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Research article

Certain subfamily of Bazilevič holomorphic functions involving the *m*-leaf polynomial

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Abstract: In this paper, we introduce and investigate a novel subfamily of Bazilevič functions, defined using the *m*-leaf polynomial and the principle of subordination. For this subfamily, we establish several properties, including subordination results, inclusion relationships, and sharp coefficient bounds. Furthermore, we derive a general Fekete–Szegö inequality, providing a solution to this classical problem for functions within the new subfamily.

Keywords: Holomorphic functions; subordination; Bazilevič function; *m*-Leaf function; Fekete–Szegö problem
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1. Introduction and preliminaries

The open unit disk $\mathbb{U}=\{\in\mathbb{C}:|z|<1\}$ serves as the domain for the space $\mathcal{H}(\mathbb{U})$ of all holomorphic functions $\chi:\mathbb{U}\to\mathbb{C}$; within $\mathcal{H}(\mathbb{U})$, the subfamily $\mathcal{A}$ consists of the functions normalized at the origin, taking the form

$$\chi(z)=z+\sum_{k=2}^{\infty}\varrho_k z^k,\quad z\in\mathbb{U}, \tag{1.1}$$

which is equivalent to requiring $\chi(0)=0$ and $\chi'(0)=1$. Geometric function theory focuses on the study of holomorphic functions based on the shapes they create when mapping the unit disk $\mathbb{U}$. At its heart lies the concept of subordination, a method for comparing two holomorphic functions by determining when one maps the disk into the image of the other. Formally, we say $\chi \in \mathcal{H}(\mathbb{U})$ is subordinate to $\rho \in \mathcal{H}(\mathbb{U})$, written $\chi(z) \prec \rho(z)$, $z \in \mathbb{U}$, if there exists a holomorphic function $\omega(z)$ in the disk $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $\chi(z) = \rho(\omega(z))$, $z \in \mathbb{U}$. When $\rho(z)$ is a univalent function in $\mathbb{U}$, this simply means $\chi(0) = \rho(0)$ and $\chi(\mathbb{U}) \subset \rho(\mathbb{U})$ (see [3, 14]). Subordination provides a way to order functions according to how dominant their image domains are. The more advanced framework of differential subordination extends this idea by incorporating derivatives. Pioneered by Miller and Mocanu, it examines conditions such as $\phi(\chi(z), z\chi'(z); z) \prec h(z)$, where ϕ is some holomorphic expression, and h is a chosen "dominant function". Solving such a relation often reveals that $\chi(z)$ itself is subordinate to an explicit function $q(z)$, called the best dominant. This technique allows researchers to convert differential inequalities into concrete geometric conclusions about $\chi(z)$, such as proving that it is starlike, convex, or close-to-convex. Subordination is intimately connected to classical geometric families. For example, starlike functions satisfy $z\chi'(z)/\chi(z) \prec (1+z)/(1-z)$, which ensures that the image domain is star-shaped with respect to the origin. Similarly, convex functions satisfy $1 + z\chi''(z)/\chi'(z) \prec (1+z)/(1-z)$, corresponding to convex image domains. These subordination characterizations are not merely reformulations; they provide a unified language that simplifies proofs of sharp growth theorems, coefficient bounds, and radius problems for these families (see [8, 13, 17]).

Bazilevič functions form a significant family of holomorphic functions in geometric function theory, introduced by Bazilevič in 1955 [2]. They generalize many classical families of univalent functions while maintaining strong geometric properties, offering a flexible framework that bridges starlike, convex, and close-to-convex mappings. A Bazilevič function of type (α, β) is defined by the integral representation:

$$\chi(z) = \left[(\alpha + i\beta) \int_0^z \rho(t)^\alpha \, h(t) t^{i\beta - 1} \, dt\right]^{1/\alpha + i\beta}, \quad z \in \mathbb{U},$$

where $\chi \in \mathcal{A}$, $\alpha > 0$, $\beta \geq 0$, $\rho \in \mathcal{A}$ is a starlike function and $h \in \mathcal{P}$, where $\mathcal{P}$ is the family of Carathéodory functions with positive real part ($\Re h(z) > 0$, $h(0) = 1$). Equivalently, they satisfy the differential equation:

$$\frac{z\chi'(z)}{\chi(z)} \left[\frac{\chi(z)}{\rho(z)}\right]^\alpha \left[\frac{\chi(z)}{z}\right]^{i\beta} = h(z), \quad z \in \mathbb{U},$$

which directly connects them to the method of differential subordination. Bazilevič's fundamental theorem established that every function of this form is univalent in the unit disk [2]. Later, Sheil-Small and Pommerenke showed that these functions are actually close-to-convex, providing stronger geometric guaranties [15, 21]. The special case $\beta = 0$ yields the family $\mathcal{B}(\alpha)$ of Bazilevič functions of type α satisfying:

$$\frac{z\chi'(z)}{\chi(z)} \left[\frac{\chi(z)}{\rho(z)}\right]^\alpha = h(z), \quad z \in \mathbb{U},$$

with specific choices of the associated parameters the family, $\mathcal{B}(\alpha)$ reduces to the well-known families of close-to-convex and starlike functions. Furthermore, if $\rho(z) = z$ in $\mathcal{B}(\alpha)$, we have the family $\mathcal{B}_1(\alpha)$ of functions satisfying:

$$\Re\left\{\chi'(z) \left[\frac{\chi(z)}{z}\right]^{\alpha - 1}\right\} > 0, \quad z \in \mathbb{U}.$$

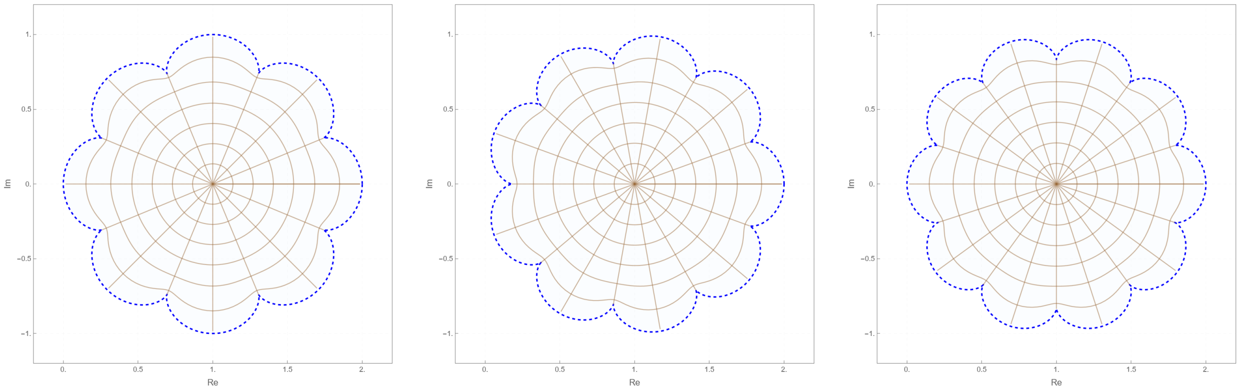
Bazilevič functions continue to be actively studied today, with recent research moving in several new directions. One important area looks at q-calculus creating q-Bazilevič functions that connect to quantum algebra and special functions [9]: Starlike, convex, or having other geometric properties [11]. Researchers are creating new subclasses of Bazilevič functions by linking them to shapes such as spirals, lemniscates, and exponentials. They have made major progress on coefficient problems—finding exact bounds for Hankel determinants and solving Fekete–Szegő inequalities with complex parameters (see [11, 16, 18, 26]). Modern studies also explore connections with operators, investigating how integral transforms, differential operators, and convolution operations affect Bazilevič functions. Additionally, researchers are extending the theory to harmonic functions and multivalent functions, showing Bazilevič's original ideas remain surprisingly flexible and applicable to contemporary mathematics (see [20, 22, 27]).

The m-leaf function, denoted by $Q_m(z)$, is a specific polynomial of degree "$m + 1$" in complex analysis defined as

$$Q_m(z) = 1 + \frac{m+1}{m+2} z + \frac{1}{m+2} z^{m+1}, \tag{1.2}$$

where $m \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ and $z \in \mathbb{U}$. It is worth observing that in the limit, this function generalizes the simple linear function: $\lim_{m\to\infty} Q_m(z) = 1 + z = Q_0(z)$. The function $Q_m(z)$ introduced by Sudharsanan et al. [24], maps the unit disk to a leaf-shaped region with m petals (see Figure 1). They proved that $Q_m(z)$ is a starlike (hence, univalent) function in $\mathbb{U}$ with respect to $z_0 = 1$, its image domain $Q_m(\mathbb{U})$ is symmetric about the real axis, and it satisfies $\Re\{Q_m(z)\} > 0$ and $|Q_m(z)| > 2$ in $z \in \mathbb{U}$. Moreover, Q_m is a convex function in the smaller disk $\mathbb{U}_c = \left\{z \in \mathbb{U} : |z| \le r_c = (1+m)^{-1/m}\right\}$.

Recent research has explored the application of the m-leaf function in defining subfamilies of holomorphic functions. Because of its simple polynomial form and its positive real part, Q_m has been adopted in recent works as a convenient dominant function in subordination conditions that define new families of holomorphic functions. For example, Gandhi [6] studied a subfamily of bounded-turning functions subordinate to the three-leaf function $\partial Q_3(z) = 1 + \frac{4}{5}z + \frac{1}{5}z^4$, while Sunthrayuth and Alshehry [1, 25] investigated families defined by subordination to the four-leaf function $Q_4(z) = 1 + \frac{5}{6}z + \frac{1}{6}z^5$.



(a) $m = 8$. **(b)** $m = 9$. **(c)** $m = 10$.

Figure 1. Illustration of the m-leaf-shaped region $Q_m(\mathbb{U})$, $m \in \{8, 9, 10\}$.

Using the standard notion of differential subordination, we introduce a new subfamily $\mathcal{B}_m(\lambda, \alpha)$ of

Bazilevič functions linked to the m-leaf function Q_m as follows:

Definition 1. A function $\chi \in \mathcal{A}$ belongs to the subfamily $\mathcal{B}_m(\lambda, \alpha)$ when it satisfies the following subordination condition:

$$(1-\lambda)\left[\frac{\chi(z)}{z}\right]^{\alpha} + \lambda\chi'(z)\left[\frac{\chi(z)}{z}\right]^{\alpha-1} \prec Q_m(z) \tag{1.3}$$

with $\left.\frac{\chi(z)}{z}\right|_{z=0} = 1$, all powers are taken as principal values, and throughout the paper, unless otherwise mentioned, the parameters are restricted to $0 \leq \lambda \leq 1$, $\alpha \in \mathbb{R}$; $m \in \mathbb{N}_0$, and $z \in \mathbb{U}$.

Example 1. For $m = 2$, $\lambda = \frac{1}{2}$, and $\alpha = 1$, consider the function $\chi(z) = z + \frac{1}{4}z^2 + \frac{1}{20}z^4$. Substituting into the left-hand side of the subordination condition (1.3) yields $\frac{1}{2}\frac{\chi(z)}{z} + \frac{1}{2}\chi'(z) = z + \frac{3}{8}z + \frac{1}{8}z^3$. Observe that this expression can be rewritten as $\frac{1+Q_2(z)}{2}$, where $Q_2(z) = 1 + \frac{3}{4}z + \frac{1}{4}z^3$ is the 2-leaf polynomial. Since $Q_2(z)$ is convex univalent in $\mathbb{U}$, its image is a convex region. The point $\frac{1+Q_2(z)}{2}$ lies on the line segment between 1 and $Q_2(z)$, and because convexity ensures this entire segment lies inside $Q_2(\mathbb{U})$, we have $\frac{1}{2}\frac{\chi(z)}{z} + \frac{1}{2}\chi'(z) \prec Q_2(z)$. Moreover, $\chi(0) = 0$ and $\chi'(0) = 1$, so $\chi(z)$ belongs to the family $\mathcal{B}_2\left(\frac{1}{2}, 1\right)$. Thus, $\chi(z) = z + \frac{1}{4}z^2 + \frac{1}{20}z^4$ provides an explicit example for the family $\mathcal{B}_m(\lambda, \alpha)$.

Special cases:

(i) $\lim_{m\to\infty}\mathcal{B}_m(\lambda, \alpha) = \mathcal{B}(\lambda, \alpha) = \left\{\chi \in \mathcal{A} : (1-\lambda)\left[\frac{\chi(z)}{z}\right]^{\alpha} + \lambda\chi'(z)\left[\frac{\chi(z)}{z}\right]^{\alpha-1} \prec 1 + z\right\} = \left\{\chi \in \mathcal{A} : \left|(1-\lambda)\left[\frac{\chi(z)}{z}\right]^{\alpha} + \lambda\chi'(z)\left[\frac{\chi(z)}{z}\right]^{\alpha-1} - 1\right| < 1\right\}$;

(ii) $\mathcal{B}_m(\lambda, 1) = \mathcal{B}_m(\lambda) = \left\{\chi \in \mathcal{A} : (1-\lambda)\frac{\chi(z)}{z} + \lambda\chi'(z) \prec Q_m(z)\right\}$ and $\lim_{m\to\infty}\mathcal{B}_m(\lambda) = \mathcal{B}(\lambda) = \left\{\chi \in \mathcal{A} : (1-\lambda)\frac{\chi(z)}{z} + \lambda\chi'(z) \prec 1 + z\right\} = \left\{\chi \in \mathcal{A} : \left|(1-\lambda)\frac{\chi(z)}{z} + \lambda\chi'(z) - 1\right| < 1\right\}$;

(iii) $\mathcal{B}_m(1, \alpha) = \mathcal{B}_m(\alpha) = \left\{\chi \in \mathcal{A} : \chi'(z)\left(\frac{\chi(z)}{z}\right)^{\alpha-1} \prec Q_m(z)\right\}$ and $\lim_{m\to\infty}\mathcal{B}_m(\alpha) = \mathcal{B}(\alpha) = \left\{\chi \in \mathcal{A} : \chi'(z)\left(\frac{\chi(z)}{z}\right)^{\alpha-1} \prec 1 + z\right\} = \left\{\chi \in \mathcal{A} : \left|\chi'(z)\left(\frac{\chi(z)}{z}\right)^{\alpha-1} - 1\right| < 1\right\}$;

(iv) $\mathcal{B}_m(0, 1) = C_m = \left\{\chi \in \mathcal{A} : \frac{\chi(z)}{z} \prec Q_m(z)\right\}$ and $\lim_{m\to\infty} C_m = C = \left\{\chi \in \mathcal{A} : \frac{\chi(z)}{z} \prec 1 + z\right\} = \left\{\chi \in \mathcal{A} : \left|\frac{\chi(z)}{z} - 1\right| < 1\right\}$;

(v) $\mathcal{B}_m(1, 1) = \mathcal{K}_m = \{\chi \in \mathcal{A} : \chi'(z) \prec Q_m(z)\}$, $\mathcal{K}_4 = \left\{\chi \in \mathcal{A} : \chi'(z) \prec Q_4(z) = 1 + \frac{5}{6}z + \frac{1}{6}z^5\right\}$ (see [25]) and $\lim_{m\to\infty}\mathcal{K}_m = \mathcal{K} = \{\chi \in \mathcal{A} : \chi'(z) \prec 1 + z\} = \{\chi \in \mathcal{A} : |\chi'(z) - 1| < 1\}$;

(vi) $\mathcal{B}_m(1, 0) = \mathcal{S}^*_m = \left\{\chi \in \mathcal{A} : \frac{z\chi'(z)}{\chi(z)} \prec Q_m(z)\right\}$, $\mathcal{S}^*_3 = \left\{\chi \in \mathcal{A} : \frac{z\chi'(z)}{\chi(z)} \prec Q_3(z) = 1 + \frac{4}{5}z + \frac{1}{5}z^4\right\}$ (see [6]) $\mathcal{S}^*_4 = \left\{\chi \in \mathcal{A} : \frac{z\chi'(z)}{\chi(z)} \prec Q_4(z) = 1 + \frac{5}{6}z + \frac{1}{6}z^5\right\}$ (see [1]) and $\lim_{m\to\infty}\mathcal{S}^*_m = \mathcal{S}^* = \left\{\chi \in \mathcal{A} : \frac{z\chi'(z)}{\chi(z)} \prec 1 + z\right\} = \left\{\chi \in \mathcal{A} : \left|\frac{z\chi'(z)}{\chi(z)} - 1\right| < 1\right\}$.

The novelty of our work lies in synthesizing the classical Bazilevič structure with the modern m-leaf function $Q_m(z)$ to define the new subfamily $\mathcal{B}_m(\lambda, \alpha)$, thus embedding the symmetric multi-leaf geometry of $Q_m(\mathbb{U})$ into the Bazilevič framework. This hybrid approach allows us to investigate previously unstudied geometric properties for such a combined family, including subordination results, inclusion relationships, sharp coefficient estimates, and Fekete–Szegö inequalities, extending the literature on Bazilevič functions which often employs linear operators by incorporating a concrete polynomial dominant with distinctive visual and features.

In order to establish our main results, we require the next lemmas.

Lemma 1. [7, Lemma 2, p. 192] Let $h(z)$ be convex (univalent) in $\mathbb{U}$ with $h(0) = 1$. Also, suppose that $\rho(z)$ defined by

$$\rho(z) = 1 + c_1 z + c_2 z^2 + ... \tag{1.4}$$

is holomorphic in $\mathbb{U}$. If

$$\rho(z) + \frac{z\rho'(z)}{\gamma} \prec h(z) \quad (\Re(\gamma) \geq 0; \gamma \neq 0; z \in \mathbb{U}), \tag{1.5}$$

then

$$\rho(z) \prec q(z) = \gamma z^{-\gamma} \int_0^z s^{\gamma-1} h(s)\ ds \prec h(z),$$

and $q(z)$ is the best dominant.

Lemma 2. [12, Lemma 2.2, p. 16] Let $\rho_1, \rho_2 \in \mathcal{H}(\mathbb{U})$ such that $\rho_1, \rho_2 \prec G$, where G is a convex function in $\mathbb{U}$. Then,

$$(1-\lambda)\rho_1 + \lambda\rho_2 \prec G \quad (0 \leq \lambda \leq 1).$$

Lemma 3. [13, Lemma 1, p. 162] Let $\rho \in \mathcal{P}$ be defined by (1.4), then

(i)

$$\left|c_2 - \nu c_1^2\right| \leq 2 \max\{1, |2\nu - 1|\} \quad for\ all\ \nu \in \mathbb{C}. \tag{1.6}$$

(ii)

$$\left|c_2 - \nu c_1^2\right| \leq \begin{cases} -4\nu + 2 & if \quad \nu \leq 0, \\ 2 & if \quad 0 \leq \nu \leq 1, \\ 4\nu - 2 & if \quad \nu \geq 1, \end{cases} \tag{1.7}$$

and the upper bound (1.7) can be improved as follows when $0 < \nu < 1$:

$$\left|c_2 - \nu c_1^2\right| + \nu |c_1|^2 \leq 2 \quad \left(0 \leq \nu \leq \frac{1}{2}\right),$$

and

$$\left|c_2 - \nu c_1^2\right| + (1-\nu)|c_1|^2 \leq 2 \quad \left(\frac{1}{2} \leq \nu \leq 1\right).$$

2. Geometric properties

Our first theorem deals with the subordination result of the functions that belong to the family $\mathcal{B}_m(\lambda, \alpha)$.

Theorem 1. If $\chi(z) \in \mathcal{B}_m(\lambda, \alpha)$ with $\frac{\lambda}{\alpha} > 0$ and $z \in \mathbb{U}_c$, then

$$\left[\frac{\chi(z)}{z}\right]^{\alpha} \prec q(z) = 1 + \frac{(m+1)\alpha}{(m+2)(\alpha+\lambda)} z + \frac{\alpha}{(m+2)[\alpha+(m+1)\lambda]} z^{m+1} \prec Q_m(z), \tag{2.1}$$

where $q(z)$ is the best dominant.

Proof. Suppose that

$$\rho(z) = \left[\frac{\chi(z)}{z}\right]^{\alpha}, \tag{2.2}$$

then $\rho(z)$ is of the form (1.4), holomorphic in $\mathbb{U}_c$, and $\rho(0) = 1$. Differentiating both sides of (2.2) gives

$$(1-\lambda)\left[\frac{\chi(z)}{z}\right]^{\alpha} + \lambda\chi'(z)\left[\frac{\chi(z)}{z}\right]^{\alpha-1} = \rho(z) + \frac{\lambda z\rho'(z)}{\alpha}. \tag{2.3}$$

Since $\chi \in \mathcal{B}_m(\lambda, \alpha)$, we have

$$\rho(z) + \frac{\lambda z\rho'(z)}{\alpha} \prec Q_m(z).$$

Now, by applying Lemma 1 for $\gamma = \frac{\alpha}{\lambda}$ and $z \in \mathbb{U}_c$, we derive that

$$\begin{aligned}\left[\frac{\chi(z)}{z}\right]^{\alpha} &\prec q(z) = \frac{\alpha}{\lambda} z^{-\frac{\alpha}{\lambda}} \int_0^z t^{\frac{\alpha}{\lambda}-1} Q_m(t)\, dt \\ &= 1 + \frac{(m+1)\alpha}{(m+2)(\alpha+\lambda)} z + \frac{\alpha}{(m+2)[\alpha+(m+1)\lambda]} z^{m+1}.\end{aligned} \tag{2.4}$$

This finishes the proof of Theorem 1. □

Letting $m \to \infty$ (or $m = 0$) in Theorem 1, we get the next corollary.

Corollary 1. If $\chi(z) \in \mathcal{B}(\lambda, \alpha)$ with $\frac{\lambda}{\alpha} > 0$, then

$$\left[\frac{\chi(z)}{z}\right]^{\alpha} \prec q_1(z) = 1 + \frac{\alpha}{\alpha+\lambda} z \prec 1 + z,$$

where $q_1(z)$ is the best dominant.

Putting $\alpha = 1$ in Theorem 1, we get the following corollary.

Corollary 2. If $\chi(z) \in \mathcal{B}_m(\lambda)$ with $\lambda > 0$ and $z \in \mathbb{U}_c$, then

$$\frac{\chi(z)}{z} \prec q_2(z) = 1 + \frac{(m+1)}{(m+2)(1+\lambda)} z + \frac{1}{(m+2)[1+(m+1)\lambda]} z^{m+1} \prec Q_m(z),$$

where $q_2(z)$ is the best dominant.

Letting $m \to \infty$ (or $m = 0$) in Corollary 2, we get the following example.

Example 2. If

$$(1-\lambda)\frac{\chi(z)}{z} + \lambda\chi'(z) \prec 1 + z \quad (\lambda > 0),$$

then

$$\frac{\chi(z)}{z} \prec 1 + \frac{1}{1+\lambda} z \prec 1 + z,$$

where the function $1 + \frac{1}{1+\lambda} z$ is the best dominant.

For $\chi \in \mathcal{A}$ as in (1.1) and $\mu > -1$, the generalized integral operator $L_\mu : \mathcal{A} \to \mathcal{A}$ is defined by (see [4])

$$L_\mu\chi(z) = \frac{\mu+1}{z^\mu} \int_0^z t^{\mu-1}\chi(t)\, dt \quad (\mu > -1). \tag{2.5}$$

It is easy to verify that, for all $\chi \in \mathcal{A}$, we have

$$z\left(L_\mu\chi(z)\right)' = (\mu+1)\chi(z) - \mu L_\mu\chi(z). \tag{2.6}$$

Theorem 2. If $\chi \in \mathcal{A}$ satisfies the next subordination condition

$$(1-\lambda)\left[\frac{L_\mu\chi(z)}{z}\right]^\alpha + \lambda\frac{\chi(z)}{z}\left[\frac{L_\mu\chi(z)}{z}\right]^{\alpha-1} \prec Q_m(z), \tag{2.7}$$

with $\frac{\lambda}{\alpha} > 0$, $z \in \mathbb{U}_c$ and L_μ is defined by (2.5), then

$$\left[\frac{L_\mu\chi(z)}{z}\right]^\alpha \prec k(z) = 1 + \frac{(m+1)(\mu+1)\alpha}{(m+2)\left[(\mu+1)\alpha+\lambda\right]}z + \frac{(\mu+1)\alpha}{(m+2)[(\mu+1)\alpha+(m+1)\lambda]}z^{m+1} \prec Q_m(z),$$

where $k(z)$ is the best dominant.

Proof. Let

$$\rho(z) = \left[\frac{L_\mu\chi(z)}{z}\right]^\alpha, \tag{2.8}$$

then ρ is a holomorphic function in $\mathbb{U}_c$. Differentiating (2.8) with respect to z and substituting (2.7) yields

$$(1-\lambda)\left[\frac{L_\mu\chi(z)}{z}\right]^\alpha + \lambda\frac{\chi(z)}{z}\left[\frac{L_\mu\chi(z)}{z}\right]^{\alpha-1} = \rho(z) + \frac{\lambda z\rho'(z)}{(\mu+1)\alpha} \prec Q_m(z).$$

The rest of the theorem follows by the same method used to prove Theorem 1. □

Letting $m \to \infty$ (or $m = 0$) in Theorem 2, we get the following corollary.

Corollary 3. If $\chi \in \mathcal{A}$ satisfies the next subordination condition

$$(1-\lambda)\left[\frac{L_\mu\chi(z)}{z}\right]^\alpha + \lambda\frac{\chi(z)}{z}\left[\frac{L_\mu\chi(z)}{z}\right]^{\alpha-1} \prec 1+z,$$

with $\frac{\lambda}{\alpha} > 0$, and L_μ is the integral operator defined by (2.5), then

$$\left[\frac{L_\mu\chi(z)}{z}\right]^\alpha \prec k_1(z) = 1 + \frac{(\mu+1)\alpha}{(\mu+1)\alpha+\lambda}z \prec 1+z,$$

where the function $k_1(z)$ is the best dominant.

Putting $\alpha = 1$ in Theorem 2, we get the following corollary.

Corollary 4. If the function $\chi \in \mathcal{A}$ satisfies the next subordination condition

$$(1-\lambda)\frac{L_\mu\chi(z)}{z} + \lambda\frac{\chi(z)}{z} \prec Q_m(z),$$

with $\lambda > 0$, $z \in \mathbb{U}_c$, and L_μ is defined by (2.5), then

$$\frac{L_\mu\chi(z)}{z} \prec k_2(z) = 1 + \frac{(m+1)(\mu+1)}{(m+2)(\mu+1+\lambda)}z + \frac{\mu+1}{(m+2)[\mu+1+(m+1)\lambda]}z^{m+1} \prec Q_m(z),$$

where the function $k_2(z)$ is the best dominant.

Letting $m \to \infty$ (or $m = 0$) and $\lambda = 1$ in Corollary 4, we get the following example.

Example 3. If $\chi \in \mathcal{A}$ satisfies the next condition

$$\frac{\chi(z)}{z} \prec 1 + z,$$

and L_μ is defined by (2.5), then

$$\frac{L_\mu \chi(z)}{z} \prec 1 + \tfrac{\mu+1}{\mu+2} z \prec 1 + z,$$

where the function $1 + \frac{\mu+1}{\mu+2} z$ is the best dominant.

Theorem 3. If $0 \leq \lambda_1 \leq \lambda_2$ and $z \in \mathbb{U}_c$, then $\mathcal{B}_m(\lambda_2, \alpha) \subset \mathcal{B}_m(\lambda_1, \alpha)$.

Proof. If we consider an arbitrary function $\chi \in \mathcal{B}_m(\lambda_2, \alpha)$, then

$$\Phi_2(z) = (1 - \lambda_2)\left[\frac{\chi(z)}{z}\right]^\alpha + \lambda_2 \chi'(z)\left[\frac{\chi(z)}{z}\right]^{\alpha-1} \prec Q_m(z).$$

Thus, the assertion of Theorem 3 holds for $\lambda_2 = \lambda_1 \geq 0$. According to Theorem 1, we have

$$\Phi_1(z) = \left[\frac{\chi(z)}{z}\right]^\alpha \prec Q_m(z).$$

A simple computation shows that

$$\begin{aligned}(1 - \lambda_1)\left[\frac{\chi(z)}{z}\right]^\alpha + \lambda_1 \chi'(z)\left[\frac{\chi(z)}{z}\right]^{\alpha-1} &= \left(1 - \frac{\lambda_1}{\lambda_2}\right)\left[\frac{\chi(z)}{z}\right]^\alpha \\ &+ \frac{\lambda_1}{\lambda_2}\left\{(1 - \lambda_2)\left[\frac{\chi(z)}{z}\right]^\alpha + \lambda_2 \chi'(z)\left[\frac{\chi(z)}{z}\right]^{\alpha-1}\right\} \\ &= \left(1 - \frac{\lambda_1}{\lambda_2}\right)\Phi_1(z) + \frac{\lambda_1}{\lambda_2}\Phi_2(z). \end{aligned} \tag{2.9}$$

Furthermore, since $0 \leq \frac{\lambda_1}{\lambda_2} < 1$, $Q_m(z)$ is a holomorphic (convex) in the disk $\mathbb{U}_c$. Applying Lemma 2 to (2.9), we find that

$$(1 - \lambda_1)\left[\frac{\chi(z)}{z}\right]^\alpha + \lambda_1 \chi'(z)\left[\frac{\chi(z)}{z}\right]^{\alpha-1} \prec Q_m(z),$$

which implies that $\chi \in \mathcal{B}_m(\lambda_1, \alpha)$. □

Letting $m \to \infty$ (or $m = 0$) in Theorem 3, we get the following corollary.

Corollary 5. If $0 \leq \lambda_1 \leq \lambda_2$, then $\mathcal{B}(\lambda_2, \alpha) \subset \mathcal{B}(\lambda_1, \alpha)$.

Putting $\alpha = 1$ in Theorem 3, we get the following corollary.

Corollary 6. If $0 \leq \lambda_1 \leq \lambda_2$ and $z \in \mathbb{U}_c$, then $\mathcal{B}(\lambda_2) \subset \mathcal{B}(\lambda_1)$.

3. Fekete-Szegö problem

The Fekete-Szegö inequality is a cornerstone of geometric function theory, providing a bound for a specific combination of the early coefficients of univalent functions. It originated from the classical work of Fekete and Szegö [5]. This section examines the Fekete–Szegö inequalities for the subfamily $\mathcal{B}_m(\lambda,\alpha)$. The Fekete–Szegö problem has been extensively studied for various subfamilies of holomorphic functions (see, e.g., [10, 19, 23]).

Theorem 4. If χ given by (1.1) belongs to $\mathcal{B}_m(\lambda,\alpha)$ with $m>1$, $\alpha\neq-\lambda$, and $\alpha\neq-2\lambda$, then

$$|\varrho_2|\leq\frac{(m+1)}{(m+2)\,|\alpha+\lambda|},$$

$$|\varrho_3|\leq\frac{(m+1)}{(m+2)\,|\alpha+2\lambda|}\max\left\{1;\left|\tfrac{(m+1)(\alpha+2\lambda)(\alpha-1)}{2(m+2)(\alpha+\lambda)^2}\right|\right\},$$

and

$$\left|\varrho_3-\mu\varrho_2^2\right|\leq\frac{(m+1)}{(m+2)\,|\alpha+2\lambda|}\max\left\{1;\left|\tfrac{(m+1)(\alpha+2\lambda)(\alpha+2\mu-1)}{2(m+2)(\alpha+\lambda)^2}\right|\right\}.$$

Proof. If $\chi\in\mathcal{B}_m(\lambda,\alpha)$, then there is a holomorphic function $\omega(z)$ in $\mathbb{U}$ with $\omega(0)=0$ and $|\omega(z)|<1$ such that

$$(1-\lambda)\left[\frac{\chi(z)}{z}\right]^{\alpha}+\lambda\chi'(z)\left[\frac{\chi(z)}{z}\right]^{\alpha-1}=1+\frac{m+1}{m+2}\omega(z)+\frac{1}{m+2}\left[\omega(z)\right]^{m+1}. \tag{3.1}$$

Define the function $g(z)$ by

$$g(z)=\frac{1+\omega(z)}{1-\omega(z)}=1+c_1z+c_2z^2+\ldots. \tag{3.2}$$

We see that $g\in\mathcal{P}$ with $g(0)=1$. Therefore,

$$1+\frac{m+1}{m+2}\omega(z)+\frac{1}{m+2}\left[\omega(z)\right]^{m+1}=1+\tfrac{(m+1)c_1}{2(m+2)}z+\tfrac{m+1}{2(m+2)}\left(c_2-\frac{c_1^2}{2}\right)z^2+\ldots. \tag{3.3}$$

Now, by substituting (3.3) in (3.1), we have

$$(1-\lambda)\left[\frac{\chi(z)}{z}\right]^{\alpha}+\lambda\chi'(z)\left[\frac{\chi(z)}{z}\right]^{\alpha-1}=1+\frac{(m+1)\,c_1}{2\,(m+2)}z+\frac{(m+1)}{2\,(m+2)}\left(c_2-\frac{c_1^2}{2}\right)z^2+\ldots.$$

Equating the coefficients of z and z^2, we obtain

$$\varrho_2=\frac{(m+1)}{2\,(m+2)\,(\alpha+\lambda)}c_1.$$

$$\varrho_3=\frac{(m+1)}{2\,(m+2)\,(\alpha+2\lambda)}\left\{c_2-\frac{1}{2}\left[1+\frac{(m+1)\,(\alpha-1)\,(\alpha+2\lambda)}{2\,(m+2)\,(\alpha+\lambda)^2}\right]c_1^2\right\}.$$

Therefore,

$$\varrho_3-\mu\varrho_2^2=\frac{(m+1)}{2\,(m+2)\,(\alpha+2\lambda)}\left\{c_2-\nu c_1^2\right\}, \tag{3.4}$$

where

$$\nu=\frac{1}{2}\left[1+\frac{(m+1)\,(\alpha+2\lambda)\,(\alpha+2\mu-1)}{2\,(m+2)\,(\alpha+\lambda)^2}\right]. \tag{3.5}$$

Applying Lemma 3(i) yields the result, completing the proof of Theorem 4. □

Letting $m \to \infty$ (or $m = 0$) in Theorem 4, we obtain the next corollary.

Corollary 7. If $\chi \in \mathcal{BN}(\lambda, \alpha)$ given by (1.1) with $\alpha \neq -\lambda$ and $\alpha \neq -2\lambda$, then

$$|\varrho_2| \leq \frac{1}{|\alpha + \lambda|},$$

$$|\varrho_3| \leq \frac{1}{|\alpha + 2\lambda|} \max\left\{1; \left|\tfrac{(\alpha+2\lambda)(\alpha-1)}{2(\alpha+\lambda)^2}\right|\right\},$$

and

$$\left|\varrho_3 - \mu\varrho_2^2\right| \leq \frac{1}{|\alpha + 2\lambda|} \max\left\{1; \left|\tfrac{(\alpha+2\lambda)(\alpha+2\mu-1)}{2(\alpha+\lambda)^2}\right|\right\}.$$

Putting $\alpha = 1$ in Theorem 4, we obtain the next result.

Corollary 8. If $\chi \in \mathcal{B}_m(\lambda)$ is given by (1.1) with $m > 1$, $\lambda \neq -1$, and $\lambda \neq -\frac{1}{2}$, then

$$|\varrho_2| \leq \frac{(m+1)}{(m+2)\,|1+\lambda|},$$

$$|\varrho_3| \leq \frac{(m+1)}{(m+2)\,|1+2\lambda|},$$

and

$$\left|\varrho_3 - \mu\varrho_2^2\right| \leq \frac{(m+1)}{(m+2)\,|1+2\lambda|} \max\left\{1; \left|\tfrac{(m+1)(1+2\lambda)\mu}{(m+2)(1+\lambda)^2}\right|\right\}.$$

Putting $m = 4$ and $\lambda = 1$ in Corollary 8, we get the following result that was obtained by [25, Theorems 4 and 5].

Corollary 9. [25] If $\chi \in \mathcal{K}_4$ given by (1.1), then

$$|\varrho_2| \leq \frac{5}{12},$$

$$|\varrho_3| \leq \frac{5}{18},$$

and

$$\left|\varrho_3 - \mu\varrho_2^2\right| \leq \max\left\{\frac{5}{18}; \frac{25}{144}\,|\mu|\right\}.$$

Letting $m \to \infty$ and $\lambda = 1$ in Corollary 8, we get the following example.

Example 4. If χ given by (1.1) belongs to $\mathcal{K}$, then

$$|\varrho_2| \leq \frac{1}{2},$$

$$|\varrho_3| \leq \frac{1}{3},$$

and

$$\left|\varrho_3 - \mu\varrho_2^2\right| \leq \frac{1}{3} \max\left\{1; \tfrac{3}{4}\,|\mu|\right\}.$$

Putting $m = 4$, $\lambda = 1$, and $\alpha = 1$ in Theorem 4, we get the following result obtained by [1, Theorems 6 and 7].

Corollary 10. [1] If χ given by (1.1) belongs to $\mathcal{S}_4^*$, then

$$|\varrho_2| \leq \frac{5}{6},$$

$$|\varrho_3| \leq \max\left\{\frac{5}{12}; \frac{25}{72}\right\},$$

and

$$\left|\varrho_3 - \mu\varrho_2^2\right| \leq \max\left\{\frac{5}{12}; \frac{25}{72}\,|2\mu - 1|\right\}.$$

Applying Lemma 3(ii) to (3.4) and (3.5), we obtain the following theorem.

Theorem 5. Let

$$\sigma_1 = \frac{1-\alpha}{2} - \frac{(m+2)(\alpha+\lambda)^2}{(m+1)(\alpha+2\lambda)}, \sigma_2 = \frac{1-\alpha}{2}, \sigma_3 = \frac{1-\alpha}{2} + \frac{(m+2)(\alpha+\lambda)^2}{(m+1)(\alpha+2\lambda)}.$$

If χ given by (1.1) belongs to $\mathcal{B}_m(\lambda, \alpha)$ with $\alpha > 0$, then

$$\left|\varrho_3 - \mu\varrho_2^2\right| \leq \begin{cases} -\frac{(m+1)^2(\alpha+2\mu-1)}{2(m+2)^2(\alpha+\lambda)^2}, & (\mu \leq \sigma_1), \\ \frac{(m+1)}{(m+2)(\alpha+2\lambda)}, & (\sigma_1 \leq \mu \leq \sigma_3), \\ \frac{(m+1)^2(\alpha+2\mu-1)}{2(m+2)^2(\alpha+\lambda)^2}, & (\mu \geq \sigma_3). \end{cases}$$

Further, if $\sigma_1 \leq \mu \leq \sigma_2$, then

$$\left|\varrho_3 - \mu\varrho_2^2\right| + \left[\frac{(m+2)(\alpha+\lambda)^2}{(m+1)(\alpha+2\lambda)} + \frac{\alpha+2\mu-1}{2}\right]|\varrho_2|^2 \leq \frac{(m+1)}{(m+2)(\alpha+2\lambda)}.$$

If $\sigma_2 \leq \mu \leq \sigma_3$, then

$$\left|\varrho_3 - \mu\varrho_2^2\right| + \left[\frac{(m+2)(\alpha+\lambda)^2}{(m+1)(\alpha+2\lambda)} - \frac{\alpha+2\mu-1}{2}\right]|\varrho_2|^2 \leq \frac{(m+1)}{(m+2)(\alpha+2\lambda)}.$$

Letting $m \to \infty$ in Theorem 5, we obtain the next result.

Corollary 11. Let

$$\sigma_4 = \frac{1-\alpha}{2} - \frac{(\alpha+\lambda)^2}{\alpha+2\lambda}, \sigma_5 = \frac{1-\alpha}{2}, \sigma_6 = \frac{1-\alpha}{2} + \frac{(\alpha+\lambda)^2}{\alpha+2\lambda}.$$

If χ given by (1.1) belongs to $\mathcal{B}(\lambda, \alpha)$ with $\alpha > 0$, then

$$\left|\varrho_3 - \mu\varrho_2^2\right| \leq \begin{cases} -\frac{\alpha+2\mu-1}{2(\alpha+\lambda)^2}, & (\mu \leq \sigma_4), \\ \frac{1}{\alpha+2\lambda}, & (\sigma_4 \leq \mu \leq \sigma_6), \\ \frac{\alpha+2\mu-1}{2(\alpha+\lambda)^2}, & (\mu \geq \sigma_6). \end{cases}$$

Further, if $\sigma_4 \leq \mu \leq \sigma_5$, then

$$\left|\varrho_3 - \mu\varrho_2^2\right| + \left[\frac{(\alpha+\lambda)^2}{\alpha+2\lambda} + \frac{\alpha+2\mu-1}{2}\right]|\varrho_2|^2 \leq \frac{1}{\alpha+2\lambda}.$$

If $\sigma_5 \leq \mu \leq \sigma_6$, then

$$\left|\varrho_3-\mu\varrho_2^2\right|+\left[\frac{(\alpha+\lambda)^2}{\alpha+2\lambda}-\frac{\alpha+2\mu-1}{2}\right]|\varrho_2|^2\leq\frac{1}{\alpha+2\lambda}.$$

Putting $\alpha = 1$ in Theorem 5, we obtain the next result.

Corollary 12. If χ given by (1.1) belongs to $\mathcal{B}_m(\lambda)$, then

$$\left|\varrho_3-\mu\varrho_2^2\right|\leq\begin{cases}-\frac{(m+1)^2\mu}{(m+2)^2(1+\lambda)^2}, & \left(\mu\leq-\frac{(m+2)(1+\lambda)^2}{(m+1)(1+2\lambda)}\right),\\ \frac{(m+1)}{(m+2)(1+2\lambda)}, & \left(-\frac{(m+2)(1+\lambda)^2}{(m+1)(1+2\lambda)}\leq\mu\leq\frac{(m+2)(1+\lambda)^2}{(m+1)(1+2\lambda)}\right),\\ \frac{(m+1)^2\mu}{(m+2)^2(1+\lambda)^2}, & \left(\mu\geq\frac{(m+2)(1+\lambda)^2}{(m+1)(1+2\lambda)}\right).\end{cases}$$

Further, if $-\frac{(m+2)(1+\lambda)^2}{(m+1)(1+2\lambda)} \leq \mu \leq 0$, then

$$\left|\varrho_3-\mu\varrho_2^2\right|+\left[\frac{(m+2)(1+\lambda)^2}{(m+1)(1+2\lambda)}+\mu\right]|\varrho_2|^2\leq\frac{m+1}{(m+2)(1+2\lambda)}.$$

If $0 \leq \mu \leq \frac{(m+2)(1+\lambda)^2}{(m+1)(1+2\lambda)}$, then

$$\left|\varrho_3-\mu\varrho_2^2\right|+\left[\frac{(m+2)(1+\lambda)^2}{(m+1)(1+2\lambda)}-\mu\right]|\varrho_2|^2\leq\frac{m+1}{(m+2)(1+2\lambda)}.$$

Letting $m \to \infty$ (or $m = 0$) and $\lambda = 1$ in Corollary 12, we get the following example.

Example 5. If χ given by (1.1) belongs to $\mathcal{K}$, then

$$\left|\varrho_3-\mu\varrho_2^2\right|\leq\begin{cases}-\frac{\mu}{4} & \left(\mu\leq-\frac{4}{3}\right),\\ \frac{1}{3} & \left(-\frac{4}{3}\leq\mu\leq\frac{4}{3}\right),\\ \frac{\mu}{4} & \left(\mu\geq\frac{4}{3}\right).\end{cases}$$

Further, if $-\frac{4}{3} \leq \mu \leq 0$, then

$$\left|\varrho_3-\mu\varrho_2^2\right|+\left(\frac{4}{3}+\mu\right)|\varrho_2|^2\leq\frac{1}{3}.$$

If $0 \leq \mu \leq \frac{4}{3}$, then

$$\left|\varrho_3-\mu\varrho_2^2\right|+\left(\frac{4}{3}-\mu\right)|\varrho_2|^2\leq\frac{1}{3}.$$

4. Conclusions

In this presentation, we have introduced and investigated the new subfamily $\mathcal{B}_m(\lambda,\alpha)$ of Bazilevič functions that are subordinate to m-leaf polynomials. For this subfamily, we derived key subordination results, established inclusion properties, and obtained a bound for the Fekete–Szegö functional $\left|\varrho 3-\mu\varrho 2^2\right|$ for the case $m > 1$. By specializing the parameters in our main results, we produced several new results, as illustrated in the corollaries.

The methodology and results presented here open several promising directions for future research:

- Fekete–Szegö problem for the subfamily $\mathcal{B}1\left(\lambda,\alpha\right)$: Extending the study to the $m = 1$ case, which remains open.
- Hankel determinants: Investigating sharp bounds for higher-order Hankel determinants for functions in $\mathcal{B}m\left(\lambda,\alpha\right)$.
- Extensions to multivalent functions: Adapting the m-leaf polynomial technique to define and study new subfamilies of p-valent functions, focusing on initial coefficient estimates and Fekete–Szegö functionals.
- Quantum calculus analogues: Developing q-extensions of the family using q-analogues of m-leaf polynomials.

Future work could also develop computer algorithms to find extremal functions for the Fekete–Szegő problem and explore links to approximation theory and special functions. Solving these open problems will strengthen the connection between geometric function theory and real-world applications.

Author contributions

Khalid M. K. Alshammari: Conceptualization, methodology, validation, formal analysis, investigation, writing–original draft preparation, writing–review and editing; Tamer M. Seoudy: Conceptualization, methodology, validation, formal analysis, investigation, writing–original draft preparation, writing–review and editing, supervision; Amr K. Amin: Conceptualization, methodology, validation, formal analysis, investigation, writing–review and editing, project administration, funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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