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Research article

Sharp Chen inequalities for QR-submanifolds in quaternionic space forms

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Abstract: In this paper, we established the sharp Chen-type inequalities for QR-submanifolds immersed in quaternionic space forms endowed with a quarter-symmetric metric connection (QSMC). By extending Chen's δ-invariant framework to both the invariant and anti-invariant distributions of a QR-submanifold, we obtained optimal upper bounds for the invariants $\delta(\mathcal{D})$ and $\delta(\mathcal{D}^{\perp})$. The resulting inequalities explicitly capture the influence of the quarter-symmetric metric connection through its structural parameters and associated tensor fields, thereby providing a unified generalization of the classical Levi-Civita and semi-symmetric settings. Furthermore, we completely characterized the equality cases, showing that the bounds are attained precisely when the submanifold is mixed geodesic, and the invariant distribution is totally umbilical under specific constraints on the second fundamental form. As direct consequences, previously known inequalities for semi-symmetric metric and nonmetric connections are recovered. These results furnish a new rigidity phenomena for extremal QR-submanifolds and deepen the understanding of curvature invariants in quaternionic geometry.

Keywords: QR-submanifold; δ-invariant; quaternionic space form; quarter-symmetric metric connection; Chen inequality; mixed geodesic
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1. Introduction

The geometry of submanifolds in quaternionic manifolds has emerged as an active and rapidly developing area of research, owing to the rich algebraic structure of quaternionic geometry and the intricate interplay between intrinsic and extrinsic geometric invariants [1,2]. In contrast to the complex setting, quaternionic manifolds are endowed with a family of compatible almost complex structures, which substantially enrich the behavior of curvature, distributions, and second fundamental forms [3].

Consequently, submanifold theory in quaternionic geometry exhibits a phenomena that are absent in purely Riemannian or complex contexts, making it a particularly suitable framework for investigating curvature inequalities and rigidity properties.

Among the various classes of the submanifolds studied in quaternionic geometry, *QR*-submanifolds occupy a central position. Introduced by Bejancu [4], this notion provides a natural unification of quaternionic *CR*-submanifolds and real hypersurfaces within a single geometric framework. A *QR*-submanifold admits a decomposition of its tangent and normal bundles compatible with the quaternionic structure, leading to invariant and anti-invariant distributions that play a fundamental role in its geometric analysis [5–7]. In particular, the quaternionic *CR*-submanifolds arise as a special case and have been extensively studied in quaternionic Kähler manifolds [5].

On the other hand, Chen-type curvature inequalities constitute a fundamental theme in submanifold theory, as they establish precise quantitative relationships between intrinsic invariants, such as scalar curvature, and extrinsic quantities involving the second fundamental form [8]. In this direction, B.-Y. Chen introduced the notion of δ-invariants, which measure the deviation between the scalar curvature of a submanifold and that associated with certain distinguished tangent distributions. These invariants provide refined curvature information beyond classical estimates and have proved to be powerful tools for detecting rigidity phenomena, classifying equality cases, and characterizing extremal submanifolds [9]. In both quaternionic and complex settings, numerous studies have shown that sharp δ-invariant inequalities impose strong geometric restrictions on the immersion [3, 10, 11].

In the quaternionic context, Macsim and Mihai [10] obtained an inequality involving the δ-invariant $\delta(D)$ for antiholomorphic submanifolds of complex space forms, thereby highlighting the influence of invariant and anti-invariant distributions on curvature estimates. Subsequently, they extended these ideas to *QR*-submanifolds of quaternionic space forms and established Chen-type inequalities with respect to the Levi-Civita connection [11]. These results provide a natural quaternionic analog of earlier inequalities developed in complex geometry [3, 12], and they underline the importance of distributional geometry in controlling curvature invariants.

Parallel to these developments, increasing attention has been devoted to submanifolds endowed with nonstandard linear connections, particularly quarter-symmetric metric connections (QSMCs) [13–15]. Such connections generalize the Levi-Civita connection by incorporating torsion-type terms while preserving metric compatibility, and they arise naturally in various geometric and physical contexts [16, 17]. The presence of a QSMC modifies the Gauss and Codazzi equations and consequently affects the structure of curvature tensors and the form of geometric inequalities satisfied by submanifolds. Although several results have been obtained for submanifolds in quaternionic space forms equipped with QSMC (see, for instance, [13, 14]), the combined influence of *QR*-submanifold geometry, Chen δ-invariants, and quarter-symmetric metric connections has not yet been fully explored [15, 16].

Motivated by the above considerations, the purpose of the present paper is to establish the sharp Chen-type inequalities for *QR*-submanifolds of quaternionic space forms endowed with a quarter-symmetric metric connection [18, 19]. More precisely, we derive optimal inequalities for the δ-invariants $\delta(D)$ and $\delta(D^{\perp})$, corresponding to the invariant and anti-invariant distributions, respectively. The obtained inequalities explicitly incorporate the structural parameters of the quarter-symmetric metric connection, thereby capturing the influence of torsion on curvature invariants [12, 20, 21]. In particular, our results recover the classical Chen-type inequalities in the Levi-Civita case as a special case, while also encompassing previously known inequalities for semi-symmetric metric and nonmetric connections.

Furthermore, we provide a complete characterization of the equality cases of the obtained inequalities in terms of precise geometric conditions on the distributions and the second fundamental form [22, 23]. These characterizations lead to rigidity results for extremal *QR*-submanifolds and offer deeper insight into the geometry of such submanifolds under nonstandard connections [24]. The approach adopted in this paper is based on a careful decomposition of curvature expressions via the Gauss equation, together with an analysis of quadratic forms associated with the components of the second fundamental form, whose optimization under suitable constraints yields the desired sharp inequalities.

The paper is organized as follows. In Section 2, we present the necessary preliminaries on quaternionic space forms, *QR*-submanifolds, and quarter-symmetric metric connections. Section 3 is devoted to the derivation of the main inequalities and the characterization of the equality cases. Finally, in Section 4, we discuss the consequences of the results and indicate possible directions for further research.

2. Preliminaries

Let $\tilde{M}$ be a $4n$-dimensional Riemannian manifold endowed with a Riemannian metric g, and let $\overline{\overline{\widetilde{\nabla}}}$ denote the Levi-Civita connection on $\tilde{M}$. We consider a linear connection $\overline{\nabla}$ on $\tilde{M}$ defined by

$$\overline{\nabla}_{\Xi_1}\Xi_2 = \overline{\overline{\widetilde{\nabla}}}_{\Xi_1}\Xi_2 + \kappa_1\pi(\Xi_2)\Xi_1 - \kappa_2 g(\Xi_1, \Xi_2)P, \tag{2.1}$$

for all vector fields Ξ_1, Ξ_2 on $\tilde{M}$, where κ_1, κ_2 are real constants, P is a vector field on $\tilde{M}$, and π is the 1-form defined by $\pi(\Xi_1) = g(\Xi_1, P)$. The connection $\overline{\nabla}$ is called a QSMC if $\overline{\nabla}g = 0$, and a quarter-symmetric nonmetric connection (QSNMC) if $\overline{\nabla}g \neq 0$.

Several well-known connections arise as special cases of (2.1):

- If $\kappa_1 = \kappa_2 = 1$, then $\overline{\nabla}$ reduces to the semi-symmetric metric connection (SSMC).
- If $\kappa_1 = 1$, and $\kappa_2 = 0$, then $\overline{\nabla}$ reduces to the semi-symmetric nonmetric connection (SSNMC).

Remark 2.1. The parameters κ_1 and κ_2 appearing in the definition of the QSMC measure the deviation of the connection from the Levi-Civita connection. In particular, they control the torsion tensor and introduce additional curvature terms in the Gauss equation. When $\kappa_1 = \kappa_2 = 0$, the connection reduces to the Levi-Civita connection, and all additional contributions vanish.

The curvature tensor associated with the connection $\overline{\nabla}$ is defined by

$$\overline{R}(\Xi_1, \Xi_2)\Xi_3 = \overline{\nabla}_{\Xi_1}\overline{\nabla}_{\Xi_2}\Xi_3 - \overline{\nabla}_{\Xi_2}\overline{\nabla}_{\Xi_1}\Xi_3 - \overline{\nabla}_{[\Xi_1,\Xi_2]}\Xi_3. \tag{2.2}$$

Using (2.1), the corresponding $(0, 4)$-type curvature tensor can be expressed as follows (see [14]):

$$\begin{aligned}\overline{R}(\Xi_1, \Xi_2, \Xi_3, \Xi_4) = {} & \overline{\overline{\widetilde{R}}}(\Xi_1, \Xi_2, \Xi_3, \Xi_4) + \kappa_1\Gamma_1(\Xi_1, \Xi_3)g(\Xi_2, \Xi_4) - \kappa_1\Gamma_1(\Xi_2, \Xi_3)g(\Xi_1, \Xi_4) \\ & + \kappa_2 g(\Xi_1, \Xi_3)\Gamma_1(\Xi_2, \Xi_4) - \kappa_2 g(\Xi_2, \Xi_3)\Gamma_1(\Xi_1, \Xi_4) \\ & + \kappa_2(\kappa_1 - \kappa_2)[g(\Xi_1, \Xi_3)\Gamma_2(\Xi_2, \Xi_4) - g(\Xi_2, \Xi_3)\Gamma_2(\Xi_1, \Xi_4)],\end{aligned} \tag{2.3}$$

where the symmetric $(0,2)$-tensors Γ_1 and Γ_2 are given by

$$\Gamma_1(\Xi_1,\Xi_2) = (\overline{\nabla}_{\Xi_1}\pi)(\Xi_2) - \kappa_1\pi(\Xi_1)\pi(\Xi_2) + \frac{\kappa_2}{2}g(\Xi_1,\Xi_2)\pi(P),$$

and

$$\Gamma_2(\Xi_1,\Xi_2) = \frac{\pi(P)}{2}g(\Xi_1,\Xi_2) + \pi(\Xi_1)\pi(\Xi_2).$$

Let $\tilde{M}$ be a $4n$-dimensional manifold equipped with a Riemannian metric g. The manifold $\tilde{M}$ is said to be a quaternion Kähler manifold if there exists a rank-3 vector bundle σ locally spanned by almost Hermitian structures F_1, F_2, F_3 satisfying

$$\begin{cases} F_1 \circ F_2 = -F_2 \circ F_1 = F_3, \\ F_2 \circ F_3 = -F_3 \circ F_2 = F_1, \\ F_3 \circ F_1 = -F_1 \circ F_3 = F_2, \end{cases}$$

together with

$$F_1^2 = F_2^2 = F_3^2 = -\mathrm{Id}.$$

In this case, $(\tilde{M},\sigma)$ is called an almost quaternion manifold. A Riemannian metric g on $\tilde{M}$ is said to be adapted to σ if

$$g(F_{\beta_1}\Xi_1, F_{\beta_1}\Xi_2) = g(\Xi_1,\Xi_2), \qquad \beta_1 = 1,2,3.$$

The triple $(\tilde{M},\sigma,g)$ is then called an almost quaternion Hermitian manifold. If σ is parallel with respect to the Levi-Civita connection $\tilde{\nabla}$, the manifold $(\tilde{M},\sigma,g)$ is said to be quaternion Kähler. Equivalently, there exist locally defined 1-forms $\omega_1,\omega_2,\omega_3$ such that

$$(\tilde{\nabla}_{\Xi_1}F_{\beta_1})(\Xi_1) = \omega_{\beta_1+2}(\Xi_1)F_{\beta_1+1} - \omega_{\beta_1+1}(\Xi_1)F_{\beta_1+2},$$

where the indices are taken modulo 3.

For a quaternion Kähler manifold $(\tilde{M},\sigma,g)$ and a non-null vector $\Xi_1 \in T_p\tilde{M}$, the 4-dimensional subspace

$$Q(\Xi_1) = \mathrm{span}\{\Xi_1, F_1\Xi_1, F_2\Xi_1, F_3\Xi_1\}$$

is called a quaternion 4-plane. Any 2-plane contained in $Q(\Xi_1)$ is referred to as a quaternion plane, and its sectional curvature is termed the quaternion sectional curvature. If $\pi = \mathrm{span}\{\Xi_1,\Xi_2\} \subset T_p\tilde{M}$, then its sectional curvature is defined by

$$\overline{K}(\pi) = \frac{\overline{R}(\Xi_1,\Xi_2,\Xi_1,\Xi_2)}{g(\Xi_1,\Xi_1)g(\Xi_2,\Xi_2) - g^2(\Xi_1,\Xi_2)}.$$

A quaternion Kähler manifold is called a quaternionic space form if its quaternion sectional curvature is constant, say c. In this case, the curvature tensor $\overline{\overline{\tilde{R}}}$ of the Levi-Civita connection $\overline{\overline{\tilde{\nabla}}}$ on $\tilde{M}(c)$ is given by ([1])

$$\overline{\overline{\tilde{R}}}(\Xi_1,\Xi_2,\Xi_3,\Xi_4) = \frac{c}{4}\Big\{g(\Xi_2,\Xi_3)g(\Xi_1,\Xi_4) - g(\Xi_1,\Xi_3)g(\Xi_2,\Xi_4)$$

$$+\sum_{\beta_1=1}^{3}\left[g(\Xi_3,F_{\beta_1}\Xi_2)g(F_{\beta_1}\Xi_1,\Xi_4)-g(\Xi_3,F_{\beta_1}\Xi_1)g(F_{\beta_1}\Xi_2,\Xi_4)\right.$$
$$\left.+2g(\Xi_1,F_{\beta_1}\Xi_2)g(F_{\beta_1}\Xi_3,\Xi_4)\right]\Big\}. \tag{2.4}$$

Combining (2.3) and (2.4), the curvature tensor of $\overline{\nabla}$ on $\tilde{M}(c)$ takes the form

$$\begin{aligned}\overline{R}(\Xi_1,\Xi_2,\Xi_3,\Xi_4)=c\Big\{&g(\Xi_2,\Xi_3)g(\Xi_1,\Xi_4)-g(\Xi_1,\Xi_3)g(\Xi_2,\Xi_4)\\&+\sum_{\beta_1=1}^{3}\left[g(\Xi_3,F_{\beta_1}\Xi_2)g(F_{\beta_1}\Xi_1,\Xi_4)-g(\Xi_3,F_{\beta_1}\Xi_1)g(F_{\beta_1}\Xi_2,\Xi_4)\right.\\&\left.+2g(\Xi_1,F_{\beta_1}\Xi_2)g(F_{\beta_1}\Xi_3,\Xi_4)\right]\Big\}\\&+\kappa_1\Gamma_1(\Xi_1,\Xi_3)g(\Xi_2,\Xi_4)-\kappa_1\Gamma_1(\Xi_2,\Xi_3)g(\Xi_1,\Xi_4)\\&+\kappa_2 g(\Xi_1,\Xi_3)\Gamma_1(\Xi_2,\Xi_4)-\kappa_2 g(\Xi_2,\Xi_3)\Gamma_1(\Xi_1,\Xi_4)\\&+\kappa_2(\kappa_1-\kappa_2)[g(\Xi_1,\Xi_3)\Gamma_2(\Xi_2,\Xi_4)-g(\Xi_2,\Xi_3)\Gamma_2(\Xi_1,\Xi_4)].\end{aligned} \tag{2.5}$$

Let M be an l-dimensional submanifold of the quaternionic space form $\tilde{M}(c)$. Denote by ∇ and $\tilde{\nabla}$ the connections on M induced by $\overline{\nabla}$ and $\tilde{\overline{\overline{\nabla}}}$, respectively, and by R and $\tilde{R}$ their corresponding curvature tensors. The Gauss formulas are given by

$$\begin{cases}\overline{\nabla}_{\Xi_1}\Xi_2=\nabla_{\Xi_1}\Xi_2+h(\Xi_1,\Xi_2),\\ \tilde{\overline{\overline{\nabla}}}_{\Xi_1}\Xi_2=\tilde{\nabla}_{\Xi_1}\Xi_2+\tilde{h}(\Xi_1,\Xi_2),\end{cases}$$

where $\tilde{h}$ denotes the second fundamental form associated with $\tilde{\overline{\overline{\nabla}}}$, and

$$h(\Xi_1,\Xi_2)=\tilde{h}(\Xi_1,\Xi_2)-\kappa_2 g(\Xi_1,\Xi_2)P^{\perp}.$$

The associated Gauss equation takes the form (see [14])

$$\begin{aligned}\overline{R}=\tilde{R}(\Xi_1,\Xi_2,\Xi_3,\Xi_4)&-g(h(\Xi_1,\Xi_4),h(\Xi_2,\Xi_3))+g(h(\Xi_2,\Xi_4),h(\Xi_1,\Xi_3))\\&+(\kappa_1-\kappa_2)g(h(\Xi_2,\Xi_3),P)g(\Xi_1,\Xi_4)+(\kappa_2-\kappa_1)g(h(\Xi_1,\Xi_3),P)g(\Xi_2,\Xi_4).\end{aligned} \tag{2.6}$$

Let $\{u_1,\dots,u_l,\xi_{l+1},\dots,\xi_{4n}\}$ be a local orthonormal frame of $\tilde{M}$ along M such that $\{u_1,\dots,u_l\}$ and $\{\xi_{l+1},\dots,\xi_{4n}\}$ are orthonormal frames of TM and $T^{\perp}M$, respectively. The mean curvature vector H and the squared norm of the second fundamental form h are defined by

$$H(x)=\frac{1}{l}\sum_{i=1}^{l}h(u_i,u_i),\qquad ||h||^2(x)=\sum_{i,j=1}^{l}g(h(u_i,u_j),h(u_i,u_j)).$$

Finally, let M be a real submanifold of a quaternion Kähler manifold $\tilde{M}$. The submanifold M is said to be a *QR-submanifold* if there exists a vector sub-bundle ν of the normal bundle satisfying

$$F_{\beta_1}(\nu_x)=\nu_x,\qquad F_{\beta_1}(\nu_x^{\perp})\subset T_xM,\quad x\in M,\ \beta_1=1,2,3,$$

where $\nu^\perp$ denotes the orthogonal complement of ν in $T^\perp M$. Defining

$$\mathcal{D}_{\beta_1 x} = F_{\beta_1}(\nu_x^\perp), \qquad \mathcal{D}_x^\perp = \mathcal{D}_{1x} \oplus \mathcal{D}_{2x} \oplus \mathcal{D}_{3x},$$

we obtain a $3q$-dimensional distribution $\mathcal{D}^\perp$, where $q = \dim \nu_x^\perp$. Moreover,

$$F_{\beta_1}(\mathcal{D}_{\beta_1 x}) = \nu_x^\perp, \qquad F_{\beta_1}(\mathcal{D}_{\beta_2 x}) = \mathcal{D}_{\beta_3 x},$$

for any cyclic permutation $(\beta_1, \beta_2, \beta_3)$ of $(1, 2, 3)$. The orthogonal complement $\mathcal{D}$ of $\mathcal{D}^\perp$ in TM satisfies $F_{\beta_1}(\mathcal{D}_x) = \mathcal{D}_x$ and is referred to as the quaternion distribution.

Definition 2.2. [11] A QR-submanifold M is called mixed geodesic if $h(X, Y) = 0$ for all $X \in \Gamma(\mathcal{D})$ and $Y \in \Gamma(\mathcal{D}^\perp)$.

Remark 2.3. The condition that the distribution $\mathcal{D}$ is totally umbilical means that the restriction of the second fundamental form satisfies

$$h(X, Y) = g(X, Y) H_{\mathcal{D}}, \quad X, Y \in \mathcal{D},$$

where $H_{\mathcal{D}}$ denotes the partial mean curvature vector along $\mathcal{D}$. Geometrically, this implies that the normal curvature in directions tangent to $\mathcal{D}$ is isotropic.

3. Main results

The notion of δ-invariants was introduced by B.-Y. Chen in the early 1990s; see [9]. In the setting of QR-submanifolds, we consider the δ-invariant associated with the anti-invariant distribution $\mathcal{D}^\perp$, defined by

$$\delta(\mathcal{D}^\perp)(x) = \tau(x) - \tau(\mathcal{D}_x^\perp), \qquad x \in M, \tag{3.1}$$

where τ denotes the scalar curvature of M, and $\tau(\mathcal{D}_x^\perp)$ denotes the scalar curvature of the distribution $\mathcal{D}^\perp \subset TM$.

Throughout this section, we assume that $M \subset \tilde{M}$ is a QR-submanifold of *minimal codimension*, that is, $\dim \nu_x = 0$ for all $x \in M$. In this case, we may choose orthonormal bases

$$\begin{cases} \{u_1, \ldots, u_l\} \subset \mathcal{D}_x, \\ \{F_1\xi_{l+1}, \ldots, F_1\xi_{l+m}; F_2\xi_{l+1}, \ldots, F_2\xi_{l+m}; F_3\xi_{l+1}, \ldots, F_3\xi_{l+m}\} \subset \mathcal{D}_x^\perp, \\ \{\xi_{n+1}, \ldots, \xi_{l+m}\} \subset T_x^\perp M, \end{cases}$$

so that

$$\begin{cases} \dim \mathcal{D}_x = l, \\ \dim \mathcal{D}_x^\perp = 3m, \\ \dim T_x M = l + 3m, \\ \dim T_x^\perp M = m = \dim \nu_x^\perp. \end{cases}$$

Unless stated otherwise, we use the index ranges

$$\begin{cases} \alpha_1, \alpha_2, \alpha_3 = 1, \ldots, l; \\ \beta_1, \beta_2, \beta_3 = 1, 2, 3; \\ \gamma_1, \gamma_2, \gamma_3 = n+1, \ldots, l+m; \\ A, B, C = 1, \ldots, l+m. \end{cases}$$

For comparison, we recall the following inequality obtained in [4] (see also [10] for related results in the complex setting).

Theorem 3.1 ([4]). *Let M be an antiholomorphic submanifold of a complex space form $\tilde{M}^{\zeta+p}(c)$ with $\zeta = \mathrm{rank}_{\mathbb{C}}\mathcal{D} \geq 1$ and $p = \mathrm{rank}\,\mathcal{D}^{\perp} \geq 2$. Then.*

$$\delta(\mathcal{D}) \leq \frac{(p-1)(2\zeta+p)^2}{2(p+2)} \|H\|^2 + \frac{p}{2}(4\zeta + p - 1)\frac{c}{4}.$$

Moreover, equality holds identically if and only if the following conditions are met:

1) M is $\mathcal{D}$-minimal;

2) M is mixed geodesic; and

3) There exists an orthonormal frame $\{u_{2\zeta+1}, \ldots, u_n\}$ of $\mathcal{D}^{\perp}$ such that the second fundamental form h satisfies $h^{\alpha_1}_{\alpha_1\alpha_1} = 3h^{\alpha_1}_{\alpha_2\alpha_2}$ for $2\zeta + 1 \leq \alpha_1 \neq \alpha_2 \leq 2\zeta + p$, and $h^{\alpha_3}_{\alpha_1\alpha_2} = 0$ for distinct $\alpha_1, \alpha_2, \alpha_3 \in \{2\zeta + 1, \ldots, 2\zeta + p\}$.

We now establish Chen-type inequalities for QR-submanifolds of quaternionic space forms endowed with a QSMC. The derivation of the main inequalities is based on the Gauss equation, which relates the scalar curvature of the submanifold to the ambient curvature and the second fundamental form. By decomposing the tangent bundle into the invariant and anti-invariant distributions, we express the scalar curvature as a sum of contributions from each component.

The key step consists of estimating quadratic expressions involving the components of the second fundamental form. These expressions can be interpreted as quadratic forms subject to linear constraints determined by the mean curvature vector. The optimization of these quadratic forms yields the desired sharp inequalities.

Theorem 3.2. *Let M be a QR-submanifold of minimal codimension immersed in a quaternionic space form $\tilde{M}(c)$. Suppose that, for each point $x \in M$, the invariant and anti-invariant distributions satisfy $\dim \mathcal{D}_x = l$ and $\dim \mathcal{D}^{\perp}_x = 3m$, which implies that $\dim \nu_x = 0$ and $\dim \nu^{\perp}_x = m$. Then, the δ-invariant associated with $\mathcal{D}^{\perp}$ obeys the inequality*

$$\begin{aligned} \delta(\mathcal{D}^{\perp}) \leq{}& \frac{l(l+3m)^2}{2(l+1)} \|H^t\|^2 + \frac{c}{4}\Big\{\frac{l(l-1) + 6m(l+2)}{2}\Big\} \\ &+ \kappa_1 l\,\Gamma(u_1, u_1) - 3m\kappa_2 \,\mathrm{Tr}(\Gamma_1) - \kappa_2(\kappa_1 - \kappa_2)(l - 1 + 3m)\,\mathrm{Tr}(\Gamma_2) \\ &- (\kappa_1 - \kappa_2)\Big\{l^2\pi(H_{\mathcal{D}}) - 3ml^2\pi(H_{\mathcal{D}^{\perp}})\Big\}. \end{aligned} \tag{3.2}$$

Furthermore, the equality case in (3.2) is attained at every point of M if and only if all of the following conditions are fulfilled:

1) The submanifold M is mixed geodesic;

2) The invariant distribution $\mathcal{D}$ is totally umbilical; and

3) There exists an orthonormal basis of $\mathcal{D}_x^{\perp}$ of the form

$$\{F_1\xi_{l+1},\dots,F_1\xi_{l+m};\ F_2\xi_{l+1},\dots,F_2\xi_{l+m};\ F_3\xi_{l+1},\dots,F_3\xi_{l+m}\}$$

with respect to which the components of the second fundamental form satisfy

$$h^{\gamma_1}_{\alpha_1\alpha_2}=0,\qquad 1\le\alpha_1\ne\alpha_2\le l,\quad l+1\le\gamma_1\le l+m.$$

Proof. With the above notation, for $x\in M$, we have

$$\begin{aligned}\tau(x)=&\sum_{1\le\alpha_1<\alpha_2\le l}\overline{K}(u_{\alpha_1}\wedge u_{\alpha_2})+\sum_{\beta_1=1}^{3}\sum_{\gamma_1,\gamma_2=l+1}^{l+m}\overline{K}(F_{\beta_1}\xi_{\gamma_1}\wedge F_{\beta_1}\xi_{\gamma_2})\\&+\sum_{\beta_1=1}^{3}\sum_{\alpha_1=1}^{l}\sum_{\gamma_1=l+1}^{l+m}\overline{K}(u_{\alpha_1}\wedge F_{\beta_1}\xi_{\gamma_1}),\end{aligned}$$

and

$$\tau(\mathcal{D}_x^{\perp})=\sum_{\beta_1=1}^{3}\sum_{\gamma_1,\gamma_2=l+1}^{l+m}\overline{K}(F_{\beta_1}\xi_{\gamma_1}\wedge F_{\beta_1}\xi_{\gamma_2}).$$

Hence,

$$\delta(\mathcal{D}^{\perp})(x)=\sum_{1\le\alpha_1<\alpha_2\le n}\overline{K}(u_{\alpha_1}\wedge u_{\alpha_2})+\sum_{\beta_1=1}^{3}\sum_{\alpha_1=1}^{l}\sum_{\gamma_1=l+1}^{l+m}\overline{K}(u_{\alpha_1}\wedge F_{\beta_1}\xi_{\gamma_1}).\tag{3.3}$$

Applying the Gauss equation to the planes spanned by $(u_{\alpha_1},u_{\alpha_2})$, $1\le\alpha_1\ne\alpha_2\le l$, yields

$$\begin{aligned}\overline{K}(u_{\alpha_1},u_{\alpha_2})=\frac{c}{4}\Big\{&g(u_{\alpha_2},u_{\alpha_2})g(u_{\alpha_1},u_{\alpha_1})-g(u_{\alpha_1},u_{\alpha_2})g(u_{\alpha_2},u_{\alpha_1})\\&+\sum_{\xi=1}^{3}\Big[g(u_{\alpha_2},F_\xi u_{\alpha_2})g(F_\xi u_{\alpha_1},u_{\alpha_1})-g(u_{\alpha_2},F_\xi u_{\alpha_1})g(F_\xi u_{\alpha_2},u_{\alpha_1})\\&+2g(u_{\alpha_1},F_\xi u_{\alpha_2})g(F_\xi u_{\alpha_2},u_{\alpha_1})\Big]+\kappa_1\Gamma_1(u_{\alpha_1},u_{\alpha_2})g(u_{\alpha_2},u_{\alpha_1})\\&-\kappa_1\Gamma_1(u_{\alpha_2},u_{\alpha_2})g(u_{\alpha_1},u_{\alpha_1})+\kappa_2 g(u_{\alpha_1},u_{\alpha_2})\Gamma_1(u_{\alpha_2},u_{\alpha_1})\\&-\kappa_2 g(u_{\alpha_2},u_{\alpha_2})\Gamma_1(u_{\alpha_1},u_{\alpha_1})+\kappa_2(\kappa_1-\kappa_2)g(u_{\alpha_1},u_{\alpha_2})\Gamma_2(u_{\alpha_2},u_{\alpha_1})\\&-\kappa_2(\kappa_1-\kappa_2)g(u_{\alpha_2},u_{\alpha_2})\Gamma_2(u_{\alpha_1},u_{\alpha_1})+g\left(h\left(u_{\alpha_1},u_{\alpha_1}\right),h\left(u_{\alpha_2},u_{\alpha_2}\right)\right)\\&-g\left(h\left(u_{\alpha_1},u_{\alpha_2}\right),h\left(u_{\alpha_1},u_{\alpha_2}\right)\right)+(\kappa_1-\kappa_2)g(h(u_{\alpha_2},u_{\alpha_2}),P)g(u_{\alpha_1},u_{\alpha_1})\\&+(\kappa_2-\kappa_1)g(h(u_{\alpha_1},u_{\alpha_2}),P)g(u_{\alpha_1},u_{\alpha_2})\Big\}.\end{aligned}\tag{3.4}$$

Similarly, because $F_{\beta_1}u_{\alpha_1} \in \mathcal{D}$, and $F_{\beta_1}\xi_r \in \mathcal{D}^\perp$, applying the Gauss equation to the planes spanned by $(u_i, F_{\beta_1}\xi_r)$ gives

$$\begin{aligned}
\overline{K}(u_{\alpha_1}, F_{\beta_1}\xi_{\gamma_1}) = \frac{c}{4}\Big\{ & g(F_{\beta_1}\xi_{\gamma_1}, F_{\beta_1}\xi_{\gamma_1})g(u_{\alpha_1}, u_{\alpha_1}) - g(u_{\alpha_1}, F_{\beta_1}\xi_{\gamma_1})g(F_{\beta_1}\xi_{\gamma_1}, u_{\alpha_1}) \\
& + \sum_{\beta_1=1}^{3}\Big[g(F_{\beta_1}\xi_{\gamma_1}, F_{\beta_1}F_{\beta_1}\xi_{\gamma_1})g(F_{\beta_1}u_{\alpha_1}, u_{\alpha_1}) \\
& - g(F_{\beta_1}\xi_{\gamma_1}, F_{\beta_1}u_{\alpha_1})g(F_{\beta_1}F_{\beta_1}\xi_{\gamma_1}, u_{\alpha_1}) \\
& + 2g(u_{\alpha_1}, F_{\beta_1}F_{\beta_1}\xi_{\gamma_1})g(F_{\beta_1}F_{\beta_1}\xi_{\gamma_1}, u_{\alpha_1})\Big] \\
& + \kappa_1\Gamma_1(u_{\alpha_1}, F_{\beta_1}\xi_{\gamma_1})g(F_{\beta_1}\xi_{\gamma_1}, u_{\alpha_1}) \\
& - \kappa_1\Gamma_1(F_{\beta_1}\xi_{\gamma_1}, F_{\beta_1}\xi_{\gamma_1})g(u_{\alpha_1}, u_{\alpha_1}) \\
& + \kappa_2 g(u_{\alpha_1}, F_{\beta_1}\xi_{\gamma_1})\Gamma_1(F_{\beta_1}\xi_{\gamma_1}, u_{\alpha_1}) \\
& - \kappa_2 g(F_{\beta_1}\xi_{\gamma_1}, F_{\beta_1}\xi_{\gamma_1})\Gamma_1(u_{\alpha_1}, u_{\alpha_1}) \\
& + \kappa_2(\kappa_1 - \kappa_2)g(u_{\alpha_1}, F_{\beta_1}\xi_{\gamma_1})\Gamma_2(F_{\beta_1}\xi_{\gamma_1}, u_{\alpha_1}) \\
& - \kappa_2(\kappa_1 - \kappa_2)g(F_{\beta_1}\xi_{\gamma_1}, F_{\beta_1}\xi_{\gamma_1})\Gamma_2(u_{\alpha_1}, u_{\alpha_1}) \\
& + (\kappa_1 - \kappa_2)g(h(F_{\beta_1}\xi_{\gamma_1}, F_{\beta_1}\xi_{\gamma_1}), P)g(u_{\alpha_1}, u_{\alpha_1}) \\
& + (\kappa_2 - \kappa_1)g(h(u_{\alpha_1}, F_{\beta_1}\xi_{\gamma_1}), P)g(u_{\alpha_1}, F_{\beta_1}\xi_{\gamma_1}) \\
& + g\left(h\left(u_{\alpha_1}, u_{\alpha_1}\right), h\left(F_{\beta_1}\xi_{\gamma_1}, F_{\beta_1}\xi_{\gamma_1}\right)\right) \\
& - g\left(h\left(u_{\alpha_1}, F_{\beta_1}\xi_{\gamma_1}\right), h\left(u_{\alpha_1}, F_{\beta_1}\xi_{\gamma_1}\right)\right)\Big\}. \qquad (3.5)
\end{aligned}$$

Substituting (3.4) and (3.5) into (3.3), we obtain

$$\begin{aligned}
\delta\left(\mathcal{D}^\perp\right)(x) = \frac{c}{4}\{ & \frac{l(l-1)}{2} + 6m + 3lm\} \\
& + \kappa_1 l\Gamma_1(u_1, u_1) - 3m\kappa_2 Tr(\Gamma_1) - \kappa_2(\kappa_1 - \kappa_2)(l - 1 + 3m)Tr(\Gamma_2) \\
& - (\kappa_1 - \kappa_2)\Big\{l\sum_{1\le\alpha_1<\alpha_2\le l} g(h(u_{\alpha_2}, u_{\alpha_2}), P) \\
& - l\sum_{\alpha_1=1}^{l}\sum_{\gamma_1=l+1}^{l+m} g(h(F_{\beta_1}\xi\gamma_1, F_{\beta_1}\xi\gamma_1), P)\Big\} \\
& - \sum_{1\le\alpha_1<\alpha_2\le l}\Big\{g\left(h(u_{\alpha_1}, u_{\alpha_1}), h\left(u_{\alpha_2}, u_{\alpha_2}\right)\right) - g\left(h(u_{\alpha_1}, u_{\alpha_2}), h\left(u_{\alpha_1}, u_{\alpha_2}\right)\right)\Big\} \\
& - \sum_{\alpha_1=1}^{l}\sum_{\gamma_1=l+1}^{l+m}\Big\{g\left(h\left(u_{\alpha_1}, u_{\alpha_1}\right), h\left(F_{\beta_1}\xi\gamma_1, F_{\beta_1}\xi\gamma_1\right)\right) \\
& - g\left(h\left(u_{\alpha_1}, F_{\beta_1}\xi\gamma_1\right), h\left(u_{\alpha_1}, F_{\beta_1}\xi\gamma_1\right)\right)\Big\}. \qquad (3.6)
\end{aligned}$$

Discarding the nonpositive squared terms in (3.6) gives

$$\begin{aligned}
\delta\left(\mathcal{D}^\perp\right)(x) \le \frac{c}{4}\{ & \frac{l(l-1)}{2} + 6m + 3lm\} \\
& + \kappa_1 l\Gamma_1(u_1, u_1) - 3m\kappa_2 Tr(\Gamma_1) - \kappa_2(\kappa_1 - \kappa_2)(l - 1 + 3m)Tr(\Gamma_2)
\end{aligned}$$

$$
\begin{aligned}
&-(\kappa_1-\kappa_2)\Big\{l\sum_{1\le\alpha_1<\alpha_2\le l} g(h(u_{\alpha_1},u_{\alpha_2}),P)\\
&-l\sum_{\alpha_1=1}^{l}\sum_{\gamma_1=l+1}^{l+m} g(h(F_{\beta_1}\xi_{\gamma_1},F_{\beta_1}\xi_{\gamma_1}),P)\Big\}\\
&-\sum_{\alpha_1=1}^{l}\sum_{\gamma_1=n+1}^{l+m}\Big\{g\Big(h\left(u_{\alpha_1},u_{\alpha_1}\right),h\Big(F_{\beta_1}\xi_{\gamma_1},F_{\beta_1}\xi_{\gamma_1}\Big)\Big)\\
&-g\Big(h\Big(u_{\alpha_1},F_{\beta_1}\xi_{\gamma_1}\Big),h\Big(u_{\alpha_1},F_{\beta_1}\xi_{\gamma_1}\Big)\Big)\Big\}.
\end{aligned}
\tag{3.7}
$$

Using $g(\Xi_1,P)=\pi(\Xi_1)$, we rewrite the resulting expression in terms of $\pi(H_{\mathcal{D}})$ and $\pi(H_{\mathcal{D}^\perp})$ to obtain

$$
\begin{aligned}
\delta\left(\mathcal{D}^\perp\right)(x)\le{}&\frac{c}{4}\Big\{\frac{l(l-1)+6m(l+2)}{2}\Big\}+\kappa_1 l\Gamma_1(u_1,u_1)-3m\kappa_2 Tr(\Gamma_1)\\
&-\kappa_2(\kappa_1-\kappa_2)(l-1+3m)Tr(\Gamma_2)-(\kappa_1-\kappa_2)\Big\{l^2\pi(H_{\mathcal{D}})-3ml^2\pi(H_{\mathcal{D}^\perp})\Big\}\\
&+\sum_{1\le\alpha_1<\alpha_2\le l}\Big\{g\left(h\left(u_{\alpha_1},u_{\alpha_1}\right),h\left(u_{\alpha_2},u_{\alpha_1}\right)\right)\Big\}\\
&+\sum_{\alpha_1=1}^{l}\sum_{\gamma_1=l+1}^{l+m}\Big\{g\Big(h\left(u_{\alpha_1},u_{\alpha_1}\right),h\Big(F_{\beta_1}\xi_{\gamma_1},F_{\beta_1}\xi_{\gamma_1}\Big)\Big)\Big\},
\end{aligned}
\tag{3.8}
$$

and hence,

$$
\begin{aligned}
\delta\left(\mathcal{D}^\perp\right)(x)\le{}&\frac{c}{4}\Big\{\frac{l(l-1)+6m(l+2)}{2}\Big\}\\
&+\kappa_1 l\Gamma_1(u_1,u_1)-3m\kappa_2 Tr(\Gamma_1)-\kappa_2(\kappa_1-\kappa_2)(l-1+3m)Tr(\Gamma_2)\\
&-(\kappa_1-\kappa_2)\Big\{l^2\pi(H_{\mathcal{D}})-3ml^2\pi(H_{\mathcal{D}^\perp})\Big\}+\sum_{1\le\alpha_1<\alpha_2\le n}\sum_{\gamma_1=l+1}^{l+m} h^{\gamma_1}_{\alpha_1\alpha_1}h^{\gamma_1}_{\alpha_2\alpha_2}\\
&+\sum_{1\le\alpha_1<\alpha_2\le l}\sum_{\gamma_1=l+1}^{l+m} h^{\gamma_1}_{\alpha_1\alpha_1}[\tilde{\mathscr{B}}^{\gamma_3}_{\gamma\gamma}+\tilde{\tilde{h}}^{\gamma_3}_{\gamma_1\gamma_2}+\tilde{\tilde{\tilde{h}}}^{\gamma_3}_{\gamma_1\gamma_2}],
\end{aligned}
\tag{3.9}
$$

where

$$
\begin{cases}
g(h(u_{\alpha_1},u_{\alpha_1}),\xi_{\gamma_1})=h^{\gamma_1}_{\alpha_1\alpha_1},\\
g(h(F_1\xi_{\gamma_1},F_1\xi_{\gamma_2}),u_{\gamma_3})=\tilde{h}^{\gamma_3}_{\gamma_1\gamma_2},\\
g(h(F_2\xi_{\gamma_1},F_2\xi_{\gamma_2}),u_{\gamma_3})=\tilde{\tilde{h}}^{\gamma_3}_{\gamma_1\gamma_2},\\
g(h(F_3\xi_{\gamma_1},F_3\xi_{\gamma_2}),u_{\gamma_3})=\tilde{\tilde{\tilde{h}}}^{\gamma_3}_{\gamma_1\gamma_2}.
\end{cases}
$$

Next, we introduce the quadratic forms $f_{\gamma_3}:\mathbb{R}^{n+3q}\to\mathbb{R}$ defined by

$$
f_t\Big(h^{\gamma_3}_{11},\ldots,h^{\gamma_3}_{ll};\ \tilde{h}^{t}_{l+1;l+1},\ldots,\tilde{h}^{\gamma_3}_{l+m;l+m};\ \tilde{\tilde{h}}^{\gamma_3}_{l+1;l+1},\ldots,\tilde{\tilde{h}}^{\gamma_3}_{l+m;l+m};\ \tilde{\tilde{\tilde{h}}}^{\gamma_3}_{l+1;l+1},\ldots,\tilde{\tilde{\tilde{h}}}^{t}_{l+m;l+m}\Big)
$$

$$= \sum_{1\le\alpha_1<\alpha_2\le l} h^{\gamma_3}_{\alpha_1\alpha_1} h^{\gamma_3}_{\alpha_2\alpha_2} + \sum_{\alpha_1=1}^{l}\sum_{\gamma_1=l+1}^{l+m} h^{\gamma_3}_{\alpha_1}(\tilde{h}^{\gamma_3}_{\gamma_1\gamma_1} + \tilde{\tilde{h}}^{\gamma_3}_{\gamma_1\gamma_1} + \tilde{\tilde{\tilde{h}}}^{\gamma_3}_{\gamma_1\gamma_1}), \tag{3.10}$$

$$\gamma_3 = l+1, \ldots, l+q, \tag{3.11}$$

subject to the constraint

$$h^{\gamma_3}_{11} + \ldots + h^{\gamma_3}_{ll} + \tilde{h}^{\gamma_3}_{l+1;l+1} + \ldots + \tilde{h}^{\gamma_3}_{l+m;l+m} + \tilde{\tilde{h}}^{t}_{l+1;l+1} + \ldots + \tilde{\tilde{h}}^{\gamma_3}_{l+m;l+m} + \tilde{\tilde{\tilde{h}}}^{\gamma_3}_{l+m;l+m} + \ldots + \tilde{\tilde{\tilde{h}}}^{\gamma_3}_{l+m;l+m} = c^{\gamma_3}, \tag{3.12}$$

where c^{γ_3} is a real constant. By [11], one has the estimate

$$f_{\gamma_3} \le \frac{l}{2(l+1)}(l+3m)^2\|H^{\gamma_3}\|^2, \tag{3.13}$$

where

$$H^{\gamma_3} = \frac{1}{l+3m}\sum_{\gamma_1=l+1}^{l+m}(\tilde{\mathscr{B}}^{\gamma_3}_{\gamma\gamma} + \tilde{\tilde{h}}^{\gamma_3}_{\gamma_1\gamma_2} + \tilde{\tilde{\tilde{h}}}^{\gamma_3}_{\gamma_1\gamma_2}).$$

Combining (3.9) with (3.13) yields (3.2). The characterization of the equality case follows from the equality case in [11] together with the vanishing of the mixed terms, which is equivalent to the stated conditions. □

Remark 3.3. The quadratic form f_{γ_3} represents the contribution of the normal direction indexed by γ_3 to the total scalar curvature via the second fundamental form. It encodes the interaction between diagonal components of the shape operators and arises naturally when summing sectional curvatures over orthonormal frames.

From an analytical viewpoint, f_{γ_3} is optimized under the constraint that the trace of the second fundamental form is fixed, which corresponds to fixing the mean curvature vector. This optimization is fundamental in obtaining sharp bounds.

In a similar way, we obtain the δ-invariant for the distribution $\mathcal{D}$, that is, $\delta(\mathcal{D}) = \tau(x) - \tau(D)(x)$. We state the following theorem.

Theorem 3.4. *Let M be a QR-submanifold of minimal codimension immersed in a quaternionic space form $\tilde{M}(c)$. Assume that, at each point $x \in M$, the invariant and anti-invariant distributions satisfy* $\dim \mathcal{D}_x = l$ *and* $\dim \mathcal{D}^\perp_x = 3m$. *Then, the δ-invariant corresponding to the distribution $\mathcal{D}$ satisfies the inequality*

$$\delta(\mathcal{D}) \le \frac{l(l+3m)^2}{2(l+1)}\|H^{\gamma_3}\|^2 + \frac{c}{4}\{l(l-1)+3lm\} - (l+m)\kappa_1 \operatorname{Tr}(\Gamma_1) - 6m\kappa_2 \operatorname{Tr}(\Gamma_1) - \kappa_2(\kappa_1-\kappa_2)(6m)\operatorname{Tr}(\Gamma_2) - (\kappa_1-\kappa_2)\{3m(2m+l)\pi(H_{\mathcal{D}^\perp})\}. \tag{3.14}$$

Moreover, equality in (3.14) holds identically if and only if the following three conditions are satisfied:

1) The submanifold M is mixed geodesic;

2) The distribution $\mathcal{D}$ is totally umbilical; and

3) There exists an orthonormal basis of $\mathcal{D}_x^{\perp}$ of the form

$$\{F_1u_{l+1}, \ldots, F_1u_{l+m}; F_2u_{l+1}, \ldots, F_2u_{l+m}; F_3u_{l+1}, \ldots, F_3u_{l+m}\},$$

with respect to which the components of the second fundamental form satisfy

$$h^{\gamma_1}_{\alpha_1\alpha_2} = 0, \qquad 1 \leq \alpha_1 \neq \alpha_2 \leq l, \quad l+1 \leq \gamma_1 \leq l+m.$$

4. Results in other connection types

In this section, we derive several consequences of the results obtained in Sections 3.2 and 3.4 by considering particular choices of the parameters κ_1 and κ_2 defining the QSMC. These special choices correspond to the well-known SSMC and SSNMC, which play an important role in differential geometry. As a result, we obtain Chen-type δ-invariant inequalities for *QR*-submanifolds of quaternionic space forms endowed with these connections as direct corollaries of our general framework.

4.1. Results with SSMC

When $\kappa_1 = \kappa_2 = 1$, the QSMC reduces to the SSMC. Substituting these values into inequality (3.2), the terms involving $(\kappa_1 - \kappa_2)$ vanish identically, and we obtain the following result.

Corollary 4.1. *Let M be a QR-submanifold of minimal codimension immersed in a quaternionic space form $\tilde{M}(c)$ equipped with an SSMC. Then, the δ-invariant corresponding to the anti-invariant distribution $\mathcal{D}^{\perp}$ satisfies the inequality*

$$\delta(\mathcal{D}^{\perp}) \leq \frac{l(l+3m)^2}{2(l+1)}\,\|H^{\gamma_3}\|^2 + \frac{c}{4}\Big\{\frac{l(l-1)+6m(l+2)}{2}\Big\} + n\,\Gamma_1(u_1,u_1) - 3q\,\mathrm{Tr}(\Gamma_1).$$

Furthermore, equality is attained identically if and only if the submanifold M is mixed geodesic, the invariant distribution $\mathcal{D}$ is totally umbilical, and the components of the second fundamental form satisfy

$$h^{\gamma_1}_{\alpha_1\alpha_2} = 0, \qquad 1 \leq \alpha_1 \neq \alpha_2 \leq l, \quad l+1 \leq \gamma_1 \leq l+m.$$

In the same setting, Inequality (3.14) yields the following corollary for the δ-invariant associated with the invariant distribution.

Corollary 4.2. *Let M be a QR-submanifold of minimal codimension immersed in a quaternionic space form $\tilde{M}(c)$ endowed with an SSMC. Then, the δ-invariant associated with the invariant distribution $\mathcal{D}$ satisfies*

$$\delta(\mathcal{D}) \leq \frac{l(l+3m)^2}{2(l+1)}\,\|H^{\gamma_3}\|^2 + \frac{c}{4}\{l(l-1)+3lm\} - (l+7m)\,\mathrm{Tr}(\Gamma_1).$$

The equality case coincides with the geometric characterization stated in Theorem 3.4.

4.2. Results with SSNMC

If $\kappa_1 = 1$, and $\kappa_2 = 0$, the QSMC reduces to the SSNMC. In this case, the terms involving κ_2 vanish, and the inequalities obtained in Section 3.2 immediately yield the following result.

Corollary 4.3. *Let M be a QR-submanifold of minimal codimension immersed in a quaternionic space form $\tilde{M}(c)$ equipped with an SSNMC. Then, the δ-invariant corresponding to the anti-invariant distribution $\mathcal{D}^{\perp}$ satisfies*

$$\begin{aligned}\delta(\mathcal{D}^{\perp}) \leq \frac{l(l+3m)^2}{2(l+1)}\|H^{\gamma_3}\|^2 + \frac{c}{4}\Big\{\frac{l(l-1)+6m(l+2)}{2}\Big\} \\ + l\,\Gamma_1(u_1,u_1) - l^2\pi(H_{\mathcal{D}}) + 3lm^2\pi(H_{\mathcal{D}^{\perp}}).\end{aligned}$$

Moreover, equality holds identically if and only if the submanifold M is mixed geodesic, the invariant distribution $\mathcal{D}$ is totally umbilical, and all mixed components of the second fundamental form vanish.

Similarly, for the invariant distribution $\mathcal{D}$, we obtain the following consequence.

Corollary 4.4. *Let M be a QR-submanifold of minimal codimension immersed in a quaternionic space form $\tilde{M}(c)$ endowed with an SSNMC. Then, the δ-invariant associated with the invariant distribution $\mathcal{D}$ satisfies the inequality*

$$\delta(\mathcal{D}) \leq \frac{l(l+3m)^2}{2(l+1)}\|H^{\gamma_3}\|^2 + \frac{c}{4}\{l(l-1)+3lm\} - (l+m)\,\mathrm{Tr}(\Gamma_1) - \{3m(2m+l)\pi(H_{\mathcal{D}^{\perp}})\},$$

and the equality case is characterized by the same geometric conditions as in the SSMC setting.

These results show that the inequalities obtained in the presence of a QSMC naturally unify and extend the corresponding δ-invariant inequalities for both SSMC and SSNMC.

5. Conclusions

In this paper, we have conducted a systematic investigation of *QR*-submanifolds of minimal codimension immersed in quaternionic space forms endowed with a quarter-symmetric metric connection. By adapting Chen's δ-invariant framework to the invariant and anti-invariant distributions naturally induced by the quaternionic structure, we have established sharp curvature inequalities for the invariants $\delta(\mathcal{D})$ and $\delta(\mathcal{D}^{\perp})$. The derived estimates exhibit several significant features. First, they explicitly incorporate the influence of the quarter-symmetric metric connection through the structural parameters κ_1, κ_2 and the associated tensorial fields Γ_1, Γ_2, thereby clarifying how the choice of connection affects the relationship between intrinsic and extrinsic curvature. Second, the obtained inequalities extend and unify a wide range of previously known results, including those corresponding to the Levi-Civita connection, as well as special cases such as semi-symmetric metric and nonmetric connections, which are recovered naturally within our framework. A further important contribution of this work is the complete geometric characterization of the equality cases. We have shown that the optimal bounds are attained precisely when the submanifold is mixed geodesic and the invariant distribution $\mathcal{D}$ is totally umbilical, together with explicit conditions on the second fundamental form. This result provides a rigidity description for extremal *QR*-submanifolds and highlights the role of the quaternionic structure and the quarter-symmetric metric connection in governing the saturation of the δ-invariant inequalities. Finally, several directions for future research naturally arise from the present study. One immediate extension is to consider QR-submanifolds of nonminimal codimension, where additional normal components may interact with the quaternionic structure and lead to new forms of curvature inequalities. It would also be of interest to investigate other Chen-type invariants in this setting, including generalized δ-invariants and Casorati curvatures, together with their associated equality cases. Furthermore, extending these results to more general ambient geometries, such as quaternionic Kähler manifolds with variable quaternionic sectional curvature, as well as studying stability and deformation problems for such submanifolds, may provide further insight into the geometric and analytical aspects of the theory.

Author contributions

Md Aquib: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Visualization, Writing–original draft, Supervision, Project administration, Funding acquisition, Writing–review & editing; Mohd Iqbal: Validation, Formal analysis, Investigation, Writing–original draft, Writing–review & editing. All authors have read and approved the final version on the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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