



Research article

A generalized topological Rudin lemma and sober spaces

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Abstract: In this paper, we introduce and study the notion of F -families in T_0 -spaces. This concept unifies several important classes of objects in domain theory and non-Hausdorff topology, including irreducible sets, filtered families of upper sets, Scott open filters, and irreducible subsets in Smyth power spaces. Using F -families, we establish a generalized topological Rudin lemma that extends the classical topological Rudin lemma to a broader setting. As an application, we obtain some characterizations of sober spaces, from which the Hofmann-Mislove theorem and Heckman-Keimel-Schalk theorem can be directly deduced.

Keywords: generalized topological Rudin lemma; F -family; Scott open filter; irreducible set; sober space

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1. Introduction

Sobriety plays a central role in the studies of domain theory and non-Hausdorff topology (see [1, 3, 5]). It has been used in the characterizations of spectral spaces by Hochster in [8] and T_0 -spaces that are determined by their open set lattices [2, 12]. With the development of domain theory and non-Hausdorff topology, another two properties also emerged as very useful and important properties for T_0 -spaces: The property of being a d -space and the well-filteredness (see [3, 5, 18]).

Rudin's Lemma plays an important role in domain theory and non-Hausdorff topology (see [1, 3–5]). In [7], Heckmann and Keimel provided a topological variant of Rudin's Lemma. Using this lemma, Xu et al. [14, 16] proved that a T_0 -space X is a well-filtered space iff its Smyth power space $P_S(X)$ is well-filtered, giving a positive answer to an open problem posed by Heckmann in [6]. Based on the topological Rudin lemma, the authors of [11, 14] introduced the notions of Rudin sets and well-filtered determined sets (shortly WD sets), which lie between the class of all closures of directed subsets and that of irreducible closed subsets, and defined the corresponding Rudin spaces and WD spaces. These

concepts have become central in recent studies (see [15–19]).

The main purpose of this paper is to establish a generalized topological Rudin lemma via the new concept of F -families. Unlike existing frameworks that treat irreducible sets, filtered families, and Scott open filters separately, F -families provide a unified framework that connects all these structures. As an important application of the generalized topological Rudin lemma, we obtain some characterizations of sober spaces, from which the Hofmann-Mislove Theorem and Heckman-Keimel-Schalk Theorem can be directly deduced. Our work shows that F -families offer a simplified and unified approach to the three classical results.

2. Preliminaries

In this section, we briefly recall some basic concepts and results about ordered structures and T_0 -spaces that will be used in the paper. For further details, we refer the reader to [1, 3, 5].

For a poset P and $A \subseteq P$, define $\uparrow A = \{x \in P : a \leq x \text{ for some } a \in A\}$ and $\downarrow A = \{x \in P : x \leq a \text{ for some } a \in A\}$. For $x \in X$, let $\uparrow x = \uparrow\{x\}$ and $\downarrow x = \downarrow\{x\}$. A subset A is called a *lower set* (resp., an *upper set*) if $A = \downarrow A$ (resp., $A = \uparrow A$). If the set of upper bounds of A has a unique smallest element, we call this element the *least upper bound* and write it as $\vee A$ or $\sup A$ (for supremum). Similarly, the greatest lower bound is written as $\wedge A$ or $\inf A$. Let $P^{(<\omega)} = \{F \subseteq P : F \text{ be a nonempty finite set}\}$ and $\mathbf{Fin}P = \{\uparrow F : F \in P^{(<\omega)}\}$.

A nonempty subset D of a poset P is called *directed* if every finite subset of D has an upper bound in D . The family of all directed subsets of P is denoted by $\mathcal{D}(P)$. Dually, we define *filtered* sets. An upper filtered subset of P is called a *filter* of P , and the poset of all filters of P is denoted by $\text{Filt}(P)$. A *dcpo* is a poset in which every directed set has a supremum.

Definition 2.1. A subset U of a poset P is *Scott open* if

- (i) $U = \uparrow U$, and
- (ii) for every directed subset D for which $\vee D$ exists, $\vee D \in U$ implies $D \cap U \neq \emptyset$.

All Scott open sets of P form the *Scott topology* $\sigma(P)$. The upper sets of P form the (*upper*) *Alexandroff topology* $\alpha(P)$.

For a T_0 -space X , we use \leq_X to represent the *specialization order* of X : $x \leq_X y$ iff $x \in \overline{\{y\}}$, and sometimes we briefly write \leq instead of \leq_X . The pair (X, \leq_X) is denoted by ΩX or simply by X if no confusion arises. The family of all open subsets (resp., closed subsets) of X is denoted by $\mathcal{O}(X)$ (resp., $\Gamma(X)$). Clearly, open sets are upper sets and closed sets are lower sets with respect to the specialization order. For a subset of $A \subseteq X$, the closure of A in X is denoted by $\text{cl}_X A$ (or simply by $\text{cl}A$ if there is no ambiguity) or \overline{A} , and the interior of A is denoted by $\text{int}_X A$ in X or simply by $\text{int}A$. Let $\mathcal{S}(X) = \{\{x\} : x \in X\}$, $\mathcal{S}_c(X) = \{\overline{\{x\}} : x \in X\}$, $\mathcal{S}^u(X) = \{\uparrow x : x \in X\}$, and $\mathbf{up}(X) = \{U \subseteq X : U \text{ be a nonempty upper set}\}$. We use 2^X to denote the family of all subsets of X . The family of all Scott open proper filters of $\mathcal{O}(X)$ is denoted by $\text{OFilt}(\mathcal{O}(X))$, namely, $\text{OFilt}(\mathcal{O}(X)) = \sigma(\mathcal{O}(X)) \cap \text{Filt}(\mathcal{O}(X)) \setminus \{\mathcal{O}(X)\}$.

A nonempty subset A of T_0 -space X is called *irreducible* if for any $F_1, F_2 \in \Gamma(X)$, $A \subseteq F_1 \cup F_2$ implies $A \subseteq F_1$ or $A \subseteq F_2$. We denote by $\text{lrr}(X)$ (resp., $\text{lrr}_c(X)$) the set of all irreducible (resp., irreducible closed) subsets of X . Clearly, every directed subset of X is irreducible, and the nonempty irreducible subsets of a poset P equipped with the Alexandroff topology are exactly the directed sets of

P (cf. [7, Fact 2.6]). The space X is called *sober* if $\text{lrr}_c(X) = \mathcal{S}_c(X)$, that is, every nonempty irreducible closed set of X is the closure of a unique point.

Remark 2.2. For a T_0 -space X and $A \subseteq X$, the following conditions are equivalent:

- (1) $A \in \text{lrr}(X)$.
- (2) For any $F_1, F_2 \in \Gamma(X)$, $\bar{A} = F_1 \cup F_2$ implies $\bar{A} = F_1$ or $\bar{A} = F_2$.
- (3) For any $U_1, U_2 \in \mathcal{O}(X)$, $A \cap U_1 \neq \emptyset$ and $A \cap U_2 \neq \emptyset$ imply $A \cap U_1 \cap U_2 \neq \emptyset$.

The following two lemmas on irreducible sets are well-known and can be easily verified.

Lemma 2.3. Let X be a space, Y be a subspace of X , and $A \subseteq Y$. Then, the following conditions are equivalent:

- (1) A is an irreducible subset of Y .
- (2) A is an irreducible subset of X .
- (3) $\text{cl}_X A$ is an irreducible closed subset of X .

Lemma 2.4. If $f : X \rightarrow Y$ is continuous and $A \in \text{lrr}(X)$, then $f(A) \in \text{lrr}(Y)$.

Theorem 2.5. ([14, Proposition 5.7]) For a T_0 -space X , the following conditions are equivalent:

- (1) X is sober.
- (2) For every $A \in \text{lrr}(X)$, $\bar{A} \cap \bigcap_{a \in A} \uparrow a \neq \emptyset$.
- (3) For every $A \in \text{lrr}_c(X)$, $A \cap \bigcap_{a \in A} \uparrow a \neq \emptyset$.
- (4) For every $A \in \text{lrr}(X)$ and $U \in \mathcal{O}(X)$, $\bigcap_{a \in A} \uparrow a \subseteq U$ implies $\uparrow a \subseteq U$ for some $a \in A$.
- (5) For every $A \in \text{lrr}_c(X)$ and $U \in \mathcal{O}(X)$, $\bigcap_{a \in A} \uparrow a \subseteq U$ implies $\uparrow a \subseteq U$ for some $a \in A$.

For a topological space X , $\mathcal{G} \subseteq 2^X$ and $A \subseteq X$, define $\diamond_{\mathcal{G}} A = \{G \in \mathcal{G} : G \cap A \neq \emptyset\}$ and $\square_{\mathcal{G}} A = \{G \in \mathcal{G} : G \subseteq A\}$. The symbols $\diamond_{\mathcal{G}} A$ and $\square_{\mathcal{G}} A$ will be simply written as $\diamond A$ and $\square A$, respectively, if there is no ambiguous occurrence. The *upper Vietoris topology* on \mathcal{G} is the topology generated by $\{\square_{\mathcal{G}} U : U \in \mathcal{O}(X)\}$, and the resulting space is denoted by $P_S(\mathcal{G})$.

A subset A of a T_0 -space X is called *saturated* if A equals the intersection of all open sets containing it (equivalently, A is an upper set in the specialization order). We denote by $K(X)$ the set of all nonempty compact saturated subsets of X and endow it with the Smyth order: $K_1 \sqsubseteq K_2$ iff $K_2 \subseteq K_1$. The space $P_S(K(X))$, denoted shortly by $P_S(X)$, is called the *Smyth power space* of X (cf. [6, 10]). It is easy to verify that the specialization order on $P_S(X)$ is the Smyth order.

Remark 2.6. Let X be a T_0 -space and $\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq 2^X$. Then,

- (1) The inclusion mapping $i : P_S(\mathcal{G}_1) \rightarrow P_S(\mathcal{G}_2)$ is a topological embedding. Indeed, for any $U \in \mathcal{O}(X)$, we have $\square_{\mathcal{G}_1} U = \square_{\mathcal{G}_2} U \cap \mathcal{G}_1$.
- (2) The *canonical mapping* $\xi_X : X \rightarrow P_S(X)$, $x \mapsto \uparrow x$, is a topological embedding (cf. [6, 7, 10]).

Lemma 2.7. ([13, Lemma 4.5]) For a T_0 -space X and $A, B \subseteq X$, we have the following conclusions:

- (1) $\downarrow_{K(X)} \xi_X(A) = \diamond_{K(X)} A = \diamond_{K(X)} \downarrow A$.
- (2) $\downarrow_{K(X)} \xi_X(A) = \downarrow_{K(X)} \xi_X(B)$ iff $\downarrow A = \downarrow B$.
- (3) $\diamond_{K(X)} A = \diamond_{K(X)} B$ iff $\downarrow A = \downarrow B$.
- (4) $\text{cl}_{P_S(X)} \xi_X(A) = \diamond_{K(X)} \bar{A}$.

$$(5) \text{cl}_{P_S(X)} \diamond_{K(X)} A = \diamond_{K(X)} \overline{A}.$$

A subset S of a T_0 -space X is called *hypercompact* if for every $U \in \mathcal{O}(X)$, $S \subseteq U$ implies there is $\uparrow F \in \mathbf{Fin}X$ such that $S \subseteq \uparrow F \subseteq U$. The family of all nonempty hypercompact saturated subsets of X is denoted by $K_s(X)$. Clearly, $\mathcal{S}''(X) \subseteq K_s(X) \subseteq K(X)$.

Definition 2.8. Let X be a T_0 -space and $A, B \in \mathbf{up}(X)$. We say that A is *relatively compact* in B , written $A \ll_o B$, if every open cover of B has a finite subcover of A . A family $\mathcal{A} \subseteq \mathbf{up}(X)$ is called *\ll_o -filtered* if for any $A_1, A_2 \in \mathcal{A}$, there exists $A_3 \in \mathcal{A}$ such that $A_3 \ll_o A_1$ and $A_3 \ll_o A_2$.

Remark 2.9. Let X be a T_0 -space and $\mathcal{A} \subseteq \mathbf{up}(X)$ be \ll_o -filtered. Then for any $A \in \mathcal{A}$, there is $B \in \mathcal{A}$ such that $B \ll_o A$. Indeed, let $A_1 = A_2 = A$. Then, as \mathcal{A} is \ll_o -filtered, there is $B \in \mathcal{A}$ such that $B \ll_o A$.

Remark 2.10. For a T_0 -space X , the relation \ll_o on $\mathbf{up}(X)$ has the following properties:

- (1) $A \ll_o B$ implies $A \subseteq B$.
- (2) $C \subseteq A \ll B \subseteq D$ implies $C \ll_o D$.
- (3) $A \ll_o C$ and $B \ll_o C$ imply $A \cup B \ll_o C$.

Lemma 2.11. For a T_0 -space X and $\mathcal{A} \subseteq \mathbf{up}(X)$, the following two conditions are equivalent:

- (1) \mathcal{A} is \ll_o -filtered.
- (2) For any $A_1, A_2 \in \mathcal{A}$, there exists $A_3 \in \mathcal{A}$ such that $A_3 \ll_o A_1 \cap A_2$.

Proof. (1) \Rightarrow (2): Let $A_1, A_2 \in \mathcal{A}$. Then there is $A_0 \in \mathcal{A}$ such that $A_0 \ll_o A_1$ and $A_0 \ll_o A_2$, so $A_0 \subseteq A_1 \cap A_2$. By Remark 2.9, there is $A_3 \in \mathcal{A}$ with $A_3 \ll_o A_0$. Hence $A_3 \ll_o A_1 \cap A_2$.

(2) \Rightarrow (1): If $A_1, A_2 \in \mathcal{A}$, then by (2) there exists $A_3 \in \mathcal{A}$ such that $A_3 \ll_o A_1 \cap A_2$. By Remark 2.10 (2), we have $A_3 \ll_o A_1$ and $A_3 \ll_o A_2$. Thus \mathcal{A} is \ll_o -filtered. \square

Proposition 2.12. Let $f : X \rightarrow Y$ be continuous and $A, B \in \mathbf{up}(X)$. If $A \ll_o B$, then $\uparrow f(A) \ll_o \uparrow f(B)$.

Proof. Let $\{V_i : i \in I\}$ be an open cover of $\uparrow f(B)$. Then $B \subseteq f^{-1}(\uparrow f(B)) \subseteq f^{-1}(\bigcup_{i \in I} V_i) = \bigcup_{i \in I} f^{-1}(V_i)$. By the continuity of f , we have $\{f^{-1}(V_i) : i \in I\} \subseteq \mathcal{O}(X)$. As $A \ll_o B$, there exists a finite subset $J \in I^{(<\omega)}$ such that $A \subseteq \bigcup_{i \in J} f^{-1}(V_i)$. Then $f(A) \subseteq f(\bigcup_{i \in J} f^{-1}(V_i)) = \bigcup_{i \in J} f(f^{-1}(V_i)) \subseteq \bigcup_{i \in J} V_i$, so $\uparrow f(A) \subseteq \bigcup_{i \in J} V_i$. Hence $\uparrow f(A) \ll_o \uparrow f(B)$. \square

3. A generalized topological Rudin lemma

Rudin's Lemma, due to Mary Rudin [9], plays a crucial role in domain theory (see [3–5]). In [7], Heckmann and Keimel presented the following topological variant of Rudin's Lemma.

Lemma 3.1. (Topological Rudin lemma) *Let X be a topological space and $\mathcal{A} \in \text{Irr}(P_S(X))$. Then, every nonempty closed set C of X that meets all members of \mathcal{A} contains a minimal irreducible closed subset A that still meets all members of \mathcal{A} .*

For a poset P , applying Lemma 3.1 to $(P, \alpha(P))$, one obtains the original Rudin's Lemma.

Lemma 3.2. (Rudin's Lemma) *Let P be a poset, C a nonempty lower subset of P , and $\mathcal{F} \subseteq \mathbf{Fin}P$ a filtered family with $\mathcal{F} \subseteq \diamond C$. Then there exists a directed subset D of C such that $\mathcal{F} \subseteq \diamond \downarrow D$.*

In order to generalize the topological Rudin lemma, we introduce the following concept.

Definition 3.3. Let X be a T_0 -space. A nonempty subfamily $\mathcal{F} \subseteq 2^X$ is called an F -family if it satisfies the following two conditions:

- (F1) For any $F \in \mathcal{F}$ and open cover $\{U_i : i \in I\}$ of F , there exist $G \in \mathcal{F}$ and $J \in I^{(<\omega)}$ such that $G \subseteq \bigcup_{i \in J} U_i$;
 (F2) For any $F_1, F_2 \in \mathcal{F}$ and $U_1, U_2 \in \mathcal{O}(X)$ with $F_1 \subseteq U_1$ and $F_2 \subseteq U_2$, there exists $F_3 \in \mathcal{F}$ such that $F_3 \subseteq U_1 \cap U_2$.

Remark 3.4. For a T_0 -space X and $\mathcal{K} \subseteq \mathcal{K}(X)$, \mathcal{K} is an F -family iff \mathcal{K} satisfies (F2).

We now present five common kinds of F -families (see Lemma 3.5, Proposition 3.6, Corollary 3.9, Corollary 3.10, and Corollary 3.11 below).

Lemma 3.5. For a T_0 -space X and a nonempty subset A of X , the following conditions are equivalent:

- (1) $A \in \text{lrr}(X)$.
- (2) $\{\uparrow a : a \in A\}$ is an F -family.
- (3) $\{\uparrow a : a \in A\}$ satisfies (F2).

Proof. (1) \Rightarrow (2): Let $a \in A$ and $\{U_i : i \in I\}$ be an open cover of $\uparrow a$. Then, there is $i \in I$ such that $\uparrow a \subseteq U_i$. Hence $\{\uparrow a : a \in A\}$ satisfies (F1). Assume that $a_1, a_2 \in A$ and $U_1, U_2 \in \mathcal{O}(X)$ with $\uparrow a_1 \subseteq U_1$ and $\uparrow a_2 \subseteq U_2$. Then by the irreducibility of A , $A \cap U_1 \cap U_2 \neq \emptyset$. Choose $a_3 \in A \cap U_1 \cap U_2$. Then $\uparrow a_3 \subseteq U_1 \cap U_2$. So $\{\uparrow a : a \in A\}$ satisfies (F2). Thus, $\{\uparrow a : a \in A\}$ is an F -family.

(2) \Rightarrow (3): Trivial.

(3) \Rightarrow (1): Assume that $U_1, U_2 \in \mathcal{O}(X)$ and $A \cap U_1 \neq \emptyset \neq A \cap U_2$. Choose $a_1 \in A \cap U_1$ and $a_2 \in A \cap U_2$. Then $\uparrow a_1 \subseteq U_1$ and $\uparrow a_2 \subseteq U_2$. As $\{\uparrow a : a \in A\}$ satisfies (F2), there exists $a_3 \in A$ such that $\uparrow a_3 \subseteq U_1 \cap U_2$, so $A \cap U_1 \cap U_2 \neq \emptyset$. Hence $A \in \text{lrr}(X)$. \square

Proposition 3.6. Let X be a T_0 -space and $\mathcal{F} \subseteq \mathbf{up}(X)$ be $\ll_{\mathcal{O}}$ -filtered. Then \mathcal{F} is an F -family.

Proof. Let $F \in \mathcal{F}$ and $\{U_i : i \in I\}$ be an open cover of F . Then by Remark 2.9, there exists $G \in \mathcal{F}$ with $G \ll_{\mathcal{O}} F$. So there is $J \in I^{(<\omega)}$ such that $G \subseteq \bigcup_{i \in J} U_i$. Hence \mathcal{F} satisfies condition (F1). Suppose that $F_1, F_2 \in \mathcal{F}$ and $U_1, U_2 \in \mathcal{O}(X)$ with $F_1 \subseteq U_1$ and $F_2 \subseteq U_2$. Then by Lemma 2.11, there is $F_3 \in \mathcal{F}$ such that $F_3 \ll_{\mathcal{O}} F_1 \cap F_2$, whence $F_3 \subseteq F_1 \cap F_2 \subseteq U_1 \cap U_2$. Thus, \mathcal{F} satisfies condition (F2), and hence \mathcal{F} is an F -family. \square

For a T_0 -space X , $A \subseteq X$, and $\mathcal{F} \subseteq 2^X$, define $\Phi(A) = \{U \in \mathcal{O}(X) : A \subseteq U\}$ and $\Phi(\mathcal{F}) = \bigcup_{F \in \mathcal{F}} \Phi(F)$.

Remark 3.7. For a T_0 -space X , $A \subseteq X$, and $\mathcal{F} \subseteq 2^X$, we have

- (1) $\bigcap \Phi(A) = \uparrow A$;
- (2) $\bigcap \Phi(\mathcal{F}) = \bigcap_{F \in \mathcal{F}} \uparrow F$.

Theorem 3.8. Let X be a T_0 -space and $\mathcal{F} \subseteq \mathbf{up}(X)$ be a nonempty family. Consider the following conditions:

- (1) \mathcal{F} is an F -family.
- (2) $\Phi(\mathcal{F}) \in \text{OFilt}(\mathcal{O}(X))$.

- (3) $\Phi(\mathcal{F})$ is a filter.
 (4) $\mathcal{F} \in \text{lrr}(P_S(\mathcal{F}))$.
 (5) For every $\mathcal{G} \subseteq \mathbf{up}(X)$ containing \mathcal{F} , $\mathcal{F} \in \text{lrr}(P_S(\mathcal{G}))$.
 (6) There is $\mathcal{G} \subseteq \mathbf{up}(X)$ containing \mathcal{F} such that $\mathcal{F} \in \text{lrr}(P_S(\mathcal{G}))$.
 (7) $\mathcal{F} \in \text{lrr}(P_S(\mathbf{up}(X)))$.
 (8) \mathcal{F} satisfies (F2).

Then (1) \Leftrightarrow (2) \Rightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5) \Leftrightarrow (6) \Leftrightarrow (7) \Leftrightarrow (8), and if $\mathcal{F} \subseteq \mathbf{K}(X)$, then all eight conditions are equivalent.

Proof. (1) \Rightarrow (2): Clearly, $X \in \Phi(\mathcal{F}) = \uparrow_{\mathcal{O}(X)}\Phi(\mathcal{F})$ and $\emptyset \notin \Phi(\mathcal{F})$. If $\{U_d : d \in D\} \in \mathcal{D}(\mathcal{O}(X))$ and $\bigcup_{d \in D} U_d \in \Phi(\mathcal{F})$, then there is $F \in \mathcal{F}$ with $F \subseteq \bigcup_{d \in D} U_d$. As \mathcal{F} satisfies (F1), there exist $G \in \mathcal{F}$ and $\{d_1, \dots, d_n\} \subseteq D$ such that $G \subseteq \bigcup_{i=1}^n U_{d_i}$. By the directedness of $\{U_d : d \in D\}$, there is $d_0 \in D$ such that $\bigcup_{i=1}^n U_{d_i} \subseteq U_{d_0}$, whence $U_{d_0} \in \Phi(G) \subseteq \Phi(\mathcal{F})$. Thus $\Phi(\mathcal{F}) \in \sigma(\mathcal{O}(X))$. Next we show that $\Phi(\mathcal{F})$ is a filter. Let $U, V \in \Phi(\mathcal{F})$. Then there are $F_1, F_2 \in \mathcal{F}$ such that $F_1 \subseteq U$ and $F_2 \subseteq V$. As \mathcal{F} satisfies (F2), there exists $F_3 \in \mathcal{F}$ with $F_3 \subseteq U \cap V$, so $U \cap V \in \Phi(F_3) \subseteq \Phi(\mathcal{F})$. Therefore, $\Phi(\mathcal{F})$ is a filter, and hence $\Phi(\mathcal{F}) \in \text{OFilt}(\mathcal{O}(X))$.

(2) \Rightarrow (1): Let $F \in \mathcal{F}$ and $\{U_i : i \in I\}$ be an open cover of F . Then $\bigcup_{i \in I} U_i \in \Phi(F) \subseteq \Phi(\mathcal{F}) \in \sigma(\mathcal{O}(X))$. So there exists $J \in I^{(<\omega)}$ such that $\bigcup_{i \in J} U_i \in \Phi(\mathcal{F})$, and consequently, there is $G \in \mathcal{F}$ with $G \subseteq \bigcup_{i \in J} U_i$. Hence \mathcal{F} satisfies (F1). Suppose that $F_1, F_2 \in \mathcal{F}$ and $U_1, U_2 \in \mathcal{O}(X)$ with $F_1 \subseteq U_1$ and $F_2 \subseteq U_2$. Then $U_1, U_2 \in \Phi(F_1) \cup \Phi(F_2) \subseteq \Phi(\mathcal{F})$. As $\Phi(\mathcal{F})$ is a filter, $U_1 \cap U_2 \in \Phi(\mathcal{F})$, so $F_3 \subseteq U_1 \cap U_2$ for some $F_3 \in \mathcal{F}$. Thus \mathcal{F} satisfies (F2), and hence it is an F -family.

(2) \Rightarrow (3): Trivial.

(3) \Rightarrow (4): Let $U, V \in \mathcal{O}(X)$ with $\mathcal{F} \cap \square U \neq \emptyset \neq \mathcal{F} \cap \square V$. Then there exist $F_1, F_2 \in \mathcal{F}$ such that $F_1 \subseteq U$ and $F_2 \subseteq V$, and consequently, $U, V \in \Phi(\mathcal{F})$. As $\Phi(\mathcal{F})$ is a filter, we have $U \cap V \in \Phi(\mathcal{F})$. Then $F_3 \subseteq U \cap V$ for some $F_3 \in \mathcal{F}$. Hence $F_3 \in \mathcal{F} \cap \square(U \cap V) = \mathcal{F} \cap \square U \cap \square V$. Thus $\mathcal{F} \in \text{lrr}(P_S(\mathcal{F}))$.

(4) \Rightarrow (5): By Lemma 2.3 and Remark 2.6.

(5) \Rightarrow (6): Trivial.

(6) \Rightarrow (7): By Lemma 2.3 and Remark 2.6.

(7) \Rightarrow (8): Let $F_1, F_2 \in \mathcal{F}$ and $(U_1, U_2) \in \Phi(F_1) \times \Phi(F_2)$. Then $\mathcal{F} \cap \square U_1 \neq \emptyset \neq \mathcal{F} \cap \square U_2$. As $\mathcal{F} \in \text{lrr}(P_S(\mathbf{up}(X)))$, we have $\mathcal{F} \cap \square(U_1 \cap U_2) = \mathcal{F} \cap \square U_1 \cap \square U_2 \neq \emptyset$, so there exists $F_3 \in \mathcal{F}$ with $F_3 \subseteq U_1 \cap U_2$. Hence \mathcal{F} satisfies (F2).

(8) \Rightarrow (3): Let $U, V \in \Phi(\mathcal{F})$. Then there are $F_1, F_2 \in \mathcal{F}$ such that $F_1 \subseteq U$ and $F_2 \subseteq V$. Since \mathcal{F} satisfies (F2), there exists $F_3 \in \mathcal{F}$ with $F_3 \subseteq U \cap V$. Then $U \cap V \in \Phi(F_3) \subseteq \Phi(\mathcal{F})$. Thus $\Phi(\mathcal{F})$ is a filter.

(8) \Rightarrow (1): Assume $\mathcal{F} \subseteq \mathbf{K}(X)$. Clearly, \mathcal{F} satisfies (F1), so by (8) it is an F -family. \square

Corollary 3.9. For a T_0 -space X and a nonempty family $\mathcal{F} \subseteq \mathbf{K}(X)$, the following conditions are equivalent:

- (1) \mathcal{F} is an F -family.
 (2) $\Phi(\mathcal{F}) \in \text{OFilt}(\mathcal{O}(X))$.
 (3) $\Phi(\mathcal{F})$ is a filter.
 (4) $\mathcal{F} \in \text{lrr}(P_S(X))$.

(5) \mathcal{F} satisfies (F2).

For a T_0 -space X and $\mathcal{F} \subseteq \mathcal{O}(X)$, it is clear that $\Phi(\mathcal{F}) = \uparrow_{\mathcal{O}(X)}\mathcal{F}$. Hence by Theorem 3.8, we have the following.

Corollary 3.10. *For a T_0 -space X and a nonempty family $\mathcal{F} \subseteq \mathcal{O}(X) \setminus \{\emptyset\}$, the following two conditions are equivalent:*

- (1) \mathcal{F} is an F -family.
- (2) $\uparrow_{\mathcal{O}(X)}\mathcal{F} \in \text{OFilt}(\mathcal{O}(X))$.

Therefore, every Scott open proper filter of $\mathcal{O}(X)$ is an F -family.

For a T_0 -space X and a nonempty subset A of X , define $\Psi_A = \{U \in \mathcal{O}(X) : U \cap A \neq \emptyset\}$. Clearly, $\Psi_A = \Phi(\{\uparrow a : a \in A\})$. So by Lemma 3.5 and Theorem 3.8 (or Corollary 3.9), we get the following.

Corollary 3.11. *For a T_0 -space X and a nonempty set A of X , the following conditions are equivalent:*

- (1) $A \in \text{lrr}(X)$.
- (2) $\Psi_A \in \text{OFilt}(\mathcal{O}(X))$.
- (3) Ψ_A is a filter.
- (4) Ψ_A is an F -family.

For a T_0 -space X and $\mathcal{F} \subseteq 2^X$, let $M(\mathcal{F}) = \{A \in \Gamma(X) : A \cap F \neq \emptyset \text{ for all } F \in \mathcal{F}\}$ (i.e., $\mathcal{F} \subseteq \diamond B$) and $m(\mathcal{F}) = \min(M(\mathcal{F}))$.

Now we give the following generalized topological Rudin lemma.

Lemma 3.12. (Generalized topological Rudin lemma) *Let X be a T_0 -space and $\mathcal{F} \subseteq \mathbf{up}(X)$ an F -family of X . Then every nonempty closed set C of X that intersects all members of \mathcal{F} contains a minimal irreducible closed subset A that still intersects all members of \mathcal{F} .*

Proof. Let $C = \{B \in \Gamma(X) : B \subseteq C \text{ and } B \in M(\mathcal{F})\}$.

Claim 1: $C \neq \emptyset$ since $C \in C$.

Claim 2: C is closed under filtered intersections.

Let $\{C_d : d \in D\} \subseteq C$ be filtered. If $\bigcap_{d \in D} C_d \notin C$, then there is $F \in \mathcal{F}$ such that $F \cap \bigcap_{d \in D} C_d = \emptyset$, or equivalently, $F \subseteq \bigcup_{d \in D} (X \setminus C_d)$. As \mathcal{F} satisfies (F1) and $\{X \setminus C_d : d \in D\} \in \mathcal{D}(\mathcal{O}(X))$, there exist $G \in \mathcal{F}$ and $d \in D$ such that $G \subseteq X \setminus C_d$, contradicting $C_d \in C$.

Claim 3: By Claim 2 and the order-dual of Zorn's Lemma, C contains a minimal element A .

Claim 4: $A \in \text{lrr}_c(X)$.

Assume that $C_1, C_2 \in \Gamma(X)$ and $A \subseteq C_1 \cup C_2$. Then $A = A \cap (C_1 \cup C_2) = (A \cap C_1) \cup (A \cap C_2)$. If $A \neq A \cap C_1$ and $A \neq A \cap C_2$, then by the minimality of A , we have $A \cap C_k \notin C$ ($k = 1, 2$). Hence there are $F_1, F_2 \in \mathcal{F}$ such that $F_1 \cap A \cap C_1 = \emptyset$ and $F_2 \cap A \cap C_2 = \emptyset$, whence $F_1 \subseteq X \setminus (A \cap C_1)$ and $F_2 \subseteq X \setminus (A \cap C_2)$. As \mathcal{F} satisfies (F2), there exists $F_3 \in \mathcal{F}$ such that $F_3 \subseteq (X \setminus (A \cap C_1)) \cap (X \setminus (A \cap C_2)) = X \setminus ((A \cap C_1) \cup (A \cap C_2)) = X \setminus A$ (using $A = (A \cap C_1) \cup (A \cap C_2)$), contradicting $A \in C$. Therefore, $A = A \cap C_1$ or $A = A \cap C_2$, so $A \subseteq C_1$ or $A \subseteq C_2$. Thus $A \in \text{lrr}_c(X)$. \square

For a T_0 -space X , applying Corollary 3.9 and Lemma 3.12 to all $\mathcal{A} \in \text{lrr}(P_S(X))$, one obtains Lemma 3.1 (the topological Rudin lemma).

4. Characterizations of sober spaces

In this section, using the generalized topological Rudin lemma, we give some characterizations of sober spaces and show that the Hofmann-Mislove theorem and Heckman-Keimel-Schalk theorem can be directly deduced from these characterizations.

Theorem 4.1. *For a T_0 -space X , the following conditions are equivalent:*

- (1) X is sober.
- (2) For every F -family $\mathcal{F} \subseteq \mathbf{up}(X)$ and $U \in \mathcal{O}(X)$, $\bigcap \mathcal{F} \subseteq U$ implies there exists $F \in \mathcal{F}$ such that $F \subseteq U$.
- (3) For every F -family $\mathcal{F} \subseteq \mathcal{O}(X) \setminus \{\emptyset\}$ and $U \in \mathcal{O}(X)$, $\bigcap \mathcal{F} \subseteq U$ implies there exists $F \in \mathcal{F}$ such that $F \subseteq U$.
- (4) For every F -family $\mathcal{F} \subseteq \mathbf{K}(X)$ and $U \in \mathcal{O}(X)$, $\bigcap \mathcal{F} \subseteq U$ implies there exists $F \in \mathcal{F}$ such that $F \subseteq U$.
- (5) For every F -family $\mathcal{F} \subseteq \mathbf{K}_s(X)$ and $U \in \mathcal{O}(X)$, $\bigcap \mathcal{F} \subseteq U$ implies there exists $F \in \mathcal{F}$ such that $F \subseteq U$.
- (6) For every F -family $\mathcal{F} \subseteq \mathbf{Fin} X$ and $U \in \mathcal{O}(X)$, $\bigcap \mathcal{F} \subseteq U$ implies there exists $F \in \mathcal{F}$ such that $F \subseteq U$.
- (7) For every F -family $\mathcal{F} \subseteq \mathcal{S}^u(X)$ and $U \in \mathcal{O}(X)$, $\bigcap \mathcal{F} \subseteq U$ implies there exists $F \in \mathcal{F}$ such that $F \subseteq U$.

Proof. (1) \Rightarrow (2): Let $\mathcal{F} \subseteq \mathbf{up}(X)$ be an F -family and U an open set containing $\bigcap \mathcal{F}$. If $F \subseteq U$ for no $F \in \mathcal{F}$, then $F \cap (X \setminus U) \neq \emptyset$ for all $F \in \mathcal{F}$. By Lemma 3.12, $X \setminus U$ contains a minimal irreducible closed set A intersecting all $F \in \mathcal{F}$. By the sobriety of X , $A = \overline{\{x\}}$ for some $x \in X$. Then $x \in \bigcap \mathcal{F} \subseteq U$, contradicting $x \in A \subseteq X \setminus U$. Hence $F \subseteq U$ for some $F \in \mathcal{F}$.

(2) \Rightarrow (3), (2) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7): By $\mathcal{O}(X) \setminus \{\emptyset\} \subseteq \mathbf{up}(X)$ and $\mathcal{S}^u(X) \subseteq \mathbf{Fin} X \subseteq \mathbf{K}_s(X) \subseteq \mathbf{K}(X) \subseteq \mathbf{up}(X)$.

(3) \Rightarrow (1): Let $A \in \text{Irr}_c(X)$. Then $\Psi_A \subseteq \mathcal{O}(X) \setminus \{\emptyset\}$, and by Lemma 3.5, Ψ_A is an F -family. Clearly, $U \not\subseteq X \setminus A$ for all $U \in \Psi_A$. It follows from (3) that $\bigcap \Psi_A \cap A \neq \emptyset$. Choose $x \in \bigcap \Psi_A \cap A$. Then $\overline{\{x\}} \subseteq A$. If $A \not\subseteq \{x\}$, then $X \setminus \{x\} \in \Psi_A$, whence $x \in X \setminus \{x\}$, a contradiction. So $A \subseteq \{x\}$, and hence $A = \{x\}$. Thus X is sober.

(7) \Rightarrow (1): Let $A \in \text{Irr}_c(X)$. Then by Lemma 3.5, $\{\uparrow a : a \in A\}$ is an F -family. Clearly, $\uparrow a \not\subseteq X \setminus A$ for all $a \in A$. Hence by (7), we have $\bigcap_{a \in A} \uparrow a \cap A \neq \emptyset$. Choose $x \in \bigcap_{a \in A} \uparrow a \cap A$. Then $A \subseteq \downarrow x = \overline{\{x\}} \subseteq A$, whence $A = \{x\}$. Therefore, X is sober. \square

Corollary 4.2. *For a sober space X and an F -family $\mathcal{F} \subseteq \mathbf{up}(X)$, $\bigcap \mathcal{F} \in \mathbf{K}(X)$. In particular, if $\mathcal{F} \in \text{Irr}(P_S(X))$, then $\bigcap \mathcal{F} \in \mathbf{K}(X)$.*

Proof. First, we claim that $\bigcap \mathcal{F} \neq \emptyset$. Assume, on the contrary, that $\bigcap \mathcal{F} = \emptyset$. Then by Theorem 4.1, there is $F \in \mathcal{F}$ such that $F = \emptyset$, a contradiction. Hence $\bigcap \mathcal{F} \neq \emptyset$. Now we show that $\bigcap \mathcal{F}$ is compact. Suppose that $\{U_i : i \in I\}$ is an open cover of $\bigcap \mathcal{F}$. Then by Theorem 4.1, there is $G \in \mathcal{F}$ such that $G \subseteq \bigcup_{i \in I} U_i$. As \mathcal{F} satisfies (F1), there exist $H \in \mathcal{F}$ and $J \in I^{(<\omega)}$ such that $H \subseteq \bigcup_{i \in J} U_i$, whence $\bigcap \mathcal{F} \subseteq H \subseteq \bigcup_{i \in J} U_i$. Therefore, $\bigcap \mathcal{F}$ is compact, and hence $\bigcap \mathcal{F} \in \mathbf{K}(X)$.

If $\mathcal{F} \in \text{Irr}(P_S(X))$, then by Corollary 3.9, \mathcal{F} is an F -family, whence $\bigcap \mathcal{F} \in \mathbf{K}(X)$. \square

By Remark 3.7, Corollary 3.10, and Theorem 4.1, we get the following.

Corollary 4.3. (Hofmann-Mislove Theorem) ([3, Theorem II-2.10 and Theorem 2.11]) For a T_0 -space X , the following conditions are equivalent:

- (1) X is sober.
- (2) For any $\mathcal{F} \in \text{OFilt}(\mathcal{O}(X))$ and $U \in \mathcal{O}(X)$, $\bigcap \mathcal{U} \subseteq U$ implies there exists $V \in \mathcal{F}$ such that $V \subseteq U$.
- (3) $\text{OFilt}(\mathcal{O}(X)) = \{\Phi(K) : K \in \mathbf{K}(X)\}$.
- (4) The mapping $\Phi : \mathbf{K}(X) \rightarrow \text{OFilt}(\mathcal{O}(X))$, $K \mapsto \{U \in \mathcal{O}(X) : K \subseteq U\}$, is an order isomorphism between $\mathbf{K}(X)$ (with the Smyth order) and $\text{OFilt}(\mathcal{O}(X))$. The inverse is $\Psi : \text{OFilt}(\mathcal{O}(X)) \rightarrow \mathbf{K}(X)$, $\mathcal{F} \mapsto \bigcap \mathcal{F}$.

Corollary 4.4. (Heckmann-Keimel-Schalk Theorem) ([7, Theorem 3.13] and [10, Lemma 7.20]) For a T_0 -space X , the following conditions are equivalent:

- (1) X is sober.
- (2) For any $\mathcal{A} \in \text{Irr}(P_S(X))$ and $U \in \mathcal{O}(X)$, $\bigcap \mathcal{A} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathcal{A}$.
- (3) $P_S(X)$ is sober.

Proof. (1) \Leftrightarrow (2): By Corollary 3.9 and Theorem 4.1.

(2) \Rightarrow (3): Assume $\mathcal{A} \in \text{Irr}_c(P_S(X))$ and let $K = \bigcap \mathcal{A}$. By (2) we have $K \in \mathbf{K}(X)$. Clearly, $\mathcal{A} \subseteq \downarrow_{\mathbf{K}(X)} K = \text{cl}_{P_S(X)}\{K\}$. If $K \notin \mathcal{A}$, then $K \in \mathbf{K}(X) \setminus \mathcal{A} \in \mathcal{O}(P_S(X))$, and consequently, there is $U \in \mathcal{U}$ such that $K \in \square U \subseteq \mathbf{K}(X) \setminus \mathcal{A}$. By $\bigcap \mathcal{A} = K \subseteq U$ and (2), there is $G \in \mathcal{A}$ such that $G \subseteq U$. Then $G \in \square U \subseteq \mathbf{K}(X) \setminus \mathcal{A}$, contradicting $G \in \mathcal{A}$. Hence $K \in \mathcal{A}$, so $\text{cl}_{P_S(X)}\{K\} \subseteq \mathcal{A}$. Therefore, $\mathcal{A} = \text{cl}_{P_S(X)}\{K\}$, and this completes the proof.

(3) \Rightarrow (1): Let $A \in \text{Irr}_c(X)$. Then by Lemma 2.4, Remark 2.6(2) and Lemma 2.7, $\{\uparrow a : a \in A\} \in \text{Irr}(P_S(X))$ and $\text{cl}_{P_S(X)}\{\uparrow a : a \in A\} = \diamond_{\mathbf{K}(X)} A$. By Theorem 2.5, we have $\bigcap_{a \in A} \uparrow_{\mathbf{K}(X)} a \cap \diamond_{\mathbf{K}(X)} A \neq \emptyset$. Choose $K \in \bigcap_{a \in A} \uparrow_{\mathbf{K}(X)} a \cap \diamond_{\mathbf{K}(X)} A$. Then $K \cap A \neq \emptyset$ and $K \subseteq \bigcap_{a \in A} \uparrow a$. Select an $x \in K \cap A$. Then $A \subseteq \downarrow x = \overline{\{x\}} \subseteq A$, so $A = \overline{\{x\}}$. Thus, X is sober. \square

5. Conclusions

In this paper, we establish a generalized topological Rudin lemma via the new concept of F -families. This concept unifies several important classes of objects in domain theory and non-Hausdorff topology, including irreducible sets, filtered families of upper sets, Scott open filters, and irreducible subsets in Smyth power spaces. As an application of the generalized topological Rudin lemma, we obtain some characterizations of sober spaces, from which the Hofmann-Mislove Theorem and Heckman-Keimel-Schalk Theorem can be directly deduced.

Following the ideas and methods presented in this paper, we will establish some new characterizations of well-filteredness of T_0 -spaces in our future work.

Author contributions

Xiaoquan Xu: Conceptualization, methodology, writing-original draft, funding acquisition; Yi Yang: Investigation, writing-review and editing; Lizi Chen: Investigation, editing the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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