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Research article

Principal normal surfaces of generalized null Cartan curves in Minkowski 3-space

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Abstract: We define the principal normal surfaces of generalized null Cartan curves in Minkowski 3-space. We give the necessary and sufficient conditions for the relationship between the singularities of these surfaces and the finite type of their base curves. An analogous result does not hold in Euclidean 3-space, where the singularities of the principal normal surface depend also on the torsion of the base curve.

Keywords: principal normal surfaces; generalized null Cartan curves; finite type; singularities
Mathematics Subject Classification: 53A04, 57R45

1. Introduction

The singularities of curves and surfaces are a central topic in differential geometry and singularity theory [1–3]. In Euclidean 3-space, the finite type of a curve at one point can be determined by its derivatives at that point [4]. It is well known that the singularities of the tangent developable of a smooth curve can be determined by the finite type of the curve at the corresponding points [5–8]. However, this theorem is not true for the principal normal surface of a Frenet-type framed base curve. This is due to the fact that the surface has two different singular sets [9, 10]. One of them does not lie on the curve and the other is related to the torsion of the curve. The Frenet-type framed base curves were firstly defined by Honda [11].

In the Minkowski 3-space, due to the absence of a corresponding geometry in Euclidean space, lightlike geometry exhibits properties that differ from those of Euclidean space, which makes it particularly interesting for mathematical research. For example, Balgetir, Bektaş, and Inoguchi showed that a null Cartan curve is a Bertrand curve if and only if it is a null Cartan helix [12]. This does not hold in Euclidean 3-space [13]. Chino and Izumiya showed that there is one more type of singularities of the tangent developable surfaces of lightlike curves than considered in Euclidean 3-space [14]. This motivates us to ask whether the finite type of a lightlike curve can determine the singularities of its principal normal surface.

Based on the description above, in this paper, we consider the principal normal surface of a generalized null Cartan curve in Minkowski 3-space. In our previous works [15, 16], we defined generalized null Cartan curves in Minkowski 3-space. They are null at regular points and are allowed to contain spacelike singularities. Then, they are natural generalizations of null Cartan curves. We remark that the singularities of this surface coincide with the generalized null Cartan curve. This means we can use the finite type of a generalized null Cartan curve to determine the singularities of its principal normal surface. Then, the necessary and sufficient relation between the singularities of the surface and the finite type of the curve is given in Theorem 4.3. The proof of this theorem is elementary, but the phenomenon is interesting.

The structure of this paper is outlined as follows. In Section 2, we briefly introduce some basic notions in Minkowski 3-space. In Section 3, we will give the definition of generalized null Cartan curves and investigate their finite type. In Section 4, we will show our main results. Specifically, we establish the relationship between the singularities of the principal normal surfaces and the finite type of their base curves. In Section 5, we will give some examples to illustrate our main results.

We shall assume that all the maps and manifolds in this paper are C^∞, unless the contrary is explicitly stated.

2. Preliminaries

Let $\mathbb{R}^3_1$ be the Minkowski 3-space with signature $(-,+,+)$. We denote $\langle , \rangle$ and $\wedge$ as the symbol of the *pseudo inner product* and the *pseudo outer product*, respectively. We say a vector $\boldsymbol{x} \in \mathbb{R}^3_1 \backslash \{\mathbf{0}\}$ is *spacelike*, *null*, or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle$ is positive, zero, or negative, respectively. Particularly, we regard $\mathbf{0}$ as a spacelike vector. For any $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^3_1$, we can have $\langle \boldsymbol{x} \wedge \boldsymbol{y}, \boldsymbol{z} \rangle = \det(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$. We denote $\|\cdot\|$ as the *norm* of vectors in $\mathbb{R}^3_1$ and NC^* as the *open nullcone* at the origin. Besides, NC^*_+ and NC^*_- are the *future and past nullcones*, respectively. For more details, see [17, 18]. Let $\gamma : I \to \mathbb{R}^3_1$ be a regular curve, where I is an open interval. Then, γ is a *spacelike*, *null*, or *timelike curve* if its tangent vector is spacelike, null, or timelike.

3. The finite type of generalized null Cartan curves

Definition 3.1. [16] We call $\gamma : I \to \mathbb{R}^3_1$ a null type curve (or generalized null curve) if there exists a smooth function $\alpha : I \to \mathbb{R}$ and a null conical curve $\mathsf{T} : I \to NC^*$ such that $\dot{\gamma}(t) = \alpha(t)\mathsf{T}(t)$, for all $t \in I$. We say that $\mathsf{T}(t)$ is the tangent vector field of $\gamma(t)$.

If $\langle \dot{\mathsf{T}}(t), \dot{\mathsf{T}}(t) \rangle \neq 0$, we take $s = \int_{t_0}^{t} \|\dot{\mathsf{T}}(u)\| \, du$ as the *pseudo arc-length parameter* of $\gamma(t)$ such that $\langle \mathsf{T}'(s), \mathsf{T}'(s) \rangle = 1$ and $F = \{\mathsf{T}, \mathsf{N}, \mathsf{B}\}$ as the *Cartan-type frame*, where $\mathsf{N}(s) = \mathsf{T}'(s)$, $\mathsf{B}(s) = -\mathsf{T}''(s) - (1/2)\langle \mathsf{T}''(s), \mathsf{T}''(s) \rangle \mathsf{T}(s)$ and $\det(\mathsf{T}(s), \mathsf{N}(s), \mathsf{B}(s)) = 1$. Then, we call γ a *generalized null Cartan curve*. The Frenet-type formula with respect to F is given by

$$\begin{pmatrix} \mathsf{T}'(s) \\ \mathsf{B}'(s) \\ \mathsf{N}'(s) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \tau(s) \\ -\tau(s) & -1 & 0 \end{pmatrix} \begin{pmatrix} \mathsf{T}(s) \\ \mathsf{B}(s) \\ \mathsf{N}(s) \end{pmatrix},$$

where $\tau(s) = (1/2)\langle \mathsf{T}''(s), \mathsf{T}''(s) \rangle$. We call (τ, α) the *curvature* of γ. Then, s_0 is a singularity of $\gamma(s)$

if and only if $\alpha(s_0) = 0$. In the present paper, we only consider that the singularities of γ are isolated unless otherwise explicitly stated, for all $s \in I$.

Remark 3.2. The Cartan-type frame is the special case of lightcone frame and more suitable for studying generalized null curves [18, 19]. The condition $\langle \dot{\mathsf{T}}(t), \dot{\mathsf{T}}(t) \rangle \neq 0$ is analogous to the non-geodesic condition for null curves [20, 21]. If γ is a regular null curve, the Cartan-type frame is the same as the Cartan frame. That is why we call F the Cartan-type frame and s the pseudo arc-length parameter.

In Euclidean 3-space $\mathbb{R}^3$, denote $\gamma(t) = (r_1(t), r_2(t), r_3(t))$. Then, the *finite type* of the smooth curve germ γ at t_0 is a triple $\mathbb{A} = (a_1, a_2, a_3)$, which satisfied that $r_i(t) = t^{a_i} + o(t^{a_i})$ and $1 \leqslant a_1 < a_2 < a_3$ for properly chosen coordinates. In brief, denote this as $\mathbb{A}(\gamma(t_0))$. Chino and Izumiya gave a method to recognize the type of a smooth curve germ by a simple calculation in their paper [14]. Then, we have the following proposition.

Proposition 3.3. *Let $\gamma : I \to \mathbb{R}^3_1$ be a generalized null Cartan curve with curvature (τ, α). Take $s_0 \in I$. Then, $\mathbb{A}(\gamma(s_0))$ can only be $(n, n+1, n+2)$, where $n \geqslant 1$.*

Proof. By the Cartan-type frame of $\gamma(s)$, we can calculate that

$$\gamma'(s) = \alpha(s)\mathsf{T}(s),\ \gamma''(s) = \alpha'(s)\mathsf{T}(s) + \alpha(s)\mathsf{N}(s)$$

and

$$\begin{aligned}\gamma^{(m)}(s) = {} & \alpha^{(m-1)}(s)\mathsf{T}(s) + (m-1)\alpha^{(m-2)}(s)\mathsf{N}(s) \\ & - \frac{(m-1)(m-2)}{2}\alpha^{(m-3)}(s)\mathsf{B}(s) + \cdots ,\end{aligned}$$

where $m \geqslant 3$. If $\gamma'(s_0) \neq 0$, $\alpha(s_0) \neq 0$, then we have

$$\gamma^{(3)}(s_0) = \alpha''(s_0)\mathsf{T}(s_0) + 2\alpha'(s_0)\mathsf{N}(s_0) - \alpha(s_0)\mathsf{B}(s_0) - \alpha(s_0)\tau(s_0)\mathsf{T}(s_0).$$

Under this condition, $\gamma'(s_0)$, $\gamma''(s_0)$ and $\gamma^{(3)}(s_0)$ are linearly independent. That is

$$\det(\gamma'(s_0), \gamma''(s_0), \gamma^{(3)}(s_0)) \neq 0.$$

This means $\mathbb{A}(\gamma(s_0)) = (1, 2, 3)$.

If $\gamma'(s_0) = \cdots = \gamma^{(i)}(s_0) = 0$ and $\gamma^{(i+1)}(s_0) \neq 0$, then $\alpha(s_0) = \cdots = \alpha^{(i-1)}(s_0) = 0$ and $\alpha^{(i)}(s_0) \neq 0$, where $i \geqslant 2$. At this time,

$$\gamma^{(i+1)}(s_0) = \alpha^{(i)}(s_0)\mathsf{T}(s_0),\ \gamma^{(i+2)}(s_0) = \alpha^{(i+1)}(s_0)\mathsf{T}(s_0) + (i+1)\alpha^{(i)}(s_0)\mathsf{N}(s_0)$$

and

$$\begin{aligned}\gamma^{(i+3)}(s_0) = {} & \alpha^{(i+2)}(s_0)\mathsf{T}(s_0) + (i+2)\alpha^{(i+1)}(s_0)\mathsf{N}(s_0) \\ & - \frac{(i+2)(i+1)}{2}\alpha^{(i)}(s_0)\mathsf{B}(s_0) \\ & - \frac{(i+2)(i+1)}{2}\alpha^{(i)}(s_0)\tau(s_0)\mathsf{T}(s_0).\end{aligned}$$

Then, $\gamma^{(i+1)}(s_0)$, $\gamma^{(i+2)}(s_0)$, and $\gamma^{(i+3)}(s_0)$ are linearly independent. This means $\mathbb{A}(\gamma(s_0)) = (i+1, i+2, i+3)$. In summary, $\mathbb{A}(\gamma(s_0))$ can only be $(n, n+1, n+2)$, where $n \geqslant 1$. □

By the proof of Proposition 3.3, we can get the following corollary.

Corollary 3.4. *Let $\gamma : I \to \mathbb{R}^3_1$ be a generalized null Cartan curve with curvature (τ, α). Take $s_0 \in I$, then $\mathbb{A}(\gamma(s_0)) = (n, n+1, n+2)$ if and only if $\alpha(s_0) = \cdots = \alpha^{(n-2)}(s_0) = 0$ and $\alpha^{(n-1)}(s_0) \neq 0$, where $n \geqslant 1$.*

4. Principal normal surfaces

In this section, we will investigate the singularities of the principal normal surface of a generalized null Cartan curve.

Let $\gamma : I \to \mathbb{R}^3_1$ be a generalized null Cartan curve with curvature (τ, α). Define a surface

$$\boldsymbol{X} : I \times \mathbb{R} \to \mathbb{R}^3_1,\ \boldsymbol{X}(s, w) = \gamma(s) + w\mathsf{N}(s).$$

We call this surface the *principal normal surface* of γ. By a calculation, we can get $\det(\gamma'(s), \mathsf{N}(s), \mathsf{N}'(s)) = -\alpha(s)$. This means $\boldsymbol{X}(s, w)$ is non-developable.

Proposition 4.1. *Let $\gamma : I \to \mathbb{R}^3_1$ be a generalized null Cartan curve with curvature (τ, α) and $\boldsymbol{X}(s, w)$ be its principal normal surface. Then, the singular set of $\boldsymbol{X}(s, w)$ is*

$$\sigma = \{(s_0, w_0) \in I \times \mathbb{R} \mid w_0 = 0,\ \alpha(s_0) = 0\}.$$

Proof. The surface $\boldsymbol{X}(s, w)$ is singular at (s_0, w_0) if and only if $\boldsymbol{X}_s(s_0, w_0)$ and $\boldsymbol{X}_w(s_0, w_0)$ are linearly dependent, equivalently, if and only if $\boldsymbol{X}_s(s_0, w_0) \wedge \boldsymbol{X}_w(s_0, w_0)$ is the zero vector, where $\boldsymbol{X}_s(s, w) = \partial\boldsymbol{X}(s, w)/\partial s$ and $\boldsymbol{X}_w(s, w) = \partial\boldsymbol{X}(s, w)/\partial w$. The normal vector of $\boldsymbol{X}(s, w)$ is

$$\boldsymbol{n}(s, w) = \boldsymbol{X}_s(s, w) \wedge \boldsymbol{X}_w(s, w) = (\alpha(s) - w\tau(s))\mathsf{T}(s) + w\mathsf{B}(s).$$

Then, (s_0, w_0) is a singularity of $\boldsymbol{X}(s, w)$ if and only if $\boldsymbol{n}(s_0, w_0) = \boldsymbol{0}$. Since $\boldsymbol{X}_w(s, w) = \mathsf{N}(s)$ and $\boldsymbol{X}_s(s, w) = (\alpha(s) - w\tau(s))\mathsf{T}(s) - w\mathsf{B}(s)$, $\boldsymbol{n}(s_0, w_0) = \boldsymbol{0}$ if and only if $\boldsymbol{X}_s(s, w) = \boldsymbol{0}$. Specifically, $\boldsymbol{X}(s, w)$ is singular at (s_0, w_0) if and only if $w_0 = 0$ and $\alpha(s_0) = 0$. □

In Euclidean 3-space, Whitney proved that a smooth map germ

$$\boldsymbol{X} : (\mathbb{R}^2, p) \to (\mathbb{R}^3, p)$$

is locally diffeomorphic to a cross cap at p if and only if there exists a local chart (s, w) such that the following conditions hold:

$$\boldsymbol{X}_w(p) \neq 0,\ \boldsymbol{X}_s(p) = 0 \text{ and } \det(\boldsymbol{X}_w(p), \boldsymbol{X}_{ws}(p), \boldsymbol{X}_{ss}(p)) \neq 0.$$

Here, a cross cap is a map germ $F : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0);\ (u, v) \mapsto (u, uv, v^2)$. Izumiya and Takeuchi proved that only cross-caps appear as singularities for non-developable ruled surfaces [22]. Then, we have the following proposition.

Proposition 4.2. *Let $\gamma : I \to \mathbb{R}^3_1$ be a generalized null Cartan curve with curvature (τ, α) and $\boldsymbol{X}(s, w)$ be its principal normal surface. Then, (s_0, w_0) is a cross cap of $\boldsymbol{X}(s, w)$ if and only if $w_0 = 0$, $\alpha(s_0) = 0$, and $\alpha'(s_0) \neq 0$.*

Proof. By calculations, we have

$$\begin{aligned}
\boldsymbol{X}_s(s, w) &= (\alpha(s) - w\tau(s))\mathsf{T}(s) - w\mathsf{B}(s),\\
\boldsymbol{X}_w(s, w) &= \mathsf{N}(s),\\
\boldsymbol{X}_{ws}(s, w) &= -\tau(s)\mathsf{T}(s) - \mathsf{B}(s),\\
\boldsymbol{X}_{ss}(s, w) &= (\alpha'(s) - w\tau'(s))\mathsf{T}(s) + (\alpha(s) - 2w\tau(s))\mathsf{N}(s).
\end{aligned}$$

Suppose that $(s_0, w_0) \in I \times \mathbb{R}$ is a singularity of $\boldsymbol{X}(s, w)$. Since $\langle \mathsf{N}(s), \mathsf{N}(s) \rangle = 1$, $\boldsymbol{X}_w(s_0, w_0) \neq \boldsymbol{0}$. Then, $\boldsymbol{X}_s(s_0, w_0) = \boldsymbol{0}$ if and only if $w_0 = 0$ and $\alpha(s_0) = 0$. From the preliminaries in Section 2, it follows that

$$\begin{aligned}\det(\boldsymbol{X}_w(s_0, w_0), \boldsymbol{X}_{ws}(s_0, w_0), \boldsymbol{X}_{ss}(s_0, w_0)) &= \langle \boldsymbol{X}_w(s_0, w_0) \wedge \boldsymbol{X}_{ws}(s_0, w_0), \boldsymbol{X}_{ss}(s_0, w_0) \rangle \\ &= \langle \mathsf{N}(s_0) \wedge (-\tau(s_0)\mathsf{T}(s_0) - \mathsf{B}(s_0)), \alpha'(s_0)\mathsf{T}(s_0) \rangle.\end{aligned} \tag{4.1}$$

From $\det(\mathsf{T}(s), \mathsf{N}(s), \mathsf{B}(s)) = \langle \mathsf{N}(s), \mathsf{N}(s) \rangle = 1$, we obtain that

$$\mathsf{T}(s) \wedge \mathsf{N}(s) = \mathsf{T}(s), \ \mathsf{N}(s) \wedge \mathsf{B}(s) = \mathsf{B}(s). \tag{4.2}$$

Substituting (4.2) into (4.1) gives

$$\det(\boldsymbol{X}_w(s_0, w_0), \boldsymbol{X}_{ws}(s_0, w_0), \boldsymbol{X}_{ss}(s_0, w_0)) = \langle \tau(s_0)\mathsf{T}(s_0) - \mathsf{B}(s_0), \alpha'(s_0)\mathsf{T}(s_0) \rangle = -\alpha'(s_0).$$

In summary, (s_0, w_0) is a cross cap of $\boldsymbol{X}(s, w)$ if and only if $w_0 = 0$, $\alpha(s_0) = 0$, and $\alpha'(s_0) \neq 0$. □

Based on the above discussion, we can get the following theorem.

Theorem 4.3. *Let $\gamma : I \to \mathbb{R}^3_1$ be a generalized null Cartan curve with curvature (τ, α) and $\boldsymbol{X}(s, w)$ be its principal normal surface. Then, we have the following:*

(1) *The singularities of $\boldsymbol{X}(s, w)$ coincide with the singularities of $\gamma(s)$.*
(2) *The germ of the principal normal surface $\boldsymbol{X}(I \times \mathbb{R})$ is locally diffeomorphic to a cross cap at $(s_0, 0)$ if and only if $\mathbb{A}(\gamma(s_0)) = (2, 3, 4)$.*

Proof. First, by Proposition 4.1 the singularities of $\boldsymbol{X}(s, w)$ coincide with the singularities of $\gamma(s)$. Next, by Corollary 3.4 and Proposition 4.2, $\boldsymbol{X}(s, w)$ is locally diffeomorphic to a cross cap at $(s_0, 0)$ if and only if $\mathbb{A}(\gamma(s_0)) = (2, 3, 4)$. □

Remark 4.4. In Euclidean 3-space, let $\bar{\gamma} : I \to \mathbb{R}^3$ be a Frenet-type framed base curve. That is, $\dot{\bar{\gamma}}(t) = \bar{\alpha}(t)\boldsymbol{T}(t)$. Take $\bar{F} = \{\boldsymbol{T}(t), \boldsymbol{N}(t), \boldsymbol{B}(t)\}$ as the Frenet-type frame along $\bar{\gamma}(t)$. Denote $\kappa(t)$ and $\bar{\tau}(t)$ as the curvature and torsion of $\bar{\gamma}(t)$, respectively. For more details see [11]. For both a Frenet-type framed base curve and a generalized null Cartan curve, the principal normal vector is given by $\boldsymbol{N}(t) = \dot{\boldsymbol{T}}(t)/\|\dot{\boldsymbol{T}}(t)\|$. Hence, their principal normal ruled surfaces are constructed identically, with the curve itself as the directrix and its principal normal vector field as the ruling. In [9] and [10, Corollary 1 and Theorem 2], the principal normal surface of a Frenet-type framed base curve $\bar{\gamma}(t)$ has a cross cap at $(t_0, 0)$ if and only if $\bar{\alpha}(t_0)\bar{\tau}(t_0) \neq 0$. This means that if $\bar{\tau}(t_0) = 0$, no matter what the finite type of $\bar{\gamma}(t)$ is at t_0, $(t_0, 0)$ cannot be a cross cap of the surface. Hence, the finite type of a Frenet-type framed base curve cannot determine the singularities of its principal normal surface. However, this holds for a generalized null Cartan curve.

Proposition 4.5. *Let $\gamma : I \to \mathbb{R}^3_1$ be a generalized null Cartan curve with curvature (τ, α) and $\boldsymbol{X}(s, w)$ be its principal normal surface. Denote $\rho(s) = \gamma(s) + (\alpha(s)\mathsf{N}(s))/(2\tau(s))$ as the striction curve of $\boldsymbol{X}(s, w)$ and take $s_0 \in I$. Assume $\tau(s_0) \neq 0$. Then, we have the following:*

(1) *If $\mathbb{A}(\gamma(s_0)) = (2, 3, 4)$, s_0 is a regular point of the striction curve $\rho(s)$.*
(2) *If $\mathbb{A}(\gamma(s_0)) = (3, 4, 5)$, s_0 is an ordinary cusp of the striction curve $\rho(s)$.*

Proof. A direct computation using the Frenet-type formula gives:

$$\begin{aligned}
\rho'(s) &= \frac{\alpha(s)}{2}\mathsf{T}(s) - \frac{\alpha(s)}{2\tau(s)}\mathsf{B}(s) + \frac{\alpha'(s)\tau(s) - \alpha(s)\tau'(s)}{2\tau^2(s)}\mathsf{N}(s),\\
\rho''(s) &= \frac{\alpha(s)\tau'(s)}{2\tau(s)}\mathsf{T}(s) - \frac{\alpha'(s)\tau(s) - \alpha(s)\tau'(s)}{\tau^2(s)}\mathsf{B}(s)\\
&\quad + \frac{\alpha''(s)\tau^2(s) - \alpha(s)\tau(s)\tau''(s) - 2\alpha'(s)\tau(s)\tau'(s) + 2\alpha(s)\tau'^2(s)}{2\tau^3(s)}\mathsf{N}(s),\\
\rho'''(s) &= \frac{-\alpha''(s)\tau^2(s) + 2\alpha(s)\tau(s)\tau''(s) + 3\alpha'(s)\tau(s)\tau'(s) - 3\alpha(s)\tau'^2(s)}{2\tau^2(s)}\mathsf{T}(s)\\
&\quad - \frac{3(\alpha''(s)\tau^2(s) - \alpha(s)\tau(s)\tau''(s) - 2\alpha'(s)\tau(s)\tau'(s) + 2\alpha(s)\tau'^2(s))}{2\tau^3(s)}\mathsf{B}(s)\\
&\quad + \Big(\frac{\alpha'''(s)\tau^3(s) - 3\alpha''(s)\tau^2(s)\tau'(s) - 3\alpha'(s)\tau^2(s)\tau''(s) + 6\alpha'(s)\tau(s)\tau'^2(s)}{2\tau^4(s)}\\
&\quad \frac{-2\alpha'(s)\tau^4(s) - \alpha(s)\tau^2(s)\tau'''(s) + 6\alpha(s)\tau(s)\tau'(s)\tau''(s) - 6\alpha(s)\tau'^3(s) + 3\alpha(s)\tau^3(s)\tau'(s)}{2\tau^4(s)}\Big)\mathsf{N}(s).
\end{aligned}$$

(1) From Corollary 3.4, we can obtain that $\mathbb{A}(\gamma(s_0)) = (2,3,4)$ if and only if $\alpha(s_0) = 0$ and $\alpha'(s_0) \neq 0$. Then, we can calculate that $\rho'(s_0) = (\alpha'(s_0)/2\tau(s_0))\mathsf{N}(s_0) \neq \mathbf{0}$. This means s_0 is a regular point of the striction curve $\rho(s)$.

(2) From Corollary 3.4, we can obtain that $\mathbb{A}(\gamma(s_0)) = (3,4,5)$ if and only if $\alpha(s_0) = \alpha'(s_0) = 0$ and $\alpha''(s_0) \neq 0$. Then, we can calculate that $\rho'(s_0) = \mathbf{0}$, $\rho''(s_0) = (\alpha''(s_0)/2\tau(s_0))\mathsf{N}(s_0)$, and

$$\rho'''(s_0) = -\frac{\alpha''(s_0)}{2}\mathsf{T}(s_0) - \frac{3\alpha''(s_0)}{2\tau(s_0)}\mathsf{B}(s_0) + \frac{\alpha'''(s_0)\tau(s_0) - 3\alpha''(s_0)\tau'(s_0)}{2\tau^2(s_0)}\mathsf{N}(s_0)$$

are linearly independent. By the definition of ordinary cusps, s_0 is an ordinary cusp of the striction curve $\rho(s)$. □

5. Examples

In the examples, we take $\tau(s) = 1/2$ for simplicity, but the same construction and conclusions hold for any $\tau(s) \neq 0$.

Example 1. Let $\gamma : I \to \mathbb{R}^3_1$ be $\gamma(t) = (r_1(t), r_2(t), r_3(t))$, where

$$r_1(t) = \frac{t^2}{2}, r_2(t) = \sin t - t\cos t, r_3(t) = \cos t + t\sin t.$$

Taking $\alpha(t) = t$, we can calculate that $s = t$ and $\tau(s) = 1/2$. So, $s_0 = 0$ is a singularity of $\gamma(s)$. By $\alpha'(s) = 1$, we have $\mathbb{A}(\gamma(0)) = (2,3,4)$. The principal normal surface $\boldsymbol{X}(s,w)$ of $\gamma(s)$ is

$$\boldsymbol{X}(s,w) = (r_1(s), r_2(s) - w\sin s, r_3(s) + w\cos s).$$

The striction curve $\rho(s)$ of $\boldsymbol{X}(s,w)$ is

$$\rho(s) = (r_1(s), r_2(s) - s\sin s, r_3(s) + s\cos s).$$

So, $(s_0, w_0) = (0, 0)$ is a cross cap of $\boldsymbol{X}(s, w)$ and $s_0 = 0$ is a regular point of $\rho(s)$. The following Figure 1 will show ρ, γ, $\boldsymbol{X}$, and their singularities.

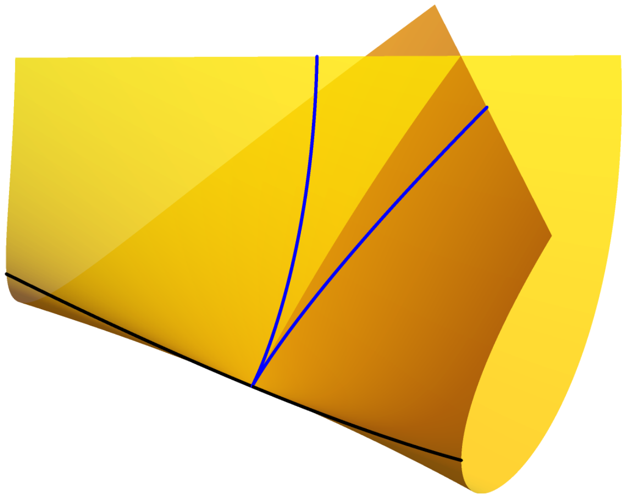

Figure 1. $\boldsymbol{X}$ (the yellow surface), ρ (the black curve), and γ (the blue curve).

Example 2. Let $\gamma : I \to \mathbb{R}^3_1$ be $\gamma(t) = (r_1(t), r_2(t), r_3(t))$, where

$$r_1(t) = \frac{t^3}{3}, r_2(t) = 2t \sin t - (t^2 - 2) \cos t, r_3(t) = (t^2 - 2) \sin t + 2t \cos t.$$

Taking $\alpha(t) = t^2$, we can calculate that $s = t$ and $\tau(s) = 1/2$. So, $s_0 = 0$ is a singularity of $\gamma(s)$. By $\alpha'(s) = 2s$ and $\alpha''(s) = 2$, we have $\mathbb{A}(\gamma(0)) = (3, 4, 5)$. Then, the principal normal surface $\boldsymbol{X}(s, w)$ of $\gamma(s)$ is

$$\boldsymbol{X}(s, w) = (r_1(s), r_2(s) - w \sin s, r_3(s) + w \cos s).$$

The striction curve $\rho(s)$ of $\boldsymbol{X}(s, w)$ is

$$\rho(s) = (r_1(s), r_2(s) - s^2 \sin s, r_3(s) + s^2 \cos s).$$

So, $(s_0, w_0) = (0, 0)$ is not a cross cap of $\boldsymbol{X}(s, w)$, and $s_0 = 0$ is an ordinary cusp of $\rho(s)$. The following Figure 2 will show ρ, γ, $\boldsymbol{X}$, and their singularities.

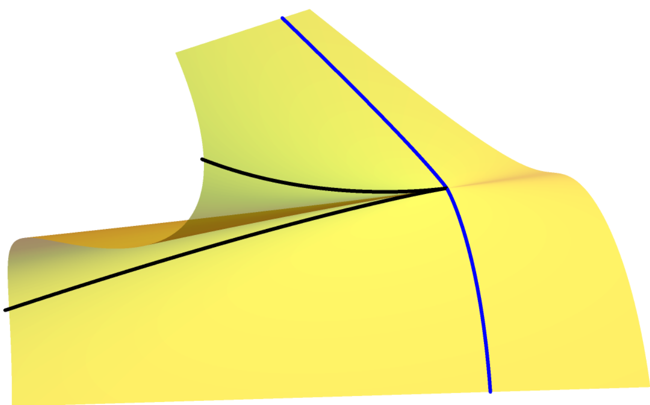

Figure 2. $\boldsymbol{X}$ (the yellow surface), ρ (the black curve), and γ (the blue curve).

6. Conclusions

In this paper, we identified the singularity types of principal normal ruled surfaces using the finite type of generalized null Cartan curves. Since the singularities of the curve and its principal normal surface coincide, the finite type of the curve alone determines the surface singularities, a phenomenon that does not occur in Euclidean space [9]. The differential geometric properties of the principal normal ruled surfaces of generalized null Cartan curves will be the subject of future work.

Use of AI tools declaration

The authors declare they have not used Artiffcial Intelligence (AI) tools in the creation of this article.

Author contributions

Boyuan Xu: Conceptualization, methodology, writing–original draft, review and editing; Donghe Pei: review, editing and funding acquisition. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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