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*Research article*

# Stability of neutral highly nonlinear hybrid stochastic delayed systems with delayed impulses

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**Abstract:** This paper investigates the stability of a class of neutral highly nonlinear hybrid stochastic delayed systems with delayed impulses. Based on the comparison principle and stochastic analysis techniques, the $l$th moment asymptotic stability and the almost sure asymptotic stability of the argumented system are proven. Finally, the effectiveness of the theoretical results is demonstrated through two examples.

**Keywords:** delayed impulses; high nonlinearity; hybrid stochastic system; neutral term; almost sure asymptotic stability
**Mathematics Subject Classification:** 93C10, 93D40

## 1. Introduction

With the swift progression of engineering and science, a class of hybrid stochastic delayed systems (HSDSs) has found extensive utilization across diverse fields, such as circuit systems, option price forecasting, and virus models (see, e.g., [1–3]). In particular, stochastic delayed systems with Markov switching require simultaneous consideration of continuous states and discrete events (also referred to as modes). Sudden changes in parameters or structures can lead to stochastic switching, which is commonly modeled using continuous-time Markov chains to characterize such abrupt variations (see, e.g., [4–6]). In addition, Wu et al. [7] considered the input-to-state stability and integral input-to-state stability of stochastic delayed systems with Markovian switching and external inputs by adopting the multiple Lyapunovlike function and Lyapunov–Krasovskii functional approaches.

Furthermore, the coefficients of certain HSDSs fail to meet the linear growth condition, but exhibit polynomial growth. Consequently, these systems are termed highly nonlinear HSDSs (HNHSDSs). In recent years, the stability of HNHSDSs has been extensively studied (see, e.g., [8–10]), along with research on the relevant control strategies in both continuous and discrete time (see, e.g., [11–13]). For instance, Dong et al. [11] focused on a class of highly nonlinear stochastic differential delay equations driven by Lévy noise and Markovian chain, where the drift and diffusion coefficients satisfy more general polynomial growth condition. Zhao et al. [12] discussed the existence and boundedness of unique global solutions for highly nonlinear switched stochastic systems with time delays under deterministic switching signals. Song et al. [13] investigated an unstable hybrid stochastic differential equation with time-varying delays whose coefficients satisfy certain polynomial growth conditions, and aim to design a time-varying-delay feedback control to stabilize the resulting closed-loop hybrid stochastic differential equation.

In practical applications, some stochastic systems evolve based on current states, past states, and the values of state derivatives at past instants. Neutral hybrid stochastic delayed systems (NHSDSs) serve as prevalent models for these practical systems, including neural networks, population ecology, and elasticity problems(see, e.g., [14–16]). In recent decades, a substantial body of theoretical research on NHSDSs has been developed. These studies have motivated numerous scholars to not only investigate the oscillatory behavior of systems (see, e.g., [16, 17]), but also systematically analyze various types of stability, such as $p$th moment stability (see, e.g., [18, 19]) and almost sure stability (see, e.g., [20–22]). Furthermore, for neutral highly nonlinear hybrid stochastic delayed systems (NHNHSDSs), stability under different control strategies has been examined (see, e.g., [23–25]). In [23], the stabilization of highly nonlinear stochastic delay systems with neutral terms was achieved via the design of a discrete-time feedback control law. In [24], periodically intermittent feedback control was applied to stabilize a given unstable highly nonlinear neutral stochastic system with Markovian switching. In [25], the stabilization was studied for a class of neutral stochastic delay differential equations driven by time-changed Lévy noise (including large and small jumps) and Markov processes via delayed feedback control. In [26], the stability analysis of highly nonlinear neutral stochastic delay differential equations driven by G-Brownian motion was conducted via delayed feedback control.

It is well known that the impulsive effect, due to its capacity to effectively characterize abrupt dynamic changes, is frequently utilized as a control scheme for control inputs (see, e.g., [15, 27, 28]). In particular, reference [15] only considers current-state-dependent impulses and investigates the $p$th moment exponential stability. Meanwhile, as the impulsive control theory has continued to evolve, impulsive systems with time delays have increasingly emerged as a prominent focus for many scholars. Such systems are described as the mixed impulsive effects which depend on both the current and historical states. Consequently, both the current state and the state at impulsive instants are dependent on historical information, thus forming a class of impulsive systems with delay-dependent characteristics. Lately, numerous methods have been introduced to process delayed impulsive, such as the Lyapunov method, the Razumikhin technique, and the average impulsive interval approach (see, e.g., [29–31]). For example, Lu et al. [31] studied the exponential stability of random impulsive delayed nonlinear systems with multiple randomly delayed impulses by adopting several analytical tools, including the average random impulsive estimation, average impulsive delay, average delay time, and Lyapunov-based techniques. Zhang et al. [32] investigated the stability of highly nonlinear hybrid stochastic delayed systems subject to multiple periodic delayed impulses by constructing comparison functions and utilizing the average impulsive interval approach.

It is noted that research regarding the stability of NHNHSDSs subject to delayed impulses remains limited. Inspired by the above discussions, the NHNHSDSs subject to delayed impulses (i.e., system (2.1) investigated in this paper incorporates delayed impulses that depend on past states, thereby generalizing the impulsive system in [15] where impulses only depend on the current state.

Moreover, we establish the more general $l$th moment asymptotic stability ($l$th MAS) and almost sure asymptotic stability (a.s. AS). The contributions of this paper are as follows:

(*i*) The system model discussed herein constitutes a further extension of the models derived in the works [6, 12, 29]. As far as we know, this is the first effort to integrate high nonlinearity, neutral terms, stochastic perturbations, Markov switching, and delayed impulses into a unified framework.

(*ii*) For the first time in such systems, we employ the Borel–Cantelli lemma and a proof by contradiction to establish almost sure convergence, without constructing a comparison system, thus reducing conservatism.

(*iii*) In numerical simulations, by comparing the dynamic behaviors of the Lotka–Volterra model under no impulse, periodic impulses, and aperiodic impulses, the mechanism by which historical dependence induced by time delay and neutral terms leads to persistent oscillations is revealed, and it is verified that both periodic and aperiodic impulses can effectively stabilize the system.

The rest of this paper is organized as follows: Section 2 introduces the model of NHNHSDSs with delayed impulses and several foundational assumptions; Section 3 proves the stability of NHNHSDSs with delayed impulses; Section 4 substantiates the validity of the main theorem through two examples; and Section 5 provides the conclusions of this paper.

**Notations:** Let $\mathbb{R}^n$ denote the $n$-dimensional real space with the Euclidean norm $\|\cdot\|$. $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{R}^+ = [0, +\infty)$. For any constant $a$ and $b$, $a \vee b = \max\{a, b\}$. $PC([-\bar{\vartheta}, 0]; \mathbb{R}^n)$ means the family of piecewise continuous functions from $[-\bar{\vartheta}, 0]$ to $\mathbb{R}^n$, and norm $\|\xi\| = \sup_{-\bar{\vartheta}\le s\le 0} |\xi(s)|$ for $\xi \in [-\bar{\vartheta}, 0]$. The complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\ge t_0}, \mathbb{P})$ is equipped with a filtration $\{\mathscr{F}_t\}_{t\ge t_0}$ which satisfies the normal conditions. On this space, an $m$-dimensional standard Wiener process $B(t)$ and a right-continuous Markov chain $r(t)$ are defined, with $r(t)$ being independent of $B(t)$. $\mathcal{S} = \{1, 2, \dots, s_0\}$ is a state space of $r(t)$, and the generator is given by $\Pi = (\gamma_{ij})_{s_0\times s_0}$, where $\gamma_{ij} \ge 0$ for $i \ne j$ and $\gamma_{ii} = -\sum_{j\ne i} \gamma_{ij}$ for each $i \in \mathcal{S}$. For a constant $l > 2$, define $L^l_{\mathscr{F}_0}([-\bar{\vartheta}, 0], \mathbb{R}^n)$ as the set of $\mathscr{F}_0$-measurable random variables $\xi$: $[-\bar{\vartheta}, 0] \to \mathbb{R}^n$ that are piecewise continuous and satisfy the condition $\mathbb{E}\|\xi\|^l < \infty$.

## 2. Model description and preliminaries

In this paper, the following NHNHSDSs with delayed impulses are considered:

$$\begin{cases} \mathrm{d}[u(t) - \Lambda(u(t-\vartheta(t)), t, r(t))] = f(u(t), u(t-\vartheta(t)), t, r(t))\mathrm{d}t + \sigma(u(t), u(t-\vartheta(t)), t, r(t))\mathrm{d}B(t), \ t \ne t_k, \\ u(t_k) = I_k(u(t_k^-), u(t_k^- - \vartheta(t_k^-))), \ k \in \mathbb{N}, \\ u(t_0+\theta) = \xi, \theta \in [-\bar{\vartheta}, 0], r(t_0) = r_0 \in \mathcal{S}, \end{cases} \tag{2.1}$$

where the initial condition $(\xi, r_0) \in L^l_{\mathscr{F}_0}([-\bar{\vartheta}, 0]; \mathbb{R}^n) \times \mathcal{S}$. $\Lambda(\cdot, \cdot, \cdot)$ is a neutral item that satisfies $\Lambda(0, t, i)$ =0. The drift coefficient $f$: $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \times \mathcal{S} \to \mathbb{R}^n$ satisfies $f(0, 0, t, i) = 0$ for $\forall t \ge t_0$. The diffusion coefficient $\sigma$: $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \times \mathcal{S} \to \mathbb{R}^{n\times m}$ satisfies $\sigma(0, 0, t, i) = 0$ for $\forall t \ge t_0$. In this paper, we let $t_0 = 0$. $\vartheta(t)$: $\mathbb{R}^+ \to [0, \bar{\vartheta}]$ is a differentiable time-varying delay with $\mathrm{d}\vartheta(t)/\mathrm{d}t \le \tilde{\vartheta} < 1$ (where constant $\tilde{\vartheta} > 0$). $I_k$: $\mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S} \to \mathbb{R}^n$ denotes the delayed impulse function. To avoid the occurrence of Zeno behavior, we assume the impulse times $\{t_k\}_{k\in\mathbb{N}}$ form a strictly increasing sequence with $\lim_{k\to\infty} t_k = \infty$ and satisfy $0 < \bar{\vartheta} < t_k - t_{k-1}$ for all $k \ge 1$. For simplicity, let $\tilde{u}(t) = u(t) - \Lambda(z(t), t, r(t))$, and $z(t) = u(t - \vartheta(t))$, and $\bar{z}(t) = u(t - 2\vartheta(t))$.

We need to impose some assumptions about functions $f$ and $\sigma$ of system (2.1).

**Assumption 2.1.** *([12]) For any $u$, $z$, $\bar{u}$, $\bar{z} \in \mathbb{R}^n$, $d > 0$, there exists a constant $L_d > 0$ that satisfies the following:*

$$|f(u,z,t,i) - f(\bar{u},\bar{z},t,i)| \vee |\sigma(u,z,t,i) - \sigma(\bar{u},\bar{z},t,i)| \le L_d\left(|u-\bar{u}| + |z-\bar{z}|\right), \tag{2.2}$$

*where $i \in \mathcal{S}$, $|u| \vee |\bar{u}| \vee |z| \vee |\bar{z}| \le d$, and $t > 0$. Moreover, for $(u,z,t,i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \times \mathcal{S}$, there exist positive constants $L$ and $\varrho_i (i = 1, ..., 4)$ that satisfy the following:*

$$\begin{aligned} |f(u,z,t,i)| &\le L\left(|u| + |z| + |u|^{\varrho_1} + |z|^{\varrho_2}\right), \\ |\sigma(u,z,t,i)| &\le L\left(|u| + |z| + |u|^{\varrho_3} + |z|^{\varrho_4}\right), \end{aligned} \tag{2.3}$$

*where $\varrho_1 > 1, \varrho_i \ge 1, i = 2, 3, 4$.*

In system (2.1), the growth of the drift coefficient $f$ and the diffusion coefficient $\sigma$ is characterized by the exponents $\varrho_i$ $(i = 1, ..., 4)$. When $0 < \varrho_i < 1$ $(i = 1, ..., 4)$, the growth of the system coefficients is slower than linear. Finite-time explosion does not occur, and the system can be handled by linearization methods. Therefore, this case is not discussed in this paper. The condition $\varrho_1 > 1$ ensures that the drift term contains a genuinely superlinear nonlinearity, which is essential to apply the Khasminskii-type condition (i.e., Assumption 2.3) to counteract the potential explosive effect of higher-order diffusion terms. Meanwhile, the conditions $\varrho_i \ge 1$ $(i = 2, 3, 4)$ prevent the delayed argument or the diffusion coefficients from growing too rapidly, thereby guaranteeing that system (2.1) remains highly nonlinear while still allowing the establishment of stability criteria via Lyapunov methods.

**Assumption 2.2.** *([33]) For $k \in \mathbb{N}$, there exist positive constants $\chi_k$ and $\psi_k$ which satisfy*

$$|u(t_k)|^l \le \chi_k |u(t_k^-)|^l + \psi_k |u(t_k^- - \vartheta(t_k^-))|^l \tag{2.4}$$

*and*

$$\sup_{k\in\mathbb{N}}(\chi_k + \psi_k) = \mu < 1, \tag{2.5}$$

*where $\mu \in (0, 1)$.*

**Remark 2.1.** *$\chi_k$ and $\psi_k$ represent the magnitudes of the impulsive effects exerted by the current state and the past state at $t_k$, respectively. This indicates that the impulsive jumps simultaneously depend on both the current and historical states, thus making the model more general than traditional impulsive systems that solely rely on the current state. The condition $\mu < 1$ ensures that the overall effect of the impulsive sequence is stabilizing; however, it is merely sufficient and can be relaxed to a more general average contraction condition, which will be a focus of our future work.*

**Remark 2.2.** *Inequality* (2.2) *and the polynomial growth condition, as opposed to the linear growth condition, are satisfied by system* (2.1). *Therefore, a Khasminskii-type condition is required to ensure the existence and uniqueness of a global solution for system* (2.1) *and to prevent its finite time explosion. The following example compares the two cases $\varrho_i = 1$ (linear growth) and $\varrho_i > 1$ (high nonlinearity) for $i = 1, ..., 4$, and illustrates why the condition $\varrho_i$ $(i = 1, ..., 4)$ in this paper necessitates* *the use of the Khasminskii-type condition to guarantee the system's stability. Consider the following one-dimensional stochastic differential equation:*

$$\mathrm{d}u = (u - u^3)\mathrm{d}t + u^2\mathrm{d}B(t),$$

*where the drift term contains* $-u^3$ *(corresponding to* $\varrho_1 = 3 > 1$*), and the diffusion term is* $u^2$ *(corresponding to* $\varrho_3 = 2 > 1$*). This is a highly nonlinear system. If the term* $-u^3$ *is omitted and only* $u\mathrm{d}t + u^2\mathrm{d}B(t)$ *remains, the solution may explode in finite time. However, the negative higher-order term* $-u^3$ *effectively counteracts the superlinear growth of the diffusion term, thus preventing explosion.*

*Conversely, when all exponents are reduced to 1, the system degenerates to the following linear case:*

$$\mathrm{d}u = (u - u)\mathrm{d}t + u\mathrm{d}B(t) = u\mathrm{d}B(t),$$

*whose solution is the geometric Brownian motion* $u(t) = u(0)\exp\left(-\frac{1}{2}t + B(t)\right)$*. In this case, no finite-time explosion occurs, and the mean-square stability only depends on the parameters.*

**Assumption 2.3.** *([23]) For* $(u, z, t, i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \times \mathcal{S}$*, assume that there exist a constant* $\alpha_1 \in \mathbb{R}$ *and positive constants* $\alpha_2$*,* $\alpha_3$*, and* $\alpha_4$ *with* $\alpha_3 > \alpha_4$ *satisfying*

$$l > (\frac{1}{2}(\hbar + \varrho_1 + 1)) \vee (2(\varrho_1 \vee \varrho_2 \vee \varrho_3 \vee \varrho_4)),$$
$$\hbar > 4(\varrho_1 \vee \varrho_2 \vee \varrho_3 \vee \varrho_4) - \varrho_1 - 1 \tag{2.6}$$

*and*

$$\tilde{u}^T(t,i)f(u,z,t,i) + \frac{l-1}{2}|\sigma(u,z,t,i)|^2 \leq \alpha_1|\tilde{u}|^2 + \alpha_2|\tilde{z}|^2 - \alpha_3|\tilde{u}|^{\hbar-l+2} + \alpha_4|\tilde{z}|^{\hbar-l+2}. \tag{2.7}$$

**Assumption 2.4.** *([22]) For any* $t \geq 0$*,* $u, \bar{u} \in \mathbb{R}^n$*,* $i \in \mathcal{S}$*, assume that there exisits a constant* $0 < \jmath < 1 \wedge \mu$ *that satisfies the following:*

$$|\Lambda(u,t,i) - \Lambda(\bar{u},t,i)| \leq \jmath|u - \bar{u}|. \tag{2.8}$$

*We define an operator* $\mathcal{L}W$ *associated with* (2.1) *as follows:*

$$\mathcal{L}W(\tilde{u},z,t,i) = W_t(\tilde{u},t,i) + W_u(\tilde{u},t,i)f(u,z,t,i) + \frac{1}{2}\,\mathrm{trace}\left[\sigma^T(u,z,t,i)W_{uu}(\tilde{u},t,i)\sigma(u,z,t,i)\right] + \sum_{j=1}^{s_0}\gamma_{ij}W(\tilde{u},z,t,j), \tag{2.9}$$

*where* $W_t = \frac{\partial W}{\partial t}$*,* $W_u = \left(\frac{\partial W}{\partial u_1}, \frac{\partial W}{\partial u_2}, \ldots, \frac{\partial W}{\partial u_n}\right)$*, and* $W_{uu} = \left(\frac{\partial^2 W}{\partial u_i \partial u_j}\right)_{n\times n}$*.*

*For the convenience of the reader, we cite the Itô formula [8]: with the operator* $\mathcal{L}W$ *defined in Eq* (2.9)*, the process* $W(\tilde{u}(t), t, r(t))$ *is an Itô process on* $t \geq 0$ *that satisfies the following stochastic differential:*

$$dW(\tilde{u}(t),t,r(t)) = \mathcal{L}W(\tilde{u}(t),z(t),t,r(t))\,dt + dM(t),$$

*where* $M(t)$ *is a continuous local martingale with* $M(0) = 0$ *(the explicit expression of* $M(t)$ *is not utilized in this paper and is therefore omitted here).*

**Lemma 2.1.** *([34]) Let Assumption 2.4 hold; for any* $l \geq 1$, $(u, t, i) \in \mathbb{R}^n \times \mathbb{R}^+ \times \mathcal{S}$, $\jmath$ *is the same as that in Assumption 2.4, and the following inequalities hold:*

$$|u - \Lambda(z)|^l \leq (1 + \jmath)^{l-1} \left(|u|^l + \jmath|z|^l\right)$$

*and*

$$|u - \Lambda(z)|^l \geq (1 - \jmath)^{l-1} \left(|u|^l - \jmath|z|^l\right).$$

**Theorem 2.1.** *Let Assumptions 2.1–2.4 hold. For any initial value*$(\xi, r_0)$*, system* (2.1) *possesses a unique global solution* $(u(t), r(t))$*. In addition, with this initial value,* $u(t)$ *exhibits almost surely continuous sample paths on* $[t_k, t_{k+1})$ *and*

$$\sup_{t \in [-\bar{\vartheta}, T]} \mathbb{E}|u(t)|^l < \infty \tag{2.10}$$

*for all* $T > 0$.

*Proof.* For any $(u, t, i) \in \mathbb{R}^n \times \mathbb{R}^+ \times \mathcal{S}$, set $W(\tilde{u}, t, i) = |\tilde{u}(t)|^l$. When $0 \leq t < t_1$, system (2.1) with the initial value $(\xi, r_0)$ reduces to an impulsive-free neutral highly nonlinear hybrid stochastic delayed system. Following the approach in [13,35], it can be deduced that the solution $(u(t), r(t))$ of system (2.1) is unique on the interval $[0, t_1)$. Furthermore, $(u(t), r(t))$ exhibits continuous sample paths on $0 \leq t < t_1$ almost surely, and

$$\sup_{t \in [-\bar{\vartheta}, t_1)} \mathbb{E}|u(t)|^l < \infty.$$

When $t = t_1$, $u(t_1) = I_1(u(t_1^-), u(t_1^- - \vartheta(t_1^-)))$. By Assumption 2.2, the following can be obtained:

$$\sup_{t \in [-\bar{\vartheta}, t_1]} \mathbb{E}|u(t)|^l < \infty.$$

For $t_1 \leq t < t_2$, the solution $(u(t), r(t))$ continues to exhibit continuous sample paths on $[t_1, t_2)$ almost surely, and

$$\sup_{t \in [t_1, t_2)} \mathbb{E}|u(t)|^l < \infty.$$

When $t = t_2$, $u(t_2) = I_2(u(t_2^-), u(t_2^- - \vartheta(t_2^-)))$. Then, using Assumption 2.3 again, the following can be obtained:

$$\sup_{t \in [t_1, t_2]} \mathbb{E}|u(t)|^l < \infty.$$

The proof is completed by iterating this process. For brevity, we omit these detailed derivations. □

## 3. Main results

In this section, with the aid of the stochastic theory and several important inequalities, we first establish a theorem on the $H_\infty$ stability of the solution about system (2.1). Subsequently, we obtain the $l$th MAS for the solution about system (2.1), and finally prove that the solution is a.s. AS.

**Theorem 3.1.** *Let Assumptions 2.1–2.4 hold and there exist constants $\varphi_k$ (where $k = 1, ..., 6$) that satisfy the following:*

$$\varphi_4 > \frac{\varphi_5}{1-\tilde{\vartheta}} + \frac{\varphi_6}{1-\tilde{\vartheta}}, \tag{3.1}$$

*where*

$$\begin{aligned}
\varphi_1 &= l\alpha_1(1-\jmath)^{l-1} + (l-2)\alpha_2(1+\jmath)^{l-1},\\
\varphi_2 &= [l\alpha_1\jmath + (2+(l-2)\jmath)\alpha_2](1+\jmath)^{l-1},\\
\varphi_3 &= 2\alpha_2\jmath(1+\jmath)^{l-1},\\
\varphi_4 &= l\alpha_3(1-\jmath)^{\hbar-1} - \frac{l(l-2)\alpha_4}{\hbar}(1+\jmath)^{\hbar-1},\\
\varphi_5 &= l\alpha_3\jmath(1-\jmath)^{\hbar-1} + \frac{l(l-2)\alpha_4\jmath}{\hbar}(1+\jmath)^{\hbar-1} + \frac{l(\hbar-l+2)\alpha_4}{\hbar}(1+\jmath)^{\hbar-1},\\
\varphi_6 &= \frac{l(\hbar-l+2)\alpha_4\jmath}{\hbar}(1+\jmath)^{\hbar-1}.
\end{aligned} \tag{3.2}$$

*Then, for any initial value $\xi \in C([-\bar{\vartheta}, 0]; \mathbb{R}^n)$, $l \geq 2$, the solution $u(t)$ of system* (2.1) *satisfies*

$$\int_0^\infty \mathbb{E}|u(t)|^l \mathrm{d}t < \infty \tag{3.3}$$

*and*

$$\int_0^\infty \mathbb{E}|\tilde{u}(t)|^l \mathrm{d}t < \infty. \tag{3.4}$$

*Proof.* Set a Lyapunov function $V(u, z, t, i)$ by the following:

$$V(u, z, t, i) = |\tilde{u}(t)|^l + c\Phi(t), \tag{3.5}$$

where

$$\Phi(t) = \int_{t-\vartheta(t)}^{t} |u(\zeta)|^l \mathrm{d}\zeta$$

and

$$c > \max\left\{-\varphi_1, \frac{\varphi_2}{1-\tilde{\vartheta}}\right\}.$$

By Assumption 2.3 and the generalized Itô formula, we can obtain the following:

$$\begin{aligned}
\mathcal{L}|\tilde{u}(t)|^l \leq & l|\tilde{u}(t)|^{l-2}\left[\tilde{u}^T(t)f(u(t), z(t), t, r(t)) + \frac{l-1}{2}|\sigma(u(t), z(t), t, r(t))|^2\right]\\
\leq & l\left[\alpha_1|\tilde{u}(t)|^l + \alpha_2|\tilde{u}(t)|^{l-2}|\tilde{z}(t)|^2 - \alpha_3|\tilde{u}(t)|^\hbar + \alpha_4|\tilde{u}(t)|^{l-2}|\tilde{z}(t)|^{\hbar-l+2}\right].
\end{aligned}$$

Applying Young's inequality, we can derive the following:

$$\begin{aligned}
|\tilde{u}(t)|^{l-2}|\tilde{z}(t)|^2 &\leq \frac{l-2}{l}|\tilde{u}(t)|^l + \frac{2}{l}|\tilde{z}(t)|^l,\\
|\tilde{u}(t)|^{l-2}|\tilde{z}(t)|^{\hbar-l+2} &\leq \frac{l-2}{\hbar}|\tilde{u}(t)|^\hbar + \frac{\hbar-l+2}{\hbar}|\tilde{z}(t)|^\hbar.
\end{aligned}$$

It follows that

$$\mathcal{L}|\tilde{u}(t)|^l \le [l\alpha_1 + (l-2)\alpha_2]\,|\tilde{u}(t)|^l + 2\alpha_2|\tilde{z}(t)|^l - \left[l\alpha_3 - \frac{l(l-2)\alpha_4}{\hbar}\right]|\tilde{u}(t)|^\hbar + \frac{l(\hbar - l + 2)\alpha_4}{\hbar}|\tilde{z}(t)|^\hbar. \tag{3.6}$$

Moreover, consider the time derivative of $\Phi(t)$, which yields the following:

$$\begin{aligned}\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t} &= |u(t)|^l - (1-\dot{\vartheta}(t))|z(t)|^l \\ &\le |u(t)|^l - (1-\tilde{\vartheta})|z(t)|^l.\end{aligned} \tag{3.7}$$

Using Lemma 2.1, we have the following:

$$\mathcal{L}V(u,z,t,i) \le (\varphi_1 + c)|u(t)|^l + [\varphi_2 - c(1-\tilde{\vartheta})]|z(t)|^l + \varphi_3|\bar{z}(t)|^l - \varphi_4|u(t)|^\hbar + \varphi_5|z(t)|^\hbar + \varphi_6|\bar{z}(t)|^\hbar, \tag{3.8}$$

where $\varphi_k$ $(k = 1, ..., 6)$ are as given in (3.2). We can further deduce that

$$\sup_{0\le t\le\infty} \mathbb{E}|\mathcal{L}V(u,z,t,i)| < \infty.$$

By applying Itô formula, it gives the following:

$$\begin{aligned}0 &\le \mathbb{E}V(u,z,t,i) \\ &\le V(u(0), z(0), 0, r(0)) + \mathbb{E}\int_0^t \mathcal{L}V(u(\zeta), z(\zeta), \zeta, r(\zeta))\mathrm{d}\zeta\end{aligned} \tag{3.9}$$

for $t \in [0, t_1)$. By combining (3.8), we can obtain the following:

$$\begin{aligned}0 &\le \mathbb{E}V(u,z,t,i) \\ &\le V(u(0), z(0), 0, r(0)) + \mathbb{E}\int_0^t \left[\bar{\varphi}_1|u(\zeta)|^l + \bar{\varphi}_2|z(\zeta)|^l + \varphi_3|\bar{z}(\zeta)|^l - \varphi_4|u(\zeta)|^\hbar + \varphi_5|z(\zeta)|^\hbar + \varphi_6|\bar{z}(\zeta)|^\hbar\right]\mathrm{d}\zeta,\end{aligned}$$

where $\bar{\varphi}_1 = \varphi_1 + c > 0$, $\bar{\varphi}_2 = \varphi_2 - c(1-\tilde{\vartheta}) < 0$. Notice that

$$\mathbb{E}\int_0^t |z(\zeta)|^l\mathrm{d}\zeta \le \frac{1}{1-\tilde{\vartheta}}\left(\mathbb{E}\int_{-\bar{\vartheta}}^0 |u(\zeta)|^l\mathrm{d}\zeta + \mathbb{E}\int_0^t |u(\zeta)|^l\mathrm{d}\zeta\right)$$

and

$$\mathbb{E}\int_0^t |\bar{z}(\zeta)|^l\mathrm{d}\zeta \le \frac{1}{1-\tilde{\vartheta}}\left(\mathbb{E}\int_{-2\bar{\vartheta}}^0 |u(\zeta)|^l\mathrm{d}\zeta + \mathbb{E}\int_0^t |u(\zeta)|^l\mathrm{d}\zeta\right).$$

Furthermore, an analogous derivation is applied to $|z(\zeta)|^\hbar$ and $|\bar{z}(\zeta)|^\hbar$. Then, it can be deduced that

$$\mathbb{E}V(u,z,t,i) \le C_1 + a\int_0^t \mathbb{E}|u(\zeta)|^l\mathrm{d}\zeta - b\int_0^t \mathbb{E}|u(\zeta)|^\hbar\mathrm{d}\zeta, \tag{3.10}$$

where

$$a = \bar{\varphi}_1 + \frac{\bar{\varphi}_2}{1-\tilde{\vartheta}} + \frac{\varphi_3}{1-\tilde{\vartheta}}, \quad b = \varphi_4 - \frac{\varphi_5}{1-\tilde{\vartheta}} - \frac{\varphi_6}{1-\tilde{\vartheta}},$$

and $C_1$ is a constant.

Recalling (3.9) yields the following:

$$b\int_0^t \mathbb{E}|u(\zeta)|^{\hbar}\mathrm{d}\zeta \le C_1 + a\int_0^t \mathbb{E}|u(\zeta)|^{l}\mathrm{d}\zeta. \tag{3.11}$$

From (2.10), there exists a positive constant $M$ that satisfies the following:

$$\sup_{t\ge -\bar{\vartheta}} \mathbb{E}|u(t)|^{l} \le M. \tag{3.12}$$

Then, we use the mathematical induction to demonstrate that the inequality

$$\int_0^{t_i^-} \mathbb{E}|u(\zeta)|^{\hbar}\mathrm{d}\zeta < \infty \tag{3.13}$$

holds for all $i \in \mathbb{N}$.

By combining (3.12), we have the following:

$$a\int_0^t \mathbb{E}|u(\zeta)|^{l}\mathrm{d}\zeta \le aMt$$

for $t \in [0, t_1)$. Then, let $t \to t_1$; by utilizing (3.11), we can obatin the following:

$$b\int_0^{t_1^-} \mathbb{E}|u(\zeta)|^{\hbar}\mathrm{d}\zeta \le C_1 + aMt_1.$$

Thus,

$$\int_0^{t_1^-} \mathbb{E}|u(\zeta)|^{\hbar}\mathrm{d}\zeta < \infty.$$

Now, assume that (3.13) holds for any $i \le m$, i.e.,

$$\int_0^{t_m^-} \mathbb{E}|u(\zeta)|^{\hbar}\mathrm{d}\zeta < \infty. \tag{3.14}$$

Then, for $t \in [t_m, t_{m+1})$, we will prove that inequality (3.13) holds. At $t = t_m$, using Assumption 2.2 and Eqs (3.12) and (3.14), it follows that

$$\int_0^{t_m} \mathbb{E}|u(\zeta)|^{\hbar}\mathrm{d}\zeta = \int_0^{t_m^-} \mathbb{E}|u(\zeta)|^{\hbar}\mathrm{d}\zeta + \int_{t_m^-}^{t_m} \mathbb{E}|u(\zeta)|^{\hbar}\mathrm{d}\zeta < \infty.$$

When $t \in [t_m, t_{m+1})$, there exist contants $C_{m+1} > 0$, $k \in \mathbb{N}$ which satisfy the following:

$$\begin{aligned} 0 &\le \mathbb{E}V(u, z, t, i) \\ &\le C_{m+1} + a\int_{t_m}^t \mathbb{E}|u(\zeta)|^{l}\mathrm{d}\zeta - b\int_{t_m}^t \mathbb{E}|u(\zeta)|^{\hbar}\mathrm{d}\zeta, \end{aligned}$$

where $C_{m+1} > 0$ is a constant. Thus,

$$b\int_{t_m}^t \mathbb{E}|u(\zeta)|^{\hbar}\mathrm{d}\zeta \le C_{m+1} + a\int_{t_m}^t \mathbb{E}|u(\zeta)|^{l}\mathrm{d}\zeta.$$

Similar to the discussion above, let $t \to t_{m+1}$; it can be inferred that

$$b \int_{t_m}^{t_{m+1}^-} \mathbb{E}|u(\zeta)|^{\hbar} \mathrm{d}\zeta \le C_{m+1} + aM(t_{m+1} - t_m).$$

Therefore,

$$\int_{t_m}^{t_{m+1}^-} \mathbb{E}|u(\zeta)|^{\hbar} \mathrm{d}\zeta < \infty.$$

Moveover, we can easily obtain the following:

$$\int_0^{t_{m+1}^-} \mathbb{E}|u(\zeta)|^{\hbar} \mathrm{d}\zeta = \int_0^{t_m} \mathbb{E}|u(\zeta)|^{\hbar} \mathrm{d}\zeta + \int_{t_m}^{t_{m+1}^-} \mathbb{E}|u(\zeta)|^{\hbar} \mathrm{d}\zeta < \infty.$$

When $i \to \infty$, we have $t_i \to \infty$. When $i$ is infinite, just let $t_{i+1} \to \infty$, that is, $t_{i+1}^- \to \infty$, which yields the following:

$$\int_0^{\infty} \mathbb{E}|u(t)|^{\hbar} \mathrm{d}t < \infty, \quad \forall \hbar > l \ge 2.$$

Therefore, we have the following:

$$\int_0^{\infty} \mathbb{E}|u(t)|^{l} \mathrm{d}t < \infty.$$

In addition, it can be shown that

$$\begin{aligned} \int_0^t |\tilde{u}(\zeta)|^l d\zeta \le & 2^{l-1} \left( \int_0^t |u(\zeta)|^l d\zeta + j^l \int_0^t |u(\zeta - \vartheta(\zeta))|^l d\zeta \right) \\ \le & 2^{l-1} j^l \int_{-\bar{\vartheta}}^0 |u(\zeta)|^l d\zeta + 2^{l-1}(1 + j^l) \int_0^t |u(\zeta)|^l d\zeta \end{aligned}$$

holds for any $t > 0$, which can be further inferred that

$$\int_0^{\infty} \mathbb{E}|\tilde{u}(t)|^{l} \mathrm{d}t < \infty.$$

This completes the proof. □

**Remark 3.1.** *Generally, we cannot obtain* (3.16) *from* (3.3) *and* (3.4). *To obtain the asymptotic properties, it is necessary to impose an additional condition, as demonstrated in* (3.15).

Theorem 3.1 establishes that the solution $u(t)$ satisfies (3.3). However, this integrability condition alone does not guarantee that $\mathbb{E}|u(t)|^l \to 0$ as $t \to \infty$. In addition, information on the growth rate of $\mathbb{E}|u(t)|^l$ is required. To obtain the $l$th MAS, we impose a stronger condition on the drift and diffusion coefficients.

**Theorem 3.2.** *Let Assumptions 2.1–2.4 hold. For any $(u, z, t, i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \times \mathcal{S}$, there exists a positive constant $K_0$ that satisfies the following:*

$$\tilde{u}^T(t, i) f(u, z, t, i) + \frac{l-1}{2} |\sigma(u, z, t, i)|^2 \le K_0(|\tilde{u}|^2 + |z|^2). \tag{3.15}$$

*Then, for any initial value $\xi \in C([-\bar{\vartheta}, 0]; \mathbb{R}^n)$, system* (2.1) *can be realized as the lth MAS ($l \ge 2$) if*

$$\lim_{t \to \infty} \mathbb{E}|u(t)|^l = 0. \tag{3.16}$$

*Proof.* First, we prove that

$$\lim_{t\to\infty}\mathbb{E}|\tilde{u}(t)|^l = 0. \tag{3.17}$$

Suppose (3.17) is false; then, there exists an $\epsilon > 0$, a impulse subsequence $\{t_{k_m}\}_{m\in\mathbb{N}}$ such that $t_{k_m+1} > t_{k_m} + 2\bar{\vartheta}$, and for any $m \in \mathbb{N}$, $E|\tilde{u}(t_{k_m})|^l \geq 3\epsilon$.

From Theorem 2.1, we have

$$\sum_{m=0}^{\infty}\int_{t_{k_m}-2\bar{\vartheta}}^{t_{k_m}}\mathbb{E}|u(\zeta)|^l d\zeta \leq \int_0^{\infty}\mathbb{E}|u(\zeta)|^l d\zeta < \infty,$$

which means

$$\lim_{m\to\infty}\int_{t_{k_m}-2\bar{\vartheta}}^{t_{k_m}}\mathbb{E}|u(\zeta)|^l d\zeta = 0.$$

Therefore, there must be some $m_0 > 0$ such that

$$\int_{t_{k_m}-2\bar{\vartheta}}^{t_{k_m}^-}\mathbb{E}|u(\zeta)|^l d\zeta < \frac{\epsilon}{2^l K_0(1+\jmath)^{l-1}}$$

holds for $m \geq m_0$.

Noting that, for any $m \geq m_0$ and $t \in [t_{k_m} - \bar{\vartheta}, t_{k_m}]$, we can obtain the following:

$$\mathbb{E}|\tilde{u}(t_{k_m})|^l - \mathbb{E}|\tilde{u}(t)|^l \leq \left(\mathbb{E}|\tilde{u}(t_{k_m})|^l - \mathbb{E}|\tilde{u}(t_{k_m}^-)|^l\right) + \left(\mathbb{E}|\tilde{u}(t_{k_m}^-)|^l - \mathbb{E}|\tilde{u}(t)|^l\right). \tag{3.18}$$

From Theorem 2.1, there exists a positive constant $M_1$ that satisfies the following:

$$\mathbb{E}|\tilde{u}(t_{k_m})|^l < M_1, \quad \mathbb{E}|\tilde{z}(t_{k_m})|^l < M_1.$$

Next, using (3.12) and Lemma 2.1, it can be derived as follows:

$$\begin{aligned}\mathbb{E}|\tilde{u}(t_{k_m})|^l - \mathbb{E}|\tilde{u}(t_{k_m}^-)|^l \leq&(1+\jmath)^{l-1}(\mathbb{E}|u(t_{k_m})|^l + \jmath\mathbb{E}|z(t_{k_m})|^l) - (1-\jmath)^{l-1}(\mathbb{E}|u(t_{k_m}^-)|^l - \jmath\mathbb{E}|z(t_{k_m}^-)|^l)\\ \leq&(1+\jmath)^{l-1}\left(\chi_{k_m}\mathbb{E}|u(t_{k_m}^-)|^l + \psi_{k_m}\mathbb{E}|z(t_{k_m}^-)|^l + \jmath\mathbb{E}|z(t_{k_m}^-)|^l\right)\\ &- (1-\jmath)^{l-1}(\mathbb{E}|u(t_{k_m}^-)|^l - \jmath\mathbb{E}|z(t_{k_m}^-)|^l)\\ \leq&(K_1+K_2)M_1,\end{aligned} \tag{3.19}$$

where $K_1 = (1+\jmath)^{l-1}\mu - (1-\jmath)^{l-1}$, and $K_2 = (1+\jmath)^{l-1}(\mu+\jmath) + (1-\jmath)^{l-1}\jmath$ . Then, take $(K_1+K_2)M_1 < \epsilon$, which implies

$$\mathbb{E}|\tilde{u}(t_{k_m})|^l - \mathbb{E}|\tilde{u}(t_{k_m}^-)|^l \leq \epsilon. \tag{3.20}$$

Using Itô formula and Lemma 2.1 leads to the following:

$$\begin{aligned}[b]\mathbb{E}|\tilde{u}(t_{k_m}^-)|^l - \mathbb{E}|\tilde{u}(t)|^l \leq&\mathbb{E}\int_t^{t_{k_m}^-} l|\tilde{u}(\zeta)|^{l-2}\left(\tilde{u}^T(\zeta)f(u(\zeta),z(\zeta),\zeta,r(\zeta)) + \frac{l-1}{2}|\sigma(u(\zeta),z(\zeta),\zeta,r(\zeta))|^2\right)d\zeta\\ \leq&K_0\mathbb{E}\int_t^{t_{k_m}^-} l|\tilde{u}(\zeta)|^{l-2}(|\tilde{u}(\zeta)|^2 + |z(\zeta)|^2)d\zeta\\ \leq&K_0\mathbb{E}\int_t^{t_{k_m}^-}\left[2(l-1)|\tilde{u}(\zeta)|^l + 2|z(\zeta)|^l\right]d\zeta\end{aligned}$$

$$
\begin{aligned}
&\leq 2^l K_0(1+\jmath)^{l-1}\mathbb{E}\int_t^{t_{k_m}^-}\left(|u(\zeta)|^l+|z(\zeta)|^l\right)d\zeta\\
&\leq 2^l K_0(1+\jmath)^{l-1}\mathbb{E}\int_{t_{k_m}^- -2\bar{\vartheta}}^{t_{k_m}^-}|u(\zeta)|^l d\zeta\\
&\leq \epsilon.
\end{aligned}
\tag{3.21}
$$

Combining (3.19)–(3.21) into (3.18) results in the following:

$$
\mathbb{E}|\tilde{u}(t_{k_m})|^l-\mathbb{E}|\tilde{u}(t)|^l\leq(K_1+K_2)M_1+\epsilon.
$$

Thus we obtain $\mathbb{E}|\tilde{u}(t)|^l\geq\mathbb{E}|\tilde{u}(t_{k_m})|^l-(K_1+K_2)M_1-\epsilon\geq\epsilon$, and then

$$
\int_0^\infty\mathbb{E}|\tilde{u}(t)|^l\mathrm{d}t\geq\sum_{m=0}^\infty\int_{t_{k_m}}^{t_{k_m+1}}\mathbb{E}|\tilde{u}(t)|^l\mathrm{d}t>\sum_{m=0}^\infty 2\bar{\vartheta}\epsilon=\infty,
$$

which contradicts the result in (3.4).

From Lemma 2.1, it is true that

$$
\mathbb{E}|u(t)|^l\leq\frac{1}{(1-\jmath)^{l-1}}\mathbb{E}|\tilde{u}(t)|^l+\jmath\mathbb{E}|u(t-\vartheta(t))|^l.
$$

Taking both sides as $t\to\infty$ yields

$$
\limsup_{t\to\infty}\mathbb{E}|u(t)|^l\leq\jmath\limsup_{t\to\infty}\mathbb{E}|u(t)|^l,
$$

and thus (3.16) holds. □

**Remark 3.2.** *Instead of constructing a comparison system, we prove Theorem 3.1 by contradiction, thereby utilizing the inherent correlations among system states at different times to establish a contradiction. Compared to the traditional comparison principle, this approach simplifies the analytical process and reduces the conservatism of the theoretical criteria.*

**Remark 3.3.** *Equivalent to the linear growth condition, condition* (3.15) *is naturally satisfied by many stochastic systems with globally Lipschitz coefficients. It should be noted that condition* (3.15) *is stronger than condition* (2.7)*, but to simplify the proof process of Theorem 3.2, the derivation from condition* (2.7) *to condition* (3.15) *is omitted. In the next theorem, a.s. AS of the solution is established without condition* (3.15).

**Theorem 3.3.** *Let Assumptions 2.1–2.4 hold. For any initial value $\xi\in C([-\bar{\vartheta},0];\mathbb{R}^n)$, system* (2.1) *can be achieved a.s. AS if*

$$
\lim_{t\to\infty}u(t)=0\quad a.s.
\tag{3.22}
$$

*Proof.* From Theorems 2.1 and 3.1, when $l=2$, we have the following:

$$
\int_0^\infty\mathbb{E}|u(\zeta)|^2\mathrm{d}\zeta<\infty,\quad\lim_{t\to\infty}\mathbb{E}|u(t)|^2=0.
\tag{3.23}
$$

Using Itô formula, it can be derived as

$$\tilde{u}(t) = \tilde{u}(t_m) + \int_{t_m}^{t} f(u(\zeta), z(\zeta), \zeta, r(\zeta))\mathrm{d}\zeta + \int_{t_m}^{t} \sigma(u(\zeta), z(\zeta), \zeta, r(\zeta))\mathrm{d}B(\zeta)$$

for $t \in [t_k, t_{k+1})$, $k = 1, 2, .., m$ ($m \in \mathbb{N}$), which implies

$$\begin{aligned} u(t) =& \Lambda(z(t)) + \tilde{u}(t_m) + \int_{t_m}^{t} f(\zeta)\mathrm{d}\zeta + \int_{t_m}^{t} \sigma(\zeta)\mathrm{d}B(\zeta) \\ =& \Lambda(z(t)) + (u(t_m) - \Lambda(z(t_m))) + \int_{t_m}^{t} f(\zeta)\mathrm{d}\zeta + \int_{t_m}^{t} \sigma(\zeta)\mathrm{d}B(\zeta), \end{aligned}$$

where $f(\zeta) = f(u(\zeta), z(\zeta), \zeta, r(\zeta))$, $\sigma(\zeta) = \sigma(u(\zeta), z(\zeta), \zeta, r(\zeta))$ for simplicity.

The inequality $(a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2)$ and mathematical expectation gives the following:

$$\begin{aligned} \mathbb{E}\left(\sup_{t_m \leq t < t_{m+1}} |u(t)|^2\right) \leq& 4\mathbb{E}\left(\sup_{t_m \leq t < t_{m+1}} |\Lambda(z(t))|^2\right) + 4\mathbb{E}\left(|u(t_m) - \Lambda(z(t_m))|^2\right) \\ &+ 4\mathbb{E}\left(\sup_{t_m \leq t < t_{m+1}} \left|\int_{t_m}^{t} f(\zeta)\mathrm{d}\zeta\right|^2\right) + 4\mathbb{E}\left(\sup_{t_m \leq t < t_{m+1}} \left|\int_{t_m}^{t} \sigma(\zeta)\mathrm{d}B(\zeta)\right|^2\right). \end{aligned} \tag{3.24}$$

Now, by Assumption 2.4, we can obtain the following:

$$\mathbb{E}\left(\sup_{t_m \leq t < t_{m+1}} |\Lambda(z(t))|^2\right) \leq \jmath^2 \mathbb{E}\left(\sup_{t_m \leq t < t_{m+1}} |z(t)|^2\right). \tag{3.25}$$

Using Lemma 2.1, we have the following:

$$\begin{aligned} \mathbb{E}\left(|u(t_m) - \Lambda(z(t_m))|^2\right) \leq& (1 + \jmath)(\mathbb{E}|u(t_m)|^2 + \jmath\mathbb{E}|z(t_m)|^2) \\ \leq& 2\mathbb{E}|u(t_m)|^2 + 2\jmath\mathbb{E}|z(t_m)|^2. \end{aligned} \tag{3.26}$$

By the Cauchy–Schwarz inequality, let $t \to t_{m+1}$; then, the following is obtained:

$$\begin{aligned} \mathbb{E}\left(\sup_{t_m \leq t < t_{m+1}} \left|\int_{t_m}^{t} f(\zeta)\mathrm{d}\zeta\right|^2\right) \leq& \mathbb{E}\left[\sup_{t_m \leq t < t_{m+1}} \left(\int_{t_m}^{t} |f(\zeta)|\mathrm{d}\zeta\right)^2\right] \\ \leq& \mathbb{E}\left[(t_{m+1} - t_m)\int_{t_m}^{t_{m+1}^-} |f(\zeta)|^2\mathrm{d}\zeta\right] \\ \leq& (t_{m+1} - t_m)\int_{t_m}^{t_{m+1}^-} \mathbb{E}|f(\zeta)|^2\mathrm{d}\zeta. \end{aligned} \tag{3.27}$$

By the Burkholder–Davis–Gundy inequality and taking $t \to t_{m+1}$, we can get

$$\mathbb{E}\left(\sup_{t_m \leq t < t_{m+1}} \left|\int_{t_m}^{t} \sigma(\zeta)\mathrm{d}B(\zeta)\right|^2\right) \leq 4\int_{t_m}^{t_{m+1}^-} \mathbb{E}|\sigma(\zeta)|^2\mathrm{d}\zeta, \tag{3.28}$$

where the Itô integral $\int_{t_m}^{t} \sigma(\zeta)\mathrm{d}B(\zeta)$ defines a continuous local martingale and zero for $t = t_m$.

Substituting (3.25)–(3.28) into (3.24) leads to the following:

$$\begin{aligned}\mathbb{E}\left(\sup_{t_m\le t<t_{m+1}}|u(t)|^2\right) &\le 4\jmath^2\mathbb{E}\left(\sup_{t_m\le t<t_{m+1}}|z(t)|^2\right)+8\mathbb{E}|u(t_m)|^2+8\jmath\mathbb{E}|z(t_m)|^2\\ &\quad+4(t_{m+1}-t_m)\int_{t_m}^{t_{m+1}^-}\mathbb{E}|f(\zeta)|^2\mathrm{d}\zeta+16\int_{t_m}^{t_{m+1}^-}\mathbb{E}|\sigma(\zeta)|^2\mathrm{d}\zeta\\ &\le 4\jmath^2\mathbb{E}\left(\sup_{t_m\le t<t_{m+1}^-}|z(t)|^2\right)+8\mathbb{E}|u(t_m)|^2+8\jmath\mathbb{E}|z(t_m)|^2\\ &\quad+4L^2(\hat{T}+4)\int_{t_m}^{t_{m+1}^-}(\mathbb{E}|u(\zeta)|^2+\mathbb{E}|z(\zeta)|^2)\mathrm{d}\zeta,\end{aligned}\tag{3.29}$$

where

$$\hat{T}=\sup_{m\in\mathbb{N}}(t_{m+1}-t_m)<\infty.$$

In addition, from Theorem 2.1 and the Fubini theorem, there exists a positive constant $M_2$ which satisfies the following:

$$\mathbb{E}\left(\sup_{t_m\le t<t_{m+1}}|z(t)|^2\right)<M_2,\quad \mathbb{E}|z(t_m)|^2<M_2.$$

Moreover, we obatin

$$\begin{aligned}\int_{t_m}^{t_{m+1}^-}\mathbb{E}|z(\zeta)|^2\mathrm{d}\zeta &\le \frac{1}{1-\tilde{\vartheta}}\int_{-\bar{\vartheta}}^{t_{m+1}^-}\mathbb{E}|u(\zeta)|^2\mathrm{d}\zeta\\ &\le \frac{1}{1-\tilde{\vartheta}}\left(K_3+\int_0^{t_{m+1}^-}\mathbb{E}|u(\zeta)|^2\mathrm{d}\zeta\right),\end{aligned}\tag{3.30}$$

where

$$K_3=\int_{-\bar{\vartheta}}^0\mathbb{E}|u(\zeta)|^2\mathrm{d}\zeta.$$

Therefore, there exist constants $K_4, K_5>0$ which depend on $L$, $\hat{T}$, $\tilde{\vartheta}$, and $\bar{\vartheta}$, where

$$\begin{aligned}\mathbb{E}\left(\sup_{t_m\le t<t_{m+1}}|u(t)|^2\right) &\le 4\jmath^2M_2+8\mathbb{E}|u(t_m)|^2+8\jmath M_2\\ &\quad+4L^2(\hat{T}+4)\left[\int_{t_m}^{t_{m+1}^-}\mathbb{E}|u(\zeta)|^2+\frac{1}{1-\tilde{\vartheta}}\left(K_3+\int_0^{t_{m+1}^-}\mathbb{E}|u(\zeta)|^2\mathrm{d}\zeta\right)\right]\\ &\le K_4\mathbb{E}|u(t_m)|^2+K_5\int_{t_m}^{t_{m+1}^-}\mathbb{E}|u(\zeta)|^2\mathrm{d}\zeta.\end{aligned}\tag{3.31}$$

For any $\varepsilon>0$, define the events as follows:

$$\mathcal{A}_{m+1}=\left\{\sup_{t_m\le t<t_{m+1}}|u(t)|>\varepsilon\right\},\quad m\in\mathbb{N}.$$

By Chebyshev's inequality and (3.31),

$$\mathbb{P}(\mathcal{A}_{m+1})\le\frac{1}{\varepsilon^2}\left(K_4\mathbb{E}|u(t_m)|^2+K_5\int_{t_m}^{t_{m+1}^-}\mathbb{E}|u(\zeta)|^2\mathrm{d}\zeta\right).\tag{3.32}$$

From (3.23), we can obtain the following:

$$\lim_{m\to\infty}\mathbb{E}|u(t_m)|^2=0,\quad \lim_{m\to\infty}\int_{t_m}^{t_{m+1}^-}\mathbb{E}|u(\zeta)|^2\mathrm{d}\zeta=0.$$

Therefore, for any $\delta>0$, there exists $N\in\mathbb{N}$; for all $m>N$, it holds that

$$\mathbb{E}|u(t_m)|^2<\delta,\quad \int_{t_m}^{t_{m+1}^-}\mathbb{E}|u(\zeta)|^2\mathrm{d}\zeta<\delta.$$

Choose

$$\delta=\frac{\varepsilon^2}{(K_4+K_5)(m+1)^2}.$$

Then, for $m>N$, it can be shown as follows:

$$\mathbb{P}(\mathcal{A}_{m+1})\le\frac{K_4\delta+K_5\delta}{\varepsilon^2}=\frac{1}{(m+1)^2}.$$

Thus, we yield the following:

$$\sum_{m=0}^{\infty}\mathbb{P}(\mathcal{A}_{m+1})\le\sum_{m=0}^{N}\mathbb{P}(\mathcal{A}_{m+1})+\sum_{m=N+1}^{\infty}\frac{1}{(m+1)^2}<\infty.$$

By the Borel–Cantelli lemma, it gives

$$\mathbb{P}(\limsup_{m\to\infty}\mathcal{A}_{m+1})=0,$$

that is, for almost all $\omega\in\Omega$ and all $m+1>Q_1(\omega)$, there exists $Q_1(\omega)\in\mathbb{N}$ that satisfies

$$\sup_{t_m\le t<t_{m+1}}|u(t,\omega)|\le\varepsilon. \tag{3.33}$$

Now, we consider impulse instants $t=t_{m+1}$; by (3.33), we know for almost all $\omega\in\Omega$, there exists $Q_1(\omega)\in\mathbb{N}$ that satisfies

$$|u(t,\omega)|\le\varepsilon.$$

For $m$ sufficiently large enough, it can easily obtain $t_{m+1}^->t_{Q_1(\omega)}$ and $t_m^--\vartheta(t_m^-)>t_{Q_1(\omega)}$. Then, it follows that

$$|u(t_{m+1}^-,\omega)|<\varepsilon,\quad |u(t_{m+1}^--\vartheta(t_{m+1}^-)),\omega)|<\varepsilon.$$

Using Assumption 2.2, for all $m>Q_2(\omega)$, it is easy to obtain

$$|u(t_{m+1},\omega)|^l\le\chi_{m+1}|u(t_{m+1}^-,\omega)|^l+\psi_{m+1}|u(t_{m+1}^--\vartheta(t_{m+1}^-)),\omega)|^l<\mu\varepsilon^l,$$

with

$$\sup_{m\in\mathbb{N}}(\chi_{m+1}+\psi_{m+1})=\mu<1,$$

that is, for all $m>Q_2(\omega)$, it is true that

$$|u(t_{m+1},\omega)|<\mu^{\frac{1}{l}}\varepsilon.$$

Therefore, for almost all $\omega$, there exists $Q(\omega) = \max\{Q_1(\omega), Q_2(\omega)\}$; for all $t > t_{Q(\omega)}$, it yields the following:

$$|u(t,\omega)| < \mu^{\frac{1}{l}}\varepsilon < \varepsilon.$$

Since the positive constant $\varepsilon$ is arbitrary, and $\lim_{m\to\infty} t_m = \infty$, it can be derived as follows:

$$\lim_{t\to\infty} u(t,\omega) = 0. \tag{3.34}$$

This completes the proof. □

**Remark 3.4.** *Instead of constructing a comparison system, we prove Theorem 3.3 by contradiction, thereby utilizing the inherent correlations among system states at different times to establish a contradiction. Compared to the traditional comparison principle, this approach simplifies the analytical process and reduces the conservatism of theoretical criteria.*

**Remark 3.5.** *In [23–25], the stability of NHNHSDSs was discussed, with stability primarily achieved through controller design. Specifically, Huang et al. [15] investigated the stability of NHNHSDSs under impulsive effects. However, the impact of delayed impulses on NHNHSDSs warrants further investigation. Therefore, this paper systematically analyzes the stability of NHNHSDSs subjected to delayed impulses. To visually demonstrate the differences between this paper and the existing results in terms of key features, Table 1 provides a detailed comparison.*

**Table 1.** Comparison table this article with [6, 12, 15, 22, 29].

| | This paper | [6] | [12] | [15] | [22] | [29] |
|---|---|---|---|---|---|---|
| Neutral term | ✓ | | | ✓ | ✓ | |
| High nonlinearity | ✓ | | ✓ | ✓ | | |
| Markovian switching | ✓ | ✓ | ✓ | ✓ | ✓ | |
| Time-varying delay | ✓ | | | ✓ | | |
| Delayed impulses | ✓ | | | | | ✓ |
| Borel–Cantelli lemma | ✓ | ✓ | | | | |
| Almost sure asymptotic stability | ✓ | ✓ | | | ✓ | ✓ |

## 4. Numerical examples

This section presents two numerical examples with the state space $\mathcal{S} = \{1, 2\}$ to demonstrate the validity and effectiveness of the theoretical results.

**Example 4.1.** *Consider the following NHNHSDSs with a delayed impulse:*

$$\begin{cases} \mathrm{d}[u(t) - \Lambda(u(t-\vartheta(t)), t, r(t))] = f(u(t), u(t-\vartheta(t)), t, r(t))\mathrm{d}t + \sigma(u(t), u(t-\vartheta(t)), t, r(t))\mathrm{d}B(t), t \neq t_k, \\ u(t_k) = 0.9u(t_k^-) + 0.05u(t_k^- - \vartheta(t_k^-)), k \in \mathbb{N}, \end{cases} \tag{4.1}$$

*with generator* $\Pi = \begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix}$, $\vartheta(t) = \frac{3}{2} - \frac{1}{2}\cos t$,

$$f(u,z,t,1) = -3u - 2u^3 + 0.3z^3, \quad \sigma(u,z,t,1) = 0.3\,(u+z), \quad \Lambda(z,t,1) = 0.1z,$$
$$f(u,z,t,2) = -u - u^3 + 0.2z^2, \quad \sigma(u,z,t,2) = 0.2\,(u+z), \quad \Lambda(z,t,2) = 0.05z.$$

*Then, it is straightforward to show that* $\jmath = 0.1$, $\bar{\vartheta} = 2$, $\tilde{\vartheta} = 0.5 < 1$, $l = 7$. *For simplicity, we let*

$$H(t,i) = \tilde{u}^T(t,i) f(u,z,t,i) + \frac{l-1}{2} |\sigma(u,z,t,i)|^2.$$

*It directly follows that*

$$H(t,1) \leq -2.31u^2 + 0.69z^2 - 1.775u^4 + 0.245z^4,$$
$$H(t,2) \leq -0.735u^2 + 0.265z^2 - 0.9125u^4 + 0.1525z^4.$$

*Therefore,*

$$H(t,i) \leq -0.735u^2 + 0.69z^2 - 0.9125u^4 + 0.245z^4,$$

*where* $\alpha_1 = -0.735$, $\alpha_2 = 0.69$, $\alpha_3 = 0.9125$, *and* $\alpha_4 = 0.245$ *with* $\alpha_3 > \alpha_4$. *For* $n \in \mathbb{N}$, *the impulsive perturbations are given by* $t_{k+1} - t_k = h$, *where constant* $h > 0$.

Figure 1 compares the dynamic behaviors of system (4.1) with impulse intervals $h = 2, 3, 4$. Subplot (a) verifies the $l$th MAS and a.s. AS. Subplot (b) shows the stochastic switching of Markovian modes, which is independent of the impulse interval. Subplot (c) confirms that the state at impulse instants satisfies the contraction condition $\mu < 1$ in Assumption 2.2. The results demonstrate that a smaller impulse interval leads to a faster convergence rate and smaller fluctuations.

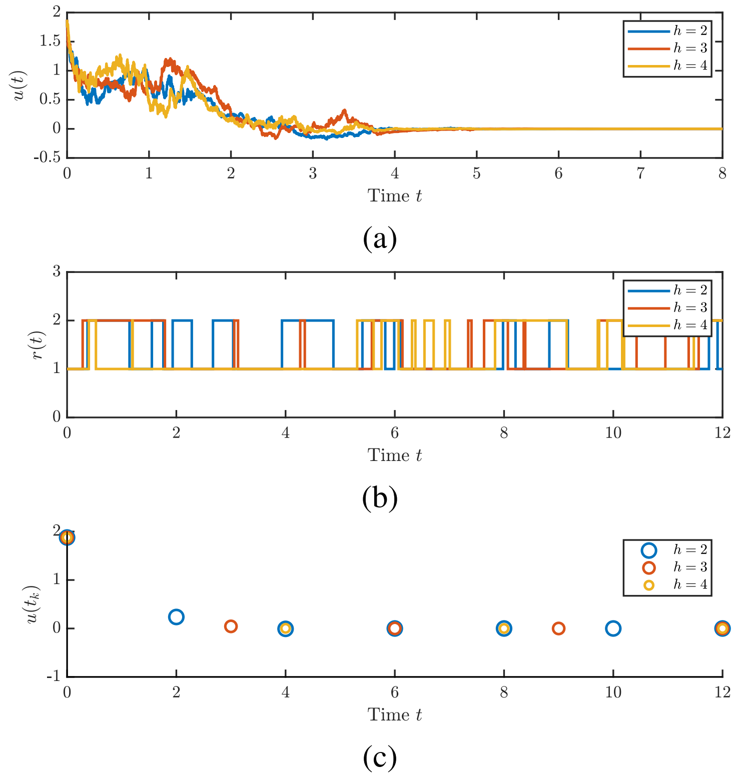


**Figure 1.** (a) State trajectories of system (4.1); (b) switching signals of Markov chains of system (4.1); (c) state trajectories of system (4.1) at impulsive instants.

**Example 4.2.** *Consider the following stochastic Lotka–Volterra population model with impulsive effects, which is primarily used to characterize the evolution of biological populations under the* *combined influence of environmental noise and historical population density delay effects:*

$$\begin{cases} \mathrm{d}[u(t) - 0.15u(t-\vartheta(t))] = \begin{cases} u(t)\left(0.5 - 6u^2(t) + 0.3u^2(t-\vartheta(t))\right)\mathrm{d}t + 0.2u^2(t-\vartheta(t))\mathrm{d}B(t), & i = 1, \\ u(t)\left(0.8 - 2u^2(t) + 0.6u^2(t-\vartheta(t))\right)\mathrm{d}t + 0.3u^2(t-\vartheta(t))\mathrm{d}B(t), & i = 2, \end{cases} \\ u(t_k) = 0.2u(t_k^-) + 0.1u(t_k^- - \vartheta(t_k^-)), k \in \mathbb{N}, \end{cases} \tag{4.2}$$

*with generator* $\Pi = \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix}$, $\vartheta(t) = 0.2(1 - \cos 1.5t)$.

*Take* $l = 10$. *Following a similar approach as in Example 4.1, we calculate that*

$$H(t,i) \leq 0.86u^2 + 0.06z^2 - 1.4525u^4 + 0.8475z^4,$$

*where* $\alpha_1 = 0.86, \alpha_2 = 0.06, \alpha_3 = 1.4525$*, and* $\alpha_4 = 0.8475$ *with* $\alpha_3 > \alpha_4$.

Figure 2 depicts the state trajectories of system (4.2) without impulse control. Affected by time delay and neutral terms, the current rate of change of the system relies excessively on historical states, thus leading to the continuous amplification of cumulative deviations, which further induces sustained oscillations and a failure to approach the equilibrium point, thus exhibiting instability. This instability arises from the introduced historical dependence, which may cause the current population change rate to be excessively influenced by past states, thus leading to oscillations.

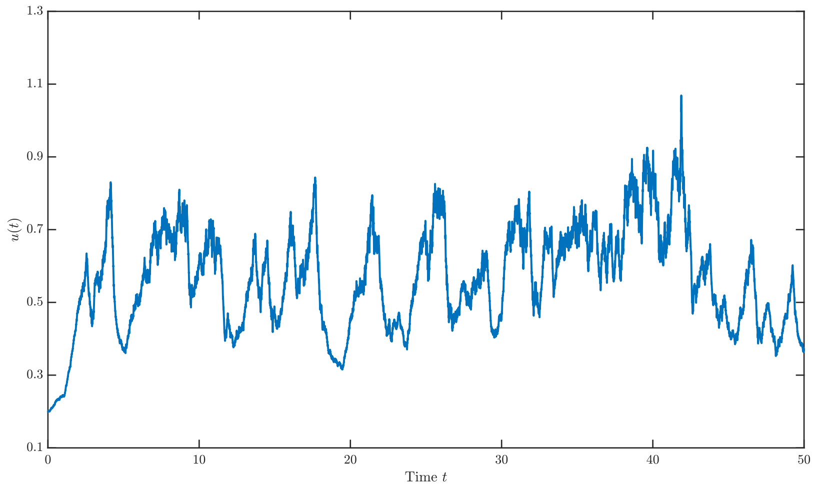


**Figure 2.** State trajectories of system (4.2) without impulse.

Figure 3 illustrates the dynamic behaviors of system (4.2) under a periodic delayed impulse control. Subplot (a) shows a rapid convergence to zero with oscillations completely suppressed. Subplot (b) demonstrates robust mode switching. Subplot (c) shows that the states at impulse instants quickly decay and satisfy $\mu = 0.3 < 1$. Compared with Figure 2, periodic delayed impulses achieve strong stabilization at low cost, thus highlighting the necessity of policy interventions.

Although theoretical analyses typically assume that a delayed impulsive control has fixed impulse intervals, it is often difficult to precisely maintain equidistant impulses in practical engineering systems. Therefore, we consider applying aperiodic delayed impulsive control(ADIC) to system (4.2).

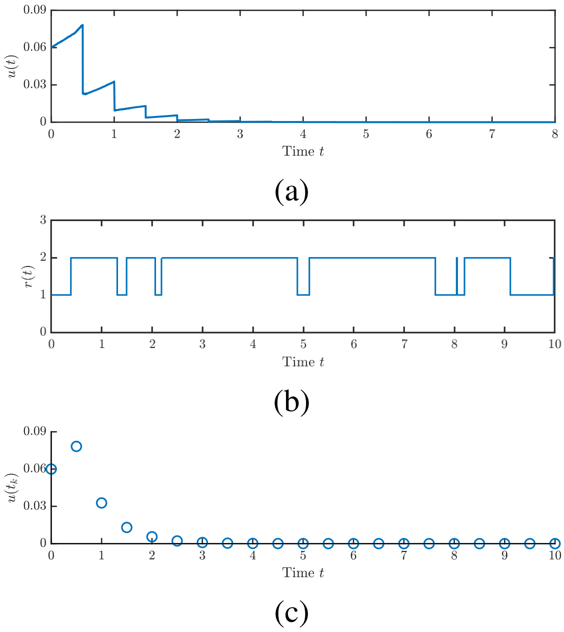


**Figure 3.** (a) State trajectories of system (4.2); (b) switching signals of Markov chains of system (4.2); (c) state trajectories of system (4.2) at impulsive instants.

Figure 4 demonstrates that the ADIC achieves a much faster convergence rate than the periodic delayed impulsive control. This is mainly attributed to its shorter impulse intervals and more frequent control actions, which accelerate the decay of system states. In practical engineering, impulse moments can be flexibly arranged, and shorter intervals in the initial stage can effectively improve the convergence efficiency. The proposed control scheme possesses superior engineering applicability.

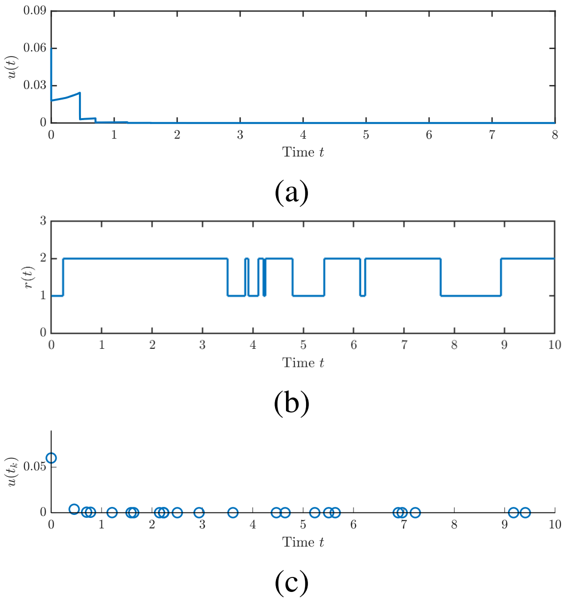


**Figure 4.** (a) State trajectories of system (4.2) under ADIC; (b) switching signals of Markov chains of system (4.2) under ADIC; (c) state trajectories of system (4.2) under ADIC at impulsive instants.

**Remark 4.1.** *In recent decades, most studies on Brownian motion-driven stochastic Lotka–Volterra population models mainly focused on cases without impulses or time delays (see, e.g., [15, 36, 37]). Differently, this paper first simultaneously considers system* (4.2) *subject to time-varying delays, neutral terms, and delayed impulses. We further investigate its stability under impulse-free, periodic delayed impulse, and aperiodic delayed impulse cases, which effectively extends the analytical approaches and applicable scenarios of such models.*

## 5. Conclusions

This paper examined the stability of a class of HHNHSDS with delayed impulses. Unlike previous models, the derivatives of the states in NHNHSDSs here depended on past states, and the states at impulsive instants also rely on historical information. Furthermore, by employing the idea of the comparison principle, the Borel–Cantelli lemma, and stochastic analysis techniques, we derived sufficient conditions for $l$th MAS and a.s. AS of the discussed system. This paper adopted a finite-state, constant-coefficient Markov chain as the basic framework. Future work will further extend to semi-Markov switching, time-varying, or infinite-state switching, and consider delayed impulses with multi-order or distributed delays, as well as more complex systems such as those with an event-triggered impulsive control.

## Author contributions

J. Li: writing–original draft; Jin-E Zhang: supervision, writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this paper.

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## Conflict of interest

The authors declare that there are no conflicts of interest.

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