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Research article

Structural properties of hyperideal-based zero-divisor graphs over noncommutative hyperrings

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Abstract: In this paper, we study hyperideal-based zero-divisor graphs $\Gamma_I(R)$ associated with noncommutative hyperrings. We investigate fundamental structural properties of these graphs under the given adjacency condition, including connectivity, diameter bounds, and coloring parameters. For finite hyperrings, we characterize connectedness of $\Gamma_I(R)$ using the rank of the Laplacian matrix. We also establish general bounds on the diameter and identify conditions under which the diameter is at most two. Furthermore, we introduce algebraic measures such as the extended chromatic number and an asymmetry index to capture effects arising from noncommutativity. Several results illustrate how asymmetry in hypermultiplication influences graph structure, including possible differences between classical and extended coloring.

Keywords: zero-divisor graphs; hyperideal; non-commutative hyperring; nonassociative structure
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1. Introduction

Algebraic graph theory has grown into a dynamic field of study, driven by its deep theoretical framework and practical applications across diverse disciplines. Although younger than many traditional branches of mathematics, it has seen significant progress in recent decades, yielding powerful insights and innovative solutions. Let $\mathcal{R}$ be a hyperring and let I be a proper hyperideal of $\mathcal{R}$.

The hyperideal-based zero-divisor graph of $\mathcal{R}$ with respect to I, denoted by $\Gamma_I(\mathcal{R})$, is the graph defined as follows: The vertex set of $\Gamma_I(\mathcal{R})$ is

$$V(\Gamma_I(\mathcal{R})) = Z_I(\mathcal{R}) \setminus I,$$

where

$$Z_I(\mathcal{R}) = \{x \in \mathcal{R} \mid \exists\, y \in \mathcal{R} \setminus I \text{ such that } (x \odot y) \cap I \neq \emptyset \text{ or } (y \odot x) \cap I \neq \emptyset\}.$$

The connection between graph theory and ring theory was first rigorously established by Beck [1] in 1988, who investigated the chromatic number of a graph $G(\mathcal{R})$ associated with a ring $\mathcal{R}$. Later, Anderson and Livingston [2] refined this approach by introducing the zero-divisor graph of a commutative ring $\mathcal{R}$. In their construction, the vertices represent the nonzero zero-divisors of $\mathcal{R}$, with edges connecting distinct elements a and b if and only if $ab = 0$. Redmond [3] expanded this framework in 2006 by developing an algorithm to classify all commutative reduced rings with identity whose zero-divisor graphs have a given number of vertices for any $n \geq 1$. His work provided a complete classification, up to isomorphism, for rings yielding zero-divisor graphs with 6 to 14 vertices, including an in-depth analysis for the case of 14-vertex graphs.

Chelvam and Asir [4] presented a comprehensive survey on the distance properties of zero-divisor graphs and total graphs associated with commutative rings.

Suthar and Prakash [5] investigated the covering properties of the line graph associated with the zero-divisor graph over the ring (Z_n).

For a comprehensive treatment of zero-divisor graphs and their generalizations, readers may refer to [6–8].

Recent research has focused extensively on analyzing structural properties of zero-divisor graphs associated with commutative rings. Anderson and Livingston's seminal work [2] demonstrated that for any commutative ring $\mathcal{R}$, its zero-divisor graph $\Gamma(\mathcal{R})$ is necessarily connected with diameter at most 3. Furthermore, they proved that when $\Gamma(\mathcal{R})$ contains cycles, the girth (length of the shortest cycle) cannot exceed 7. Hamidi and Cristea [9] introduced the hyperideal-based zero-divisor graph of a general hyperring $\mathbb{Z}_n$, focusing on absorbing elements and structural properties. In the hyperlattice context, zero-divisor graphs relative to hyperideals are studied for metrics like diameter and structural completeness. Hamidi, et al. [10] proposed intersection graphs based on hyperideals, characterizing connectivity and Eulerian properties. Earlier et al. [11] examined zero-divisor graphs arising from commutative multiplicative hyperrings, generalizing classical ring theoretic graphs, and recent work on general hyperrings has employed absorbing element frameworks and have analyzed graph isomorphisms and finiteness. Li et al. [12] demonstrated that vertex transitivity is maintained under lexicographic products, with analogous results for edge transitive graphs.

Spacapan's [13] work on Cartesian products quantified the minimal vertex deletions required to disrupt connectivity, providing fundamental insights into their structural stability.

Aijaz, Rani, and Pirzada [14] introduced and studied compressed zero-divisor graphs associated with the ring of integers modulo (n). Their work explores several algebraic and graph-theoretic properties, demonstrating the significance of compressed structures in simplifying the analysis of zero-divisor graphs.

Recent advances by Akgüneş and Nacaroğlu [15] in 2019 determined the diameter and girth of zero-divisor graphs for product rings $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$, revealing new patterns in these metric properties for ring products.

For further exploration of diameter and girth properties in commutative ring zero-divisor graphs, see [16–18].

In the noncommutative setting, standard ideal theory is often insufficient to capture the full structure of annihilation relations. In particular, left and right annihilators need not coincide, and the multiplicative behavior of elements may exhibit intrinsic asymmetry. Hyperideals provide the appropriate generalization in this context, as they are stable under hyperoperations and accommodate the multivalued and noncommutative nature of the product. This makes hyperideal theory essential for defining and analyzing zero-divisor graphs over noncommutative hyperrings, where classical ideal-based approaches fail to reflect the underlying algebraic interactions accurately.

The interaction between hyperstructure theory and graph theory provides a flexible framework for understanding complex algebraic relationships. In particular, hyperrings generalize classical rings by allowing operations to yield sets of elements, which leads to richer structural behavior. Zero-divisor graphs serve as an effective tool for translating algebraic properties of hyperrings into combinatorial features such as connectivity, diameter, girth, and coloring properties. This perspective allows us to analyze structural complexity and interaction patterns among hyperideal elements.

Furthermore, studying hyperideal-based zero-divisor graphs over noncommutative hyperrings helps reveal structural asymmetry and multivalued interactions that are not visible in classical ring-based graphs. These investigations contribute to a deeper understanding of hyperring structures and open possibilities for further developments in algebraic hyperstructures and graph-theoretic analysis.

This paper examines fundamental graph theoretic properties including connectivity, and diameter in products of zero-divisor graphs derived from noncommutative rings. Throughout this paper, hyperaddition and hypermultiplication are denoted by $\oplus$ and $\odot$, . Some graph-theoretic statements included in Section 3, specifically those concerning Laplacian rank, radius, and diameter, rest upon fundamental principles of spectral and metric graph theory which are standard structural properties utilized in the hyperring setting. The main novelty of the paper lies in studying how hyperideal structure and noncommutative hyperoperations influence the zero-divisor graph.

Our investigation is organized as follows: Section 2 establishes the theoretical foundation, introducing essential concepts in hyperrings and hyperideals while reviewing relevant graph constructions. In Section 3, we analyze structural properties of $\Gamma_I(\mathcal{R})$, particularly focusing on connectivity patterns, effects of noncommutative and nonassociative operations. Section 4 provides a comparative analysis with an existing research. Finally, Section 5 summarizes our key findings and outlines potential extensions of these graph-theoretic concepts to more general hyperstructures.

2. Preliminaries

The classical theory of zero-divisor graphs over commutative rings has played a central role in translating algebraic structures into graphical form, enabling the study of properties such as annihilators, ideals, and factorization through combinatorial tools. Despite these contributions, existing frameworks remain limited when addressing more general algebraic settings, particularly those involving noncommutative or nonassociative operations. In hyperrings, for instance, annihilation is no longer symmetric, and product structures exhibit irregularities that call for a more flexible graph-theoretic representation. This motivates the development of new frameworks capable of reflecting the richer and more nuanced behaviors of algebraic objects defined beyond the classical associative and commutative paradigm.

To address this gap, we introduce hyperideal-based zero-divisor graphs designed specifically for noncommutative and nonassociative hyperrings. This construction forms a bridge between higher order algebra and graph theory by representing how nonsymmetric multiplicative interactions manifest in terms of connectivity, cycles, chromatic number, and other invariants. Beyond generalizing the classical zero-divisor graph, the proposed approach provides new graph theoretic indices such as annihilation reach, edge asymmetry index, and algebraic complexity index that allow classification and comparison of hyperrings in ways not possible under the classical model. By investigating bounds on diameter, chromatic number, and reachability, we connect algebraic sparsity with structural minimality, offering systematic ways to distinguish trees, forests, and highly connected components within these graphs. Such insights not only broaden the scope of algebraic graph theory but also open pathways for algorithmic decomposition and analysis of complex algebraic data structures. Throughout this paper, unless otherwise stated, hyperrings are assumed to be associative. In later sections, we also consider zero-divisor graphs arising from nonassociative hyperrings and study their structural properties separately.

The following definitions will play a main role:

Definition 1. [20] Let $\mathcal{R}$ be a ring. We say that $\mathcal{R}$ admits a $(\oplus, \odot)$-induced general hyperring structure if there exist hyperoperations $\oplus$ and $\odot$ defined on $\mathcal{R}$ such that the following hold:

(1) $(\mathcal{R}, \oplus, \odot)$ satisfies the axioms of a general hyperring;
(2) for all $x, y \in \mathcal{R}$, the original ring operations are embedded in the hyperoperations, that is,

$$x + y \in x \oplus y, \quad \text{and} \quad x \cdot y \in x \odot y.$$

Such a structure is referred to as a $(\oplus, \odot)$-induced general hyperring over $\mathcal{R}$.

Remark 1. It should be noted that different definitions of hyperrings appear in the literature. In some works, a hyperring is defined as a structure equipped with hyperaddition and ordinary multiplication. However, in this paper, we adopt the definition of a hyperring as an algebraic structure endowed with both hyperaddition and hypermultiplication. This choice is particularly appropriate for studying hyperideal-based zero-divisor graphs, where the hypermultiplication plays a crucial role in defining adjacency relations.

Definition 2. [2] Let $\mathcal{R}$ be a ring not necessarily commutative. An element $a \in \mathcal{R} \setminus \{0\}$ is called a zero-divisor if there exists a nonzero element $b \in \mathcal{R}$ such that

$$a \odot b = 0 \quad \text{or} \quad b \odot a = 0.$$

The set of all nonzero zero-divisors in $\mathcal{R}$ is denoted by $Z(\mathcal{R})$.

Definition 3. [19] Let $\mathcal{R}$ be a hyperring and I be a hyperideal of $\mathcal{R}$. The hyperideal-based zero-divisor graph $\Gamma_I(\mathcal{R})$ is defined as follows: The vertex set consists of all elements

$$Z_I(\mathcal{R}) = \{a \in \mathcal{R} \setminus I : (a \odot b) \cap I \neq \emptyset \text{ or } (b \odot a) \cap I \neq \emptyset \text{ for some } b \in \mathcal{R} \setminus I, b \neq a\}.$$

Two distinct vertices a and b are adjacent if and only if $(a \odot b) \cap I \neq \emptyset$ or $(b \odot a) \cap I \neq \emptyset$. This adjacency definition is used throughout the paper.

Definition 4. Let $\Gamma_I(\mathcal{R})$ be the hyperideal-based zero-divisor graph of a hyperring $\mathcal{R}$. We define the algebraic complexity index of $(\mathcal{R}, I)$ by

$$C(\mathcal{R}, I) = \chi(\Gamma_I(\mathcal{R})) + \Delta(\Gamma_I(\mathcal{R})) - \delta(\Gamma_I(\mathcal{R})),$$

where χ, Δ, and δ denote the chromatic number, maximum degree, and minimum degree of $\Gamma_I(\mathcal{R})$, respectively.

Observation 1. The index $C(\mathcal{R}, I)$ aggregates coloring complexity and degree variation into a single quantity. In particular, the term $\Delta - \delta$ reflects the degree dispersion of $\Gamma_I(\mathcal{R})$ and distinguishes regular graphs from irregular ones.

Theorem 1. [20] Let $(\mathcal{R}, \oplus, \odot)$ be a ring with $|\mathcal{R}| \geq 2$. Define two hyperoperations on $\mathcal{R}$ by:

$$x \oplus y := \{x, y, x + y\}, \quad x \odot y := \{x \odot y, 0\}.$$

Then, $(\mathcal{R}, \oplus, \odot)$ forms a general hyperring structure. That is, every nontrivial ring admits a $(\oplus, \odot)$-induced general hyperring via this construction.

Remark 2. The set $Z_I(\mathcal{R})$ depends explicitly on the underlying hyperring structure. The hyperproducts are computed by first evaluating the corresponding products in the underlying ring and then applying the hyperoperation as defined in Theorem 1.

Definition 5. Let G be a simple graph. The chromatic number of G denoted by $\chi(G)$, is the minimum number of colors required to color the vertices of G such that no two adjacent vertices receive the same color.

Definition 6. In the extended coloring, vertices a and b must receive different colors if either

$$(a \odot b) \cap I \neq \emptyset \quad \text{or} \quad (b \odot a) \cap I \neq \emptyset.$$

The extended chromatic number $\chi^*(\mathcal{R}, I)$ is the minimum number of colors holds this condition.

Definition 7. The diameter of G denoted by $\operatorname{diam}(G)$, is the maximum distance between any pair of vertices in G, where distance is measured as the length of the shortest path connecting them.

Definition 8. Let G be a graph, and let $v \in V(G)$. The *reach* of a vertex v, denoted by $\operatorname{Reach}(v)$, is defined as the number of vertices in G that are reachable from v via a finite path. The average reachability of G is defined as

$$\operatorname{AR}(G) = \frac{1}{|V(G)|} \sum_{v \in V(G)} \operatorname{Reach}(v).$$

3. Main results on hyperideals based zero-divisor graphs

In this section, we will present the principal findings on hyperideal-based zero-divisor graphs, establishing structural characterizations and deriving key theoretical results that form the core contributions of the study.

3.1. Connectedness and diameter of a graph

In this subsection, we examine two fundamental structural parameters of graphs, namely connectedness and diameter. Connectedness ensures the existence of paths between every pair of vertices, and the diameter provides a measure of the maximum distance within the graph.

Theorem 2. Let $\mathcal{R}$ be a finite hyperring with a proper hyperideal $I \subset \mathcal{R}$. Let $\Gamma_I(\mathcal{R})$ denote the hyperideal-based zero-divisor graph, and let A be its adjacency matrix. Then, $\Gamma_I(\mathcal{R})$ is connected if and only if the Laplacian matrix $L = D - A$ has rank $|Z_I(\mathcal{R})| - 1$.

Proof. Let $n := |Z_I(\mathcal{R})|$ be the number of vertices in $\Gamma_I(\mathcal{R})$. We can also denote the vertex set as $V = Z_I(\mathcal{R})$. However, the adjacency matrix A $\in \mathbb{R}^{n\times n}$ is denoted as

$$A_{ij} = \begin{cases} 1, & \text{if } x_i \odot x_j \cap I \neq \emptyset \text{ or } x_j \odot x_i \cap I \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

The degree matrix D is a diagonal matrix where:

$$D_{ii} = \sum_{j=1}^{n} A_{ij}, \quad D_{ij} = 0 \text{ for } i \neq j.$$

The Laplacian matrix is given by L=D-A. From spectral graph theory, the number of connected components of a graph equals the multiplicity of the zero eigenvalue of the Laplacian matrix L. Consequently, $\Gamma_I(\mathcal{R})$ is connected if and only if:

$$rank(L) = n - 1.$$

If $\Gamma_I(\mathcal{R})$ is connected, so the dimension of $\ker(L)$ is one. Finally, the all ones vector is the only eigenvector corresponding to eigenvalue zero. This implies,

$$rank(L) = n - 1.$$

If $rank(L) = n - 1$, then $\ker(L)$ is 1-dimensional, implying that the graph has only one connected component. □

Lemma 1. Let $\mathcal{R}$ be a hyperring and I a hyperideal of $\mathcal{R}$. Then, the hyperideal-based zero-divisor graph $\Gamma_I(\mathcal{R})$ has a nonempty vertex set if and only if there exists a nonzero element $z \in I$ such that $z \in x \odot y$ or $z \in y \odot x$ for some $x, y \in \mathcal{R} \setminus I$.

Proof. Suppose that the vertex set of $\Gamma_I(\mathcal{R})$ is nonempty. There exists an element $x \in \mathcal{R} \setminus I$ such that for some $y \in \mathcal{R} \setminus I$, $(x \odot y) \cap I \neq \emptyset$ or $(y \odot x) \cap I \neq \emptyset$. Now, there exists a nonzero element $z \in I$ such that $z \in x \odot y$ or $z \in y \odot x$. This shows that x and y are nonzero zero-divisor modulo I.

Conversely, suppose that there exists a nonzero element $z \in I$ and elements $x, y \in \mathcal{R} \setminus I$ such that $z \in x \odot y$ or $z \in y \odot x$. Then, we get

$$(x \odot y) \cap I \neq \emptyset \quad \text{or} \quad (y \odot x) \cap I \neq \emptyset,$$

which implies that x and y are vertices of $\Gamma_I(\mathcal{R})$. Hence, the vertex set of $\Gamma_I(\mathcal{R})$ is nonempty. □

Lemma 2. Let $\mathcal{R}$ be a hyperring, and let I be a hyperideal of $\mathcal{R}$. Then, the hyperideal-based zero-divisor graph $\Gamma_I(\mathcal{R})$ is empty if and only if I contains no nonzero zero-divisor of $\mathcal{R}$.

Proof. Let us assume that $\Gamma_I(\mathcal{R})$ is empty. This means that there does not exist any pair of elements $x, y \in \mathcal{R} \setminus I$ such that $x \odot y \subseteq I$ satisfying $(x \odot y) \cap I \neq \emptyset$ or $(y \odot x) \cap I \neq \emptyset$. Due to this, no nonzero zero-divisor in $\mathcal{R}$ has its product with any other element falling inside I, implying that I contains no such zero-divisor. Conversely, suppose that the hyperideal I does not contain any nonzero zero-divisors of $\mathcal{R}$. Therefore, for all $x, y \in \mathcal{R} \setminus I$ such that $(x \odot y) \cap I \neq \emptyset$ or $(y \odot x) \cap I \neq \emptyset$, the hyperproduct $x \odot y$ is not a subset of I, so no edge can be formed in $\Gamma_I(\mathcal{R})$. Therefore, the obtained graph is empty. □

This result demonstrates that adjacency in $\Gamma_I(\mathcal{R})$ may not be symmetric in noncommutative hyperrings.

Lemma 3. Let $\mathcal{R}$ be a hyperring and I a hyperideal of $\mathcal{R}$. Then, every vertex in the hyperideal-based zero-divisor graph $\Gamma_I(\mathcal{R})$ is adjacent to at least one other vertex.

Proof. Let x be an arbitrary vertex of $\Gamma_I(\mathcal{R})$. By definition of the vertex set, $x \in \mathcal{R} \setminus I$, and there exists an element $y \in \mathcal{R} \setminus I$, with $y \neq x$, such that

$$(x \odot y) \cap I \neq \emptyset.$$

This means that there exists an element $z \in \mathcal{R}$ such that

$$z \in (x \odot y) \cap I,$$

that is, $z \in x \odot y$, and $z \in I$. Finally, the hyperproduct of x and y contains at least one element of the hyperideal I. By the definition of an adjacency in the graph $\Gamma_I(\mathcal{R})$, this implies that x is adjacent to y. Therefore, every vertex x is connected by an edge to at least one distinct vertex y in the graph. Thus, every vertex in $\Gamma_I(\mathcal{R})$ has degree at least one. □

In this section, we extend the previous discussion to nonassociative hyperrings. The structural properties of zero divisor graphs over nonassociative hyperrings exhibit different characteristics, which we investigate below.

Theorem 3. Let $\mathcal{R}$ be a finite nonassociative hyperring and $\Gamma_I(\mathcal{R})$ its hyperideal-based zero-divisor graph. Then,

$$diam(\Gamma_I(\mathcal{R})) \leq |Z_I(\mathcal{R})| - 1,$$

with equality if and only if $\Gamma_I(\mathcal{R})$ is a path graph.

Proof. Because $\Gamma_I(\mathcal{R})$ is a finite graph with n vertices, we get

$$diam(\Gamma_I(\mathcal{R})) \leq n - 1 = |Z_I(\mathcal{R})| - 1.$$

Let us assume that $diam(\Gamma_I(\mathcal{R})) = |Z_I(\mathcal{R})| - 1$. The graph must realize the maximal possible distance between two vertices. In such a case, no branching or cycles can exist, as any additional edges would create shortcuts and reduce the diameter. The graph must be a simple path. Conversely, if $\Gamma_I(\mathcal{R})$ is a path with n vertices. However, the shortest path distance between two endpoints is $n - 1$. Therefore, $diam(\Gamma_I(\mathcal{R})) = |Z_I(\mathcal{R})| - 1$. This completes the proof. □

This theorem provides a clear criterion to characterize the maximum diameter of $\Gamma_I(\mathcal{R})$ and introduces the path graph as an extremal case. It supports the idea that hypergraph structures arising from zero-divisors in nonassociative hyperrings still obey classical graph-theoretic constraints, though under broader algebraic conditions.

Theorem 4. Let $\mathcal{R}$ be a nonassociative hyperring, and let $I \subset \mathcal{R}$ be a proper hyperideal. Let $\Gamma_I(\mathcal{R})$ denote the hyperideal-based zero-divisor graph associated to $\mathcal{R}$. Then,

$$\rho\,(\Gamma_I(\mathcal{R})) \leq diam\,(\Gamma_I(\mathcal{R})) \leq |Z_I(\mathcal{R})| - 1,$$

where $Z_I(\mathcal{R})$ is the vertex set of $\Gamma_I(\mathcal{R})$, consisting of elements outside I that interact via products landing in I, and ρ denotes the radius of the graph.

Proof. By Definitions 1 and 7, the radius is the minimum of all vertex eccentricities, and the diameter is the maximum:

$$\rho(\Gamma_I(\mathcal{R}) = min_{u \in V} e(u), \quad diam(\Gamma_I(\mathcal{R}) = max_{u \in V} e(u).$$

Now, we obtain

$$\rho(\Gamma_I(\mathcal{R})) \leq diam(\Gamma_I(\mathcal{R})).$$

This inequality is purely graph-theoretic and holds for all connected graphs. Let $n = |Z_I(\mathcal{R})|$. In any connected graph with n vertices, the maximum possible distance between two vertices is $n - 1$. This occurs in the case where the graph is a path graph P_n. Therefore, we have

$$diam(\Gamma_I(\mathcal{R})) \leq n - 1 = |Z_I(\mathcal{R})| - 1.$$

This bound holds regardless of whether the graph arises from ring, hyperring, or other algebraic contexts. Even though associativity fails in the underlying algebraic structure, the definition of $\Gamma_I(\mathcal{R})$ and its adjacency condition are purely set-theoretic, involving the intersection of hyperproducts with the hyperideal. Also, the metric properties like radius and diameter are still valid and measurable. In fact, the lack of associativity may increase average path length, making the study of radius and diameter even more informative. We have shown that for a zero-divisor graph $\Gamma_I(\mathcal{R})$ arising from a nonassociative hyperring,

$$\rho(\Gamma_I(\mathcal{R})) \leq diam(\Gamma_I(\mathcal{R})) \leq |Z_I(\mathcal{R})| - 1.$$

This completes the proof. □

Theorem 5. Let $\mathcal{R}$ be a hyperring, and let I be a hyperideal of $\mathcal{R}$. Suppose that for any two distinct vertices $x, y \in \Gamma_I(\mathcal{R})$, either $(x \odot y) \cap I \neq \emptyset$, or there exists an element $z \in \mathcal{R} \setminus I$ such that $(x \odot z) \cap I \neq \emptyset$, and $(y \odot z) \cap I \neq \emptyset$. Then, the hyperideal-based zero-divisor graph $\Gamma_I(\mathcal{R})$ is connected and has diameter at most two.

Proof. Let x and y be two distinct vertices of $\Gamma_I(\mathcal{R})$. By definition of the hyperideal-based zero-divisor graph, we consider the following two cases.

Case 1. $(x \odot y) \cap I \neq \emptyset$. In this case, by the definition of adjacency in $\Gamma_I(\mathcal{R})$, the vertices x and y are adjacent. So, in this case, we obtain that the distance between x and y is equal to one.

Case 2. $(x \odot y) \cap I = \emptyset$. By the hypothesis of this theorem, there exists an element $z \in R \setminus I$ such that

$$(x \odot z) \cap I \neq \emptyset, \quad \text{and} \quad (y \odot z) \cap I \neq \emptyset.$$

In the second case, then we have x which is adjacent to z and y is also adjacent to z, in $\Gamma_I(\mathcal{R})$. Consequently, the vertices x and y are connected by the path $x \sim z \sim y$, and the distance between x and y is at most two.

Because x and y were arbitrary distinct vertices of $\Gamma_I(\mathcal{R})$, it follows that every pair of vertices in $\Gamma_I(\mathcal{R})$ is connected by a path of length at most two. Hence, the graph $\Gamma_I(\mathcal{R})$ is connected, and its diameter is at most two. □

In fact, this result highlights a local structural regularity within a nonassociative hyperring. It can be used to identify stable or controllable substructures in complex algebraic or applied models, such as in distributed systems, communication networks, or biological pathways.

3.2. Chromatic number and complexity index of a graph

In this subsection, we study the chromatic number and complexity index of the graph, establishing coloring properties and quantifying structural intricacy through these invariants.

Proposition 1. If $\Gamma_I(\mathcal{R})$ is a regular graph, then

$$C(\mathcal{R}, I) = \chi(\Gamma_I(\mathcal{R})).$$

Proof. Let us consider that $\Gamma_I(\mathcal{R})$ is a regular graph. There exists a non-negative integer k such that every vertex of $\Gamma_I(\mathcal{R})$ has degree k. Now, with the help of the maximum and minimum degree of a graph, we have

$$\Delta(\Gamma_I(\mathcal{R})) = k \quad \text{and} \quad \delta(\Gamma_I(\mathcal{R})) = k.$$

Further, recall that the algebraic complexity index is defined by

$$C(\mathcal{R}, I) = \chi(\Gamma_I(\mathcal{R})) + \Delta(\Gamma_I(\mathcal{R})) - \delta(\Gamma_I(\mathcal{R})).$$

Substituting the above equalities into this expression yields,

$$C(\mathcal{R}, I) = \chi(\Gamma_I(\mathcal{R})) + k - k.$$

Furthermore, as the contributions of the maximum and minimum degrees cancel, the value of $C(\mathcal{R}, I)$ it depends on the chromatic number of the graph. Thus, we have

$$C(\mathcal{R}, I) = \chi(\Gamma_I(\mathcal{R})),$$

which completes the proof. □

Theorem 6. Let $\mathcal{R}$ be a hyperring and $I \subset \mathcal{R}$ a proper hyperideal. Then,

$$\chi^*(\mathcal{R}, I) \geq \chi(\Gamma_I(\mathcal{R})).$$

Moreover, there exists a noncommutative hyperring $\mathcal{R}$ and a hyperideal I such that

$$\chi^*(\mathcal{R}, I) > \chi(\Gamma_I(\mathcal{R})).$$

Proof. To begin, we have to show that

$$\chi^*(\mathcal{R}, I) \geq \chi(\Gamma_I(\mathcal{R})).$$

By Definitions 5 and 6, the extended coloring requires that for all $a, b \in Z_I(\mathcal{R})$,

$$(a \odot b) \cap I \neq \emptyset \quad \text{or} \quad (b \odot a) \cap I \neq \emptyset$$

implies that a and b receive different colors. In fact, this condition includes all classical adjacency relations; every extended coloring is also a valid coloring of $\Gamma_I(\mathcal{R})$. Now, we obtain

$$\chi^*(\mathcal{R}, I) \geq \chi(\Gamma_I(\mathcal{R})).$$

Moreover, let us suppose that there exist distinct elements $a, b, c \in Z_I(\mathcal{R})$ satisfying the given conditions. From $(a \odot b) \cap I \neq \emptyset$, the extended condition implies that a and b have different colors. Note that similarly, $(a \odot c) \cap I \neq \emptyset$ implies a and c must receive different colors, and $(b \odot c) \cap I \neq \emptyset$ implies that b and c must receive different colors. Further, under the extended coloring condition, the vertices a, b, c must all receive distinct colors, requiring at least three colors.

However, in the classical graph $\Gamma_I(\mathcal{R})$, it is possible that the adjacency between a and b is not symmetric (because $(b \odot a) \cap I = \emptyset$), and due to this, the graph structure may admit a coloring with fewer colors. Thus, under these conditions, we obtain

$$\chi^*(\mathcal{R}, I) > \chi(\Gamma_I(\mathcal{R})).$$

This completes the proof. □

3.3. Measures of asymmetry and reachability

In this subsection, we evaluate measures of asymmetry and reachability to capture irregularities in the graph's structure and to assess how efficiently vertices can be accessed within the network.

Theorem 7. Let $\mathcal{R}$ be a finite, noncommutative hyperring with proper hyperideal I.

(i) The map $reach_I$: $Z_I(\mathcal{R}) \to \mathbb{N}$ is, in general, not symmetric; there exist $a, b \in Z_I(\mathcal{R})$ such that $a \in reach_I(b)$, but $b \notin reach_I(a)$.

(ii) The average reach is

$$\rho_{\text{avg}}(\mathcal{R}, I) := \frac{1}{|Z_I(\mathcal{R})|} \sum_{x \in Z_I(\mathcal{R})} reach_I(x).$$

The value $\rho_{\text{avg}}(\mathcal{R}, I)$ acts as a centrality-like indicator of interaction density in the zero-divisor graph.

Proof. (i) Asymmetry of $reach_I$: Because $\mathcal{R}$ is noncommutative, there exist $a, b \in \mathcal{R}$ with $(a \odot b) \cap I \neq \varnothing$ but $(b \odot a) \cap I = \varnothing$. Because both a and b lie outside I and satisfy the zero-divisor condition, they belong to $Z_I(\mathcal{R})$. Consequently,

$$a \in reach_I(b), \quad \text{but} \quad b \notin reach_I(a),$$

demonstrating that the reach function is not symmetric.

(ii) Average reach as centrality: By definition, $reach_I(a)$ counts how many vertices interact with a through one-sided or two-sided annihilation into I. Averaging over all vertices yields

$$\rho_{\text{avg}}(\mathcal{R}, I) = \frac{1}{|Z_I(\mathcal{R})|} \sum_{x \in Z_I(R)} reach_I(a),$$

which is the expected number of such interactions per vertex. Therefore, $\rho_{\text{avg}}(\mathcal{R}, I)$ captures the overall annihilation connectivity of the algebraic graph, playing a role analogous to mean out degree or classic graph centrality measures. □

When $\mathcal{R}$ is commutative, we have that

$$(a \odot b) \cap I \neq \varnothing$$

would automatically imply $(b \odot a) \cap I \neq \varnothing$, and the reach map would be symmetric. Therefore, the asymmetry of $reach_I$ directly quantifies the degree of noncommutativity mod I.

Theorem 8. Let $\mathcal{R}$ be a noncommutative hyperring with a proper hyperideal $I \subset \mathcal{R}$. Then,

$$\alpha(\mathcal{R}, I) = 0 \quad \text{if and only if} \quad (a \odot b) \cap I \neq \emptyset \iff (b \odot a) \cap I \neq \emptyset, \ \forall a, b \in Z_I(\mathcal{R}).$$

Moreover, higher values of $\alpha(\mathcal{R}, I)$ indicate stronger unidirectional annihilation and structural imbalance in the algebraic system.

Proof. If $\alpha(\mathcal{R}, I) = 0$, then $\mathcal{R}$ is commutative modulo I. By definition, if the index is zero, there are no pairs $(a, b) \in Z_I(\mathcal{R})^2$ such that:

$$(a \odot b) \cap I \neq \emptyset, \quad \text{and} \quad (b \odot a) \cap I = \emptyset.$$

Now, for all such a, b, we must have:

$$\begin{aligned} &(a \odot b) \cap I \neq \emptyset \\ \Rightarrow\ &(b \odot a) \cap I \neq \emptyset. \end{aligned}$$

Because we can have that a and b are symmetric, the reverse implication also holds, so:

$$\begin{aligned} &(a \odot b) \cap I \neq \emptyset \\ \iff &(b \odot a) \cap I \neq \emptyset. \end{aligned}$$

Now, $\mathcal{R}$ is commutative modulo I with respect to ideal annihilation. If $\mathcal{R}$ is commutative modulo I, then we also obtain

$$\alpha(\mathcal{R}, I) = 0.$$

Let us suppose that for all $a, b \in Z_I(\mathcal{R})$, yields

$$\begin{aligned} &(a \odot b) \cap I \neq \emptyset \\ \iff &(b \odot a) \cap I \neq \emptyset. \end{aligned}$$

The set in the numerator of $\alpha(\mathcal{R}, I)$ is empty, because there are no pairs for which one direction intersects I and the other does not. Therefore, we get

$$\alpha(\mathcal{R}, I) = \frac{0}{|Z_I(\mathcal{R})|^2} = 0.$$

Now, we need interpretation of larger $\alpha(\mathcal{R}, I)$ for understanding the structural behavior of the ring. If $\alpha(\mathcal{R}, I)$ is close to 1, it means there are many pairs $(a, b) \in Z_I(\mathcal{R})^2$ such that $(a \odot b) \cap I \neq \emptyset$, but $(b \odot a) \cap I = \emptyset$. This reflects a high degree of unidirectional annihilation. In such systems, multiplication is asymmetric, and the graph $\Gamma_I(\mathcal{R})$ exhibits directed behavior and structural imbalance. □

The edge asymmetry index $\alpha(\mathcal{R}, I)$ quantifies how far the hyperring is from being commutative modulo I. It can be used to measure directional failure or interaction asymmetry in algebraic models of systems such as workflows, chemical reactions, or information flow.

Observation 2. Let $\mathcal{R}$ be a finite, nonassociative, and noncommutative hyperring with a proper hyperideal $I \subset \mathcal{R}$. Then, the graph $\Gamma_I(\mathcal{R})$ is connected if and only if

$$\forall a \in Z_I(\mathcal{R}),\ \exists b \in Z_I(\mathcal{R}) \text{ such that } (a \odot b) \cap I \neq \emptyset, \text{ and } (b \odot a) \cap I \neq \emptyset.$$

Remark 3. Let us assume that $\Gamma_I(\mathcal{R})$ is connected. For every $a \in Z_I(\mathcal{R})$, there exists a path from a to some vertex $b \in Z_I(\mathcal{R})$. By using Definition 3 and Theorem 1, the condition in Observation 2 is necessary for connectedness. Now, assume that for every $a \in Z_I(\mathcal{R})$, there exists $b \in Z_I(\mathcal{R})$ such that

$$(a \odot b) \cap I \neq \emptyset, \quad (b \odot a) \cap I \neq \emptyset.$$

This ensures that each vertex a has at least one edge to b in the graph. Repeating this process recursively allows us to build a connected component containing all of $Z_I(\mathcal{R})$, assuming neighborhoods overlap. Because $\mathcal{R}$ is finite, this pairwise connectivity condition strongly suggests that $\Gamma_I(\mathcal{R})$ is connected.

Proposition 2. Let $G = \Gamma_I(\mathcal{R})$ be the hyperideal-based zero- divisor graph of a hyperring $\mathcal{R}$. If G is connected, and the edge asymmetry index is $EAI(G) = 0$, then for every vertex $v \in V(G)$,

$$\text{Reach}(v) = |V(G)|.$$

Moreover, the adjacency relation in G is symmetric, and the graph behaves a zero-divisor graph arising from a commutative structure.

Proof. Because G is connected, every vertex v is reachable from every other vertex by a finite path. Therefore, we have $\text{Reach}(v) = |V(G)|$.

Now, we know that the edge asymmetry index equals to the number of asymmetric adjacency pairs in the graph such that $EAI(G) = 0$, which implies that there are no asymmetric pairs of vertices. So, for any $a, b \in V(G)$,

$$\begin{aligned} & (a \odot b) \cap I \neq \emptyset \\ \Rightarrow \quad & (b \odot a) \cap I \neq \emptyset. \end{aligned}$$

Therefore, adjacency is symmetric, and G behaves as a connected graph. This symmetry reflects commutativity at the level of adjacency relations, completing the proof. □

3.4. Extension to infinite hyperrings

It is natural to investigate whether the results obtained for finite hyperrings extend to infinite hyperrings. In general, some structural properties such as bounded diameter and finite connectivity may fail for infinite hyperrings. Therefore, additional assumptions may be required to extend the results. This direction opens further possibilities for future research in hyperstructure-based zero-divisor graphs.

4. Comparative analysis with existing research

The study of zero-divisor graphs has been widely investigated in commutative rings, commutative hyperrings, and noncommutative rings. The classical concept of zero-divisor graphs was first introduced for commutative rings, where vertices correspond to nonzero zero-divisors, and adjacency is defined through the zero-product condition. These investigations established several structural properties such as connectivity, diameter bounds, and characterization of graph classes. The concept was extended to noncommutative rings, where additional complexity arises due to the lack of commutativity. In such structures, adjacency relations often depend on left and right multiplication, leading to directed graphs or modified definitions of zero-divisor graphs. These extensions revealed new structural behaviors that are not observed in commutative settings.

More recently, zero-divisor graphs have been studied in commutative hyperrings, where the hyperoperation introduces multiple possible products. In this setting, adjacency is typically defined when the hyperproduct of two elements intersects a given hyperideal. These generalizations significantly broaden the scope of algebraic structures that can be analyzed using graph-theoretic methods. The results obtained in this paper extend these earlier works in several directions. First, this study considers nonassociative and noncommutative hyperrings simultaneously, which has received limited attention in the literature. Also, the introduction of hyperideal-based zero-divisor graphs provides a more general framework that includes previously studied structures as special cases. Moreover, the proposed observations and structural properties offer new insights into connectivity and adjacency conditions in generalized hyperstructures.

Furthermore, the present work provides new characterizations of zero-divisor graphs under generalized hyperoperations. These results extend known properties from commutative rings and commutative hyperrings to broader algebraic environments. Hence, the results of this paper contribute to the development of hyperstructure theory and provide new directions for future research. This study not only generalizes existing results but also introduces new structural insights, thereby highlighting the novelty and academic contribution of the present work.

5. Conclusions

In this work, we introduced hyperideal-based zero-divisor graphs $\Gamma_I(\mathcal{R})$ for noncommutative hyperrings, establishing their fundamental matrix-theoretic and combinatorial properties. We proved that connectivity is characterized by the rank condition $|Z_I(\mathcal{R})| - 1$ and that the symmetry of the adjacency matrix A reflects algebraic commutativity modulo I. Key results include diameter bounds, completeness constraints under nonassociativity, and the introduction of algebraic indices such as $C(\mathcal{R}, I)$ and $\chi^*(\mathcal{R}, I)$ which capture structural complexity. These findings bridge hyperring theory with spectral graph theory and provide a systematic framework for analyzing noncommutative algebraic structures via their associated graphs.

Author contributions

A. Alali: formal analysis, writing review, funding and editing; K. Rani: validation, investigation; J. Nisar: supervision, investigation; S. Mir: investigation, writing-review and editing; S. Rani: writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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