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Research article

Mild solutions and controllability of (k, φ)-Hilfer fractional delay differential equations with history-dependent operators

Doha A. Kattan[1], **Hasanen A. Hammad**[2,*] **and Najat Almutairi**[2,*]

[1] Department of Mathematics, College of Sciences and Art, King Abdulaziz University, Rabigh, Saudi Arabia

[2] Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia

* **Correspondence:** Email: h.abdelwareth@qu.edu.sa, nbalmutairi@qu.edu.sa.

Abstract: The objective of this work was to investigate a class of Hilfer-type fractional differential equations governed by the interaction of semigroup operators, history-dependent mechanisms, and probability density functions, with particular attention to their analytical and control properties. First, the existence of mild solutions was established through semigroup theory and fixed-point arguments tailored to fractional dynamics. We then addressed the controllability problem for the corresponding (k, φ)-Hilfer fractional delay differential equation, taking into account the influence of memory and delay effects induced by history-dependent operators. By combining Mönch's fixed-point theorem with the measure of noncompactness, a set of sufficient conditions for controllability was obtained. This approach not only captures the complexity of the system but also deepens the understanding of how fractional-order behavior and past-state dependence affect controllability.

Keywords: fractional derivative; Hilfer operator; fixed-point technique; history-dependent operator; measure of noncompactness; delay term
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1. Introduction

In recent years, fractional calculus has attracted considerable attention as a rapidly developing field that extends the classical notions of differentiation and integration to non-integer orders, thereby providing a richer and more flexible analytical framework. This generalization has proven highly effective in modeling real-world phenomena characterized by memory, hereditary properties, and complex dynamics that cannot be adequately described by traditional integer-order models, which has led to the widespread use of fractional differential equations as powerful mathematical tools offering more accurate and realistic descriptions in various disciplines, including engineering, physics, biology, finance, and image processing [1–3]. Their ability to capture anomalous diffusion, viscoelastic behavior, and long-range temporal dependence underscores their importance in both theoretical and applied research. For a comprehensive treatment of the theory and its applications, the reader is referred to the standard monographs [4–6], together with the growing body of research articles [7, 8], which highlight the depth and scope of this active area of study. Fractional derivatives are particularly significant because they extend classical differentiation to non-integer orders, naturally incorporating memory and hereditary effects so that past states influence current dynamics, making them especially suitable for describing complex phenomena such as viscoelasticity and anomalous diffusion [9–11]; as a result, they provide more accurate and flexible models in fields such as physics, engineering, and biology, while also unifying various classical operators and enabling deeper theoretical analysis of dynamical systems [12, 13].

Moreover, control theory is devoted to the systematic analysis and design of dynamical systems whose behavior can be influenced or regulated through suitable external inputs. A fundamental question in this discipline is whether it is possible to steer a system from a given initial state to a prescribed final configuration within a finite time interval using admissible control functions. This leads to the concept of controllability, which plays a central role in understanding the inherent capabilities and limitations of a system, as well as in guiding the construction of effective control strategies [14, 15]. In particular, the notions of exact and approximate controllability have been extensively studied, depending on whether the target state is reached precisely or within an arbitrarily small neighborhood. These concepts are especially significant in infinite-dimensional systems, where practical constraints often necessitate weaker forms of controllability. Over the past decades, substantial progress has been made in analyzing controllability for a wide range of systems, including those governed by differential equations, delay effects, and fractional dynamics. For further developments and detailed discussions, the reader may consult the extensive literature, including [16–18], where various theoretical approaches and applications are explored.

Hilfer fractional differential equations constitute a significant generalization within the theory of fractional calculus, providing a unified framework that bridges the classical Riemann–Liouville and Caputo formulations. This generality stems from the presence of an additional parameter, $\eta \in [0, 1]$, which enables a smooth interpolation between these two standard fractional derivatives, thereby offering greater flexibility in modeling. Owing to this feature, Hilfer-type operators are particularly well-suited for describing systems with memory and hereditary characteristics, where the underlying dynamics cannot be captured adequately by integer-order models. As a result, Hilfer fractional differential equations have found extensive applications in diverse areas such as viscoelasticity, diffusion processes, control systems, and other branches of applied science and engineering. Their ability to incorporate both local and nonlocal effects makes them a powerful and versatile tool for analyzing complex phenomena exhibiting intermediate behaviors between classical fractional models, see [19, 20].

Kucche and Mali [21] established, in 2021, important results concerning the existence and uniqueness of solutions for nonlinear (k, φ)-Hilfer fractional differential equations, thereby contributing significantly to the theoretical foundation of this class of problems. A key advantage of this generalized fractional derivative lies in the flexibility of its kernel function φ, which can be appropriately chosen to suit different modeling requirements. This adaptability enables the (k, φ)-Hilfer operator to unify and extend a wide range of previously studied fractional derivatives within a single coherent framework. In particular, as shown in [22], several well-known fractional derivatives can be recovered as special cases of the (k, φ)-Hilfer formulation, highlighting its generality and its potential as a powerful tool for advancing the study of fractional differential equations.

History-dependent operators are the mathematical equivalent of a system having a memory, meaning their current output depends on the entire history of their input, not just the present state. This crucial feature makes them indispensable for modeling real-world phenomena like hysteresis in materials or the time-delayed responses in biological processes. Their inclusion in advanced analytical frameworks, such as establishing conditions for variational inequalities and designing optimal feedback control for systems with fractional (non-integer order) dynamics, is essential for accurately capturing the full, progressive influence of past events on a system's current behavior. For more details, see [23–25].

The ability to control a linear system lies at the heart of many fundamental problems in control theory, including pole assignment, structural stabilization, and the formulation of optimal control strategies [26, 27]. In this context, the present work is devoted to the study of controllability for (k, φ)-Hilfer fractional delay differential equations involving history-dependent operators, where both memory effects and delayed responses significantly influence the system dynamics. The inclusion of the (k, φ)-Hilfer derivative provides a highly flexible modeling framework that unifies and extends several existing fractional operators, thereby enabling a more comprehensive analysis of complex systems. A notable feature of this study is its ability to recover and generalize earlier controllability results established for Hilfer and φ-Hilfer fractional differential equations through appropriate selections of the functions k and φ [28–30]. Furthermore, the presence of history-dependent operators introduces additional analytical challenges, as the system evolution depends not only on the current state but also on its past behavior. By addressing these aspects, this paper contributes to a deeper understanding of controllability in fractional systems with memory and delay. For broader perspectives on controllability theory and its applications, the reader may refer to [31–33], while related results on the existence of mild solutions can be found in [9, 34, 35].

The study of controllability in fractional systems has attracted considerable attention in recent years, particularly for models involving Hilfer-type derivatives and delay effects. In this direction, Kavitha et al. [28] investigated both controllability and approximate controllability for Hilfer fractional neutral differential inclusions with infinite delay, highlighting the role of memory and hereditary properties in system dynamics. In their subsequent work [36], they further examined the controllability of Hilfer fractional differential equations with infinite delay by employing the measure of noncompactness in combination with Mönch's fixed-point theorem, providing a robust analytical framework for such problems. Parallel contributions by the same authors in [37] continued this line of research, focusing on Hilfer fractional neutral differential equations and consistently utilizing measure of noncompactness techniques to establish controllability results under more general conditions. Extending these developments, Thakur and Ali [38] studied the existence of mild solutions and the controllability of (k, ϕ)-Hilfer fractional differential equations with infinite delay, thereby incorporating an additional level of generalization through the (k, ϕ) framework. Moreover, Haque et al. [33] addressed the controllability of fractional dynamical systems governed by the (k, ψ)-Hilfer fractional derivative, further enriching the literature by exploring more generalized operators. These contributions collectively demonstrate the progressive evolution of controllability theory in fractional settings and underscore the importance of advanced analytical tools in handling increasingly complex models with delay and memory effects.

Recent developments in fractional calculus have witnessed a growing interest in generalized k-fractional derivatives and integrals, which substantially extend classical formulations and offer a unified platform for analyzing complex systems with memory and nonlocal characteristics. These operators have proven to be highly effective in capturing intricate dynamical behaviors that cannot be addressed by standard fractional models. However, despite the richness of this emerging theory, a notable gap persists in the literature. In particular, the controllability of systems governed by the generalized (k,φ)-Hilfer fractional derivative in infinite-dimensional settings has not been adequately explored, especially when history-dependent operators are involved. Such operators introduce additional analytical challenges due to their reliance on past states of the system. Motivated by foundational results in controllability theory and fixed-point analysis, as well as recent contributions reported in [28, 38], this work seeks to fill this gap by investigating the controllability of a class of (k,φ)-Hilfer fractional differential equations with infinite delay and history-dependent effects. The considered problem is stated as follows:

$$\begin{cases} \left({}^{H}D_{k,0^+}^{\rho,\eta;\varphi}\xi\right)(s) = \Upsilon\xi(s) + h(s,\xi_s) + \Xi\upsilon(s) + (R\xi)(s), \ s \in V = (0,t], \\ \left(J_{k,0^+}^{(k-\rho)(1-\eta);\varphi}\xi\right)(s) = g(s) \in Z_g, \end{cases} \tag{1.1}$$

where ${}^{H}D_{k,0^+}^{\rho,\eta;\varphi}$ represents the (k,φ)-Hilfer fractional derivative operator of order $\rho \in (0,1)$ and type $\eta \in [0,1]$. Let A and W be Banach spaces with the norms $\|.\|_A$ and $\|.\|_W$, respectively. The operator Υ: $D(\Upsilon) \subset A \to A$ is the infinitesimal generator of an analytic semigroup $\{P(s)\}_{s\geq 0}$ of uniformly bounded linear operators on A such that there is $N \geq 1$ such that

$$N = \sup_{s\in[0,+\infty)} |P(s)|.$$

Moreover, Ξ: $L^2(W,A) \to L^2(V,A)$ is a bounded linear operator such that $\|\Xi\| \leq L_c$, $\xi(.)$ takes the value in A, and the control function is $\upsilon(.) \in L^2(V,W)$. Further, $J_{k,0^+}^{(k-\rho)(1-\eta)}$ is the (k,φ)-Hilfer fractional integral operator of order $(k-\rho)(1-\eta)$, R is referred as the history-dependent operator, and the mapping h will be defined later.

History-dependent operators play a fundamental role in modeling fractional delay systems because they provide a rigorous mathematical representation of memory effects inherent in Hilfer fractional differential equations. In particular, they enable the reformulation of complex evolution problems into equivalent fixed-point problems defined on suitable function spaces, thereby allowing the use of functional analytic tools for their investigation. By carefully analyzing key properties such as continuity and boundedness, and in some cases compactness or condensing behavior, one can establish the existence of mild solutions in a systematic and structured way. Moreover, these operators are especially important in controllability analysis, as they explicitly describe how past states of the system influence the present dynamics and, consequently, the required control actions. Motivated by the increasing need to understand complete controllability for fractional evolution systems with memory and delay, this study focuses on systems driven by the highly generalized (k,φ)-Hilfer fractional derivative in conjunction with history-dependent operators. This combination allows the model to incorporate both generalized fractional dynamics and complex memory effects in a unified framework. The main contribution of this work is the derivation of new sufficient conditions ensuring controllability without imposing the classical and often restrictive assumption of compactness on the associated solution operators, thereby significantly extending the applicability of the results. This is achieved through a refined analytical approach that integrates the concept of the measure of noncompactness with Mönch's fixed-point theorem, a powerful technique developed within the noncompactness framework pioneered by Banas and Goebel.

Remark 1.1. It should be noted that the measure of noncompactness plays a crucial role in our setting, since the solution operator is generally not compact due to the existence of infinite delay and history-dependent terms. Classical fixed-point theorems, such as those of Banach or Schauder in their standard forms, require either a contraction condition or compactness assumptions, which are not satisfied in this framework. The measure of noncompactness overcomes this difficulty by compensating for the lack of compactness and enabling us to work within a broader setting in which such operators can still be effectively controlled.

The remainder of this paper is structured as follows. In Section 2, we introduce the necessary preliminaries, including the fundamental definitions, notation, and auxiliary lemmas that form the analytical basis of our study. These preliminary results are essential for the development of the main theoretical framework presented in the subsequent sections. Sections 3 and 4 are devoted to deriving sufficient conditions for the controllability of the (k, φ)-Hilfer fractional differential equation with infinite delay and history-dependent operators. The analysis in these sections is carried out using the measure-of-noncompactness technique combined with appropriate fixed-point arguments. To illustrate the applicability and effectiveness of the obtained results, Section 5 provides a detailed example that verifies the theoretical conditions through a concrete formulation, either analytically or computationally. Finally, Section 6 concludes the paper by summarizing the main contributions and outlining possible directions for future research in this area.

2. Preliminary work

This section reviews fundamental definitions, notations, and preliminary results from fractional calculus, including concepts related to weighted spaces and relevant lemmas. We also incorporate key findings from the measure of noncompactness that will be applied throughout this paper.

Let $C(V, A)$ be the space consisting of continuous functions from V to A. We further define the weighted space $C_{1-\frac{\varrho}{\eta}}(V, A)$ as

$$C_{1-\frac{\varrho}{\eta}}(V, A) = \left\{\xi : [\varphi(s) - \varphi(0)]^{1-\frac{\varrho}{\eta}} \xi(s) \in C(V, A)\right\},$$

equipped with the norm

$$\|\xi\|_{\frac{\varrho}{\eta},\varphi} = \sup\left\{[\varphi(s) - \varphi(0)]^{1-\frac{\varrho}{\eta}} \|\xi\|\right\}.$$

Also, we define $L^p(V, A)$ as the Banach space of functions h: $V \times Z_g \to A$ that are Bochner integrable. For any $h \in L^p(V, A)$, where $p \in [1, \infty)$, its norm is given by $\|h\|_{L^p(V,A)}$. The symbol Z_g refers to the abstract phase space.

Suppose g: $(-\infty, 0] \to (0, +\infty)$ is continuous, and its improper integral

$$\mu = \int_{-\infty}^{0} g(\beta)\, d\beta < \infty$$

exists and is finite. Then, for any $b > 0$,

$$Z = \{\xi : [-b, 0] \to A \text{ such that } \xi(\beta) \text{ is bounded and measurable}\},$$

endowed with the norm

$$\|\xi\|_{[-b,0]} = \sup_{s\in[-b,0]} \|\xi(s)\|, \ \forall \xi \in Z.$$

Next, we assume that

$$Z_g = \left\{\xi : (-\infty, 0] \to A \text{ such that for any } b > 0, \ \xi|_{[-b,0]} \in Z \text{ and } \int_{-\infty}^{0} g(s)\|\xi\|_{[s,0]} ds < +\infty\right\}$$

under the norm

$$\|\xi\|_{Z_g} = \int_{-\infty}^{0} g(s)\|\xi\|_{[s,0]} ds, \quad \forall \xi \in Z_g.$$

Therefore, $\left(Z_g, \|.\|_{Z_g}\right)$ forms a Banach space. We proceed to discuss

$$Z_g' = \left\{z : (-\infty, r] \to A \text{ such that } r > 0, \ z|_V \in C(V, A) \text{ and, } z_0 = g \in Z_g\right\}.$$

Consider the seminorm $\|.\|_r$ on Z_g' defined as

$$\|z\|_r = \|g\|_{Z_g} + \sup_{s\in[0,r]} \left\{z(s) : z \in Z_g'\right\}.$$

Now, we recall key definitions from fractional calculus.

Definition 2.1. [39] The k-gamma function $\Gamma_k(\omega)$ for $\omega \in \mathbb{C}$ with $Re(\omega) > 0$ and $k > 0$ ($k \in \mathbb{R}$) is defined by

$$\Gamma_k(\omega) = \int_0^{\infty} \zeta^{\omega-1} e^{-\frac{\zeta^k}{k}} d\zeta.$$

Definition 2.2. [40] For an integrable function G on (b, c) and an increasing function φ having a continuous, non-zero derivative φ' on (b, c), (i.e., $\varphi'(s) \neq 0$, $\forall s \in V$), the left-sided fractional integral of order ρ of G with respect to φ is described as

$$I_{b^+}^{\rho,\varphi} G(s) = \frac{1}{\Gamma(\rho)} \int_b^s [\varphi(s) - \varphi(u)]^{\rho-1} \varphi'(u) G(u) du. \tag{2.1}$$

Remark 2.3. When $\varphi(s) = s$, we obtain the well-known classical Riemann–Liouville fractional integral.

Definition 2.4. [40] For $\rho \in (m-1, m)$, $m \in \mathbb{N}$ and $\eta \in [0, 1]$, let $G, \varphi \in \mathbb{C}^m(V, \mathbb{R})$ be functions where φ is increasing and $\varphi'(s) \neq 0$ for all $s \in V$. The left-sided φ-Hilfer fractional derivative of order ρ and type η of G is given by

$$D_{b^+}^{\rho,\eta;\varphi} G(s) = I_{b^+}^{\kappa(m-\rho);\varphi} \left[\frac{1}{\varphi'(s)} \frac{d}{ds}\right]^m I_{b^+}^{(1-\eta)(m-\rho);\varphi} G(s). \tag{2.2}$$

Remark 2.5. It should be noted that:

(i) Equation (2.2) can be written as

$$D_{b^+}^{\rho,\eta;\varphi} G(s) = I_{b^+}^{\kappa(m-\rho);\varphi} D_{b^+}^{\varrho;\varphi} G(s),$$

where $\varrho = \rho + \eta(m-\rho)$ and

$$D_{b^+}^{\varrho,\varphi} G(s) = \left[\frac{1}{\varphi'(s)} \frac{d}{ds}\right]^m I_{b^+}^{(1-\kappa)(m-\rho);\varphi} G(s).$$

(ii) If $\rho \in (0,1)$, the Eq (2.2) reduces to

$$D_{b^+}^{\rho,\eta;\varphi} G(s) = \frac{1}{\Gamma(\varrho-\rho)} \int_b^s [\varphi(s) - \varphi(u)]^{\varrho-\rho-1} D_{b^+}^{\varrho;\varphi} G(u)\, du = I_{b^+}^{\varrho-\rho;\varphi} D_{b^+}^{\varrho;\varphi} G(s),$$

where $\varrho = \rho + \eta(1-\rho)$, $I_{b^+}^{\varrho-\rho;\varphi}(.)$ is described as (2.1) and

$$D_{b^+}^{\varrho,\varphi} G(s) = \left[\frac{1}{\varphi'(s)} \frac{d}{ds}\right] I_{b^+}^{1-\varrho;\varphi} G(s).$$

Definition 2.6. [41] For $G \in L^1[b,c]$ and $k, \rho \in \mathbb{R}$, let φ be a positively defined increasing function on $(b,c]$ that has a continuous derivative φ' on (b,c). The left-sided fractional integral of G with respect to φ on $[b,c]$ of order ρ is given by

$$\mathfrak{I}_{k,b^+}^{\rho,\varphi} G(s) = \frac{1}{k\Gamma_k(\rho)} \int_b^s [\varphi(s) - \varphi(\zeta)]^{\frac{\rho}{k}-1} \varphi'(\zeta) G(\zeta) d\zeta, \ s > b,$$

where $\Gamma_k(.)$ is the k-gamma function. Clearly, if $\rho = 0$, we have

$$\mathfrak{I}_{k,b^+}^{0,\varphi} G(s) = G(s).$$

Definition 2.7. [21] For $k, \rho \in \mathbb{R}$ and $\eta \in [0,1]$, let $G, \varphi \in \mathbb{C}^m[b,c]$ be functions such that $\varphi'(s) > 0$ on $[b,c]$. The (k,φ)-Hilfer fractional derivative of G with respect to φ on $[b,c]$ of order ρ is described as

$$D_{k,b^+}^{\rho,\eta;\varphi} G(s) = \mathfrak{I}_{k,b^+}^{\kappa(km-\rho),\varphi} \left[\frac{k}{\varphi'(s)} \frac{d}{ds}\right]^m \mathfrak{I}_{k,b^+}^{(1-\eta)(km-\rho);\varphi} G(s), \quad m = \left[\frac{\rho}{k}\right].$$

Remark 2.8. Clearly, if we put $\eta = 0$, one can write

$$\begin{aligned} D_{k,b^+}^{\rho,0;\varphi} G(s) = D_{k,b^+}^{\rho;\varphi} G(s) &= \left[\frac{k}{\varphi'(s)} \frac{d}{ds}\right]^m \mathfrak{I}_{k,b^+}^{km-\rho;\varphi} G(s) \\ &= \frac{\left[\frac{k}{\varphi'(s)} \frac{d}{ds}\right]^m}{k\Gamma_k(km-\rho)} \int_b^s [\varphi(s) - \varphi(\zeta)]^{m-\frac{\rho}{k}-1} \varphi'(\zeta) G(\zeta) d\zeta. \end{aligned}$$

Lemma 2.9. *[21] Assume that $k, \rho \in \mathbb{R}^+$ and $\delta \in \mathbb{R}$ such that $\frac{\rho}{k} + 1 > 0$. Then,*

$$\mathfrak{I}_{k,b^+}^{\rho,\varphi} [\varphi(s) - \varphi(b)]^{\frac{\rho}{k}} = \frac{\Gamma_k(\delta+k)}{\Gamma_k(\delta+k+\rho)} [\varphi(s) - \varphi(b)]^{\frac{\rho+\delta}{k}}.$$

Theorem 2.10. *[42] Suppose that $G \in C^n_{\delta,\varphi}[a,b]$, $\rho \in (m-1,m)$, $\eta \in [0,1]$, where $m \in \mathbb{N}$ and $k > 0$. Then,*

$$\left(\Im^{\rho,\varphi}_{k,b^+} \, {}^H D^{\rho,\kappa;\varphi}_{k,b^+} G\right)(s) = G(s) - \sum_{j=1}^{m} \frac{(\varphi(s) - \varphi(b))^{\delta-j}}{k^{j-m}\Gamma_k(k(\delta - j + 1))} \left[\delta^{m-j}_{\varphi}\left(\Im^{k(m-\delta);\varphi}_{k,b^+} G(b)\right)\right], \tag{2.3}$$

where $\delta = \frac{1}{k}(\eta(km-\rho)+\rho)$.

Remark 2.11. In the case of $m = 1$, Eq (2.3) takes the form

$$\left(\Im^{\rho,\varphi}_{k,b^+} \, {}^H D^{\rho,\kappa;\varphi}_{k,b^+} G\right)(s) = G(s) - \frac{(\varphi(s) - \varphi(b))^{\delta-1}}{\Gamma_k(\eta(k-\rho)+\rho)} \left\{\left(\Im^{(1-\eta)(k-\rho);\varphi}_{k,b^+}\right) G(b)\right\}.$$

Definition 2.12. [43] For $k, \vartheta > 0$, and real-valued functions G, φ: $[b,\infty) \to \mathbb{R}$ such that φ is continuous and $\varphi'(s) > 0$ on (b,c), the (k,φ)-generalized Laplace transform of G is defined as

$$L^{\vartheta,\varphi}_{k,b^+}\{G(s)\}(u) = \int_b^\infty \exp\left[-uk^{1-\frac{\vartheta}{k}(\varphi(s)-\varphi(b))}\right]\varphi'(s)G(s)ds, \text{ for all } s \in \mathbb{R}.$$

Lemma 2.13. *[43] Consider a function $G(s)$ that is piecewise continuous over every finite interval $[b,S]$ and of exponential order $\varphi(s)$. If $\varrho > 0$ and $\varphi'(s) > 0$, then*

$$L^{\vartheta,\varphi}_{k,b^+}\left\{\Im^{\rho,\varphi}_{k,b^+} G(s)\right\}(u) = \frac{L^{\vartheta,\varphi}_{k,b^+}\{G(s)\}}{\left(uk^{1-\frac{\vartheta}{k}}\right)^{\frac{\rho}{k}} k^{\frac{\rho}{k}}}.$$

Now, to facilitate our discussion, we briefly recall some fundamental concepts related to the Kuratowski measure of noncompactness. This tool is essential for analyzing the existence of solutions in various functional spaces.

Definition 2.14. [44] For any bounded subset M of a metric space (Ω, d), the Kuratowski measure of noncompactness is described as

$$\xi(M) = \left\{\epsilon > 0 : M = \cup_{j=1}^m M_j, \ diam(M_j) \le \epsilon, \ 1 \le j \le m < \infty\right\},$$

where $diam(M_j)$ refers to the diameter of set $M_j \subset \Omega$.

Definition 2.15. [44] The Hausdorff measure of noncompactness E for a bounded subset K of a Banach space A is given by

$$E(K) = \{\gamma > 0 : K \text{ is finitely coverable by balls of radius } \gamma\}.$$

Due to the limited number of Banach spaces where the Hausdorff measure of noncompactness can be directly expressed using its definition, Banas and Goebel [44] proposed an axiomatic definition of the measure of noncompactness.

Definition 2.16. [44] For any Banach space A, we denote the set of all its bounded subsets by ϕ_ε. We define a measure of noncompactness for A as a mapping E: $\phi_\varepsilon \to \mathbb{R}^+$ that fulfills the subsequent conditions:

(i) $\ker(E) = \{K \in \phi_\varepsilon : \mho(K) = 0\}$ is a nonempty family subset of ϕ_ε;
(ii) $K_1 \subset K_2$ implies $E(K_1) \leq \mho(K_2)$;
(iii) $E(conv\, B) = E(B)$;
(iv) $E(sK_1 + (1-s)K_2) \leq sE(K_1) + (1-s)E(K_2)$;
(v) If $\{K_l\}$ is a sequence of closed sets in ϕ_ε such that $K_{l+1} \subset K_l$, $l \in \mathbb{N}$, and $\lim_{l\to\infty} E(K_l) = 0$, then $K_\infty = \bigcap_{l=1}^{\infty} K_l$ is non-empty.

Definition 2.17. [44] In an ordered Banach space $(A, \leq)$ with positive cone $\wp^+$, a function E that assigns a value from $\wp^+$ to each bounded subset of A (i.e., $E\colon \phi_\varepsilon \to \wp^+$) is defined as a measure of noncompactness if and only if it satisfies the property that $E(conv\, K) = E(K)$ for any bounded set $K \in \phi_\varepsilon$, where $conv\, K$ refers to the closed convex hull of K.

Lemma 2.18. *[44] For bounded subsets K_1, K_2 of a Banach space, the following properties hold:*

1) $E(K) = 0$ if and only if K is precompact;
2) $E(K) = E\left(\overline{K}\right) = \mho(conv(K))$, where $\overline{K}$ denotes the closure of K and $conv(K)$ represents its convex hull;
3) If $K_1 \subseteq K_2$, then $E(K_1) \leq E(K_2)$;
4) $E(K_1 + K_2) \leq E(K_1) + E(K_2)$, where $K_1 + K_2 = \{c_1 + c_2 : c_1 \in K_1,\ c_2 \in K_2\}$;
5) $E(K_1 \cup K_2) \leq \max\{E(K_1), E(K_2)\}$;
6) If $E(\varsigma K) = |\varsigma| E(K)$ holds true for all $\varsigma \in \mathbb{R}$, then A is a real Banach space;
7) If $T\colon D(T) \subseteq A \to A_1$ is a Lipschitz continuous operator with constant σ, then for any bounded subset $K \subset D(T)$, the Hausdorff measure of noncompactness E in the Banach space A_1 satisfies $E(T(K)) \leq \sigma E(K)$.

Lemma 2.19. *We can express system (1.1) as an equivalent integral equation:*

$$\begin{aligned}\xi(s) = {} & \frac{[\varphi(s) - \varphi(0)]^{\frac{\rho}{k}-1}}{\Gamma_k(\rho)} g(0) + \frac{\Upsilon}{k\Gamma_k(\rho)} \int_0^s [\varphi(s) - \varphi(\zeta)]^{\frac{\rho}{k}-1} \varphi'(\zeta)\xi(\zeta)\, d\zeta \\ & + \frac{1}{k\Gamma_k(\rho)} \int_0^s [\varphi(s) - \varphi(\zeta)]^{\frac{\rho}{k}-1} \varphi'(\zeta) h\left(\zeta, \xi_\zeta\right) d\zeta \\ & + \frac{\Xi}{k\Gamma_k(\rho)} \int_0^s [\varphi(s) - \varphi(\zeta)]^{\frac{\rho}{k}-1} \varphi'(\zeta)\upsilon(\zeta)\, d\zeta \\ & + \frac{1}{k\Gamma_k(\rho)} \int_0^s [\varphi(s) - \varphi(\zeta)]^{\frac{\rho}{k}-1} \varphi'(\zeta) (R\xi)(\zeta) d\zeta. \end{aligned} \tag{2.4}$$

Lemma 2.20. *[30] Assume that $\xi \in Z'_g$, and then $\xi_s \in Z_g$ for $s \in V$. Furthermore,*

$$\mu |\xi(s)| \leq \|\xi_s\|_{Z_g} + \mu \sup_{\varsigma \in V} |\xi(s)|,$$

where $\mu = \int_{-\infty}^0 g(\beta)\, d\beta < \infty$.

Lemma 2.21. *[45] For parameters $a \in (0,1)$ and $r \in \mathbb{C}$, the Wright function $U_a(r)$ is given by the series:*

$$U_a(r) = \sum_{m=1}^{\infty} \frac{(-r)^{m-1}}{(m-1)!\Gamma(1 - am)}.$$

This function is known to have the following properties:

(p_1) For $r>0,\ U_a(r)>0$*;*

(p_2) For $\varsigma>0,\ \int_0^\infty r^{\varsigma}U_a(r)\,dr=\frac{\Gamma(1+\varsigma)}{\Gamma(1+a\varsigma)}$*;*

(p_3) For $\zeta>0,\ \int_0^\infty \frac{a}{r^{(1+a)}}e^{-\zeta r}U_a\left(\frac{1}{r^a}\right)dr=e^{-\zeta a}$.

Theorem 2.22. *[46] Assume that the sequence* $\{\varpi_l\}_{l=1}^{\infty}$ *is a set of Bochner-integrable functions from* V *to* A *such that* $\|\varpi_l(s)\|\le\delta(s)$ *for almost every* $s\in V$ *and all* $l\ge1$ *and* $\delta\in L^1(V,\mathbb{R})$*. If* $\Phi(s)=E\{\varpi_l(s):l\ge1\}$*, then*

(i) $\Phi(s)\in L^1(V,\mathbb{R})$;

(ii) $E\left(\int_0^s\varpi_l(u)du:l\ge1\right)\le2\int_0^s\Phi(u)\,du$.

Lemma 2.23. *[46] Let* χ *be an equicontinuous and bounded subset of* $C(V,A)$*. The function* $s\longmapsto E(\chi(s))$ *is continuous on* $[0,t]$*, and*

$$E(\chi)=\sup\{E(\chi(s))\},$$

where $\chi(s)=\{\xi(s):\xi\in\chi,\ s\in[0,t]\}$.

Lemma 2.24. *[46] A continuous map* Φ*:* $Z\to A$ *is guaranteed to have a fixed-point in* Z*, provided that* Z *is a closed, convex subset of a Banach space* A *containing the origin* ($0\in Z$)*, and* Φ *satisfies the Mönch condition, that is, if* $Z_1\subset Z$ *is countable and* $Z_1\subset conv(\{0\}\cup\Phi(Z_1))$*, it implies that* Z_1 *is compact.*

3. Integral representation and mild solutions

In this section, we provide the integral and mild solutions to the problem at hand, starting with the proof of the following lemma:

Lemma 3.1. *The solution to the system (1.1) is given by the integral representation:*

$$\xi(s)=\int_0^\infty\frac{[\varphi(s)-\varphi(0)]^{\frac{\varrho}{k}-2}}{\Gamma_k\left(k\left(\frac{\varrho}{k}-2\right)\right)}T_{\frac{\varrho}{k};\varphi}(s)g(0)ds+\int_0^\infty T_{\frac{\varrho}{k};\varphi}(s)J_{\eta,0^+}^{\rho;\varphi}\left[h(s,\xi_s)\right]ds$$
$$+\,\Xi\int_0^\infty T_{\frac{\varrho}{k};\varphi}(s)J_{\eta,0^+}^{\rho;\varphi}\left[\upsilon(s)\right]ds+\int_0^\infty T_{\frac{\varrho}{k};\varphi}(s)J_{\eta,0^+}^{\rho;\varphi}\left[(R\xi)(\zeta)\right]ds,$$

where

$$T_{\frac{\varrho}{k};\varphi}(s)=\int_0^\infty e^{-\varsigma ks}\vartheta_{\frac{\varrho}{k}}(r)\,P\left(s^{\frac{\varrho}{k}}r\right)dr,$$

and $\vartheta_{\frac{\varrho}{k}}(r)$ *is a probability density function defined on* $(0,\infty)$*, i.e.,* $\vartheta_{\frac{\varrho}{k}}(r)\ge0,\ r\in(0,\infty)$*, and*

$$\int_0^\infty\vartheta_{\frac{\varrho}{k}}(r)\,dr=1.$$

Proof. Applying the integral operator $J_{\eta,0^+}^{\rho;\varphi}$ to the Eq (1.1) and invoking Theorem 2.10, we obtain that

$$\xi(s)=\frac{[\varphi(s)-\varphi(0)]^{\frac{\varrho}{k}-1}}{\Gamma_k(\rho)}g(0)+J_{\eta,0^+}^{\rho;\varphi}\left[\Upsilon\xi(s)\right]+J_{\eta,0^+}^{\rho;\varphi}\left[h(s,\xi,\xi_s)\right]+J_{\eta,0^+}^{\rho;\varphi}\left[\Xi\upsilon(s)\right]+J_{\eta,0^+}^{\rho;\varphi}\left[(R\xi)(s)\right].\quad(3.1)$$

By applying the generalized Laplace transform to (3.1), we obtain that

$$L_{k,0^+}^{k,\varphi}\left\{\xi(s)\right\}(u)=\frac{g(0)}{\Gamma_k(\rho)}L_{k,0^+}^{k,\varphi}\left\{\left[\varphi(s)-\varphi(0)\right]^{\frac{\rho}{k}-1}\right\}+L_{k,0^+}^{k,\varphi}\left\{J_{\eta,0^+}^{\rho;\varphi}\left[\Upsilon\xi(s)\right]\right\}L_{k,0^+}^{k;\varphi}\left\{J_{\eta,0^+}^{\rho;\varphi}\left[h(s,\xi_s)\right]\right\}$$
$$+L_{k,0^+}^{k,\varphi}\left\{J_{\eta,0^+}^{\rho;\varphi}\left[\Xi\upsilon(s)\right]\right\}+L_{k,0^+}^{k,\varphi}\left\{J_{\eta,0^+}^{\rho;\varphi}\left[(R\xi)(s)\right]\right\}.$$

According to [43], we have

$$L_{k,0^+}^{k,\varphi}\left\{\xi(s)\right\}(u)=\frac{g(0)}{u^{\frac{\rho}{k}}k^{\frac{\rho}{k}-1}}+\frac{L_{k,0^+}^{k,\varphi}\left\{\Upsilon\xi(s)\right\}(u)}{(uk)^{\frac{\rho}{k}}}+\frac{L_{k,0^+}^{k,\varphi}\left\{h(s,\xi_s)\right\}(u)}{(uk)^{\frac{\rho}{k}}}\frac{L_{k,0^+}^{k,\varphi}\left\{\Xi\upsilon(s)\right\}(u)}{(uk)^{\frac{\rho}{k}}}+\frac{L_{k,0^+}^{k,\varphi}\left\{(R\xi)(s)\right\}(u)}{(uk)^{\frac{\rho}{k}}},$$

and it follows that

$$\xi(\varsigma)=\frac{g(0)}{\varsigma^{\frac{\rho}{k}}k^{\frac{\rho}{k}-1}}+\frac{\Upsilon\xi(\varsigma)}{(\varsigma k)^{\frac{\rho}{k}}}+\frac{h(\varsigma,\xi_\varsigma)}{(\varsigma k)^{\frac{\rho}{k}}}+\frac{\Xi\upsilon(\varsigma)}{(\varsigma k)^{\frac{\rho}{k}}}+\frac{(R\xi)(\varsigma)}{(\varsigma k)^{\frac{\rho}{k}}},$$

or, equivalently,

$$\left[(\varsigma k)^{\frac{\rho}{k}}I-\Upsilon\right]\xi(\varsigma)=kg(0)+h(\varsigma,\xi_\varsigma)+\Xi\upsilon(\varsigma)+(R\xi)(\varsigma),$$

which implies that

$$\xi(\varsigma)=k\left[(\varsigma k)^{\frac{\rho}{k}}I-\Upsilon\right]^{-1}g(0)+\left[(\varsigma k)^{\frac{\rho}{k}}I-\Upsilon\right]^{-1}h(\varsigma,\xi_\varsigma)$$
$$+\left[(\varsigma k)^{\frac{\rho}{k}}I-\Upsilon\right]^{-1}\Xi\upsilon(\varsigma)+\left[(\varsigma k)^{\frac{\rho}{k}}I-\Upsilon\right]^{-1}(R\xi)(\varsigma).$$

Hence,

$$\xi(\varsigma)=k\int_0^\infty e^{-(\varsigma k)^{\frac{\rho}{k}}.u}P(u)g(0)du+\int_0^\infty e^{-(\varsigma k)^{\frac{\rho}{k}}.u}P(u)h(\varsigma,\xi_\varsigma)\,du$$
$$+\int_0^\infty e^{-(\varsigma k)^{\frac{\rho}{k}}.u}P(u)\Xi\upsilon(\varsigma)\,du+\int_0^\infty e^{-(\varsigma k)^{\frac{\rho}{k}}.u}P(u)(R\xi)(\varsigma)\,du.$$

Substituting u with $s^{\frac{\rho}{k}}$ gives us

$$\xi(\varsigma)=k\int_0^\infty e^{-(\varsigma ks)^{\frac{\rho}{k}}}P(s^{\frac{\rho}{k}})g(0)\frac{\rho}{k}s^{\frac{\rho}{k}-1}ds+\int_0^\infty e^{-(\varsigma ks)^{\frac{\rho}{k}}}P(s^{\frac{\rho}{k}})h(\varsigma,\xi_\varsigma)\frac{\rho}{k}s^{\frac{\rho}{k}-1}ds$$
$$+\int_0^\infty e^{-(\varsigma ks)^{\frac{\rho}{k}}}P(s^{\frac{\rho}{k}})\Xi\upsilon(\varsigma)\frac{\rho}{k}s^{\frac{\rho}{k}-1}ds+\int_0^\infty e^{-(\varsigma ks)^{\frac{\rho}{k}}}P(s^{\frac{\rho}{k}})(R\xi)(\varsigma)\frac{\rho}{k}s^{\frac{\rho}{k}-1}ds$$
$$=k\int_0^\infty e^{-(\varsigma ks)^{\frac{\rho}{k}}}P(s^{\frac{\rho}{k}})g(0)\frac{\rho}{k}s^{\frac{\rho}{k}-1}ds$$
$$+\int_0^\infty\int_0^\infty e^{-(\varsigma ks)^{\frac{\rho}{k}}}P(s^{\frac{\rho}{k}})\frac{\rho}{k}s^{\frac{\rho}{k}-1}e^{-\varsigma[\varphi(u)-\varphi(0)]}\varphi'(u)h(u,\xi_u)\,duds$$
$$+\int_0^\infty\int_0^\infty e^{-(\varsigma ks)^{\frac{\rho}{k}}}P(s^{\frac{\rho}{k}})\frac{\rho}{k}s^{\frac{\rho}{k}-1}e^{-\varsigma[\varphi(u)-\varphi(0)]}\varphi'(u)\Xi\upsilon(u)\,duds$$
$$+\int_0^\infty\int_0^\infty e^{-(\varsigma ks)^{\frac{\rho}{k}}}P(s^{\frac{\rho}{k}})\frac{\rho}{k}s^{\frac{\rho}{k}-1}e^{-\varsigma[\varphi(u)-\varphi(0)]}\varphi'(u)(R\xi)(u)\,duds.$$

Using the fact that

$$\frac{d}{ds}\left(e^{-(\varsigma ks)^{\frac{\rho}{k}}}\right)=e^{-(\varsigma ks)^{\frac{\rho}{k}}}\left(-\frac{\rho}{k}\right)(\varsigma ks)^{\frac{\rho}{k}-1}(\varsigma k),$$

one can write

$$
\begin{aligned}
\xi(\varsigma) &= k\int_0^\infty -\frac{1}{(\varsigma k)^{\frac{\varrho}{k}}}\frac{d}{ds}\left(e^{-(\varsigma k s)^{\frac{\varrho}{k}}}\right)P(s^{\frac{\varrho}{k}})g(0)ds \\
&+\int_0^\infty\int_0^\infty -\frac{1}{(\varsigma k)^{\frac{\varrho}{k}}}\frac{d}{ds}\left(e^{-(\varsigma k s)^{\frac{\varrho}{k}}}\right)P(s^{\frac{\varrho}{k}})e^{-\varsigma[\varphi(u)-\varphi(0)]}\varphi'(u)h(u,\xi_u)\,duds \\
&+\int_0^\infty\int_0^\infty -\frac{1}{(\varsigma k)^{\frac{\varrho}{k}}}\frac{d}{ds}\left(e^{-(\varsigma k s)^{\frac{\varrho}{k}}}\right)P(s^{\frac{\varrho}{k}})e^{-\varsigma[\varphi(u)-\varphi(0)]}\varphi'(u)\Xi\upsilon(u)\,duds \\
&+\int_0^\infty\int_0^\infty -\frac{1}{(\varsigma k)^{\frac{\varrho}{k}}}\frac{d}{ds}\left(e^{-(\varsigma k s)^{\frac{\varrho}{k}}}\right)P(s^{\frac{\varrho}{k}})e^{-\varsigma[\varphi(u)-\varphi(0)]}\varphi'(u)(R\xi)(u)\,duds \\
&= k\int_0^\infty -\frac{1}{(\varsigma k)^{\frac{\varrho}{k}}}\frac{d}{ds}\left(\int_0^\infty e^{-\varsigma k s r}U_{\frac{\varrho}{k}}(r)\,dr\right)P(s^{\frac{\varrho}{k}})g(0)ds \\
&+\int_0^\infty\int_0^\infty -\frac{1}{(\varsigma k)^{\frac{\varrho}{k}}}\frac{d}{ds}\left(\int_0^\infty e^{-\varsigma k s r}U_{\frac{\varrho}{k}}(r)\,dr\right)P(s^{\frac{\varrho}{k}})e^{-\varsigma[\varphi(u)-\varphi(0)]}\varphi'(u)h(u,\xi_u)\,duds \\
&+\int_0^\infty\int_0^\infty -\frac{1}{(\varsigma k)^{\frac{\varrho}{k}}}\frac{d}{ds}\left(\int_0^\infty e^{-\varsigma k s r}U_{\frac{\varrho}{k}}(r)\,dr\right)P(s^{\frac{\varrho}{k}})e^{-\varsigma[\varphi(u)-\varphi(0)]}\varphi'(u)\Xi\upsilon(u)\,duds \\
&+\int_0^\infty\int_0^\infty -\frac{1}{(\varsigma k)^{\frac{\varrho}{k}}}\frac{d}{ds}\left(\int_0^\infty e^{-\varsigma k s r}U_{\frac{\varrho}{k}}(r)\,dr\right)P(s^{\frac{\varrho}{k}})e^{-\varsigma[\varphi(u)-\varphi(0)]}\varphi'(u)(R\xi)(u)\,duds \\
&= k\int_0^\infty\int_0^\infty \frac{\varsigma k r}{(\varsigma k)^{\frac{\varrho}{k}}}e^{-\varsigma k s r}U_{\frac{\varrho}{k}}(r)\,P(s^{\frac{\varrho}{k}})g(0)drds \\
&+\int_0^\infty\int_0^\infty\int_0^\infty \frac{\varsigma k r}{(\varsigma k)^{\frac{\varrho}{k}}}e^{-\varsigma k s r}U_{\frac{\varrho}{k}}(r)\,P(s^{\frac{\varrho}{k}})e^{-\varsigma[\varphi(u)-\varphi(0)]}\varphi'(u)h(u,\xi_u)\,drduds \\
&+\int_0^\infty\int_0^\infty\int_0^\infty \frac{\varsigma k r}{(\varsigma k)^{\frac{\varrho}{k}}}e^{-\varsigma k s r}U_{\frac{\varrho}{k}}(r)\,P(s^{\frac{\varrho}{k}})e^{-\varsigma[\varphi(u)-\varphi(0)]}\varphi'(u)\Xi\upsilon(u)\,drduds \\
&+\int_0^\infty\int_0^\infty\int_0^\infty \frac{\varsigma k r}{(\varsigma k)^{\frac{\varrho}{k}}}e^{-\varsigma k s r}U_{\frac{\varrho}{k}}(r)\,P(s^{\frac{\varrho}{k}})e^{-\varsigma[\varphi(u)-\varphi(0)]}\varphi'(u)(R\xi)(u)\,drduds.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\xi(\varsigma) &= \int_0^\infty\int_0^\infty \frac{r}{\varsigma^{\frac{\varrho}{k}-1}k^{\frac{\varrho}{k}-2}}e^{-\varsigma k s r}U_{\frac{\varrho}{k}}(r)\,P(s^{\frac{\varrho}{k}})g(0)drds \\
&+\int_0^\infty\int_0^\infty\int_0^\infty \frac{r}{(\varsigma k)^{\frac{\varrho}{k}-1}}e^{-\varsigma k s r}U_{\frac{\varrho}{k}}(r)\,P(s^{\frac{\varrho}{k}})e^{-\varsigma[\varphi(u)-\varphi(0)]}\varphi'(u)h(u,\xi_u)\,drduds \\
&+\int_0^\infty\int_0^\infty\int_0^\infty \frac{r}{(\varsigma k)^{\frac{\varrho}{k}-1}}e^{-\varsigma k s r}U_{\frac{\varrho}{k}}(r)\,P(s^{\frac{\varrho}{k}})e^{-\varsigma[\varphi(u)-\varphi(0)]}\varphi'(u)\Xi\upsilon(u)\,drduds \\
&+\int_0^\infty\int_0^\infty\int_0^\infty \frac{r}{(\varsigma k)^{\frac{\varrho}{k}-1}}e^{-\varsigma k s r}U_{\frac{\varrho}{k}}(r)\,P(s^{\frac{\varrho}{k}})e^{-\varsigma[\varphi(u)-\varphi(0)]}\varphi'(u)(R\xi)(u)\,drduds \\
&= \int_0^\infty\int_0^\infty \frac{1}{\varsigma^{\frac{\varrho}{k}-1}k^{\frac{\varrho}{k}-2}}e^{-\varsigma k s}U_{\frac{\varrho}{k}}(r)\,P\left(\frac{s^{\frac{\varrho}{k}}}{r^{\frac{\varrho}{k}}}\right)g(0)drds \\
&+\int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{(\varsigma k)^{\frac{\varrho}{k}-1}}e^{-\varsigma k s}U_{\frac{\varrho}{k}}(r)\,P\left(\frac{s^{\frac{\varrho}{k}}}{r^{\frac{\varrho}{k}}}\right)e^{-\varsigma[\varphi(u)-\varphi(0)]}\varphi'(u)h(u,\xi_u)\,drduds
\end{aligned}
$$

$$+\int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{(\varsigma k)^{\frac{\rho}{k}-1}} e^{-\varsigma ks} U_{\frac{\rho}{k}}(r) P\left(\frac{s^{\frac{\rho}{k}}}{r^{\frac{\rho}{k}}}\right) e^{-\varsigma[\varphi(u)-\varphi(0)]}\varphi'(u)\Xi\upsilon(u)\, drduds$$
$$+\int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{(\varsigma k)^{\frac{\rho}{k}-1}} e^{-\varsigma ks} U_{\frac{\rho}{k}}(r) P\left(\frac{s^{\frac{\rho}{k}}}{r^{\frac{\rho}{k}}}\right) e^{-\varsigma[\varphi(u)-\varphi(0)]}\varphi'(u)(R\xi)(u)\, drduds.$$

For simplicity, substitute $\frac{\rho}{k}$ with a, and we get

$$\begin{aligned}\xi(\varsigma) = \int_0^\infty e^{-\varsigma ks}\Bigg\{&\int_0^\infty \frac{1}{\varsigma^{a-1}k^{a-2}} e^{-\varsigma ks} U_a(r) P\left(\frac{s^a}{r^a}\right) g(0)dr\\
&+\int_0^\infty e^{-\varsigma[\varphi(u)-\varphi(0)]}\varphi'(u)h(u,\xi_u)\,du\int_0^\infty \frac{1}{(\varsigma k)^{a-1}} U_a(r) P\left(\frac{s^a}{r^a}\right)dr\\
&+\int_0^\infty e^{-\varsigma[\varphi(u)-\varphi(0)]}\varphi'(u)\Xi\upsilon(u)\,du\int_0^\infty \frac{1}{(\varsigma k)^{\frac{\rho}{k}-1}} U_a(r) P\left(\frac{s^a}{r^a}\right)dr\\
&+\int_0^\infty e^{-\varsigma[\varphi(u)-\varphi(0)]}\varphi'(u)(R\xi)(u)\,du\int_0^\infty \frac{1}{(\varsigma k)^{a-1}} U_a(r) P\left(\frac{s^a}{r^a}\right)dr\Bigg\}ds.\end{aligned}$$

Again, set $r = r^{-\frac{1}{a}}$, $dr = -\frac{1}{a}r^{-\frac{1}{a}-1}dr$, and $-\frac{1}{a}r^{-\frac{1}{a}-1}U_a\left(r^{-\frac{1}{a}}\right) = \vartheta_a(r)$, and one has

$$\begin{aligned}\xi(\varsigma) = \int_0^\infty e^{-\varsigma ks}\Bigg\{&\int_0^\infty \frac{1}{\varsigma^{a-1}k^{a-2}}\vartheta_a(r)P(s^a r)\, g(0)dr\\
&+\int_0^\infty e^{-\varsigma[\varphi(u)-\varphi(0)]}\varphi'(u)h(u,\xi_u)\,du\int_0^\infty \frac{1}{(\varsigma k)^{a-1}}\vartheta_a(r)P(s^a r)\,dr\\
&+\int_0^\infty e^{-\varsigma[\varphi(u)-\varphi(0)]}\varphi'(u)\Xi\upsilon(u)\,du\int_0^\infty \frac{1}{(\varsigma k)^{\frac{\rho}{k}-1}}\vartheta_a(r)P(s^a r)\,dr\\
&+\int_0^\infty e^{-\varsigma[\varphi(u)-\varphi(0)]}\varphi'(u)(R\xi)(u)\,du\int_0^\infty \frac{1}{(\varsigma k)^{a-1}}\vartheta_a(r)P(s^a r)\,dr\Bigg\}ds.\end{aligned}$$

By applying $L_{k,0^+}^{k,\varphi}$ to both sides of the preceding equation, we obtain

$$\begin{aligned}L_{k,0^+}^{k;\varphi}\{\xi(s)\} =& \int_0^\infty\int_0^\infty \frac{L_{k,0^+}^{k,\varphi}\left\{[\varphi(u)-\varphi(0)]^{a-2}\right\}}{\Gamma_k((a-1)k)} e^{-\varsigma ks}\vartheta_a(r)P(s^a r)\, g(0)drds\\
&+\int_0^\infty\int_0^\infty \frac{e^{-\varsigma ks}}{(\varsigma k)^{a-1}}\vartheta_a(r)P(s^a r)\, L_{k,0^+}^{k,\varphi}\{h(s,\xi_s)\}\,drds\\
&+\Xi\int_0^\infty\int_0^\infty \frac{e^{-\varsigma ks}}{(\varsigma k)^{a-1}}\vartheta_a(r)P(s^a r)\, L_{k,0^+}^{k,\varphi}\{\upsilon(s)\}drds\\
&+\int_0^\infty\int_0^\infty \frac{e^{-\varsigma ks}}{(\varsigma k)^{a-1}}\vartheta_a(r)P(s^a r)\, L_{k,0^+}^{k,\varphi}\{(R\xi)(s)\}\,drds\\
=& \int_0^\infty\int_0^\infty \frac{L_{k,0^+}^{k,\varphi}\left\{[\varphi(u)-\varphi(0)]^{\frac{\rho}{k}-2}\right\}}{\Gamma_k\left(\left(\frac{\rho}{k}-1\right)k\right)} e^{-\varsigma ks}\vartheta_{\frac{\rho}{k}}(r)P\left(s^{\frac{\rho}{k}}r\right) g(0)drds\\
&+\int_0^\infty\int_0^\infty e^{-\varsigma ks}\vartheta_{\frac{\rho}{k}}(r)P\left(s^{\frac{\rho}{k}}r\right) L_{k,0^+}^{k,\varphi}\left\{J_{\eta,0^+}^{\rho,\varphi}[h(s,\xi_s)]\right\}drds\end{aligned}$$

$$+\,\Xi\int_0^{\infty}\int_0^{\infty} e^{-\varsigma ks}\vartheta_{\frac{\rho}{k}}(r)P\left(s^{\frac{\rho}{k}}r\right)L_{k,0^+}^{k,\varphi}\{J_{\eta,0^+}^{\rho,\varphi}\left[\upsilon\left(s\right)\right]\}drds$$
$$+\int_0^{\infty}\int_0^{\infty} e^{-\varsigma ks}\vartheta_{\frac{\rho}{k}}(r)P\left(s^{\frac{\rho}{k}}r\right)\left\{J_{\eta,0^+}^{\rho,\varphi}\left[\left(R\xi\right)(s)\right]\right\}drds.$$

Now, performing the inverse Laplace transform on both sides, and using

$$T_{\frac{\rho}{k};\varphi}(s)=\int_0^{\infty} e^{-\varsigma ks}\vartheta_{\frac{\rho}{k}}\left(r\right)P\left(s^{\frac{\rho}{k}}r\right)dr,$$

we get

$$\begin{aligned}
\xi(s) &= \int_0^{\infty}\frac{\left[\varphi\left(u\right)-\varphi\left(0\right)\right]^{\frac{\rho}{k}-2}}{\Gamma_k\left(\left(\frac{\rho}{k}-1\right)k\right)}T_{\frac{\rho}{k};\varphi}(s)g(0)ds+\int_0^{s}T_{\frac{\rho}{k};\varphi}(s)J_{\eta,0^+}^{\rho;\varphi}\left[h\left(s,\xi,\xi_s\right)\right]du \\
&+\Xi\int_0^{s}T_{\frac{\rho}{k};\varphi}(s)J_{\eta,0^+}^{\rho;\varphi}\left[\upsilon\left(s\right)\right]du+\int_0^{s}T_{\frac{\rho}{k};\varphi}(s)J_{\eta,0^+}^{\rho;\varphi}\left(R\xi\right)(s)\,du \\
&= Q_{\frac{\rho}{k};\varphi}\left(s\right)g(0)+\int_0^{s}T_{\frac{\rho}{k},\varphi}(s)J_{\eta,0^+}^{\rho;\varphi}\left[h\left(s,\xi,\xi_s\right)\right]du+\Xi\int_0^{s}T_{\frac{\rho}{k};\varphi}(s)J_{\eta,0^+}^{\rho;\varphi}\left[\upsilon\left(s\right)\right]du \\
&+\int_0^{s}T_{\frac{\rho}{k};\varphi}(s)J_{\eta,0^+}^{\rho;\varphi}\left(R\xi\right)(s)\,du,
\end{aligned}$$

where

$$Q_{\frac{\rho}{k};\varphi}\left(s\right)=\int_0^{\infty}\frac{\left[\varphi\left(s\right)-\varphi\left(0\right)\right]^{\frac{\rho}{k}-2}}{\Gamma_k\left(\left(\frac{\rho}{k}-1\right)k\right)}T_{\frac{\rho}{k};\varphi}(s)ds.$$

This completes the proof. □

Now, we can provide a definition of the mild solution to the problem under study as follows:

Definition 3.2. A function ξ: $(-\infty,t]\to A$ is considered a mild solution of the problem (1.1) if ξ is continuous, satisfies $\xi(0)=g(0)\in Z_g$, and also meets the following condition:

$$\begin{aligned}
\xi\left(s\right) &= Q_{\frac{\rho}{k};\varphi}\left(s\right)g(0)+\int_0^{s}\left[\varphi\left(s\right)-\varphi\left(\zeta\right)\right]^{\frac{\rho}{k}-1}\varphi'(\zeta)M_{\frac{\rho}{k};\varphi}(s)h\left(\zeta,\xi_\zeta\right)d\zeta \\
&+\int_0^{s}\left[\varphi\left(s\right)-\varphi\left(\zeta\right)\right]^{\frac{\rho}{k}-1}\varphi'(\zeta)M_{\frac{\rho}{k};\varphi}(s)\Xi\upsilon\left(\zeta\right)d\zeta \\
&+\int_0^{s}\left[\varphi\left(s\right)-\varphi\left(\zeta\right)\right]^{\frac{\rho}{k}-1}\varphi'(\zeta)M_{\frac{\rho}{k};\varphi}(s)\left(R\xi\right)(\zeta)d\zeta,
\end{aligned}\tag{3.2}$$

where

$$M_{\frac{\rho}{k};\varphi}(s)=\int_0^{\infty}T_{\frac{\rho}{k};\varphi}(s)ds,$$
$$Q_{\frac{\rho}{k};\varphi}\left(s\right)=\int_0^{\infty}\frac{\left[\varphi\left(s\right)-\varphi\left(0\right)\right]^{\frac{\rho}{k}-2}}{\Gamma_k\left(\left(\frac{\rho}{k}-1\right)k\right)}T_{\frac{\rho}{k};\varphi}(s)ds,$$

and

$$T_{\frac{\rho}{k};\varphi}(s)=\int_0^{\infty} e^{-\varsigma ks}\vartheta_{\frac{\rho}{k}}\left(r\right)P\left(s^{\frac{\rho}{k}}r\right)dr.$$

Lemma 3.3. *The following properties apply to the operators $Q_{\frac{\varrho}{k};\varphi}(s)$ and $T_{\frac{\varrho}{k};\varphi}(s)$:*

(i) For all $s \geq 0$, the operators $Q_{\frac{\varrho}{k};\varphi}(s)$ and $T_{\frac{\varrho}{k};\varphi}(s)$ are strongly continuous;
(ii) For all $s \geq 0$, $Q_{\frac{\varrho}{k};\varphi}(s)$ and $T_{\frac{\varrho}{k};\varphi}(s)$ bounded linear operators are characterized by:

$$\left\|T_{\frac{\varrho}{k};\varphi}(s)\xi\right\| = \left\|\int_0^\infty e^{-\varsigma ks}\vartheta_{\frac{\varrho}{k}}(r)\, P\left(s^{\frac{\varrho}{k}}r\right)\xi dr\right\|$$
$$\leq \left\|\int_0^\infty \vartheta_{\frac{\varrho}{k}}(r)\, P\left(s^{\frac{\varrho}{k}}r\right)\xi dr\right\| \leq N\|\xi\|$$

and

$$\left\|Q_{\frac{\varrho}{k};\varphi}(s)\,\xi\right\| \leq N\frac{\left[\varphi(s)-\varphi(0)\right]^{\frac{\varrho}{k}-1}}{\left(\frac{\varrho}{k}-1\right)\Gamma_k\left(\left(\frac{\varrho}{k}-1\right)k\right)}\|\xi\|.$$

Proof. For each $\xi \in A$ and $0 \leq s_1 \leq s_2 \leq t$, we have

$$\left\|Q_{\frac{\varrho}{k};\varphi}(s_2)\,\xi - Q_{\frac{\varrho}{k};\varphi}(s_1)\,\xi\right\| \to 0, \text{ as } s_1 \to s_2$$

and

$$\left\|T_{\frac{\varrho}{k};\varphi}(s_2)\xi - T_{\frac{\varrho}{k};\varphi}(s_1)\xi\right\| = \left\|e^{-\varsigma ks}\int_0^\infty \vartheta_{\frac{\varrho}{k}}(r)\left[P\left(s_2^{\frac{\varrho}{k}}r\right)-P\left(s_1^{\frac{\varrho}{k}}r\right)\right]\xi dr\right\| \to 0, \text{ as } s_1 \to s_2.$$

This completes the proof. □

To investigate the controllability of system (1.1), it is essential to establish and employ the following assertions, which provide the fundamental analytical framework required for the subsequent development of the main results. These assertions play a crucial role in verifying the necessary properties of the associated operators, deriving suitable estimates, and ensuring that the imposed conditions are sufficient for the application of the adopted fixed-point and measure-of-noncompactness techniques. Moreover, they facilitate the rigorous treatment of the system dynamics and the construction of appropriate control functions, thereby paving the way for proving the controllability of system (1.1):

(A_1) The mapping h: $V \times Z_g \to A$ fulfills:

1) For all $g \in Z_g$, $h(., g)$ is measurable and for a.e. $s \in V$, $h(s, .)$ is continuous;
2) Z_g and $h(g, .) : [0, t] \to A$ are strongly measurable;
3) For all $(s, g) \in V \times Z_g$, there exist $\eta_1 \in (0, \rho)$, $\rho \in \left(\frac{1}{2}, 1\right)$, $\varkappa_1 \in L^{\frac{1}{\eta_1}}(I, \mathbb{R}^+)$, and a nondecreasing continuous function $\hbar$: $\mathbb{R}^+ \to \mathbb{R}^+$ such that $\|h(s, g)\| \leq \varkappa_1(s)\,\hbar(\|(g)\|)$ and $\hbar$ is true on $\liminf_{\zeta\to\infty}\frac{\hbar(\zeta)}{\zeta} = 0$;
4) For bounded subsets $Z_1 \in Z_g$, there exist $\eta_2 \in (0, \rho)$, $\varkappa_2 \in L^{\frac{1}{\eta_2}}(I, \mathbb{R}^+)$ such that

$$E(h(s, Z_1)) \leq \varkappa_2(s) \sup_{\zeta' \in (-\infty, 0]} E(Z_1(\zeta')).$$

(A_2) The bounded operator F: $L^2(V, W) \to A$ is defined by

$$F(\upsilon) = \int_0^t [\varphi(t)-\varphi(\zeta)]^{\frac{\varrho}{k}-1}\varphi'(\zeta)M_{\frac{\varrho}{k};\varphi}(s)\Xi\upsilon(\zeta)d\zeta,$$

and fulfills:

1) The inverse of the bounded linear operator F is F^{-1}, which takes values in $L^2(V,W)/Ker(F)$;
2) There exist $L_\Xi, L_F > 0$ such that

$$\|\Xi\| \le L_\Xi \text{ and } \left\|F^{-1}\right\| \le L_{F^{-1}}.$$

(A$_3$) For any bounded subset $\chi \subset A$, there exist $\eta_3 \in (0,\rho)$, $\varkappa_3 \in L^{\frac{1}{\eta_3}}(I,\mathbb{R}^+)$ such that

$$E\left(\left(F^{-1}\chi\right)(s)\right) \le \varkappa_3(s)\, E(\chi).$$

(A$_4$) R: $L^2(V,W) \to L^2(V,W)$ is a history-dependent operator, that is, for all $z_1, z_2 \in W$, there exist $L_R > 0$ such that

$$\|(Rz_1)(s) - (Rz_2)(s)\|_W \le L_R \int_0^s \|z_1(\zeta) - z_2(\zeta)\|_W \, d\zeta, \text{ for a.e. } s \in [0,t].$$

Clearly, if we consider $(R0)(s) = B(s)$, then we have $B(s) \in L^{\frac{1}{\eta_4}}(0,\mathbb{R}^+)$, $\eta_4 \in (0,\rho)$, and

$$\|(Rz)(s)\|_W \le B(s) + L_R \int_0^s \|z(\zeta)\|_W \, d\zeta = B(s) + L_R B_1(s) = B^*(s),$$

for a.e. $s \in V$ and all $z \in [0,t]$, where $B_1(s) =_R \int_0^s \|z(\zeta)\|_W \, d\zeta$ and $B(s) + L_R B_1(s) = B^*(s)$ with $\liminf_{\zeta\to\infty} \frac{B^*(\zeta)}{\zeta} = 0$.

For brevity, we take

$$L_1 = l_1 \|\varkappa_1\|_{L^{\frac{1}{\eta_1}}(I,\mathbb{R}^+)}, \quad L_2 = l_2 \|\varkappa_1\|_{L^{\frac{1}{\eta_1}}(I,\mathbb{R}^+)}$$

and

$$l_j = \left[\left(\frac{1-\eta_j}{\rho-\eta_j}\right)\left([\varphi(t)-\varphi(\zeta)]^{\frac{\rho-\eta_j}{1-\eta_j}}\right)\right]^{1-\eta_j}, \quad j = 1,2. \tag{3.3}$$

Now, our main result in this part is as follows:

Theorem 3.4. *Under the hypotheses (A$_1$)–(A$_4$), the problem (1.1) is controllable, provided that*

$$\widetilde{M} = 2NL_1 [\varphi(t) - \varphi(0)]^{1-\frac{\rho}{k}} (1 + 2NL_\Xi L_{F^{-1}} L_R L_2) < 1, \tag{3.4}$$

for some $\rho \in \left(\frac{1}{2}, 1\right)$.

Proof. We define the control function υ according to (A$_2$) as

$$\upsilon(s) = F^{-1}\left\{\xi_1 - Q_{\frac{\rho}{k};\varphi}(t)\, g(0) - \int_0^t [\varphi(t)-\varphi(\zeta)]^{\frac{\rho}{k}-1} \varphi'(\zeta) M_{\frac{\rho}{k};\varphi}(t) \left[h(t,\xi_t) + (R\xi)(t)\right] d\zeta\right\}.$$

Our goal is to demonstrate that the operator Φ: $Z'_g \to Z'_g$ defined by

$$\Phi\xi(s) = \begin{cases} g(s), \ s \in (-\infty, 0], \\ Q_{\frac{\rho}{k};\varphi}(s)\, g(0) + \int_0^s [\varphi(s)-\varphi(\zeta)]^{\frac{\rho}{k}-1} \varphi'(\zeta) M_{\frac{\rho}{k};\varphi}(s) h(s,\xi_s)\, d\zeta \\ + \int_0^s [\varphi(s)-\varphi(\zeta)]^{\frac{\rho}{k}-1} \varphi'(\zeta) M_{\frac{\rho}{k};\varphi}(s) (R\xi)(s)\, d\zeta \\ + \int_0^s [\varphi(s)-\varphi(\zeta)]^{\frac{\rho}{k}-1} \varphi'(\zeta) M_{\frac{\rho}{k};\varphi}(s) \Xi \upsilon_\xi(\zeta) d\zeta, \ s \in V, \end{cases}$$

has a fixed-point. This fixed-point represents a mild solution to system (1.1). The fact that $\Phi\xi(t) = \xi_1$ directly implies the controllability of system (1.1) on $[0,t]$. □

Now, for $g \in Z_g$, we describe $\widetilde{g}(s)$ as

$$\widetilde{g}(s) = \begin{cases} g(s), & s \in (-\infty, 0], \\ Q_{\frac{\varrho}{k};\varphi}(s)\, g(0), & s \in V, \end{cases}$$

and then $\widetilde{g}(s) \in Z'_g$. Assume that $\xi(s) = (z(s) + \widetilde{g}(s))$, $s \in (-\infty, t]$. Evidently, ξ satisfies the Eq (3.2) if and only if z satisfies $z_0 = 0$ and

$$\begin{aligned} z(s) = & \int_0^s [\varphi(s) - \varphi(\zeta)]^{\frac{\varrho}{k}-1} \varphi'(\zeta) M_{\frac{\varrho}{k};\varphi}(s) h\left(s, \left(z_\zeta + \widetilde{g}\right)\right) d\zeta \\ & + \int_0^s [\varphi(s) - \varphi(\zeta)]^{\frac{\varrho}{k}-1} \varphi'(\zeta) M_{\frac{\varrho}{k};\varphi}(s) \left(R\left(z + \widetilde{g}\right)\right)(\zeta)\, d\zeta \\ & + \int_0^s [\varphi(s) - \varphi(\zeta)]^{\frac{\varrho}{k}-1} \varphi'(\zeta) M_{\frac{\varrho}{k};\varphi}(s) \Xi \upsilon_z(\zeta) d\zeta, \end{aligned}$$

where

$$\begin{aligned} \upsilon_z(\zeta) = F^{-1} \Bigg\{ & \xi_1 - Q_{\frac{\varrho}{k};\varphi}(t)\, g(0) - \int_0^t [\varphi(t) - \varphi(\zeta)]^{\frac{\varrho}{k}-1} \varphi'(\zeta) M_{\frac{\varrho}{k};\varphi}(t) \\ & \times \left[h\left(s, \left(z_\zeta + \widetilde{g}\right)\right) + \left(R\left(z + \widetilde{g}\right)\right)(\zeta) \right] d\zeta \Bigg\}(\zeta). \end{aligned}$$

Define $Z''_g = \left\{ \xi \in Z'_g : \xi_0 = 0 \in Z_g \right\}$. For $\xi \in Z''_g$, define the norm

$$\|\xi\|_t = \|\xi_0\|_{Z_g} + \sup\left\{ \|\xi(\eta)\| : \kappa \in [0, t] \right\} = \sup\left\{ \|\xi(\eta)\| : \eta \in [0, t] \right\}.$$

Then, $\left(Z''_g, \|.\|\right)$ is a Banach space. Consider for $\zeta_0 > 0$ that

$$T_{\zeta_0} = \left\{ \xi \in Z''_g : \|\xi\|_t \leq \zeta_0 \right\}.$$

Hence, $T_{\zeta_0} \subseteq Z''_g$ is uniformly bounded and for $\xi \in T_{\zeta_0}$ from Lemma 2.20, one has

$$\|z_s + \widetilde{g}_s\| \leq \|z_s\|_{Z_g} + \|\widetilde{g}_s\|_{Z_g}.$$

We describe an operator F: $Z''_g \to Z''_g$ as

$$Fz(s) = \begin{cases} 0, \ s \in (-\infty, 0], \\ \int_0^s [\varphi(s) - \varphi(\zeta)]^{\frac{\varrho}{k}-1} \varphi'(\zeta) M_{\frac{\varrho}{k};\varphi}(s) h\left(\zeta, \left[z_\zeta + \widetilde{g}\right]\right) d\zeta \\ + \int_0^s [\varphi(s) - \varphi(\zeta)]^{\frac{\varrho}{k}-1} \varphi'(\zeta) M_{\frac{\varrho}{k};\varphi}(s) \left(R\left[z + \widetilde{g}\right]\right)(s)\, d\zeta \\ + \int_0^s [\varphi(s) - \varphi(\zeta)]^{\frac{\varrho}{k}-1} \varphi'(\zeta) M_{\frac{\varrho}{k};\varphi}(s) \Xi \upsilon_z(\zeta) d\zeta, \ s \in V. \end{cases} \tag{3.5}$$

We can readily see that the fixed-point of operator Φ is equivalent to F. Our next step is to prove that F possesses a fixed-point. This will be achieved by applying the Mönch fixed-point theorem, as outlined in the subsequent steps:

Step (1): We prove that $F\left(\Xi_\zeta\right) \subseteq \Xi_\zeta$ for $\zeta > 0$. Indeed, define the set Ξ_ζ as

$$\Xi_\zeta = \left\{ \xi \in A : \|\xi\|_{\frac{\varrho}{k};\varphi} \leq \zeta \right\},$$

for $\zeta > 0$. Evidently, Ξ_ζ forms a closed, bounded, and convex set in A. If it is not true, then for each $\zeta > 0$, there exists $z^\zeta \in \Xi_\zeta$. But $F\left(z^\zeta\right) \notin z^\zeta$, i.e., $\left\|Fz^\zeta(s)\right\| > \zeta$ for all $s \in V$. Hence,

$$\left\|Fz^\zeta(s)\right\|_{\frac{\rho}{k};\varphi} = \sup\left\{[\varphi(s)-\varphi(0)]^{1-\frac{\rho}{k}} \left\|Fz^\zeta(s)\right\| : \left\|Fz^\zeta(s)\right\| > \zeta,\ \forall s \in V\right\}.$$

Thus,

$$\begin{aligned}
\zeta &< [\varphi(s)-\varphi(0)]^{1-\frac{\rho}{k}} \left\|Fz^\zeta(s)\right\| \\
&< [\varphi(s)-\varphi(0)]^{1-\frac{\rho}{k}} \left\|\int_0^s [\varphi(s)-\varphi(\zeta)]^{\frac{\rho}{k}-1} \varphi'(\zeta) M_{\frac{\rho}{k};\varphi}(s) h\left(\zeta, \left[z_\zeta + \widetilde{g}\right]\right) d\zeta\right\| \\
&+ [\varphi(s)-\varphi(0)]^{1-\frac{\rho}{k}} \left\|\int_0^s [\varphi(s)-\varphi(\zeta)]^{\frac{\rho}{k}-1} \varphi'(\zeta) M_{\frac{\rho}{k};\varphi}(s) \left(R\left[z+\widetilde{g}\right]\right)(s)\, d\zeta\right\| \\
&+ [\varphi(s)-\varphi(0)]^{1-\frac{\rho}{k}} \left\|\int_0^s [\varphi(s)-\varphi(\zeta)]^{\frac{\rho}{k}-1} \varphi'(\zeta) M_{\frac{\rho}{k};\varphi}(s) \Xi \upsilon_z(\zeta) d\zeta\right\| \\
&\leq N[\varphi(s)-\varphi(0)]^{1-\frac{\rho}{k}} \left\|\int_0^s [\varphi(s)-\varphi(\zeta)]^{\frac{\rho}{k}-1} \varphi'(\zeta) h\left(\zeta, \left[z_\zeta + \widetilde{g}\right]\right) d\zeta\right\| \\
&+ N[\varphi(s)-\varphi(0)]^{1-\frac{\rho}{k}} \left\|\int_0^s [\varphi(s)-\varphi(\zeta)]^{\frac{\rho}{k}-1} \varphi'(\zeta) \left(R\left[z+\widetilde{g}\right]\right)(\zeta)\, d\zeta\right\| \\
&+ N[\varphi(s)-\varphi(0)]^{1-\frac{\rho}{k}} \left\|\int_0^s [\varphi(s)-\varphi(\zeta)]^{\frac{\rho}{k}-1} \varphi'(\zeta) \Xi \upsilon_z(\zeta) d\zeta\right\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
\zeta &< N[\varphi(s)-\varphi(0)]^{1-\frac{\rho}{k}} \left\|\int_0^s [\varphi(s)-\varphi(\zeta)]^{\frac{\rho}{k}-1} \varphi'(\zeta) \varkappa_1(\zeta) \hbar(\zeta)\, d\zeta\right\| \\
&+ N[\varphi(s)-\varphi(0)]^{1-\frac{\rho}{k}} \left\|\int_0^s [\varphi(s)-\varphi(\zeta)]^{\frac{\rho}{k}-1} \varphi'(\zeta) \left(B(s) + L_R B_1(s)\right) d\zeta\right\| \\
&+ N[\varphi(s)-\varphi(0)]^{1-\frac{\rho}{k}} \int_0^s [\varphi(s)-\varphi(\zeta)]^{\frac{\rho}{k}-1} \Bigg\| \Xi F^{-1} \Bigg[\xi_1 - Q_{\frac{\rho}{k};\varphi}(t)\, g(0) \\
&- \int_0^t [\varphi(t)-\varphi(\zeta)]^{\frac{\rho}{k}-1} \varphi'(\zeta) M_{\frac{\rho}{k};\varphi}(t) \left[h\left(s, \left(z_\zeta + \widetilde{g}\right)\right) + \left(R\left(z+\widetilde{g}\right)\right)(\zeta)\right]\Bigg]\Bigg\| \varphi'(\zeta) d\zeta \\
&\leq N[\varphi(s)-\varphi(0)]^{1-\frac{\rho}{k}} \left\|\int_0^s [\varphi(s)-\varphi(\zeta)]^{\frac{\rho}{k}-1} \varphi'(\zeta) \varkappa_1(\zeta) \hbar(\zeta)\, d\zeta\right\| \\
&+ N[\varphi(s)-\varphi(0)]^{1-\frac{\rho}{k}} \left\|\int_0^s [\varphi(s)-\varphi(\zeta)]^{\frac{\rho}{k}-1} \varphi'(\zeta) \left(B(s) + L_R B_1(s)\right) d\zeta\right\| \\
&+ N L_\Xi L_{F^{-1}} [\varphi(t)-\varphi(0)]^{1-\frac{\rho}{k}} \int_0^s [\varphi(s)-\varphi(\zeta)]^{\frac{\rho}{k}-1} \Bigg[\|\xi_1\| + \frac{N[\varphi(s)-\varphi(0)]^{\frac{\rho}{k}-1}}{\left(\frac{\rho}{k}-1\right)\Gamma_k\left(\left(\frac{\rho}{k}-1\right)k\right)} \|g(0)\| \\
&+ N \int_0^t [\varphi(t)-\varphi(\zeta)]^{\frac{\rho}{k}-1} \varphi'(\zeta) \left[\varkappa_1(\zeta)\hbar(\zeta) + \left(B(s) + L_R B_1(s)\right)(\zeta)\right]\Bigg] \varphi'(\zeta) d\zeta \\
&\leq N L_1 [\varphi(t)-\varphi(0)]^{1-\frac{\rho}{k}} \hbar(\zeta) + N L_1 [\varphi(d)-\varphi(0)]^{1-\frac{\rho}{k}} B^*(\zeta) + N L_\Xi L_{F^{-1}} [\varphi(t)-\varphi(0)]^{1-\frac{\rho}{k}} \\
&\left(\|\xi_1\| + \frac{N[\varphi(s)-\varphi(0)]^{\frac{\rho}{k}-1}}{\left(\frac{\rho}{k}-1\right)\Gamma_k\left(\left(\frac{\rho}{k}-1\right)k\right)} \|g(0)\| + N\left(L_1 \hbar(\zeta) + B^*(\zeta)\right)\right).
\end{aligned}$$

It follows that

$$\zeta \leq NL_1 \left[\varphi(t)-\varphi(0)\right]^{1-\frac{\varrho}{k}} \hbar(\zeta) + NL_1 \left[\varphi(d)-\varphi(0)\right]^{1-\frac{\varrho}{k}} B^*(\zeta) + NL_{\Xi} L_{F^{-1}} \left[\varphi(t)-\varphi(0)\right]^{1-\frac{\varrho}{k}}$$
$$\left(\|\xi_1\| + \frac{N\left[\varphi(s)-\varphi(0)\right]^{\frac{\varrho}{k}-1}}{\left(\frac{\varrho}{k}-1\right)\Gamma_k\left(\left(\frac{\varrho}{k}-1\right)k\right)} \|g(0)\| + N\left(L_1 \hbar(\zeta) + B^*(\zeta)\right) \right)$$
$$\leq NL_1 \left[\varphi(t)-\varphi(0)\right]^{1-\frac{\varrho}{k}} \left(\hbar(\zeta) + B^*(\zeta)\right) + NL_{\Xi} L_{F^{-1}} \left[\varphi(t)-\varphi(0)\right]^{1-\frac{\varrho}{k}} \left[\left(\|\xi_1\| + N\left(L_1 \hbar(\zeta) + B^*(\zeta)\right)\right)\right].$$

Dividing both side by ζ and letting $\zeta \to \infty$, we have $1 \leq 0$. This is a contradiction. Hence, $F\left(\Xi_\zeta\right) \subseteq \Xi_\zeta$ for $\zeta \geq 0$.

Step (2): Claim that $Fz(s)$ is continuous on Ξ_ζ. Indeed, for any $z^m, z \in \Xi_\zeta$ with $\lim_{m\to\infty} z^m = z$, then we get $\lim_{m\to\infty} z^m(s) = z(s)$ and

$$\lim_{m\to\infty} \left[\varphi(s)-\varphi(0)\right]^{1-\frac{\varrho}{k}} z^m(s) = \left[\varphi(s)-\varphi(0)\right]^{1-\frac{\varrho}{k}} z(s).$$

Assume that $\xi(s) = \left[\varphi(s)-\varphi(0)\right]^{1-\frac{\varrho}{k}} (z(s)+\widetilde{g}(s))$ and $(z^m+\widetilde{g}) \subset \Xi_\zeta$ such that $\lim_{m\to\infty}(z^m+\widetilde{g}) = (z+\widetilde{g})$ in Ξ_ζ. Then, one has

$$\begin{aligned} h\left(s, \xi^m(s)\right) &= h\left(s, \left[\varphi(s)-\varphi(0)\right]^{1-\frac{\varrho}{k}} (z^m(s)+\widetilde{g}(s))\right) \\ &\longrightarrow h\left(s, \left[\varphi(s)-\varphi(0)\right]^{1-\frac{\varrho}{k}} (z(s)+\widetilde{g}(s))\right) \\ &\longrightarrow h\left(s, \xi(s)\right), \text{ as } m \longrightarrow \infty, \end{aligned}$$

where $h\left(s, \left[\varphi(s)-\varphi(0)\right]^{1-\frac{\varrho}{k}} (z^m(s)+\widetilde{g}(s))\right) = Z_m(s)$ and $h\left(s, \left[\varphi(s)-\varphi(0)\right]^{1-\frac{\varrho}{k}} (z(s)+\widetilde{g}(s))\right) = Z(s)$. The application of (A_1) and the Lebesgue dominated convergence theorem then yield

$$\int_0^s \left[\varphi(s)-\varphi(0)\right]^{1-\frac{\varrho}{k}} \|Z_m(s)-Z(s)\| \varphi'(\zeta) d\zeta \longrightarrow 0, \text{ as } m \longrightarrow \infty. \tag{3.6}$$

Now, from (A_1), we can write

$$\begin{aligned} \|Fz^m - Fz\|_{\frac{\varrho}{k};\varphi} &\leq \left[\varphi(s)-\varphi(0)\right]^{1-\frac{\varrho}{k}} \left\| \int_0^s \left[\varphi(s)-\varphi(\zeta)\right]^{\frac{\varrho}{k}-1} M_{\frac{\varrho}{k};\varphi}(s) \left[h\left(\zeta, \left[z_\zeta^m+\widetilde{g}\right]\right) - h\left(\zeta, \left[z_\zeta+\widetilde{g}\right]\right)\right] \varphi'(\zeta) d\zeta \right\| \\ &+ \left[\varphi(s)-\varphi(0)\right]^{1-\frac{\varrho}{k}} \left\| \int_0^s \left[\varphi(s)-\varphi(\zeta)\right]^{\frac{\varrho}{k}-1} M_{\frac{\varrho}{k};\varphi}(s) \left[\left(R\left[z^m+\widetilde{g}\right] - R\left[z+\widetilde{g}\right]\right)(s)\right] \varphi'(\zeta) d\zeta \right\| \\ &+ \left[\varphi(s)-\varphi(0)\right]^{1-\frac{\varrho}{k}} \left\| \int_0^s \left[\varphi(s)-\varphi(\zeta)\right]^{\frac{\varrho}{k}-1} M_{\frac{\varrho}{k};\varphi}(s) \left[\Xi \upsilon_{z^m}(\zeta) - \Xi \upsilon_z(\zeta)\right] \varphi'(\zeta) d\zeta \right\| \\ &\leq N\left[\varphi(s)-\varphi(0)\right]^{1-\frac{\varrho}{k}} \left\| \int_0^s \left[\varphi(s)-\varphi(\zeta)\right]^{\frac{\varrho}{k}-1} \left[h\left(\zeta, \left[z_\zeta^m+\widetilde{g}\right]\right) - h\left(\zeta, \left[z_\zeta+\widetilde{g}\right]\right)\right] \varphi'(\zeta) d\zeta \right\| \\ &+ N\left[\varphi(s)-\varphi(0)\right]^{1-\frac{\varrho}{k}} \left\| \int_0^s \left[\varphi(s)-\varphi(\zeta)\right]^{\frac{\varrho}{k}-1} \left[\left(R\left[z^m+\widetilde{g}\right] - R\left[z+\widetilde{g}\right]\right)(s)\right] \varphi'(\zeta) d\zeta \right\| \\ &+ NL_{\Xi}\left[\varphi(s)-\varphi(0)\right]^{1-\frac{\varrho}{k}} \left\| \int_0^s \left[\varphi(s)-\varphi(\zeta)\right]^{\frac{\varrho}{k}-1} \left[\upsilon_{z^m}(\zeta) - \upsilon_z(\zeta)\right] \varphi'(\zeta) d\zeta \right\| \\ &\leq N\left[\varphi(s)-\varphi(0)\right]^{1-\frac{\varrho}{k}} \left\| \int_0^s \left[\varphi(s)-\varphi(\zeta)\right]^{\frac{\varrho}{k}-1} \left[Z_m(s) - Z(s)\right] \varphi'(\zeta) d\zeta \right\| \\ &+ L_R N\left[\varphi(s)-\varphi(0)\right]^{1-\frac{\varrho}{k}} \left\| \int_0^s \left[\varphi(s)-\varphi(\zeta)\right]^{\frac{\varrho}{k}-1} \left[(z^m - z)(s)\right] \varphi'(\zeta) d\zeta \right\| \\ &+ NL_{\Xi} \int_0^s \left[\varphi(s)-\varphi(\zeta)\right]^{1-\frac{\varrho}{k}} \varphi'(\zeta) d\zeta \left\| \int_0^t \left[\varphi(t)-\varphi(\zeta)\right]^{\frac{\varrho}{k}-1} \left[Z_m(s) - Z(s)\right] \varphi'(\zeta) d\zeta \right\|. \end{aligned}$$

Based on (3.6), we have

$$\|Fz^m - Fz\|_{\frac{\rho}{k};\varphi} \longrightarrow 0, \text{ as } m \longrightarrow \infty.$$

This proves that $Fz(s)$ is continuous on Ξ_ζ.

Step (3): Show that $F\left(\Xi_\zeta\right)$ is equicontinuous on V. For this, assume that $\xi \in \Xi_\zeta$ and $0 \le s_1 \le s_2 \le t$. It is sufficient to prove that

$$\|\xi(s_2) - \xi(s_1)\|_{\frac{\rho}{k};\varphi} \longrightarrow 0, \text{ as } s_2 \longrightarrow s_1,$$

where

$$\begin{aligned}\xi(s) = {} & Q_{\frac{\rho}{k};\varphi}(s)\, g(0) + \int_0^s [\varphi(s) - \varphi(\zeta)]^{\frac{\rho}{k}-1} \varphi'(\zeta) M_{\frac{\rho}{k};\varphi}(s) Z(\zeta) d\zeta \\ & + \int_0^s [\varphi(s) - \varphi(\zeta)]^{\frac{\rho}{k}-1} \varphi'(\zeta) M_{\frac{\rho}{k};\varphi}(s) \Xi \upsilon_\xi(\zeta)\, d\zeta \\ & + \int_0^s [\varphi(s) - \varphi(\zeta)]^{\frac{\rho}{k}-1} \varphi'(\zeta) M_{\frac{\rho}{k};\varphi}(s) (R\xi)(\zeta) d\zeta.\end{aligned}$$

Hence,

$$\begin{aligned}\|\xi(s_2) - \xi(s_1)\|_{\frac{\rho}{k};\varphi} = {} & \Bigg\| [\varphi(s_2) - \varphi(0)]^{1-\frac{\rho}{k}} \int_0^{s_2} [\varphi(s_2) - \varphi(\zeta)]^{\frac{\rho}{k}-1} M_{\frac{\rho}{k};\varphi}(s_2) \\ & \times \left(Z(\zeta) + \Xi\upsilon_\xi(\zeta) + (R\xi)(\zeta)\right) \varphi'(\zeta) d\zeta \\ & - [\varphi(s_1) - \varphi(0)]^{1-\frac{\rho}{k}} \int_0^{s_1} [\varphi(s_1) - \varphi(\zeta)]^{\frac{\rho}{k}-1} M_{\frac{\rho}{k};\varphi}(s_1) \\ & \times \left(Z(\zeta) + \Xi\upsilon_\xi(\zeta) + (R\xi)(\zeta)\right) \varphi'(\zeta) d\zeta \Bigg\| \\ \le {} & \Bigg\| [\varphi(s_2) - \varphi(0)]^{1-\frac{\rho}{k}} \int_{s_1}^{s_2} [\varphi(s_2) - \varphi(\zeta)]^{\frac{\rho}{k}-1} M_{\frac{\rho}{k};\varphi}(s_2) \\ & \left(Z(\zeta) + \Xi\upsilon_\xi(\zeta) + (R\xi)(\zeta)\right) \varphi'(\zeta) d\zeta \Bigg\| \\ & + \Bigg\| [\varphi(s_2) - \varphi(0)]^{1-\frac{\rho}{k}} \int_{s_1-\varepsilon}^{s_1} [\varphi(s_2) - \varphi(\zeta)]^{\frac{\rho}{k}-1} \left[M_{\frac{\rho}{k};\varphi}(s_2) - M_{\frac{\rho}{k};\varphi}(s_1)\right] \\ & \left(Z(\zeta) + \Xi\upsilon_\xi(\zeta) + (R\xi)(\zeta)\right) \varphi'(\zeta) d\zeta \Bigg\| \\ & + \Bigg\| [\varphi(s_2) - \varphi(0)]^{1-\frac{\rho}{k}} \int_{s_1-\varepsilon}^{s_1} \left([\varphi(s_2) - \varphi(\zeta)]^{\frac{\rho}{k}-1} - [\varphi(s_1) - \varphi(\zeta)]^{\frac{\rho}{k}-1}\right) M_{\frac{\rho}{k};\varphi}(s_1) \\ & \left(Z(\zeta) + \Xi\upsilon_\xi(\zeta) + (R\xi)(\zeta)\right) \varphi'(\zeta) d\zeta \Bigg\| \\ & + \Bigg\| [\varphi(s_2) - \varphi(0)]^{1-\frac{\rho}{k}} \int_0^{s_1-\varepsilon} [\varphi(s_2) - \varphi(\zeta)]^{\frac{\rho}{k}-1} \left[M_{\frac{\rho}{k};\varphi}(s_2) - M_{\frac{\rho}{k};\varphi}(s_1)\right] \\ & \left(Z(\zeta) + \Xi\upsilon_\xi(\zeta) + (R\xi)(\zeta)\right) \varphi'(\zeta) d\zeta \Bigg\| \\ & + \Bigg\| [\varphi(s_2) - \varphi(0)]^{1-\frac{\rho}{k}} \int_0^{s_1-\varepsilon} \left([\varphi(s_2) - \varphi(\zeta)]^{\frac{\rho}{k}-1} - [\varphi(s_1) - \varphi(\zeta)]^{\frac{\rho}{k}-1}\right) M_{\frac{\rho}{k};\varphi}(s_1) \\ & \left(Z(\zeta) + \Xi\upsilon_\xi(\zeta) + (R\xi)(\zeta)\right) \varphi'(\zeta) d\zeta \Bigg\|.\end{aligned}$$

Provided ε is sufficiently small, the dominated convergence theorem implies that

$$\|\xi(s_2) - \xi(s_1)\|_{\frac{\rho}{k};\varphi} \longrightarrow 0, \text{ as } s_2 \longrightarrow s_1.$$

Therefore, $F\left(\Xi_\zeta\right)$ is equicontinuous on V.

Step (4): Illustrate that the Mönch condition is true. For this regard, assume that $z^0(s) + \widetilde{g}(s) = [\varphi(s) - \varphi(0)]^{1-\frac{\rho}{k}} \widetilde{g}_0$ for all $s \in V$, $z^{m+1}(s) + \widetilde{g}(s) = F\left[z^m(s) + \widetilde{g}(s)\right]$, $m = 0, 1, \cdots$, and F is relatively compact.

Suppose that χ is a countable subset of Ξ_ζ. If χ is also a subset of $conv(0 \cup F(\chi))$, then we demonstrate that $E(\chi) = 0$, where E represents the measure of noncompactness.

Given $\chi = \{z^m + \widetilde{g}\}_{m=1}^{\infty}$, our aim is to claim that $E(\chi(s))$ is relatively compact in A for every $s \in V$. Then, Theorem 2.22 allows us to conclude that

$$\begin{aligned}
E(\chi(s)) &= E\left(\{z^m + \widetilde{g}\}_{m=0}^{\infty}\right) \\
&= E\left(\{z^0 + \widetilde{g}\} + \{z^m + \widetilde{g}\}_{m=1}^{\infty}\right) \\
&= E\left(\{z^m(s) + \widetilde{g}(s)\}_{m=1}^{\infty}\right)
\end{aligned}$$

and

$$\begin{aligned}
E\left(\{Fz^m(s)\}_{m=1}^{\infty}\right) = E\Bigg(\Bigg\{&[\varphi(s) - \varphi(0)]^{1-\frac{\rho}{k}} \int_0^s [\varphi(s) - \varphi(\zeta)]^{\frac{\rho}{k}-1} M_{\frac{\rho}{k};\varphi}(s) \\
&\left(Z_m(\zeta) + \Xi\upsilon_{\xi^m}(\zeta) + (R\xi^m)(\zeta)\right)\varphi'(\zeta)d\zeta\Bigg\}_{m=1}^{\infty}\Bigg) \\
\le\ & D_1 + D_2 + D_3,
\end{aligned}$$

where

$$\begin{aligned}
D_1 &= E\left([\varphi(s) - \varphi(0)]^{1-\frac{\rho}{k}} \int_0^s [\varphi(s) - \varphi(\zeta)]^{\frac{\rho}{k}-1} M_{\frac{\rho}{k};\varphi}(s) Z_m(\zeta)\varphi'(\zeta)d\zeta\right) \\
&\le 2N[\varphi(t) - \varphi(0)]^{1-\frac{\rho}{k}} \int_0^s [\varphi(s) - \varphi(\zeta)]^{\frac{\rho}{k}-1} E\left(\{Z_m(\zeta)\}_{m=1}^{\infty}\right)\varphi'(\zeta)d\zeta \\
&\le 2N[\varphi(t) - \varphi(0)]^{1-\frac{\rho}{k}} \int_0^s [\varphi(s) - \varphi(\zeta)]^{\frac{\rho}{k}-1} E\left(h\left(\zeta, \left[z_\zeta^m + \widetilde{g}\right]\right)\right)\varphi'(\zeta)d\zeta \\
&\le 2N[\varphi(t) - \varphi(0)]^{1-\frac{\rho}{k}} \int_0^s [\varphi(s) - \varphi(\zeta)]^{\frac{\rho}{k}-1} \varkappa_1(\zeta) \\
&\times \sup_{-\infty < r \le 0} E\left([z^m(\zeta + r) + \widetilde{g}(\zeta + r)]\right)\varphi'(\zeta)d\zeta \\
&\le 2N[\varphi(t) - \varphi(0)]^{1-\frac{\rho}{k}} \int_0^s [\varphi(s) - \varphi(\zeta)]^{\frac{\rho}{k}-1} \varkappa_1(\zeta) \sup_{-\infty < r \le 0} E\left(\chi(r')\right)\varphi'(\zeta)d\zeta, \\
D_2 &= E\left([\varphi(s) - \varphi(0)]^{1-\frac{\rho}{k}} \int_0^s [\varphi(s) - \varphi(\zeta)]^{\frac{\rho}{k}-1} M_{\frac{\rho}{k};\varphi}(s) \Xi\upsilon_{\xi^m}(\zeta)\varphi'(\zeta)d\zeta\right) \\
&\le 2NL_\Xi\left([\varphi(t) - \varphi(0)]^{1-\frac{\rho}{k}} \int_0^s [\varphi(s) - \varphi(\zeta)]^{\frac{\rho}{k}-1} \mho\left(\left\{\upsilon_{\xi^m}(\zeta)\right\}_{m=1}^{\infty}\right)\varphi'(\zeta)d\zeta\right) \\
&\le 2NL_\Xi\Bigg([\varphi(t) - \varphi(0)]^{1-\frac{\rho}{k}} \int_0^s [\varphi(s) - \varphi(\zeta)]^{\frac{\rho}{k}-1} \varphi'(\zeta)d\zeta
\end{aligned}$$

$$\times\left[2NL_{F^{-1}}L_R\int_0^t[\varphi(t)-\varphi(\zeta)]^{\frac{\varrho}{k}-1}\varkappa_2(\zeta)\left(E\left(h\left(\zeta,\left[z_\zeta^m+\widetilde{g}\right]\right)\right)+E\left(\left[z_\zeta^m+\widetilde{g}\right]\right)\right)\varphi'(\zeta)d\zeta\right]\Bigg)$$
$$\leq 4NL_\Xi L_{F^{-1}}L_R\left([\varphi(t)-\varphi(0)]^{1-\frac{\varrho}{k}}\int_0^s[\varphi(s)-\varphi(\zeta)]^{\frac{\varrho}{k}-1}\varphi'(\zeta)d\zeta\right.$$
$$\left.\times\left[\int_0^t[\varphi(t)-\varphi(\zeta)]^{\frac{\varrho}{k}-1}\varkappa_2(\zeta)E(\chi(\zeta'))\varphi'(\zeta)d\zeta\right]\right.$$

and

$$\begin{aligned}D_3&=E\left([\varphi(s)-\varphi(0)]^{1-\frac{\varrho}{k}}\int_0^s[\varphi(s)-\varphi(\zeta)]^{\frac{\varrho}{k}-1}M_{\frac{\varrho}{k};\varphi}(s)(R\xi^m)(\zeta)\varphi'(\zeta)d\zeta\right)\\&\leq 2NL_R\left([\varphi(t)-\varphi(0)]^{1-\frac{\varrho}{k}}\int_0^s[\varphi(s)-\varphi(\zeta)]^{\frac{\varrho}{k}-1}\mho(\{\xi^m\}_{m=1}^\infty)\varphi'(\zeta)d\zeta\right)\\&\leq 2NL_R[\varphi(t)-\varphi(0)]^{1-\frac{\varrho}{k}}\int_0^s[\varphi(s)-\varphi(\zeta)]^{\frac{\varrho}{k}-1}\varkappa_1(\zeta)\sup_{-\infty<r\leq 0}E(\chi(r'))\varphi'(\zeta)d\zeta.\end{aligned}$$

By collecting D_1–D_3, we have

$$D_1+D_2+D_3\leq 2NL_1[\varphi(t)-\varphi(0)]^{1-\frac{\varrho}{k}}(1+2NL_\Xi L_{F^{-1}}L_RL_2)\sup E(\chi(r')),$$

which implies that

$$E(F(\chi))\leq\widetilde{M}\mho(\chi),$$

where $\widetilde{M}$ is defined in (3.4). As a result of Mönch's condition, we find that

$$E(\chi)\leq E(conv(0\cup F(\chi)))\leq E(F(\chi))\leq\widetilde{M}E(\chi).$$

This implies that $E(\chi)=0$, which in turn means χ is relatively compact. Therefore, F has a fixed-point and this completes the proof.

4. An illustrative example

In this part, we illustrate how the obtained results can be applied to the following problem, with the goal of connecting theoretical developments to practical implementation. Assume the following fractional problem:

$$\begin{cases}\left({}^HD_{1,0^+}^{\frac{6}{7},0,s^3}\xi\right)(s,r)=\frac{\partial^2}{\partial s^2}\xi(s,r)+\frac{\sin s}{16}e^{-\xi(s,r)}+\Delta(s,r)+\frac{1}{9}\int\limits_0^s\xi(s,r)dr,\ s\in[0,1],\\\left(J_{1,0^+}^{\frac{1}{7};s^3}\xi\right)(s,r)|_{s=0}=g(r),\ r\in[0,\pi],\end{cases}\tag{4.1}$$

where $\rho=\frac{6}{7}\in\left(\frac{1}{2},1\right)$, $\eta=0\in[0,1]$, $k=1$, $\varphi(s)=s^3$, ${}^HD_{1,0^+}^{\frac{6}{7},0,s^3}$ is the (k,φ)-Hilfer fractional derivative operator, $J_{1,0^+}^{\frac{1}{7};s^3}$ is the (k,φ)-Hilfer fractional integral operator, $g\in Z_g$, $\Delta(s,r)$: $[0,1]\times[0,\pi]\to\mathbb{R}$ is continuous, and $(R\xi)(s,r)=\frac{1}{9}\int\limits_0^s\xi(s,r)dr$ is the history-dependent operator.

Assume that $A = W = L^2\,[0,\pi]$ and the operator Υ: $D(\Upsilon) \subset A \to A$ is described as

$$\Upsilon\xi = \left\{ \frac{\partial^2}{\partial r^2}\xi\,(s,r) : \xi \in D(\Upsilon) \right\},$$

where

$$D(\Upsilon) = \{\xi \in A : \xi'' \in A \text{ and } \xi\,(s,0) = \xi\,(s,\pi) = 0\}.$$

Clearly, the infinitesimal generator of the analytic semigroup $\{P\,(s)\}_{s\geq 0}$ in A is Υ with $P\,(s)\,z\,(\varrho) = z\,(s+\varrho)$ for $z \in A$. Although $P\,(s)$ is not compact on A, it holds that $E\,(P\,(s)\,D) \leq E\,(D)$, where E denotes the Hausdorff measure of noncompactness. Furthermore, we can find an $N \geq 1$ such that $\sup_{s\in[0,1]} \|P(s)\| \leq N$.

Let the control operator Ξ: $A \to A$ be defined as

$$(\Xi\upsilon)\,(s,r) = \Delta\,(s,r)\,,\;\; r \in [0,\pi].$$

For $r \in (0,\pi)$, F is given by

$$F\,(\upsilon) = \int_0^1 \left[1-\zeta^3\right]^{\frac{-1}{7}} (3\zeta^2) M_{\frac{6}{7};s^3}(s,r)\Delta\,(s,r)\,ds,$$

where

$$M_{\frac{6}{7};s^3}(s,r) = \int_0^\infty \vartheta_{\frac{6}{7}}(r)\xi\left(s^{\frac{6}{7}}+r\right)dr.$$

We assume that $h\,(s,\xi\,(s,r)) = \frac{\sin s}{16}e^{-\xi(s,r)}$ with

$$\|h\,(s,\xi\,(s,r))\| \leq \frac{\sin s}{16}\left\|\frac{1}{\xi\,(s,r)}\right\| = \frac{\sin s}{16}\frac{1}{\|\xi\,(s,r)\|_{Z_g}}, \quad \xi\,(s) \neq 0.$$

Also, for any bounded subset $Z_1 \subset Z_g$, one has

$$E\,(h\,(s,Z_1)) \leq \frac{\sin s}{16}\left[\sup_{-\infty<\zeta'\leq 0} E\,(Z_1\,(\zeta'))\right].$$

Hence, $L_\Xi = L_{F^{-1}} = 1$ and $\varkappa_1(s) = \varkappa_2(s) = \frac{\sin s}{16}$. This implies that (A_1) and (A_2) are fulfilled.

Furthermore, from the definition of R, the assertion (A_4) is true with $B(s) = 0$, $B(s) = 1$, and $B^*(s) = L_R = \frac{1}{9}$ for each $s \in [0,1]$.

Now, we define the phase space Z_g using the norm

$$\|\xi\|_{Z_g} = \int_{-\infty}^0 g\,(s)\,\|\xi\|_{[s,0]}\,ds, \quad \forall\xi \in Z_g,$$

where $g\,(s) = e^{6s}$ for $s < 0$ and $\int_{-\infty}^0 g\,(s)\,ds = \frac{1}{6}$. Hence, $\|\xi\|_{Z_g} = \frac{1}{6}\,\|\xi\|$ and the hypotheses (A_3) holds with $\varkappa_3(s) = \frac{\sin s}{16}$.

Therefore, the conditions (A_1)–(A_4) are fulfilled. By taking $N = 1$ and $\eta_1 = \eta_2 = \eta_3 = \frac{3}{4}$, and from the above, $\varkappa_1(s) = \varkappa_2(s) = \varkappa_3(s) = \frac{\sin s}{16} \in L^2\,([0,1],\mathbb{R}^+)$, and we find that $\eta_j \in \left(\frac{1}{2},\frac{6}{7}\right)$.

By simple calculations, we have $L_1 = L_2 \approx 0.258214$. Form (3.3), we have $l_1 = l_2 \approx 1.528781$ and by condition (3.4), we obtain that $\widetilde{M} \approx 0.546061 < 1$. Consequently, all requirements of Theorem 3.4 are fulfilled. Hence, the problem (4.1) is controllable on $[0,1]$.

5. Conclusions

Controllability, defined as the ability to steer a dynamical system from any initial state to a desired final state within a finite time, is a fundamental concept in control theory. In the context of (k, φ)-Hilfer fractional delay differential equations with history-dependent operators, establishing controllability is particularly important due to the presence of memory effects and nonlocal dependencies, where the system's future evolution is influenced by its entire past through a generalized fractional derivative. Such features make these models highly relevant in practical applications across engineering and biological systems, including areas such as robotic control, chemical processes, drug delivery, and disease management. In this work, a mild solution is obtained for the considered Hilfer-type system, involving key operators associated with semigroup theory, history-dependent terms, and certain probability density functions. By employing Mönch's fixed-point theorem together with the measure of noncompactness, sufficient conditions for controllability are established for the (k, φ)-Hilfer fractional delay differential equation with history-dependent operators. The proposed approach enables a rigorous analysis of the complex interaction between fractional dynamics, delay, and memory effects, leading to new theoretical insights. Nevertheless, the study presents notable challenges and limitations: the coexistence of fractional order, infinite delay, and history dependence introduces strong nonlocal behavior, complicating the formulation of suitable functional settings and the proof of existence results. Moreover, the associated solution operators are generally noncompact, restricting the applicability of classical fixed-point methods and requiring more advanced techniques. The verification of controllability conditions is further hindered by the intricate coupling between delay, memory, and control terms, often leading to restrictive assumptions that may limit the generality and direct applicability of the results.

Author contributions

Doha A. Kattan: validation, formal analysis, writing–review and editing; Hasanen A. Hammad: formal analysis, methodology, writing-original draft, writing–review and editing; Najat Almutairi: methodology, funding acquisition, writing–review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they do not have any conflicts.

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