021114
0.6	3.0	0.015036	0.024907	0.043194	0.009870	0.028158
0.6	4.0	0.010572	0.019253	0.043194	0.008681	0.032622
1.0	1.5	0.000862	0.036293	0.043403	0.035431	0.042541
1.0	2.0	0.000000	0.032346	0.043403	0.032346	0.043403
1.0	3.0	0.000000	0.025059	0.043403	0.025059	0.043403
1.0	4.0	0.000926	0.019526	0.043403	0.018600	0.042477
1.5	1.5	0.023450	0.039641	0.047313	0.016191	0.023863
1.5	2.0	0.016931	0.035394	0.047313	0.018462	0.030382
1.5	3.0	0.011258	0.027589	0.047313	0.016331	0.036055
1.5	4.0	0.009343	0.021707	0.047313	0.012364	0.037970
2.0	1.5	0.038139	0.042852	0.051094	0.004713	0.012955
2.0	2.0	0.027778	0.038293	0.051094	0.010515	0.023317
2.0	3.0	0.018313	0.029930	0.051094	0.011617	0.032782
2.0	4.0	0.014506	0.023639	0.051094	0.009133	0.036588

In addition, the validity of both theorems can be verified by means of Figure 3, which shows the LHS with a green color, while the RHSs for Theorems 3.1 and 4 are depicted with blue and red colors, respectively.



(a) LHS and RHS vs α for fixed $p = 2$.

(b) LHS and RHS vs p for fixed $\alpha = 1$.

Figure 3. Comparison of Theorem 3.1 (blue) and Theorem 4 of [13] (red). The graphs in (a) and (b) show that our bound is smaller than the classical bound, hence our result provides a sharper estimate.

Theorem 3.4. *Assume that the conditions of Lemma 2.4 hold. If* $|F'|$ *is a hyperbolic p-convex function on* $[\kappa_1, \kappa_2]$ *and* $v \geq 1$, *then, the following Newton-type inequality holds:*

$$\begin{aligned}
&\left|\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[\Im^{\alpha}_{\kappa_1^+}F\left(\frac{2\kappa_1+\kappa_2}{3}\right)+\Im^{\alpha}_{\frac{2\kappa_1+\kappa_2}{3}^+}F\left(\frac{\kappa_1+2\kappa_2}{3}\right)+\Im^{\alpha}_{\frac{\kappa_1+2\kappa_2}{3}^+}F(\kappa_2)\right]\right.\\
&\left.-\frac{1}{8}\left[F(\kappa_1)+3F\left(\frac{2\kappa_1+\kappa_2}{3}\right)+3F\left(\frac{\kappa_1+2\kappa_2}{3}\right)+F(\kappa_2)\right]\right|\\
&\leq \frac{\kappa_2-\kappa_1}{9}\left[(C_1)^{1-\frac{1}{v}}\left(|F'(\kappa_2)|^{v}C_2+|F'(\kappa_1)|^{v}C_3\right)^{\frac{1}{v}}\right.\\
&\quad+(C_4)^{1-\frac{1}{v}}\left(|F'(\kappa_2)|^{v}C_5+|F'(\kappa_1)|^{v}C_6\right)^{\frac{1}{v}}\\
&\quad\left.+(C_7)^{1-\frac{1}{v}}\left(|F'(\kappa_2)|^{v}C_8+|F'(\kappa_1)|^{v}C_9\right)^{\frac{1}{v}}\right],
\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}
C_1 &= \int_0^1\left|\psi^{\alpha}-\frac{5}{8}\right|d\psi,\\
C_2 &= \int_0^1\left|\psi^{\alpha}-\frac{5}{8}\right|\left(\frac{\sinh\left(p\left(\frac{1-\psi}{3}\right)(\kappa_2-\kappa_1)\right)}{\sinh(p(\kappa_2-\kappa_1))}\right)d\psi,\\
C_3 &= \int_0^1\left|\psi^{\alpha}-\frac{5}{8}\right|\left(\frac{\sinh\left(p\left(\frac{2+\psi}{3}\right)(\kappa_2-\kappa_1)\right)}{\sinh(p(\kappa_2-\kappa_1))}\right)d\psi,\\
C_4 &= \int_0^1\left|\psi^{\alpha}-\frac{1}{2}\right|d\psi,\\
C_5 &= \int_0^1\left|\psi^{\alpha}-\frac{1}{2}\right|\left(\frac{\sinh\left(p\left(\frac{2-\psi}{3}\right)(\kappa_2-\kappa_1)\right)}{\sinh(p(\kappa_2-\kappa_1))}\right)d\psi,\\
C_6 &= \int_0^1\left|\psi^{\alpha}-\frac{1}{2}\right|\left(\frac{\sinh\left(p\left(\frac{1+\psi}{3}\right)(\kappa_2-\kappa_1)\right)}{\sinh(p(\kappa_2-\kappa_1))}\right)d\psi,\\
C_7 &= \int_0^1\left|\psi^{\alpha}-\frac{3}{8}\right|d\psi,\\
C_8 &= \int_0^1\left|\psi^{\alpha}-\frac{3}{8}\right|\left(\frac{\sinh\left(p\left(\frac{3-\psi}{3}\right)(\kappa_2-\kappa_1)\right)}{\sinh(p(\kappa_2-\kappa_1))}\right)d\psi,\\
C_9 &= \int_0^1\left|\psi^{\alpha}-\frac{3}{8}\right|\left(\frac{\sinh\left(p\left(\frac{\psi}{3}\right)(\kappa_2-\kappa_1)\right)}{\sinh(p(\kappa_2-\kappa_1))}\right)d\psi.
\end{aligned}$$

Proof. Define the LHS follows:

$$\begin{aligned}
LHS = &\left|\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[\Im^{\alpha}_{\kappa_1^+}F\left(\frac{2\kappa_1+\kappa_2}{3}\right)+\Im^{\alpha}_{\frac{2\kappa_1+\kappa_2}{3}^+}F\left(\frac{\kappa_1+2\kappa_2}{3}\right)+\Im^{\alpha}_{\frac{\kappa_1+2\kappa_2}{3}^+}F(\kappa_2)\right]\right.\\
&\left.-\frac{1}{8}\left[F(\kappa_1)+3F\left(\frac{2\kappa_1+\kappa_2}{3}\right)+3F\left(\frac{\kappa_1+2\kappa_2}{3}\right)+F(\kappa_2)\right]\right|.
\end{aligned}$$

From Lemma 2.4 and the triangle inequality, we obtain the following:

$$\begin{aligned} LHS \leq \frac{\kappa_2 - \kappa_1}{9}\Bigg[&\int_0^1 \left|\psi^\alpha - \frac{5}{8}\right| \left|F'\left(\psi\kappa_1 + (1-\psi)\frac{2\kappa_1 + \kappa_2}{3}\right)\right| d\psi \\ &+ \int_0^1 \left|\psi^\alpha - \frac{1}{2}\right| \left|F'\left(\psi\frac{2\kappa_1 + \kappa_2}{3} + (1-\psi)\frac{\kappa_1 + 2\kappa_2}{3}\right)\right| d\psi \\ &+ \int_0^1 \left|\psi^\alpha - \frac{3}{8}\right| \left|F'\left(\psi\frac{\kappa_1 + 2\kappa_2}{3} + (1-\psi)\kappa_2\right)\right| d\psi\Bigg]. \end{aligned}$$

Now, rewrite the arguments as convex combinations of κ_1 and κ_2 as follows:

$$\begin{aligned} \psi\kappa_1 + (1-\psi)\frac{2\kappa_1 + \kappa_2}{3} &= \frac{1-\psi}{3}\kappa_2 + \frac{2+\psi}{3}\kappa_1, \\ \psi\frac{2\kappa_1 + \kappa_2}{3} + (1-\psi)\frac{\kappa_1 + 2\kappa_2}{3} &= \frac{2-\psi}{3}\kappa_2 + \frac{1+\psi}{3}\kappa_1, \\ \psi\frac{\kappa_1 + 2\kappa_2}{3} + (1-\psi)\kappa_2 &= \frac{3-\psi}{3}\kappa_2 + \frac{\psi}{3}\kappa_1. \end{aligned}$$

Thus,

$$\begin{aligned} LHS \leq \frac{\kappa_2 - \kappa_1}{9}\Bigg[&\int_0^1 \left|\psi^\alpha - \frac{5}{8}\right| \left|F'\left(\frac{1-\psi}{3}\kappa_2 + \frac{2+\psi}{3}\kappa_1\right)\right| d\psi \\ &+ \int_0^1 \left|\psi^\alpha - \frac{1}{2}\right| \left|F'\left(\frac{2-\psi}{3}\kappa_2 + \frac{1+\psi}{3}\kappa_1\right)\right| d\psi \\ &+ \int_0^1 \left|\psi^\alpha - \frac{3}{8}\right| \left|F'\left(\frac{3-\psi}{3}\kappa_2 + \frac{\psi}{3}\kappa_1\right)\right| d\psi\Bigg]. \end{aligned}$$

Applying the power-mean inequality (with exponent $v \geq 1$) to each integral gives the following:

$$\begin{aligned} LHS \leq \frac{\kappa_2 - \kappa_1}{9}\Bigg[&\left(\int_0^1 \left|\psi^\alpha - \frac{5}{8}\right| d\psi\right)^{1-\frac{1}{v}} \left(\int_0^1 \left|\psi^\alpha - \frac{5}{8}\right| \left|F'\left(\frac{1-\psi}{3}\kappa_2 + \frac{2+\psi}{3}\kappa_1\right)\right|^v d\psi\right)^{\frac{1}{v}} \\ &+ \left(\int_0^1 \left|\psi^\alpha - \frac{1}{2}\right| d\psi\right)^{1-\frac{1}{v}} \left(\int_0^1 \left|\psi^\alpha - \frac{1}{2}\right| \left|F'\left(\frac{2-\psi}{3}\kappa_2 + \frac{1+\psi}{3}\kappa_1\right)\right|^v d\psi\right)^{\frac{1}{v}} \\ &+ \left(\int_0^1 \left|\psi^\alpha - \frac{3}{8}\right| d\psi\right)^{1-\frac{1}{v}} \left(\int_0^1 \left|\psi^\alpha - \frac{3}{8}\right| \left|F'\left(\frac{3-\psi}{3}\kappa_2 + \frac{\psi}{3}\kappa_1\right)\right|^v d\psi\right)^{\frac{1}{v}}\Bigg]. \end{aligned}$$

Since $v \geq 1$ and $t \mapsto t^v$ is convex and increasing on $[0, \infty)$, $|F'|^v$ is also hyperbolic p-convex. Using Definition 2.2, we obtain the following:

$$\begin{aligned} \left|F'\left(\frac{1-\psi}{3}\kappa_2 + \frac{2+\psi}{3}\kappa_1\right)\right|^v &\leq \frac{\sinh\left(p\frac{1-\psi}{3}(\kappa_2 - \kappa_1)\right)}{\sinh\left(p(\kappa_2 - \kappa_1)\right)}|F'(\kappa_2)|^v + \frac{\sinh\left(p\frac{2+\psi}{3}(\kappa_2 - \kappa_1)\right)}{\sinh\left(p(\kappa_2 - \kappa_1)\right)}|F'(\kappa_1)|^v, \\ \left|F'\left(\frac{2-\psi}{3}\kappa_2 + \frac{1+\psi}{3}\kappa_1\right)\right|^v &\leq \frac{\sinh\left(p\frac{2-\psi}{3}(\kappa_2 - \kappa_1)\right)}{\sinh\left(p(\kappa_2 - \kappa_1)\right)}|F'(\kappa_2)|^v + \frac{\sinh\left(p\frac{1+\psi}{3}(\kappa_2 - \kappa_1)\right)}{\sinh\left(p(\kappa_2 - \kappa_1)\right)}|F'(\kappa_1)|^v, \end{aligned}$$

$$\left|F'\left(\frac{3-\psi}{3}\kappa_2+\frac{\psi}{3}\kappa_1\right)\right|^{\nu} \le \frac{\sinh\left(p\frac{3-\psi}{3}(\kappa_2-\kappa_1)\right)}{\sinh(p(\kappa_2-\kappa_1))}|F'(\kappa_2)|^{\nu}+\frac{\sinh\left(p\frac{\psi}{3}(\kappa_2-\kappa_1)\right)}{\sinh(p(\kappa_2-\kappa_1))}|F'(\kappa_1)|^{\nu}.$$

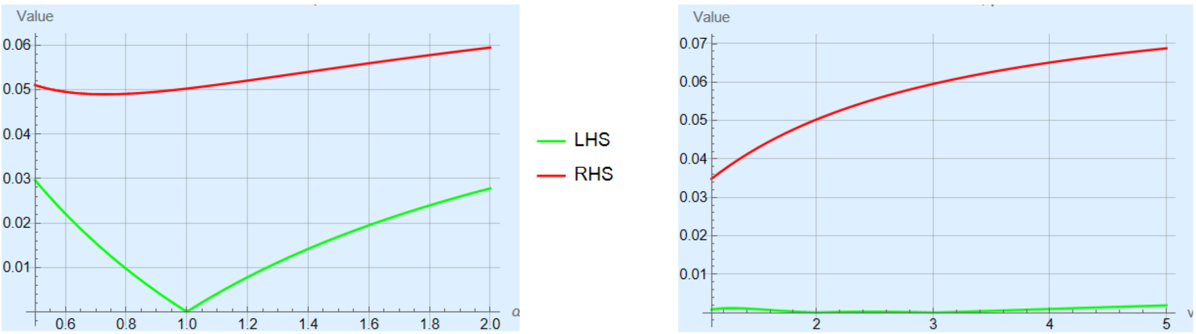
Substituting these estimates and using the definitions of $C_1,\dots,C_9$ yields the following:

$$\begin{aligned} LHS \le \frac{\kappa_2-\kappa_1}{9}\Big[&(C_1)^{1-\frac{1}{\nu}}\left(|F'(\kappa_2)|^{\nu}C_2+|F'(\kappa_1)|^{\nu}C_3\right)^{\frac{1}{\nu}} \\ &+(C_4)^{1-\frac{1}{\nu}}\left(|F'(\kappa_2)|^{\nu}C_5+|F'(\kappa_1)|^{\nu}C_6\right)^{\frac{1}{\nu}} \\ &+(C_7)^{1-\frac{1}{\nu}}\left(|F'(\kappa_2)|^{\nu}C_8+|F'(\kappa_1)|^{\nu}C_9\right)^{\frac{1}{\nu}}\Big], \end{aligned}$$

which is exactly Inequality (3.2). This completes the proof. □

Remark 3.5. *By substituting* $\sinh(px)=px$ *for* $p\to 0$ *in Theorem 3.4, one obtains Theorem* 5 *in [13].*

Example 3.6. *Let* $[\kappa_1,\kappa_2]=[0,1]$*, and define the function* $F:[0,1]\to\mathbb{R}$ *by* $F(\psi)=\frac{\psi^p}{p}$ *such that* $F'(\psi)=\psi^{p-1}$ *and* $|F'|$ *is hyperbolic p-convex on* $[0,1]$*. By taking* $\kappa_2=1,\kappa_1=0,p=2,\nu=2$*, and* $\alpha=0.4$*, under these assumptions, Inequality* (3.2) *reduces to* $0.0383597\le 0.0541255$*. The validity of the inequality* $LHS\le RHS$ *is illustrated in Figures 4 and 5 for different values of* $\alpha>0$ *and* $\nu\ge 1$*.*



(a) LHS and RHS vs α for fixed $p=2$.

(b) LHS and RHS vs ν for fixed $\alpha=1$.

Figure 4. Graphical verification of Theorem 3.1 for Example 3.6 (function $F(\psi)=\psi^p/p$ on $[0,1]$). (a) and (b) are 2D slices showing that the RHS always lies above the LHS.

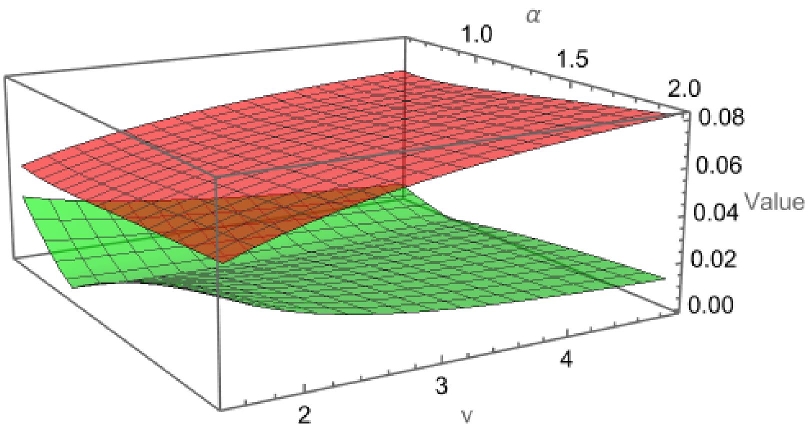


Figure 5. 3D surfaces of the LHS (green) and RHS (red) for the same example.

3.2. Comparison 2

We shall prove Theorem 3.4 by means of Lemma 2.4, and Thanin Sitthiwirattham *et al.* employed the same lemma to prove Theorem 5 in their paper [13]. Now, let us compare the two theorems.

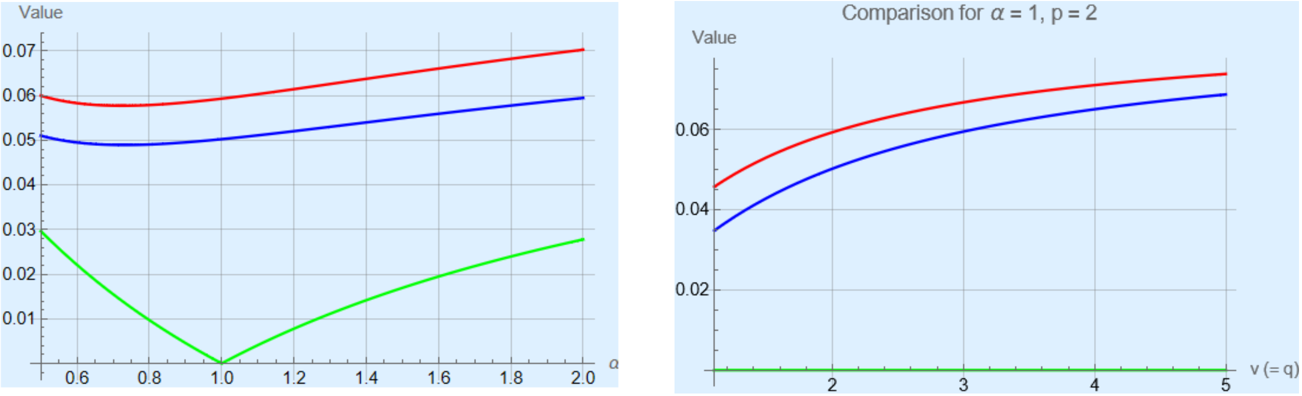
Consider the function $F(\psi) = \dfrac{\psi^p}{p}$ on the interval $[0, 1]$ (so $\kappa_1 = 0$, $\kappa_2 = 1$) with the derivative $F'(\psi) = \psi^{p-1}$. Taking $v = p = 2$, and $\alpha = 0.6$, Inequality (12) becomes $0.0220798 \leq 0.0583121$, while inequality (3.2) becomes $0.0220798 \leq 0.0495039$. The bound of difference for Theorem 3.4 is given by 0.0274241, which is smaller than the corresponding bound of difference for Theorem 5 in [13], (i.e., 0.0362323). Therefore, we can conclude that Theorem 3.4 gives a tighter bound than Theorem 5.

Table 2 shows that Theorem 3.4 yields better bounds than Theorem 5 of [13].

Table 2. Comparison between Theorem 3.4 and Theorem 5 of [13] for the function $F(\psi) = \psi^p/p$ on $[0, 1]$. The data show that our bound of (3.2) is always smaller than the paper's bound of (12) of [13], and both satisfy the inequality.

α	v	*LHS*	*RHS* of (3.2)	*RHS* of (12) of [13]	*Bounds* of (3.2)	*Bounds* of (12) of [13]
0.6	1.2	0.022080	0.036826	0.047417	0.014747	0.025337
0.6	2.0	0.022080	0.049504	0.058312	0.027424	0.036232
0.6	3.0	0.022080	0.058396	0.065430	0.036316	0.043351
0.6	4.0	0.022080	0.063737	0.069538	0.041657	0.047458
1.0	1.2	0.000000	0.037085	0.047848	0.037085	0.047848
1.0	2.0	0.000000	0.050244	0.059321	0.050244	0.059321
1.0	3.0	0.000000	0.059504	0.066792	0.059504	0.066792
1.0	4.0	0.000000	0.065065	0.071087	0.065065	0.071087
1.5	1.2	0.016931	0.040544	0.052212	0.023613	0.035280
1.5	2.0	0.016931	0.054973	0.064926	0.038042	0.047995
1.5	3.0	0.016931	0.065204	0.073240	0.048273	0.056309
1.5	4.0	0.016931	0.071370	0.078029	0.054439	0.061097
2.0	1.2	0.027778	0.043838	0.056410	0.016060	0.028632
2.0	2.0	0.027778	0.059480	0.070284	0.031702	0.042506
2.0	3.0	0.027778	0.070647	0.079404	0.042869	0.051626
2.0	4.0	0.027778	0.077400	0.084670	0.049622	0.056892

In addition, we can see that the validity of both theorems can be verified by means of the Figure 6, which shows the LHS with a green color, while the RHSs for Theorems 3.4 and 5 are depicted with blue and red colors, respectively.



(a) LHS and RHS vs α for fixed $p = 2$. **(b)** LHS and RHS vs v for fixed $\alpha = 1$.

Figure 6. The graphs show that the bound of Theorem 3.4 is smaller than the bound of Theorem 5 in [13] for both (a) and (b); hence, our result provides a sharper estimate.

Theorem 3.7. *Assume that the conditions of Lemma 2.4 hold. If* $|F'|$ *is a hyperbolic p-convex function on* $[\kappa_1, \kappa_2]$ *and* $l, v > 1$ *with* $\frac{1}{l} + \frac{1}{v} = 1$*; then, the following Newton-type inequality holds:*

$$\begin{aligned}
&\Bigg|\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[\Im^{\alpha}_{\kappa_1^+}F\left(\frac{2\kappa_1+\kappa_2}{3}\right)+\Im^{\alpha}_{\frac{2\kappa_1+\kappa_2}{3}^+}F\left(\frac{\kappa_1+2\kappa_2}{3}\right)+\Im^{\alpha}_{\frac{\kappa_1+2\kappa_2}{3}^+}F(\kappa_2)\right]\\
&-\frac{1}{8}\left[F(\kappa_1)+3F\left(\frac{2\kappa_1+\kappa_2}{3}\right)+3F\left(\frac{\kappa_1+2\kappa_2}{3}\right)+F(\kappa_2)\right]\Bigg|\\
&\leq\frac{\kappa_2-\kappa_1}{9}\Bigg[(D_1)^{\frac{1}{l}}\left(|F'(\kappa_2)|^{v}D_2+|F'(\kappa_1)|^{v}D_3\right)^{\frac{1}{v}}\\
&\quad+(D_4)^{\frac{1}{l}}\left(|F'(\kappa_2)|^{v}D_5+|F'(\kappa_1)|^{v}D_6\right)^{\frac{1}{v}}\\
&\quad+(D_7)^{\frac{1}{l}}\left(|F'(\kappa_2)|^{v}D_8+|F'(\kappa_1)|^{v}D_9\right)^{\frac{1}{v}}\Bigg],
\end{aligned}\tag{3.3}$$

where

$$D_1=\int_0^1\left|\psi^{\alpha}-\frac{5}{8}\right|^{l}d\psi,\qquad D_2=\int_0^1\frac{\sinh\left(p\frac{1-\psi}{3}(\kappa_2-\kappa_1)\right)}{\sinh(p(\kappa_2-\kappa_1))}d\psi,$$

$$D_3=\int_0^1\frac{\sinh\left(p\frac{2+\psi}{3}(\kappa_2-\kappa_1)\right)}{\sinh(p(\kappa_2-\kappa_1))}d\psi,\qquad D_4=\int_0^1\left|\psi^{\alpha}-\frac{1}{2}\right|^{l}d\psi,$$

$$D_5=\int_0^1\frac{\sinh\left(p\frac{2-\psi}{3}(\kappa_2-\kappa_1)\right)}{\sinh(p(\kappa_2-\kappa_1))}d\psi,\qquad D_6=\int_0^1\frac{\sinh\left(p\frac{1+\psi}{3}(\kappa_2-\kappa_1)\right)}{\sinh(p(\kappa_2-\kappa_1))}d\psi,$$

$$D_7=\int_0^1\left|\psi^{\alpha}-\frac{3}{8}\right|^{l}d\psi,\qquad D_8=\int_0^1\frac{\sinh\left(p\frac{3-\psi}{3}(\kappa_2-\kappa_1)\right)}{\sinh(p(\kappa_2-\kappa_1))}d\psi,$$

$$D_9=\int_0^1\frac{\sinh\left(p\frac{\psi}{3}(\kappa_2-\kappa_1)\right)}{\sinh(p(\kappa_2-\kappa_1))}d\psi.$$

Proof. Set

$$LHS=\left|\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[\Im^{\alpha}_{\kappa_1^+}F\left(\frac{2\kappa_1+\kappa_2}{3}\right)+\Im^{\alpha}_{\frac{2\kappa_1+\kappa_2}{3}^+}F\left(\frac{\kappa_1+2\kappa_2}{3}\right)+\Im^{\alpha}_{\frac{\kappa_1+2\kappa_2}{3}^+}F(\kappa_2)\right]\right.$$

$$-\frac{1}{8}\left[F(\kappa_1)+3F\left(\frac{2\kappa_1+\kappa_2}{3}\right)+3F\left(\frac{\kappa_1+2\kappa_2}{3}\right)+F(\kappa_2)\right]\Bigg|.$$

From Lemma 2.4 and the triangle inequality, we obtain the following:

$$\begin{aligned}LHS \leq \frac{\kappa_2-\kappa_1}{9}\Bigg[&\int_0^1\left|\psi^{\alpha}-\frac{5}{8}\right|\left|F'\left(\psi\kappa_1+(1-\psi)\frac{2\kappa_1+\kappa_2}{3}\right)\right|d\psi\\&+\int_0^1\left|\psi^{\alpha}-\frac{1}{2}\right|\left|F'\left(\psi\frac{2\kappa_1+\kappa_2}{3}+(1-\psi)\frac{\kappa_1+2\kappa_2}{3}\right)\right|d\psi\\&+\int_0^1\left|\psi^{\alpha}-\frac{3}{8}\right|\left|F'\left(\psi\frac{\kappa_1+2\kappa_2}{3}+(1-\psi)\kappa_2\right)\right|d\psi\Bigg].\end{aligned}$$

Now, rewrite the arguments as convex combinations of κ_1 and κ_2 as follows:

$$\begin{aligned}\psi\kappa_1+(1-\psi)\frac{2\kappa_1+\kappa_2}{3}&=\frac{1-\psi}{3}\kappa_2+\frac{2+\psi}{3}\kappa_1,\\\psi\frac{2\kappa_1+\kappa_2}{3}+(1-\psi)\frac{\kappa_1+2\kappa_2}{3}&=\frac{2-\psi}{3}\kappa_2+\frac{1+\psi}{3}\kappa_1,\\\psi\frac{\kappa_1+2\kappa_2}{3}+(1-\psi)\kappa_2&=\frac{3-\psi}{3}\kappa_2+\frac{\psi}{3}\kappa_1.\end{aligned}$$

Thus,

$$\begin{aligned}LHS \leq \frac{\kappa_2-\kappa_1}{9}\Bigg[&\int_0^1\left|\psi^{\alpha}-\frac{5}{8}\right|\left|F'\left(\frac{1-\psi}{3}\kappa_2+\frac{2+\psi}{3}\kappa_1\right)\right|d\psi\\&+\int_0^1\left|\psi^{\alpha}-\frac{1}{2}\right|\left|F'\left(\frac{2-\psi}{3}\kappa_2+\frac{1+\psi}{3}\kappa_1\right)\right|d\psi\\&+\int_0^1\left|\psi^{\alpha}-\frac{3}{8}\right|\left|F'\left(\frac{3-\psi}{3}\kappa_2+\frac{\psi}{3}\kappa_1\right)\right|d\psi\Bigg].\end{aligned}$$

Applying Hölder's inequality with conjugate exponents $l, v > 1$ ($\frac{1}{l}+\frac{1}{v}=1$) gives the following:

$$\begin{aligned}LHS \leq \frac{\kappa_2-\kappa_1}{9}\Bigg[&\left(\int_0^1\left|\psi^{\alpha}-\frac{5}{8}\right|^l d\psi\right)^{\frac{1}{l}}\left(\int_0^1\left|F'\left(\frac{1-\psi}{3}\kappa_2+\frac{2+\psi}{3}\kappa_1\right)\right|^v d\psi\right)^{\frac{1}{v}}\\&+\left(\int_0^1\left|\psi^{\alpha}-\frac{1}{2}\right|^l d\psi\right)^{\frac{1}{l}}\left(\int_0^1\left|F'\left(\frac{2-\psi}{3}\kappa_2+\frac{1+\psi}{3}\kappa_1\right)\right|^v d\psi\right)^{\frac{1}{v}}\\&+\left(\int_0^1\left|\psi^{\alpha}-\frac{3}{8}\right|^l d\psi\right)^{\frac{1}{l}}\left(\int_0^1\left|F'\left(\frac{3-\psi}{3}\kappa_2+\frac{\psi}{3}\kappa_1\right)\right|^v d\psi\right)^{\frac{1}{v}}\Bigg].\end{aligned}$$

Because $v \geq 1$ and $t \mapsto t^v$ is convex and increasing on $[0,\infty)$, $|F'|^v$ is also hyperbolic p-convex. Using Definition 2.2, we obtain the following:

$$\left|F'\left(\frac{1-\psi}{3}\kappa_2+\frac{2+\psi}{3}\kappa_1\right)\right|^v \leq \frac{\sinh\left(p\frac{1-\psi}{3}(\kappa_2-\kappa_1)\right)}{\sinh\left(p(\kappa_2-\kappa_1)\right)}|F'(\kappa_2)|^v+\frac{\sinh\left(p\frac{2+\psi}{3}(\kappa_2-\kappa_1)\right)}{\sinh\left(p(\kappa_2-\kappa_1)\right)}|F'(\kappa_1)|^v,$$

$$\left|F'\left(\frac{2-\psi}{3}\kappa_2+\frac{1+\psi}{3}\kappa_1\right)\right|^{\nu} \leq \frac{\sinh\left(p\frac{2-\psi}{3}(\kappa_2-\kappa_1)\right)}{\sinh\left(p(\kappa_2-\kappa_1)\right)}|F'(\kappa_2)|^{\nu}+\frac{\sinh\left(p\frac{1+\psi}{3}(\kappa_2-\kappa_1)\right)}{\sinh\left(p(\kappa_2-\kappa_1)\right)}|F'(\kappa_1)|^{\nu},$$
$$\left|F'\left(\frac{3-\psi}{3}\kappa_2+\frac{\psi}{3}\kappa_1\right)\right|^{\nu} \leq \frac{\sinh\left(p\frac{3-\psi}{3}(\kappa_2-\kappa_1)\right)}{\sinh\left(p(\kappa_2-\kappa_1)\right)}|F'(\kappa_2)|^{\nu}+\frac{\sinh\left(p\frac{\psi}{3}(\kappa_2-\kappa_1)\right)}{\sinh\left(p(\kappa_2-\kappa_1)\right)}|F'(\kappa_1)|^{\nu}.$$

Integrating these inequalities over ψ from 0 to 1 and using the definitions of $D_2,\ldots,D_9$ yields the following:

$$\int_0^1\left|F'\left(\frac{1-\psi}{3}\kappa_2+\frac{2+\psi}{3}\kappa_1\right)\right|^{\nu}d\psi \leq |F'(\kappa_2)|^{\nu}D_2+|F'(\kappa_1)|^{\nu}D_3,$$
$$\int_0^1\left|F'\left(\frac{2-\psi}{3}\kappa_2+\frac{1+\psi}{3}\kappa_1\right)\right|^{\nu}d\psi \leq |F'(\kappa_2)|^{\nu}D_5+|F'(\kappa_1)|^{\nu}D_6,$$
$$\int_0^1\left|F'\left(\frac{3-\psi}{3}\kappa_2+\frac{\psi}{3}\kappa_1\right)\right|^{\nu}d\psi \leq |F'(\kappa_2)|^{\nu}D_8+|F'(\kappa_1)|^{\nu}D_9.$$

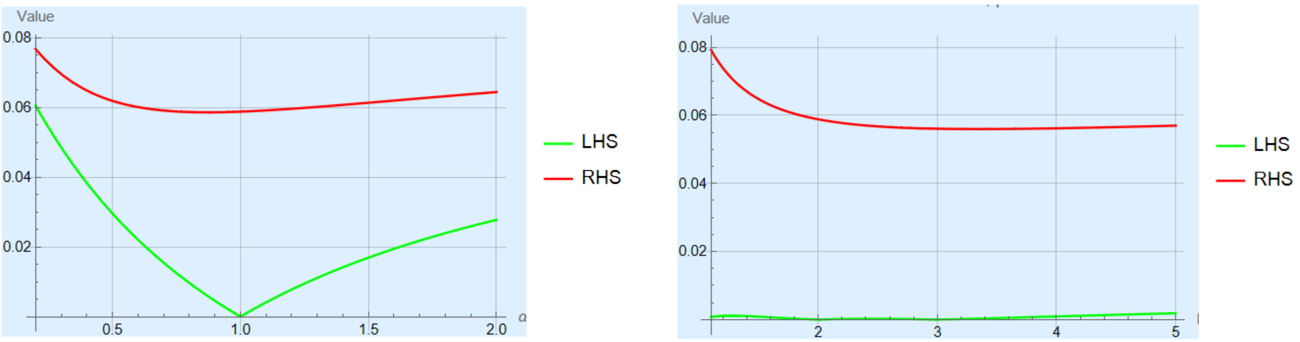
By substituting these estimates together with the definitions of D_1, D_4, D_7, we obtain the following:

$$\begin{aligned} LHS \leq \frac{\kappa_2-\kappa_1}{9}\Big[&(D_1)^{\frac{1}{l}}\left(|F'(\kappa_2)|^{\nu}D_2+|F'(\kappa_1)|^{\nu}D_3\right)^{\frac{1}{\nu}}\\ &+(D_4)^{\frac{1}{l}}\left(|F'(\kappa_2)|^{\nu}D_5+|F'(\kappa_1)|^{\nu}D_6\right)^{\frac{1}{\nu}}\\ &+(D_7)^{\frac{1}{l}}\left(|F'(\kappa_2)|^{\nu}D_8+|F'(\kappa_1)|^{\nu}D_9\right)^{\frac{1}{\nu}}\Big], \end{aligned}$$

which is exactly Inequality (3.3). This completes the proof. □

Remark 3.8. *By substituting* $\sinh(px) = px$ *for* $p \to 0$ *in Theorem 3.7, one obtains Theorem* 6 *in [13].*

Example 3.9. *Let* $[\kappa_1, \kappa_2] = [0, 1]$*, and define the function* $F : [0, 1] \to \mathbb{R}$ *by* $F(\psi) = \dfrac{\psi^p}{p}$ *such that* $F'(\psi) = \psi^{p-1}$ *and* $|F'|$ *is hyperbolic p-convex on* $[0, 1]$*. By taking* $\kappa_2 = 1, \kappa_1 = 0, p = 2, l = 2$*, and* $\alpha = 0.2$*, under these assumptions, Inequality* (3.3) *reduces to* $0.0605479 \leq 0.0768403$*. The validity of the inequality* $LHS \leq RHS$ *is illustrated in Figures 7 and 8 for different values of* $\alpha > 0$ *and* $\nu \geq 1$.



(a) LHS and RHS vs α for fixed $p = 2$.

(a) LHS and RHS vs l for fixed $\alpha = 1$.

Figure 7. Graphical verification of Theorem 3.1 for Example 3.9 (function $F(\psi) = \psi^p/p$ on $[0, 1]$). (a) and (b) are 2D slices showing that the RHS always lies above the LHS.

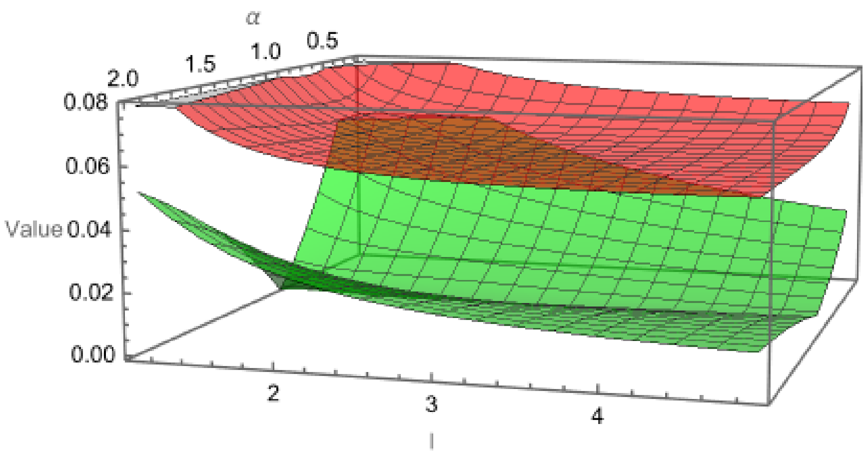


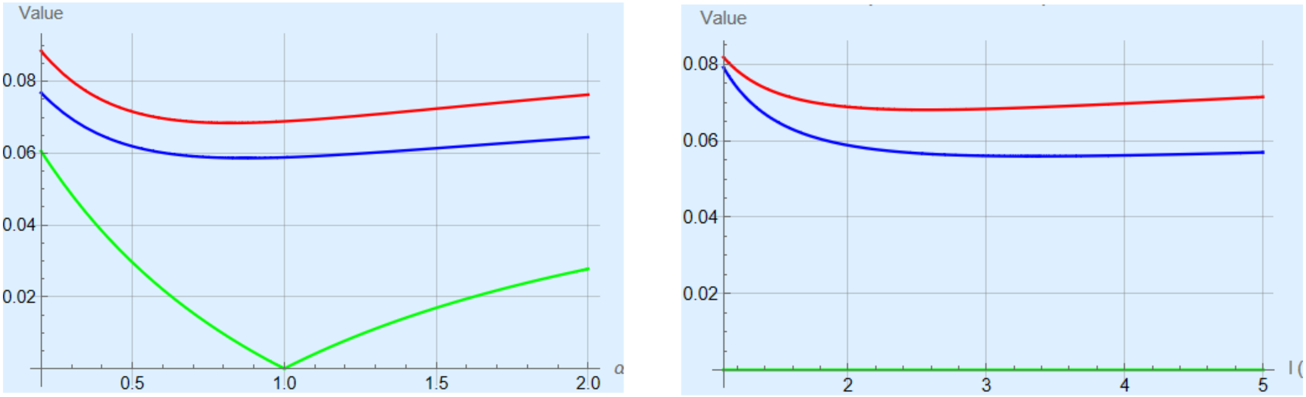
Figure 8. 3D surfaces of LHS (green) and RHS (red) for the same example.

3.3. *Comparison 3*

We shall prove Theorem 3.7 by means of Lemma 2.4, and Thanin Sitthiwirattham *et al.* employed the same lemma to prove Theorem 6 in their paper [13]. Now, let us compare the two theorems.

Consider the function $F(\psi) = \dfrac{\psi^p}{p}$ on the interval $[0, 1]$ (so $\kappa_1 = 0$, $\kappa_2 = 1$) with the derivative $F'(\psi) = \psi^{p-1}$. Taking $l = p = 2$, and $\alpha = 0.2$, Inequality (13) becomes $0.0605479 \leq 0.0768403$, while Inequality (3.3) becomes $0.0605479 \leq 0.0884737$. The bound of difference for Theorem 3.7 is given by 0.0162924, which is smaller than the corresponding bound of difference for Theorem 6 in [13], (i.e., 0.0279258). Therefore, we can conclude that Theorem 3.7 gives a tighter bound than Theorem 6.

In addition, we can see that the validity of both theorems can be verified by means of the Figure 9, which shows the LHS with a green color, while the RHSs for Theorems 3.7 and 6 are depicted with blue and red colors, respectively.



(a) LHS and RHS vs α for fixed $p = 2$.

(b) LHS and RHS vs v for fixed $\alpha = 1$.

Figure 9. The graphs show that the bound of Theorem 3.7 is smaller than the bound of Theorem 6 in [13] for both (a) and (b); hence, our result provides a sharper estimate.

Table 3 shows that Theorem 3.7 yields better bounds than Theorem 6 of [13].

Table 3. Comparison between Theorem 3.7 and Theorem 6 of [13] for the function $F(\psi) = \psi^p/p$ on $[0, 1]$. The data show that our bound of (3.3) is always smaller than the paper's bound of (13) of [13], and both satisfy the inequality.

α	l	*LHS*	*RHS* of (3.3)	*RHS* of (13) of [13]	*Bounds* of (3.3)	*Bounds* of (13) of [13]
0.2	1.2	0.060548	0.098138	0.103367	0.037590	0.042819
0.2	2.0	0.060548	0.076840	0.088474	0.016292	0.027926
0.2	3.0	0.060548	0.070142	0.083952	0.009594	0.023404
0.2	4.0	0.060548	0.067767	0.082608	0.007219	0.022060
1.0	1.2	0.000000	0.073839	0.078322	0.073839	0.078322
1.0	2.0	0.000000	0.058817	0.068858	0.058817	0.068858
1.0	3.0	0.000000	0.056050	0.068340	0.056050	0.068340
1.0	4.0	0.000000	0.056179	0.069761	0.056179	0.069761
1.5	1.2	0.016931	0.079968	0.085044	0.063037	0.068112
1.5	2.0	0.016931	0.061376	0.072390	0.044444	0.055459
1.5	3.0	0.016931	0.057235	0.070428	0.040304	0.053497
1.5	4.0	0.016931	0.056729	0.071117	0.039798	0.054185
2.0	1.2	0.027778	0.086163	0.091746	0.058385	0.063968
2.0	2.0	0.027778	0.064437	0.076290	0.036659	0.048512
2.0	3.0	0.027778	0.058989	0.072950	0.031212	0.045172
2.0	4.0	0.027778	0.057792	0.072846	0.030015	0.045068

4. Conclusions

Newton-type inequalities are widely used in many areas of mathematics, particularly algebra, combinatorics, and optimization. They help calculate simple symmetric polynomial bounds in algebra, which are used to estimate polynomial roots. They aid in the proof of inequalities in combinatorics by utilizing probability distributions and binomial coefficients. They support function minimization and constraint analyses in optimization problems. These inequalities are a reliable way to solve mathematical problems because of their numerous applications. This dissertation explored new variants of Newton-type inequalities for differentiable hyperbolic p-convex functions using Riemann-Liouville fractional integrals. Some refined results were obtained by applying the power-mean inequality and the Hölder inequality. The study of Newton-type inequalities with fractional integrals opens up new possibilities for mathematical analyses. This line of investigation can be expanded by examining different fractional operators and classes of convex functions. Furthermore, frameworks such as quantum calculus, time-scale calculus, alternative coordinate systems and fractal calculus offer methods to establish pertinent convexity related inequalities with fractional integrals. Potential engineering applications include viscoelasticity (creep compliance), heat transfer (exponential temperature profiles), and certain control problems.

Authors contributions

Saima Riaz initiated the draft, performed validation and prepared the figures. Khuram Ali Khan edited the draft, performed supervision, and methodology. Tamador Alihia performed formal analysis and finalized the manuscript. Ghulam Abbas edited the draft and performed investigations. Tahreem Akram made investigations.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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