AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(6): 15324–15360.
DOI:10.3934/math.2026630
Received: 18 March 2026
Revised: 13 May 2026
Accepted: 15 May 2026
Published: 01 June 2026

Research article

Boundary peak solutions with residual mass of a slightly subcritical Hénon type problem

Dalal Almutairi and Mohamed Ben Ayed *

Department of Mathematics, College of Science, Qassim University, Buraydah 51542, Saudi Arabia

* **Correspondence:** Email: M.BenAyed@qu.edu.sa.

Abstract: We investigated the existence of boundary blow-up solutions for a slightly subcritical semilinear elliptic problem posed in a smooth bounded domain of $\mathbb{R}^n$ with $n \in \{4, 5, 6\}$. We constructed solutions that have a nonzero weak limit while simultaneously blowing up at a boundary point. This behavior stands in contrast to the well-known compactness properties on manifolds without boundary for small dimensions, where such weak convergence would force strong convergence. Our construction shows that, in low dimensions, the presence of a boundary allows blow-up and residual mass to coexist. Moreover, we identified the precise boundary points where this concentration occurs. The results are proved by means of delicate asymptotic estimates of the gradient of the associated Euler–Lagrange functional.

Keywords: partial differential equations; critical Sobolev exponent; blowing-up solution
Mathematics Subject Classification: 35A15, 35J20, 35J25

1. Introduction and main results

In this paper, we construct some boundary blowing up solutions with residual mass of the following slightly subcritical Hénon type problem:

$$(P_\varepsilon) \quad \begin{cases} -\Delta u &= K|u|^{p-1-\varepsilon}u \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 3$, ε is a real positive parameter, $p + 1 = 2n/(n-2)$ is the critical Sobolev exponent for the embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$, and K is a C^3-positive function on $\overline{\Omega}$.

Problem (P_ε) is a classic model in nonlinear analysis, with applications spanning several fields, including nonlinear optics [12, 18] and conformal geometry (see [15]) and the references therein).

For $\varepsilon = 0$, the variational problem associated with (P_ε) lacks compactness. Many papers have investigated conditions on the function K and the domain Ω under which (P_0) admits a positive solution ω, for example; we refer the reader to [7] for the case $K \equiv 1$ and to Theorem 1.4 of [17] for nonconstant K. For $\varepsilon > 0$, much work has been devoted to studying this problem; in particular, many blowing up solutions (as $\varepsilon \to 0$) are constructed. These solutions blow up at some precise points in $\overline{\Omega}$; see; for example, [8, 13, 14, 23]. In [8], the authors studied (P_ε) with $K \equiv 1$ and they introduced a new method to construct some positive solutions of (P_ε). They reduced the existence of solutions to solving a finite dimensional system. This method is used in many perturbed noncompact problems and it has become standard. For example, one can see [3, 4, 9, 16, 23].

For $K \equiv 1$, an important result (proved by R. Schoen [25]) gives the following alternative: Let u_ε be a family of positive solutions of (P_ε). Then the following hold

- In one case, the solutions u_ε converge weakly to a positive solution ω of (P_0). In this case, they have to be L^∞-bounded and, therefore, they converge strongly to ω.
- Alternatively, they blow up. In this case, they have to converge weakly to 0. Moreover, the blow-up points have to be in Ω and isolated simple (see [19] for the definition). Conversely, in [8], the authors constructed solutions which blow up at k points $y_1, \cdots, y_k$ in Ω (satisfying some precise assumptions).

When K is a positive function (not a constant), the problem is an extension of the prescribing scalar curvature problem which is introduced for the compact manifolds (M, g) without boundary. Notice that the prescribing scalar curvature problem on (M, g) is given by

$$(SC) \qquad -\Delta u + \underline{c}_n R_g u = K u^p, \quad u > 0 \quad \text{in } M \quad \Big(\text{with } \underline{c}_n := \frac{n-2}{4(n-1)}\Big),$$

where R_g is the scalar curvature of (M, g). Assume that K has only nondegenerate critical points with $\Delta K \neq 0$ and let (SC_ε) be the perturbed problem (by taking $p - \varepsilon$ instead of p, with ε being small). For a family of positive solutions (u_ε) of (SC_ε) which converges weakly to a positive function ω, we have the following.

- If $n \leq 5$, then u_ε has to converge strongly to ω; therefore, it is an L^∞-bounded solution.
- If $n \geq 7$, we have the following alternative: Either u_ε converges strongly to ω, or u_ε blows up. In the second case, the blow-up points are isolated and simple and they belong to the set $\{y \in M : \nabla K(y) = 0 \text{ and } \Delta K(y) < 0\}$; see [19].

However, concerning the problem (P_ε), in [5], Aubin showed that any concentration point in $\partial\Omega$ of a sequence of solutions has to satisfy $\partial K/\partial\nu \geq 0$ (see Proposition 6.44). Hence, assuming that $\partial K/\partial\nu < 0$ on $\partial\Omega$, the blow-up points are in Ω. However, taking Ω to be the ball $B(O, 1)$ and taking $K = |x|^\gamma$ with $\gamma > 0$ (which implies that $\partial K/\partial\nu > 0$ on $\partial\Omega$), in [13], the authors showed that the ground state solution blows up at a point $\overline{a} \in \partial\Omega$. Notice that the ground state solution converges weakly to 0. In addition, blow-up solutions at multiple isolated simple boundary points were constructed in [20]. Lastly, in [9, 14, 16], the authors constructed some solutions u_ε (which are the sum of many bubbles) that blow up at the same point $y \in \partial\Omega$. This point is characterized as a critical point of $K_1 := K_{|\partial\Omega}$ with $(\partial K/\partial\nu)(y) > 0$. This result is very important, since it shows that y is not an isolated simple blow-up point. Hence, a new phenomenon appears for the problem (P_ε) compared with the problem

(SC_ε). These constructed solutions converge weakly to zero. A natural question arises: Is it possible to get a similar result with a residual mass, i.e., when u_ε converges weakly to a positive function ω? For a compact manifold without boundary, the answer is negative for the small dimensions $n \leq 5$. The aim of this paper is to prove that for the small dimensions $n \in \{4, 5, 6\}$, we are able to construct some solutions which converge weakly to a positive function ω and blow up at some points at the boundary.

To be more precise, for $a \in \mathbb{R}^n$ and $\lambda > 0$, let

$$\delta_{a,\lambda}(x) := c_0 \frac{\lambda^{(n-2)/2}}{(1 + \lambda^2|x-a|^2)^{(n-2)/2}}, \qquad \text{with } c_0 := (n(n-2))^{(n-2)/4}. \tag{1.1}$$

These functions are the positive solutions of the Yamabe problem on $\mathbb{R}^n$ for $n \geq 3$, i.e., these functions satisfy

$$-\Delta\delta_{a,\lambda} = \delta_{a,\lambda}^{\frac{n+2}{n-2}} \qquad \text{in} \quad \mathbb{R}^n.$$

Since $\delta_{a,\lambda} \notin H_0^1(\Omega)$, we introduce its projection, denoted $P\delta_{a,\lambda}$, onto this space. This projection $P\delta_{a,\lambda}$ is defined by the following, for $a \in \Omega$:

$$-\Delta P\delta_{a,\lambda} = -\Delta\delta_{a,\lambda} = \delta_{a,\lambda}^{\frac{n+2}{n-2}} \text{ in } \Omega, \quad \text{ and } \quad P\delta_{a,\lambda} = 0 \text{ on } \partial\Omega. \tag{1.2}$$

Our first result deals with one blow-up point $\xi^* \in \partial\Omega$, and our aim is to construct some solutions u_ε having the form

$$u_\varepsilon = \omega + \sum_{i=1}^{N} P\delta_{a_{i,\varepsilon},\lambda_{i,\varepsilon}} + \phi_\varepsilon$$

such that $\lim a_{i,\varepsilon} = \xi^*$ for each i, where ω is a nondegenerate solution of (P_0).

We remark that for $\omega = 0$, the authors in [14] constructed positive blowing-up solutions at a boundary point $y \in \partial\Omega$ that converge weakly to 0, using the projection $P\delta_{a,\lambda}$ as a bubble. Their result is restricted to dimensions $n \leq 6$. This result was extended in [9] to dimensions $n \geq 7$ by using a more refined bubble, denoted $\widetilde{P\delta}_{a,\lambda}$, which allows better control of the remainder terms. However, this refined bubble requires more delicate expansions. Consequently, we believe that our result also holds for $n \geq 7$, provided that one uses $\widetilde{P\delta}_{a,\lambda}$ instead of $P\delta_{a,\lambda}$. In this paper, we focus on $n \in \{4, 5, 6\}$ because for these low dimensions, the blow-up phenomenon with residual mass does not occur on compact manifolds without boundary. Hence, our result reveals a genuinely new phenomenon. By contrast, for higher dimensions $n \geq 7$, the coexistence of a residual mass and bubbles has already been established for compact manifolds without boundary [2].

To introduce our results, let us define the following function:

$$\mathbb{F}_{N,\xi^*}(y_1, \cdots, y_N) := \frac{(n-2)^2 S_n 2^{n-2}}{2n\tau^{2n/(n-2)}} \sum_{i=1}^{N} \sum_{k \neq i} \frac{-1}{|y_i - y_k|^n} - \sum_{i=1}^{N} \frac{c_3}{2K(\xi^*)} D^2(K_{|\partial\Omega})(\xi^*)(y_i, y_i), \tag{1.3}$$

defined in $X := \{(y_1, \cdots, y_N) \in (T_{\xi^*}\partial\Omega)^N : y_i \neq y_j \text{ if } i \neq j\}$, and where S_n, c_3, and τ are defined by

$$S_n := c_0^{2n/(n-2)} \int_{\mathbb{R}^n} \frac{dy}{(1+|y|^2)^n}, \quad c_3 = \frac{n-2}{n} c_0^{2n/(n-2)} \int_{\mathbb{R}^n} \frac{|y|^2}{(1+|y|^2)^{n+1}} dy, \tag{1.4}$$

$$\tau := \left(\frac{8nc_3}{(n-2)^2 S_n K(\xi^*)} \frac{\partial K}{\partial \nu}(\xi^*) \right)^{(n-2)/2}.$$

We point out that similar Hamiltonian-type functions arise in characterizing the location of concentration points for mean field type-equations in mathematical physics and for Euler equations in fluid dynamics. The relevance of such vortex-type problems to Yamabe-type equations was first discovered by Thizy and Vétois [22] and Pistoia and Vaira [21].

Our result can then be stated as follows.

Theorem 1.1. *Let $n \in \{4,5,6\}$, ω be a positive nondegenerate solution of (P_0) and $\xi^* \in \partial\Omega$ be a nondegenerate critical point of $K_{|\partial\Omega}$ satisfying*

$$\frac{\partial K}{\partial \nu}(\xi^*) > 0, \tag{1.5}$$

where ν is the outward normal vector field on $\partial\Omega$. In addition, for each $N \in \mathbb{N}\backslash\{0\}$, we assume that $\mathbb{F}_{N,\xi^*}$ has a nondegenerate critical point $(y_1^*, \cdots, y_N^*)$. Then, there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$, there exists solution u_ε of (P_ε) satisfying

$$u_\varepsilon = \omega + \sum_{i=1}^{N} P\delta_{a_{i,\varepsilon},\lambda_{i,\varepsilon}} + \phi_\varepsilon, \tag{1.6}$$

with $\|\phi_\varepsilon\| \to 0$ as ε tends to 0 and

$$c_1'\varepsilon^{(n-1)/(n-2)} \le \lambda_{i,\varepsilon}^{-1} \le c_1''\varepsilon^{(n-1)/(n-2)} \quad \forall\ i \ge 1, \tag{1.7}$$

$$c_1'\varepsilon^{(n+1)/(n+2)} \le |a_{i,\varepsilon} - a_{j,\varepsilon}| \le c_1''\varepsilon^{(n+1)/(n+2)} \quad \forall\ i \ne j \ge 1, \tag{1.8}$$

$$c_1'\varepsilon \le d_{i,\varepsilon} := d(a_{i,\varepsilon}, \partial\Omega) \le c_1''\varepsilon \quad \forall\ i \ge 1, \tag{1.9}$$

$$c_1'\varepsilon^{(n+1)/(n+2)} \le |a_{i,\varepsilon} - \xi^*| \le c_1''\varepsilon^{(n+1)/(n+2)} \quad \forall\ i \ge 1, \tag{1.10}$$

where c_1' and c_1'' are some positive constants.

Furthermore, for each $i \in \{1, \cdots, N\}$, it holds: $b_{i,\varepsilon} := \varepsilon^{-(n+1)/(n+2)}(a_{i,\varepsilon} - \xi^*)$ converges to y_i^*.

We remark that Eq (1.10) implies that the points $a_{i,\varepsilon}$ converge to $\xi^* \in \partial\Omega$ as $\varepsilon \to 0$; therefore, ξ^* is a blow-up point of u_ε. Furthermore, it is not an isolated simple blow-up point (see [19] for the definition).

The positivity assumption on ω is necessary, since we aim to construct positive solutions of (P_ε), and thus their weak limit must be positive. On the other hand, the nondegeneracy assumption on ω is generic: By perturbing the function K slightly, one can obtain a new function $\widetilde{K}$ for which the corresponding solution becomes nondegenerate.

Remark 1.1. *(1) Observe that if the point ξ^* satisfies*

$$D^2(K_{|\partial\Omega})(\xi^*)(x,x) > 0 \qquad \forall\ x \ne 0,$$

it follows that for each $N \in \mathbb{N}$, $\mathbb{F}_{N,\xi^*}$ goes to $-\infty$ when the point $(y_1, \cdots, y_N)$ goes to the boundary of X; therefore, the function $\mathbb{F}_{N,\xi^*}$ achieves its maximum, which implies the existence of a critical point of $\mathbb{F}_{N,\xi^*}$.

(2) For an example where $\mathbb{F}_{N,\xi^*}$ possesses a nondegenerate critical point, we refer the reader to Proposition 4.2 of [1].

We next consider the case of multiple blow-up points on $\partial\Omega$, for which we obtain the following results.

Theorem 1.2. *Let $n \in \{4, 5, 6\}$, ω be a positive nondegenerate solution of (P_0) and let $\xi_1^*, \cdots, \xi_m^*$ in $\partial\Omega$ be nondegenerate critical points of $K_{|\partial\Omega}$ satisfying the same assumptions as ξ^* (in Theorem 1.1). Then, for each $N_1, \cdots, N_m$ in $\mathbb{N} \setminus \{0\}$, there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$, there is a solution u_ε of (P_ε) satisfying*

$$u_\varepsilon = \omega + \sum_{i=1}^{N_1} P\delta_{a_{1,i,\varepsilon},\lambda_{1,i,\varepsilon}} + \cdots + \sum_{i=1}^{N_m} P\delta_{a_{m,i,\varepsilon},\lambda_{m,i,\varepsilon}} + \phi_\varepsilon, \tag{1.11}$$

with $\|\phi_\varepsilon\| \to 0$ and for each $k \leq m$, the points $a_{k,i,\varepsilon}$ converge to ξ_k^* as ε tends to 0. Moreover, for each $k = 1, \cdots, m$, the points $a_{k,i,\varepsilon}$ and the speeds $\lambda_{k,i,\varepsilon}$ satisfy the properties in (1.7)–(1.10).

Remark 1.2. *Theorems 1.1 and 1.2 are stated for $n \in \{4, 5, 6\}$. Nevertheless, we believe that they remain valid for $n = 3$. The main difficulty lies in improving the estimate of the odd part of the error term. Section 5 deals with the odd component $(x-a)_k$ of the error term for $k \geq 2$ (see Lemmas 5.3–5.6). To extend our results to the case $n = 3$, we would need to study the case $k = 1$, which requires more delicate estimates. We plan to return to this challenging estimates in future work.*

The proof of our results relies on a Lyapunov–Schmidt reduction. Specifically, we seek a solution u_ε of the form given in (1.6) (for Theorem 1.1) and (1.11) (for Theorem 1.2), where the function ϕ lies in an infinite-dimensional space. By deriving a precise expansion of the associated variational functional I_ε (defined in (2.1)) with respect to ϕ, we establish the existence of a function $\overline{\phi}$ that extremizes I_ε in this infinite-dimensional setting. The remaining parameters of the solution are then determined through a detailed analysis of the expansion of the gradient ∇I_ε. This approach, introduced in [8], has become standard and hinges on the accurate expansion of ∇I_ε. The main challenges in the present work are the following.

- Controlling the contribution of the function ω in these expansions;
- Controlling the contribution of the function $\overline{\phi}$;
- Handling the interactions between bubbles whose concentration points are close to one another and ultimately converge to the same boundary point.

It is worth noting that in previous applications of this method, the term $\|\overline{\phi}\|^2$ is typically small compared with the leading-order terms in the expansions. In our setting, however, this is not the case (see Remark 4.1). Nevertheless, we observed that in the integrals where this issue arises, we can refine the analysis by exploiting certain oddness properties. This motivates the study of the odd part of $\overline{\phi}$, as detailed in Section 5.

Finally, we remark that classical blow-up analysis methods often depend on delicate pointwise C^0-estimates and the use of Pohozaev identities. The presence of the function ω significantly complicates such an analysis. In contrast, the approach adopted here circumvents these classical techniques, relying instead on precise expansions of the scalar products of I_ε with carefully chosen test functions adapted to the bubbles under consideration. In addition, compared with the case $\omega = 0$ studied in [9, 14], the presence of a nonzero function ω in the ansatz for the desired solution introduces many additional integrals that must be controlled. In the setting of compact manifolds without boundary or of interior blow-up solutions, the function ω is bounded below by a positive constant. In our situation, however, ω vanishes on $\partial\Omega$ and the concentration points a_i lie close to the boundary. Consequently, for any x near the boundary, $\omega(x) \sim cd(x, \partial\Omega)$. This fact plays a crucial role in our analysis, allowing us to control the contribution of ω in all integrals where it appears (see, for example, Lemma 4.1 below).

Our paper is organized as follows. In Section 2, we introduce some basic tools. Section 3 is devoted to reducing the problem to a finite dimensional problem where we prove the existence of a function $\overline{\phi}$ minimizing the functional I_ε with respect to the infinite dimensional variable. The expansion of the gradient of the variational functional I_ε is given in Section 4. To improve some results presented in Section 4, we will study the odd part of the remainder function $\overline{\phi}$ in Section 5 and we improve the expansion of the gradient of I_ε (with respect to the concentration point a_i) in Section 6. Finally, we prove the theorems in Section 7. We end the paper by some useful results in the Appendix.

2. Basic tools

First, we notice that

$$\langle u, v\rangle := \int_\Omega \nabla u \cdot \nabla v \quad \text{and} \quad \|u\| := \Big(\int_\Omega |\nabla u|^2\Big)^{1/2} \qquad \text{for } \ u, v \in H_0^1(\Omega)$$

are the standard scalar product of $H_0^1(\Omega)$ and its associated norm.

Second, we note that (P_ε) is a variational problem; that is, its solutions are the critical points of the following functional:

$$I_\varepsilon(u) := \frac{1}{2}\|u\|^2 - \frac{1}{p+1-\varepsilon}\int_\Omega K|u|^{p+1-\varepsilon} \qquad \text{for} \quad u \in H_0^1(\Omega). \tag{2.1}$$

Easy computations imply that I_ε is a differentiable functional on $H_0^1(\Omega)$, and we have

$$\langle \nabla I_\varepsilon(u), h\rangle = \langle u, h\rangle - \int_\Omega |u|^{p-1-\varepsilon} uh \qquad \text{for each} \quad u, h \in H_0^1(\Omega). \tag{2.2}$$

While the functional I_ε is standard, a broader methodological perspective on constructing variational principles is provided by the so-called semi-inverse method [26]. This technique offers a systematic way to derive a variational functional directly from the field equations, which may find applications in other physical contexts where no obvious Lagrangian exists.

Now, recall that the projection $P\delta_{a,\lambda}$ is defined in (1.2). Using the maximum principle, it is easy to see that this function satisfies the following.

Proposition 2.1. *[11, 24] For $a \in \Omega$ and a large λ such that $\lambda d_a := \lambda d(a, \partial\Omega)$ is large, the following holds:*

$$P\delta_{a,\lambda} := \delta_{a,\lambda} - \theta_{a,\lambda} = \delta_{a,\lambda} - c_0\frac{H(a,.)}{\lambda^{(n-2)/2}} + f_{a,\lambda}, \tag{2.3}$$

where $H(.,.)$ is the regular part of the Green function $G(.,.)$ with the Dirichlet boundary condition ($H(a,.) := G(a,.) - |a - .|^{2-n}$) and $f_{a,\lambda}$ satisfies

$$\|f_{a,\lambda}\|_\infty \le \frac{c}{\lambda^{(n+2)/2} d_a^n}, \quad \Big\|\lambda\frac{\partial f_{a,\lambda}}{\partial\lambda}\Big\|_\infty \le \frac{c}{\lambda^{(n+2)/2} d_a^n}, \quad \Big\|\frac{1}{\lambda}\frac{\partial f_{a,\lambda}}{\partial a}\Big\|_\infty \le \frac{c}{\lambda^{(n+4)/2} d_a^{n+1}}. \tag{2.4}$$

In addition, we have

$$0 \le P\delta_{a,\lambda} \le \delta_{a,\lambda}, \quad \Big|\lambda \frac{\partial P\delta_{a,\lambda}}{\partial \lambda}\Big| \le cP\delta_{a,\lambda}, \quad \Big|\frac{1}{\lambda}\frac{\partial P\delta_{a,\lambda}}{\partial a}\Big| \le cP\delta_{a,\lambda} \qquad \text{in } \Omega, \tag{2.5}$$

$$\delta_{a,\lambda} \le cP\delta_{a,\lambda} \qquad \text{in } \ B(a, d_a/2). \tag{2.6}$$

For $a \in \Omega$ and $\lambda > 0$ such that $\lambda d_a := \lambda d(a, \partial\Omega)$ is very large, we write

$$E_{a,\lambda} := \ \mathrm{span}\{P\delta_{a,\lambda}, \partial P\delta_{a,\lambda}/\partial\lambda, \partial P\delta_{a,\lambda}/\partial a_1, \cdots, \partial P\delta_{a,\lambda}/\partial a_n\}. \tag{2.7}$$

Now, let $N \in \mathbb{N}$, m_1 be a small positive constant and let m_2 be a large positive constant. We define the set $\mathbb{A}_{N,\varepsilon;\xi^*}$ as the set of the elements $(\alpha, a, \lambda) \in (0,\infty)^N \times \Omega^N \times (0,\infty)^N$ satisfying

$$|\alpha_i^{4/(n-2)} K(a_i) - 1| \le m_2\varepsilon |\ln\varepsilon| \quad \forall\, i \in \{1, \cdots, N\}, \tag{2.8}$$

$$m_1 \le \lambda_i \varepsilon^{(n-1)/(n-2)} \le m_2 \quad \forall\ i \in \{1, \cdots, N\}, \tag{2.9}$$

$$m_1 \le |a_i - a_j| \varepsilon^{-(n+1)/(n+2)} \le m_2 \quad \forall\ i \ne j \in \{1, \cdots, N\}, \tag{2.10}$$

$$m_1\varepsilon \le d_i := d(a_i, \partial\Omega) \le m_2\varepsilon \quad \forall\ i \in \{1, \cdots, N\}, \tag{2.11}$$

$$m_1 \le |a_i - \xi^*| \varepsilon^{-(n+1)/(n+2)} \le m_2 \quad \forall\ i \in \{1, \cdots, N\}. \tag{2.12}$$

Notice that $\mathbb{A}_{N,\varepsilon,\xi^*} = \widetilde{\mathcal{V}}(N)$, which is defined in Eq (4.1) of [9]. Furthermore, for $i \ne j$, using (2.9) and (2.10), easy computation implies that

$$\varepsilon_{ij} := \Big(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j|a_i - a_j|^2\Big)^{(2-n)/2} \le \frac{1}{\lambda_i^{(n-2)/2}\lambda_j^{(n-2)/2}|a_i - a_j|^{n-2}} = O\Big(\varepsilon^{2n/(n+2)}\Big). \tag{2.13}$$

In addition, for each $j \ne i$, it is easy to see that $d_i \le c\,\varepsilon \le c\,\varepsilon^{1/(n+2)}\,|a_i - a_j|$; therefore, we have the following (see Lemma 4.1 of [9] for the first inequality of the second assertion):

$$\delta_{a_j,\lambda_j} \le \frac{c\lambda_j^{(n-2)/2}}{(\lambda_j|a_j - a_i|)^{n-2}} \le c\varepsilon^{\frac{5-n}{2} - \frac{4}{n+2}} \quad \text{ in } \ B(a_i, d_i/2), \tag{2.14}$$

$$P\delta_{a_j,\lambda_j} \le c\,\varepsilon^{2/(n+2)}\delta_{a_j,\lambda_j} \le c\varepsilon^{-\gamma_1} \quad \text{ in } \ B(a_i, d_i/2) \ \text{ with } \gamma_1 := \frac{n-5}{2} + \frac{2}{n+2}, \tag{2.15}$$

$$P\delta_{a_i,\lambda_i} \ge c\delta_{a_i,\lambda_i} \ge \frac{c\lambda_i^{(n-2)/2}}{(\lambda_i d_i)^{n-2}} \ge c\varepsilon^{-\gamma_2} \quad \text{ in } \ B(a_i, d_i/2) \ \text{ with } \gamma_2 := \frac{n-3}{2}. \tag{2.16}$$

Hence, it is easy to see that

$$\delta_{a_j,\lambda_j} = o(P\delta_{a_i,\lambda_i}) \quad \text{ in } \ B(a_i, d_i/2) \quad \forall\, j \ne i. \tag{2.17}$$

Moreover, since $\varepsilon \ln \lambda_i$ is small and Ω is bounded, it follows that $\varepsilon \ln(1 + \lambda_i^2|x - a_i|^2)$ is small uniformly in Ω; therefore, by using the expansion $e^t = 1 + t + O(t^2)$ near zero, we deduce that

$$\delta_{a_i,\lambda_i}^{-\varepsilon} = c_0^{-\varepsilon}\lambda_i^{-\varepsilon\frac{n-2}{2}}\Big(1 + \frac{n-2}{2}\,\varepsilon\,\ln(1 + \lambda_i^2|x - a_i|^2)\Big) + O\Big(\varepsilon^2 \ln^2(1 + \lambda_i^2|x - a_i|^2)\Big) \tag{2.18}$$

$$= 1 + o(1). \tag{2.19}$$

In the sequel, we use σ to denote the following value:

$$\sigma := 2 + \frac{1}{n-2} - \frac{1}{n+2} = 2 + \frac{4}{n^2-4}. \tag{2.20}$$

Furthermore, σ_0 denotes a positive real number which can be different from line to line.

Our aim is to construct some critical points of the functional I_ε having the form

$$u = \omega + \sum_{i=1}^{N} \alpha_i P\delta_{a_i,\lambda_i} + \phi \quad \text{with} \quad (\alpha, a, \lambda) \in \mathbb{A}_{N,\varepsilon,\xi^*} \text{ and } \phi \in \mathcal{F}_{a,\lambda} := \Big(\cup_{i=1}^{N} E_{a_i,\lambda_i} \Big)^{\perp}, \tag{2.21}$$

where E_{a_i,λ_i} is defined in (2.7).

In the sequel, to simplify the presentation, we write

$$\widetilde{u} := \sum_{i=1}^{N} \alpha_i P\delta_{a_i,\lambda_i}. \tag{2.22}$$

Using (2.5), (2.9), (2.15), and (2.17), it is easy to see that

$$\widetilde{u} = \alpha_i P\delta_{a_i,\lambda_i} + O(\varepsilon^{-\gamma_1}) = \alpha_i P\delta_{a_i,\lambda_i}(1 + o(1)) \quad \text{ in } B(a_i, d_i/2), \tag{2.23}$$

$$(\widetilde{u})^{-\varepsilon} = 1 + O(\varepsilon|\ln\varepsilon|) \quad \text{ in } \cup B(a_j, d_j/2). \tag{2.24}$$

Concerning the function ω, we remark that since $\omega = 0$ on $\partial\Omega$, and ω is bounded in the C^2-norm, using (2.11), we deduce that

$$0 < \omega \leq c\, d_j \leq c\,\varepsilon \qquad \text{in } \Omega_b := \{x \in \Omega : d(x, \partial\Omega) \leq M\varepsilon\}, \tag{2.25}$$

$$C \geq \omega \geq c\varepsilon \qquad \text{in } \Omega \setminus \Omega_b, \tag{2.26}$$

where M is a large positive constant. Thus, since $p - 1 \geq 1$, we deduce that

$$\omega^{p-1-\varepsilon} = \omega^{p-1} + O(\varepsilon + \varepsilon^{p-1-\varepsilon}) = \omega^{p-1} + O(\varepsilon) \qquad \text{in} \quad \Omega. \tag{2.27}$$

3. Finite dimensional reduction

Observe that, the function u defined in (2.21) is parametrized by the element $(\alpha, \lambda, a) \in \mathbb{A}_{N,\varepsilon,\xi^*}$ and a function $\phi \in \mathcal{F}_{a,\lambda}$. Let us introduce

$$\mathcal{I}_\varepsilon : \{(\alpha, \lambda, a, \phi) \in \mathbb{A}_{N,\varepsilon,\xi^*} \times H_0^1(\Omega) : \phi \in \mathcal{F}_{a,\lambda}\} \to \mathbb{R}; \quad \mathcal{I}_\varepsilon(\alpha, \lambda, a, \phi) := I_\varepsilon\Big(\omega + \sum_{i=1}^{N} \alpha_i P\delta_{a_i,\lambda_i} + \phi\Big).$$

To find a critical point of I_ε, we need to find a critical point of $\mathcal{I}_\varepsilon$. We start by extremizing the function $\mathcal{I}_\varepsilon$ with respect to the variable ϕ.

Proposition 3.1. *Let $n \in \{4, 5, 6\}$ and $(\alpha, a, \lambda) \in \mathbb{A}_{N,\varepsilon,\xi^*}$, there exists a unique $\overline{\phi} := \overline{\phi}_{\varepsilon,\alpha,a,\lambda}$ such that*

$$\overline{\phi} \in \mathcal{F}_{a,\lambda}, \tag{3.1}$$

$$\langle \nabla I_\varepsilon(\omega + \widetilde{u} + \overline{\phi}), h \rangle = 0 \quad \forall\, h \in \mathcal{F}_{a,\lambda}, \tag{3.2}$$

$$\|\overline{\phi}\| \leq c\varepsilon|\ln\varepsilon|, \tag{3.3}$$

where $\widetilde{u}$ is introduced in (2.22).

Proof. First, we need to expand the functional I_ε with respect to $\phi \in \mathcal{F}_{a,\lambda}$. Indeed, we have

$$I_\varepsilon(\omega+\widetilde{u}+\phi) = \frac{1}{2}\|\omega+\widetilde{u}+\phi\|^2 - \frac{1}{p+1-\varepsilon}\int K|\omega+\widetilde{u}+\phi|^{p+1-\varepsilon}. \tag{3.4}$$

Observe that since $\phi \in \mathcal{F}_{a,\lambda}$, it holds that

$$\|\omega+\widetilde{u}+\phi\|^2 = \|\omega+\widetilde{u}\|^2 + 2\langle\omega,\phi\rangle + \|\phi\|^2.$$

Concerning the second part of (3.4), observe that for $\gamma > 2$, $s,t \in \mathbb{R}$, we have

$$|s+t|^\gamma = |s|^\gamma + \gamma|s|^{\gamma-2}st + \frac{1}{2}\gamma(\gamma-1)|s|^{\gamma-2}t^2 + \begin{cases} O(|s|^{\gamma-3}|t|^3) & \text{if } |t|<|s|, \\ O(|t|^\gamma) & \text{if } |s|\le|t|. \end{cases}$$

Thus, the integral in (3.4) becomes

$$\begin{aligned}\int K|\omega+\widetilde{u}+\phi|^{p+1-\varepsilon} = &\int K(\omega+\widetilde{u})^{p+1-\varepsilon} + (p+1-\varepsilon)\int K(\omega+\widetilde{u})^{p-\varepsilon}\phi \\ &+ \frac{1}{2}(p+1-\varepsilon)(p-\varepsilon)\int K(\omega+\widetilde{u})^{p-1-\varepsilon}\phi^2 \\ &+ O\Big(\int_{[|\phi|\le\omega+\widetilde{u}]}(\omega+\widetilde{u})^{p-2}\phi^{3-\varepsilon} + \int|\phi|^{p+1-\varepsilon}\Big).\end{aligned} \tag{3.5}$$

Notice that

$$\int_{[|\phi|\le\omega+\widetilde{u}]}(\omega+\widetilde{u})^{p-2}\phi^{3-\varepsilon} + \int|\phi|^{p+1-\varepsilon} = o(\|\phi\|^2).$$

Therefore, for each $\phi \in \mathcal{F}_{a,\lambda}$, we obtain

$$I_\varepsilon(\omega+\widetilde{u}+\phi) = I_\varepsilon(\omega+\widetilde{u}) + L_\varepsilon(\phi) + \frac{1}{2}Q_\varepsilon(\phi) + o(\|\phi\|^2) \tag{3.6}$$

$$\text{where } Q_\varepsilon(\phi) := \|\phi\|^2 - (p-\varepsilon)\int K(\omega+\widetilde{u})^{p-1-\varepsilon}\phi^2 \tag{3.7}$$

$$L_\varepsilon(\phi) := \langle\omega,\phi\rangle - \int K(\omega+\widetilde{u})^{p-\varepsilon}\phi. \tag{3.8}$$

We claim that Q_ε is a nondegenerate quadratic form on $\mathcal{F}_{a,\lambda}$. In fact, using the fact that

$$|s+t|^\gamma = |s|^\gamma + |t|^\gamma + O(|s|^{\gamma-1}|t| + |s||t|^{\gamma-1}) \qquad \text{for} \quad \gamma>0,\ s,t\in\mathbb{R},$$

and the fact that $\alpha_i^{4/(n-2)}K(a_i) = 1 + o(1)$, we deduce that

$$\begin{aligned}\int K(\omega+\widetilde{u})^{p-1-\varepsilon}\phi^2 &= \int K\omega^{p-1-\varepsilon}\phi^2 + \sum_{i=1}^N\int\delta_{a_i,\lambda_i}^{p-1-\varepsilon}\phi^2 + o(\|\phi\|^2) \\ &= \int K\omega^{p-1}\phi^2 + \sum_{i=1}^N\int\delta_{a_i,\lambda_i}^{p-1}\phi^2 + o(\|\phi\|^2),\end{aligned}$$

where we have used (2.19) and (2.27). Thus, we get

$$Q_\varepsilon(\phi) = Q(\phi) + o(\|\phi\|^2) \qquad \text{where} \quad Q(\phi) := \|\phi\|^2 - p\int K\omega^{p-1}\phi^2 - p\sum_{i=1}^{N}\int \delta_{a_i,\lambda_i}^{p-1}\phi^2.$$

However, by arguing as in the proof of Proposition 1 of [4], it follows that the quadratic form Q is nondegenerate on $\mathcal{F}_{a,\lambda}$. Hence, our claim follows.

Now, since Q_ε is a nondegenerate quadratic form and L_ε is a linear form, using the implicit function theorem near 0, we deduce the existence of a unique function $\overline{\phi}$ satisfying (3.1), and (3.2), and we have $\|\overline{\phi}\| \le c\|L_\varepsilon\|$. Thus, to prove (3.3), we need to estimate $\|L_\varepsilon\|$.

Using the fact that $P\delta_{a,\lambda} \le c\delta_{a,\lambda}$ and the fact that for each $\gamma > 0$, $s, t \in \mathbb{R}$, we have

$$|s+t|^\gamma(s+t) = |s|^\gamma s + |t|^\gamma t + \begin{cases} O(|s|^\gamma|t|) & \text{if } |t| \le |s|, \\ O(|s||t|^\gamma) & \text{if } |s| \le |t|, \end{cases} \tag{3.9}$$

we derive the following (since ω is bounded):

$$\int_\Omega K(\omega+\widetilde{u})^{p-\varepsilon}\phi = \int_\Omega K\omega^{p-\varepsilon}\phi + \int_\Omega K(\widetilde{u})^{p-\varepsilon}\phi + \sum O\Big(\int_\Omega \delta_{a_i,\lambda_i}^{p-\varepsilon-1}|\phi| + \int_\Omega \delta_{a_i,\lambda_i}|\phi|\Big). \tag{3.10}$$

To complete the estimate of (3.10), using the fact that $\delta_{a_i,\lambda_i}^{-\varepsilon} \le c$ and (1.1), it holds that

$$\int_\Omega \delta_{a_i,\lambda_i}^{p-\varepsilon-1}|\phi| + \int_\Omega \delta_{a_i,\lambda_i}|\phi| \le c\|\phi\|\Big[\Big(\int_\Omega \delta_{a_i,\lambda_i}^{\frac{8n}{n^2-4}}\Big)^{\frac{n+2}{2n}} + \Big(\int_\Omega \delta_{a_i,\lambda_i}^{\frac{2n}{n+2}}\Big)^{\frac{n+2}{2n}}\Big] \le c\chi_1(\lambda_i)\|\phi\|, \tag{3.11}$$

where χ_1 is defined in (1.2). Concerning the first integral in (3.10), using (2.27), we get

$$\int_\Omega K\omega^{p-\varepsilon}\phi = \int_\Omega K\omega^p\phi + O\Big(\varepsilon\int_\Omega|\phi|\Big) = \int_\Omega(-\Delta\omega)\phi + O\Big(\varepsilon\|\phi\|\Big) = \langle\omega,\phi\rangle + O\Big(\varepsilon\|\phi\|\Big). \tag{3.12}$$

Regarding the second integral of (3.10), from Lemmas 2.1 and A.1 of [10] (these results are proved for $p+\varepsilon$ instead of $p-\varepsilon$; however, since, for each i, $\varepsilon\ln\lambda_i$ is small, the proof is exactly the same), we deduce that

$$\Big|\int_\Omega K(\widetilde{u})^{p-\varepsilon}\phi\Big| \le c\|\phi\|\sum_i\Big(\varepsilon + \frac{1}{\lambda_i} + \frac{\ln(\lambda_i d_i)}{(\lambda_i d_i)^{n-2}} + \sum_{j\ne i}\varepsilon_{ij}\ln(\varepsilon_{ij}^{-1})^{(n-2)/n}\Big).$$

Combining the previous estimate and (2.9), (2.11), and (2.13), we get

$$\Big|\int_\Omega K(\widetilde{u})^{p-\varepsilon}\phi\Big| \le c\varepsilon|\ln\varepsilon|\|\phi\|. \tag{3.13}$$

Combining (3.11), (3.12), and (3.13), Eq (3.10) becomes

$$\int_\Omega K(\omega+\widetilde{u})^{p-\varepsilon}\phi = \langle\omega,\phi\rangle + O\Big(\varepsilon|\ln\varepsilon|\|\phi\|\Big). \tag{3.14}$$

Therefore, the linear form is estimated to be

$$|L_\varepsilon(\phi)| \le c\,\varepsilon|\ln\varepsilon|\|\phi\|,$$

which implies that $\|L_\varepsilon\| \le c\,\varepsilon|\ln\varepsilon|$. This achieves the proof of the proposition.

Remark 3.1. *From* (3.2)*, using the multiplier Lagrange theorem, we deduce that some constants A_j, B_j, and $C_{j,\ell}$ exist such that*

$$\begin{aligned}-\Delta\overline{\phi} =&K|\omega+\widetilde{u}+\overline{\phi}|^{p-\varepsilon-1}(\omega+\widetilde{u}+\overline{\phi})-(-\Delta)(\omega+\widetilde{u})\\ &+(-\Delta)\Big[\sum_{j=1}^{N}\Big(A_jP\delta_j+B_j\lambda_j\frac{\partial P\delta_j}{\partial\lambda_j}+\sum_{\ell=1}^{n}C_{j,\ell}\frac{1}{\lambda_j}\frac{\partial P\delta_j}{\partial a_{j,\ell}}\Big)\Big].\end{aligned}\tag{3.15}$$

In fact, for $(\alpha,a,\lambda)\in\mathbb{A}_{N,\varepsilon,\xi^*}$, from Equation (3.2), we deduce that $\overline{\phi}$ is a critical point of the functional

$$\psi:\mathcal{F}_{a,\lambda}\to\mathbb{R}\quad,\quad \phi\mapsto\psi(\phi):=I_\varepsilon(\omega+\widetilde{u}+\phi),$$

where $\widetilde{u}$ is introduced in (2.22). Recall that the set $\mathcal{F}_{a,\lambda}$ is defined by some orthogonality constraints which are as follows, for $j\in\{1,\cdots,N\}$ and $\ell\in\{1,\cdots,n\}$:

$$\varphi_{1,j}(\phi):=\langle v,P\delta_{a_j,\lambda_j}\rangle;\quad \varphi_{2,j}(\phi):=\langle v,\lambda_j\frac{\partial P\delta_{a_j,\lambda_j}}{\partial\lambda_j}\rangle;\quad \varphi_{3,j,\ell}(\phi):=\langle v,\frac{1}{\lambda_j}\frac{\partial P\delta_{a_j,\lambda_j}}{\partial a_{j,\ell}}\rangle.$$

Therefore, using the multiplier Lagrange theorem, we deduce the existence of some constants $(A,B,C)\in\mathbb{R}^N\times\mathbb{R}^N\times(\mathbb{R}^n)^N$ such that

$$\nabla\psi(\overline{\phi})=\sum_{j=1}^{N}\Big(A_j\nabla\varphi_{1,j}(\overline{\phi})+B_j\nabla\varphi_{2,j}(\overline{\phi})+\sum_{\ell=1}^{n}C_{j,\ell}\nabla\varphi_{3,j,\ell}(\overline{\phi})\Big).$$

Thus, using (2.2), we deduce that for each $h\in H_0^1(\Omega)$, we have

$$\begin{aligned}&\int_\Omega\nabla(\omega+\widetilde{u}+\overline{\phi})\cdot\nabla h-\int_\Omega K|\omega+\widetilde{u}+\overline{\phi}|^{p-\varepsilon-1}(\omega+\widetilde{u}+\overline{\phi})h\\ &=\sum_{j=1}^{N}\Big(A_j\int_\Omega\nabla P\delta_{a_j,\lambda_j}\nabla h+B_j\int_\Omega\nabla[\lambda_j\frac{\partial P\delta_{a_j,\lambda_j}}{\partial\lambda_j}]\cdot\nabla h+\sum_{\ell=1}^{n}C_{j,\ell}\int_\Omega\nabla[\frac{1}{\lambda_j}\frac{\partial P\delta_{a_j,\lambda_j}}{\partial a_{j,\ell}}]\cdot\nabla h\Big).\end{aligned}$$

This completes the proof of (3.15).

4. Expansion of the gradient

The aim of this section is to perform delicate asymptotic estimates of the gradient of the functional I_ε given by (2.2). To this end, we compute the scalar product of the gradient of I_ε with certain functions. We start by providing the following lemma.

Lemma 4.1. *Let $i\in\{1,\cdots,N\}$ and $\varphi_i\in\{P\delta_{a_i,\lambda_i},(\lambda_i\partial P\delta_{a_i,\lambda_i}/\partial\lambda_i),\lambda_i^{-1}(\partial P\delta_{a_i,\lambda_i}/\partial a_i)\}$. It holds that*

$$\langle\nabla I_\varepsilon(\omega+\widetilde{u}+\overline{\phi}),\varphi_i\rangle=\langle\nabla I_\varepsilon(\widetilde{u}+\overline{\phi}),\varphi_i\rangle+O\Big(\varepsilon^{\frac{5}{2}}|\ln\varepsilon|\ \text{ if } n=4,\ \ \varepsilon^{2+\frac{2}{n-2}}\ \text{ if } n=5,6\Big).$$

Proof. Observe that, by using (3.9), we have

$$\int_\Omega K|\omega+\widetilde{u}+\overline{\phi}|^{p-\varepsilon-1}(\omega+\widetilde{u}+\overline{\phi})\varphi_i=\int_\Omega K\omega^{p-\varepsilon}\varphi_i+\int_\Omega K|\widetilde{u}+\overline{\phi}|^{p-\varepsilon-1}(\widetilde{u}+\overline{\phi})\varphi_i$$

$$+ O\Big(\int_{[|\widetilde{u}+\overline{\phi}| \le \omega]} \omega^{p-1-\varepsilon} |\widetilde{u} + \overline{\phi}||\varphi_i| + \int_{[\omega \le |\widetilde{u}+\overline{\phi}|]} |\widetilde{u} + \overline{\phi}|^{p-\varepsilon-1} \omega |\varphi_i| \Big). \tag{4.1}$$

We need to estimate the integrals involved in (4.1). Regarding the last integral in (4.1), we have

$$\int_{[\omega \le |\widetilde{u}+\overline{\phi}|]} |\widetilde{u} + \overline{\phi}|^{p-\varepsilon-1} \omega |\varphi_i| \le c \int_\Omega |\overline{\phi}|^{p-1} \delta_{a_i,\lambda_i} + \sum \int_\Omega \delta^p_{a_j,\lambda_j} \omega.$$

However,

$$\begin{aligned} \int_\Omega |\overline{\phi}|^{p-1} \delta_{a_i,\lambda_i} &\le c \|\overline{\phi}\|^{p-1} \Big(\int_\Omega \delta^{n/(n-2)}_{a_i,\lambda_i} \Big)^{(n-2)/n} \le c(\varepsilon |\ln \varepsilon|)^{p-1} \frac{(\ln \lambda_i)^{(n-2)/n}}{\lambda^{(n-2)/2}} \le c\, \varepsilon^{\frac{n-1}{2} + \frac{2}{n-2}}, \\ \int_\Omega \delta^p_{a_j,\lambda_j} \omega &\le c\varepsilon \int_{B_j} \delta^p_{a_j,\lambda_j} + c \int_{\Omega \setminus B_j} \delta^p_{a_j,\lambda_j} \le \frac{c\, \varepsilon}{\lambda_j^{(n-2)/2}} + \frac{c}{\lambda_j^{(n-2)/2} (\lambda_j d_j)^2} \le c\, \varepsilon^{\frac{n-1}{2} + \frac{2}{n-2}}, \end{aligned} \tag{4.2}$$

where we have used (2.25) in (4.2). Therefore, the last integral in (4.1) becomes

$$\int_{[\omega \le |\widetilde{u}+\overline{\phi}|]} |\widetilde{u} + \overline{\phi}|^{p-\varepsilon-1} \omega |\varphi_i| \le c\, \varepsilon^{\frac{n-1}{2} + \frac{2}{n-2}}. \tag{4.3}$$

In the same way, we have

$$\int_{[|\widetilde{u}+\overline{\phi}| \le \omega]} \omega^{p-1-\varepsilon} |\widetilde{u} + \overline{\phi}||\varphi_i| \le c \sum \int_\Omega \delta^2_{a_j,\lambda_j} + \int_\Omega |\overline{\phi}| \delta_{a_i,\lambda_i}.$$

However

$$\int_\Omega \delta^2_{a_j,\lambda_j} \le c \begin{cases} \frac{\ln \lambda_j}{\lambda_j^2} & \text{if } n = 4 \\ \frac{1}{\lambda_j^2} & \text{if } n \ge 5 \end{cases} \Bigg\} \le c \begin{cases} \varepsilon^3 |\ln \varepsilon| & \text{if } n = 4, \\ \varepsilon^{2 + \frac{2}{n-2}} & \text{if } n \ge 5, \end{cases} \tag{4.4}$$

and using (1.1), it holds that

$$\int_\Omega |\overline{\phi}| \delta_{a_i,\lambda_i} \le c \|\overline{\phi}\| \Big(\int_\Omega \delta^{\frac{2n}{n+2}}_{a_i,\lambda_i} \Big)^{\frac{n+2}{2n}} \le c \chi_1(\lambda_i) \|\overline{\phi}\| \le c \begin{cases} \varepsilon^{\frac{n+1}{2}} |\ln \varepsilon| & \text{if } n = 4, 5, \\ \varepsilon^{7/2} |\ln \varepsilon|^{5/3} & \text{if } n = 6. \end{cases}$$

Thus, the third integral on the right hand-side of (4.1) becomes

$$\int_{[|\widetilde{u}+\overline{\phi}| \le \omega]} \omega^{p-1-\varepsilon} |\widetilde{u} + \overline{\phi}||\varphi_i| \le c \begin{cases} \varepsilon^{5/2} |\ln \varepsilon| & \text{if } n = 4, \\ \varepsilon^{2 + \frac{2}{n-2}} & \text{if } n = 5, 6. \end{cases} \tag{4.5}$$

Concerning the first integral on the right-hand side of (4.1), using (2.27), we have

$$\int_\Omega K \omega^{p-\varepsilon} \varphi_i = \int_\Omega K \omega^p \varphi_i + O\Big(\varepsilon \int_\Omega \delta_{a_i,\lambda_i} \Big).$$

On the other hand,

$$\varepsilon \int_\Omega \delta_{a_i,\lambda_i} \le c\, \varepsilon \int_\Omega \delta_{a_i,\lambda_i} \le \frac{c\, \varepsilon}{\lambda_i^{(n-2)/2}} \le c\, \varepsilon^{(n+1)/2},$$

$$\int_\Omega K\omega^p\varphi_i = \int_\Omega(-\Delta\omega)\varphi_i = \int_\Omega \omega(-\Delta\varphi_i) = O\Big(\int_\Omega \omega\delta^p_{a_i,\lambda_i}\Big) = O(\varepsilon^{\frac{n-1}{2}+\frac{2}{n-2}}), \tag{4.6}$$

(by using (4.2)). Therefore, the first integral on the right hand-side of (4.1) becomes

$$\int_\Omega K\omega^{p-\varepsilon}\varphi_i = O\Big(\varepsilon^{\frac{n-1}{2}+\frac{2}{n-2}}\Big). \tag{4.7}$$

Combining, (4.3), (4.5), and (4.7), (4.1) becomes

$$\begin{aligned}\int_\Omega K|\omega+\widetilde{u}+\overline{\phi}|^{p-\varepsilon-1}(\omega+\widetilde{u}+\overline{\phi})\varphi_i &= \int_\Omega K|\widetilde{u}+\overline{\phi}|^{p-\varepsilon-1}(\widetilde{u}+\overline{\phi})\varphi_i \\ &\quad + O\Big(\varepsilon^{5/2}|\ln\varepsilon| \text{ if } n=4, \quad \varepsilon^{2+\frac{2}{n-2}} \text{ if } n=5,6\Big).\end{aligned} \tag{4.8}$$

Finally, using (2.2), (4.6), and (4.8), we obtain

$$\begin{aligned}&\langle\nabla I_\varepsilon(\omega+\widetilde{u}+\overline{\phi}),\varphi_i\rangle \\ &= \langle\omega,\varphi_i\rangle + \langle\widetilde{u}+\overline{\phi},\varphi_i\rangle - \int_\Omega K|\omega+\widetilde{u}+\overline{\phi}|^{p-\varepsilon-1}(\omega+\widetilde{u}+\overline{\phi})\varphi_i \\ &= \langle\widetilde{u}+\overline{\phi},\varphi_i\rangle - \int_\Omega K|\widetilde{u}+\overline{\phi}|^{p-\varepsilon-1}(\widetilde{u}+\overline{\phi})\varphi_i + O\Big(\varepsilon^{\frac{5}{2}}|\ln\varepsilon| \text{ if } n=4, \ \varepsilon^{2+\frac{2}{n-2}} \text{ if } n=5,6\Big) \\ &= \langle\nabla I_\varepsilon(\widetilde{u}+\overline{\phi}),\varphi_i\rangle + O\Big(\varepsilon^{\frac{5}{2}}|\ln\varepsilon| \text{ if } n=4, \ \varepsilon^{2+\frac{2}{n-2}} \text{ if } n=5,6\Big).\end{aligned}$$

This completes the proof of the lemma.

Proposition 4.1. *Let $i\in\{1,\cdots,N\}$, it holds that*

$$\langle\nabla I_\varepsilon(\omega+\widetilde{u}+\overline{\phi}), P\delta_{a_i,\lambda_i}\rangle = \alpha_i S_n(1-\alpha_i^{4/(n-2)}K(a_i)) + O(\varepsilon|\ln\varepsilon|),$$

where S_n is defined in (1.4).

Proof. Take $\varphi_i = P\delta_{a_i,\lambda_i}$ in Lemma 4.1. The proof follows from Lemma 4.1, (3.3), and Proposition 2.4 of [16] by using the behavior of the parameters λ_j, d_j, and ε_{jk} given by (2.9), (2.11), and (2.13).

Proposition 4.2. *Let $i\in\{1,\cdots,N\}$, it holds that*

$$\begin{aligned}\langle\nabla I_\varepsilon(\omega+\widetilde{u}+\overline{\phi}), \lambda_i\frac{\partial P\delta_{a_i,\lambda_i}}{\partial\lambda_i}\rangle = -\frac{n-2}{2}\alpha_i\left(c_1\frac{H(a_i,a_i)}{\lambda_i^{n-2}} - \frac{n-2}{2n}S_n\varepsilon\right) \\ + O(\varepsilon^{\inf(1+n/(n+2);1+2/(n-2))}),\end{aligned}$$

where S_n is defined in (1.4) and

$$c_1 := c_0^{2n/(n-2)}\int_{\mathbb{R}^n}\frac{dx}{(1+|x|^2)^{(n+2)/2}}. \tag{4.9}$$

Proof. Take $\varphi_i = \lambda_i\partial P\delta_{a_i,\lambda_i}/\partial\lambda_i$ in Lemma 4.1. The proof follows from Lemma 4.1, (3.3), and Proposition 2.5 of [16] by using the behavior of the parameters λ_j, d_j, and ε_{jk} given by (2.9), (2.11), and (2.13).

Proposition 4.3. *Let $i \in \{1, \cdots, N\}$, it holds that*

$$\langle \nabla I_\varepsilon(\omega + \widetilde{u} + \overline{\phi}), \frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_i}\rangle = \frac{1}{2}\alpha_i \frac{c_1}{\lambda_i^{n-1}}\frac{\partial H}{\partial a_i}(a_i, a_i) - \alpha_i^p \frac{c_3}{\lambda_i}\nabla K(a_i)$$
$$- c_1 \sum_{j=1, j\neq i}^{N} \alpha_j \frac{1}{\lambda_i}\frac{1}{(\lambda_i\lambda_j)^{(n-2)/2}}\frac{\partial G(a_i, a_j)}{\partial a_i} + O(\varepsilon^{1+\frac{2}{n-2}}), \tag{4.10}$$

where c_1 is defined in (4.9) and c_3 is defined in (1.4).
Proof. Take $\varphi_i = \lambda_i^{-1}\partial P\delta_{a_i,\lambda_i}/\partial a_i$ in Lemma 4.1. The proof follows from Lemma 4.1, (3.3), and Proposition 2.6 of [16] by using the behavior of the parameters λ_j, d_j, and ε_{jk} given by (2.9), (2.11), and (2.13).

Remark 4.1. *Notice that a_i is close to the boundary $\partial\Omega$. Let $\overline{a}_i$ be the projection of a_i on the boundary and let ν_i be the outnormal vector to the boundary at the point $\overline{a}_i$.*

- Multiplying (4.10) by the vector ν_i, we observe that the principal terms in (4.10) become of the order $\varepsilon^{1+1/(n-2)}$. Hence, the remainder term in this equation is small with respect to these principal terms.
- However, if we multiply (4.10) by a tangential vector, we observe that the principal terms in (4.10) become of the order ε^σ, with $\sigma := 2 + \frac{4}{n^2-4}$. Hence, we need to improve Proposition 4.3 in this case.
- We observe that in the expansions of the gradient, particularly in Proposition 4.3, the remainder term $\|\overline{\phi}\|^2$ appears and is estimated by $\|\overline{\phi}\|^2 \le c\varepsilon^2\ln^2(\varepsilon)$. Hence, we need to improve this estimate. We believe, however, that the bound on $\|\overline{\phi}\|$ itself is optimal. Nevertheless, the main part of the integrals that we need to refine involves the product of $\overline{\phi}$ with the function $\delta^\gamma_{a_i,\lambda_i}(\partial\delta_{a_i,\lambda_i}/\partial a_{i,k})$ (where $a_{i,k}$ is the k^{th}-component of a_i), and this function is odd with respect to $(x - a_i)_k$. Consequently, the even part of $\overline{\phi}$ does not contribute to these integrals. Moreover, near a_i, the function $\overline{\phi}$ is close to an even function. Therefore, the norm of its odd component is small compared with $\|\overline{\phi}\|$. This improvement will be very useful when refining the computations to obtain a better version of Proposition 4.3.

5. The odd part of the error term

As stated in Remark 4.1, we need to improve the remainder terms in Proposition 4.3. We notice that $\|\overline{\phi}\|^2$ appears in this remainder term, and we cannot improve the estimate of this norm. Nevertheless, we see that $\partial\delta_{a_i,\lambda_i}/\partial a_{i,k}$ is an odd function with respect to the k^{th}-component of the variable $(x - a_i)$. Therefore, the contribution of the even part of $\overline{\phi}$, in some integrals, will be 0. Hence, we need to improve the estimate of the odd part of $\overline{\phi}$. This is the aim of this section. First, we start by estimating the constants A_i, B_i, and $C_{i,j}$ introduced in Remark 3.1.

Lemma 5.1. *Let $(\alpha, a, \lambda) \in \mathbb{A}_{N,\varepsilon,\xi^*}$. The constants A_j, B_j, and $C_{j,\ell}$, introduced in Remark 3.1, satisfy the following: for each $k \in \{1, \cdots, N\}$ and for each $j \in \{1, \cdots, n\}$, we have*

$$A_k = O(\varepsilon|\ln\varepsilon|), \qquad B_k = O(\varepsilon), \qquad C_{k,j} = O(\varepsilon^{1+\frac{1}{n-2}}). \tag{5.1}$$

Proof. Multiplying (3.15) by $P\delta_{a_k,\lambda_k}$, $\lambda_k\partial P\delta_{a_k,\lambda_k}/\partial\lambda_k$, and $\lambda_k^{-1}\partial P\delta_{a_k,\lambda_k}/\partial a_{k,j}$, respectively, we obtain the following system:

$$\begin{aligned}
cA_k + O\Big(\varepsilon[|A| + |B| + |C|]\Big) &= \langle \nabla I_\varepsilon(\omega + \widetilde{u} + \overline{\phi}), P\delta_{a_k,\lambda_k}\rangle = O(\varepsilon|\ln\varepsilon|),\\
cB_k + O\Big(\varepsilon[|A| + |B| + |C|]\Big) &= \langle \nabla I_\varepsilon(\omega + \widetilde{u} + \overline{\phi}), \lambda_k\partial P\delta_{a_k,\lambda_k}/\partial\lambda_k\rangle = O(\varepsilon),\\
cC_{k,j} + O\Big(\varepsilon[|A| + |B| + |C|]\Big) &= \langle \nabla I_\varepsilon(\omega + \widetilde{u} + \overline{\phi}), \lambda_k^{-1}\partial P\delta_{a_k,\lambda_k}/\partial a_{k,j}\rangle = O(\varepsilon^{1+\frac{1}{n-2}}),
\end{aligned}$$

where c is a positive constant which differs from line to line.

Taking the sum of the previous equations, we derive

$$|A| + |B| + |C| = O(\varepsilon|\ln\varepsilon|).$$

Putting this information into the previous system, the result follows.

Let $(\alpha, a, \lambda) \in \mathbb{A}_{N,\varepsilon,\xi^*}$ and $\overline{\phi}$ be the function introduced in Proposition 3.1. Recall that for each $j \neq i$, by using (2.10) and (2.11), we have

$$|a_j - a_i| \geq c\varepsilon^{(n+1)/(n+2)} >>> \varepsilon \geq cd_i. \tag{5.2}$$

For each $i \in \{1, \cdots, N\}$, we write $B_i := B(a_i, d_i/2) \subset \Omega$ and let ϕ_i be the projection of $\overline{\phi}$ onto $H_0^1(B_i)$. It satisfies the following partial differential equation (PDE)

$$\begin{cases} \Delta\phi_i = \Delta\overline{\phi}, & \text{in } B_i,\\ \phi_i = 0, & \text{on } \partial B_i. \end{cases} \tag{5.3}$$

Lemma 5.2. *It holds that*

$$\|\phi_i\|_{H_0^1(B_i)} \leq \|\overline{\phi}\|, \tag{5.4}$$

$$|(\overline{\phi} - \phi_i)(x)| \leq c\varepsilon^{-(n-4)/2}|\ln\varepsilon| \qquad \forall\ x \in B(a_i, d_i/4). \tag{5.5}$$

Proof. Multiplying the first equation of (5.3) by ϕ_i and integrating over B_i, we get

$$\int_{B_i} |\nabla\phi_i|^2 = \int_{B_i} \nabla\phi_i\nabla\overline{\phi} \leq \Big(\int_{B_i} |\nabla\phi_i|^2\Big)^{1/2}\Big(\int_{B_i} |\nabla\overline{\phi}|^2\Big)^{1/2},$$

which implies that

$$\|\phi_i\|_{H_0^1(B_i)} := \Big(\int_{B_i} |\nabla\phi_i|^2\Big)^{1/2} \leq \|\overline{\phi}\|.$$

Hence, the proof of (5.4) is completed. Concerning (5.5), from (5.3), we deduce that

$$\begin{cases} \Delta(\overline{\phi} - \phi_i) = 0, & \text{in } B_i,\\ (\overline{\phi} - \phi_i) = \overline{\phi}, & \text{on } \partial B_i. \end{cases}$$

Thus, let G_i be the Green function in B_i with a Dirichlet boundary condition. It follows that

$$(\overline{\phi} - \phi_i)(x) = -\int_{\partial B_i} \frac{\partial G_i}{\partial\nu}(x, y)\overline{\phi}(y)dy.$$

Taking $x \in B(a_i, d_i/4)$, it follows that for each $y \in \partial B_i$, we have $|y - x| \geq c d_i$; therefore, we obtain

$$|(\overline{\phi} - \phi_i)(x)| \leq \frac{c}{d_i^{n-1}} \int_{\partial B_i} |\overline{\phi}(y)| dy \leq c\, \varepsilon^{1-n} \int_{\partial B_i} |\overline{\phi}(y)| dy. \tag{5.6}$$

However, using the continuity of the embedding

$$H^1(B_i) \hookrightarrow L^{\frac{2n-2}{n-2}}(\partial B_i),$$

we obtain the following, by using (3.3):

$$\begin{aligned} \int_{\partial B_i} |\overline{\phi}(y)| dy \leq (d_i^{n-1})^{\frac{n}{2n-2}} \Big(\int_{\partial B_i} |\overline{\phi}(y)|^{\frac{2n-2}{n-2}} dy \Big)^{\frac{n-2}{2n-2}} &\leq c d_i^{n/2} \|\overline{\phi}\|_{H_0^1(B_i)} \\ &\leq c \varepsilon^{(n+2)/2} |\ln \varepsilon|. \end{aligned} \tag{5.7}$$

Hence, combining (5.7) and (5.6), the proof of (5.5) is as follows.

Since $d(a_i, \partial\Omega)$ is small and Ω is assumed to be regular, there is a unique $\overline{a}_i$ satisfying $|a_i - \overline{a}_i| = d(a_i, \partial\Omega)$. Without loss of generality, we can assume

$$\nu_i := \nu(\overline{a}_i) = -e_1, \tag{5.8}$$

where $(e_1, \cdots, e_n)$ is the canonical basis of $\mathbb{R}^n$, and $\nu(a)$ denotes the outnormal vector at the point $a \in \partial\Omega$ to the boundary $\partial\Omega$. Furthermore, for $k \in \{1, \cdots, n\}$, we write

$$\phi_i := \phi_{i,k}^o + \phi_{i,k}^e \quad \text{where } \phi_{i,k}^o \text{ is odd and } \phi_{i,k}^e \text{ is even with respect to } (x - a_i)_k. \tag{5.9}$$

Lemma 5.3. *Under the assumption of* (5.8)*, for each* $k \in \{2, \cdots, n\}$*, it holds that*

$$\int_{B_i} K (P\delta_{a_i,\lambda_i})^{p-\varepsilon} \phi_{i,k}^o = O\Big(\varepsilon^{1+\frac{1}{n-2}} \|\phi_i^o\|_{H_0^1(B_i)}\Big).$$

Proof. First, expanding K around the point a_i and using (2.5) and (2.19), we get

$$\int_{B_i} K (P\delta_{a_i,\lambda_i})^{p-\varepsilon} \phi_{i,k}^o = K(a_i) \int_{B_i} (P\delta_{a_i,\lambda_i})^{p-\varepsilon} \phi_{i,k}^o + O\Big(\int_{B_i} |x - a_i| \delta_{a_i,\lambda_i}^p |\phi_{i,k}^o| \Big). \tag{5.10}$$

Using Holder's inequality and the embedding $H_0^1(B_i) \hookrightarrow L^{p+1}(B_i)$, we get

$$\int_{B_i} |x - a_i| \delta_{a_i,\lambda_i}^p |\phi_{i,k}^o| \leq c \|\phi_i^o\|_{H_0^1(B_i)} \Big(\int_{B_i} |x - a_i|^{\frac{2n}{n+2}} \delta_{a_i,\lambda_i}^{p+1} \Big)^{\frac{n+2}{2n}} \leq \frac{c}{\lambda_i} \|\phi_i^o\|_{H_0^1(B_i)}.$$

Now, using the change of variables $y := \lambda_i (x - a_i)$ and passing to the polar coordinates, we get

$$\begin{aligned} \int_{B_i} |x - a_i|^{\frac{2n}{n+2}} \delta_{a_i,\lambda_i}^{p+1} &\leq c \int_{B(a_i, d_i/2)} \frac{\lambda_i^n |x - a_i|^{\frac{2n}{n+2}}}{(1 + \lambda_i^2 |x - a_i|^2)^n} dx \\ &\leq \frac{c}{\lambda_i^{2n/(n+2)}} \int_{B(0, \lambda_i d_i/2)} \frac{|y|^{2n/(n+2)}}{(1 + |y|^2)^n} dy \end{aligned}$$

$$\leq \frac{c}{\lambda_i^{2n/(n+2)}} \int_0^{\lambda_i d_i/2} \frac{r^{\frac{2n}{n+2}+n-1}}{(1+r^2)^n} dr \leq \frac{c}{\lambda_i^{2n/(n+2)}}.$$

Thus, using (2.9), which implies that $\frac{1}{\lambda_i} \leq c\varepsilon^{(n-1)/(n-2)}$, we deduce that

$$\int_{B_i} |x-a_i| \delta_{a_i,\lambda_i}^p |\phi_{i,k}^o| \leq \frac{c}{\lambda_i} \|\phi_i^o\|_{H_0^1(B_i)} \leq c\varepsilon^{1+\frac{1}{n-2}} \|\phi_i^o\|_{H_0^1(B_i)}. \tag{5.11}$$

Concerning the other integral, let $\widetilde{a}_i$ satisfies $a_i - \widetilde{a}_i = -2d_i\nu_i = 2d_i e_1$. From Claim (a) of Lemma 7.1 of [9], we know that

$$H(a_i,x) = \frac{1}{|x-\widetilde{a}_i|^{n-2}} + O\Big(\frac{d_i}{|x-\widetilde{a}_i|^{n-2}}\Big) = \frac{1}{|x-\widetilde{a}_i|^{n-2}} + O\Big(\frac{1}{d_i^{n-3}}\Big) \quad \text{in } B_i. \tag{5.12}$$

In addition, by taking $y = x - a_i$, it follows that

$$|x-\widetilde{a}_i| = |y + (a_i - \widetilde{a}_i)| = |y + 2d_i e_1|, \tag{5.13}$$

which implies that this function is even with respect to the variable y_j for each $j \geq 2$. Thus, using (2.3), we derive that

$$\begin{aligned} P\delta_{a_i,\lambda_i}(x) &= \delta_{a_i,\lambda_i}(x) - c_0 \frac{H(a_i,x)}{\lambda_i^{(n-2)/2}} + O\Big(\frac{1}{\lambda_i^{(n+2)/2} d_i^n}\Big) \\ &= \underbrace{\delta_{a_i,\lambda_i}(x) - c_0 \frac{1}{\lambda_i^{(n-2)/2} |x-\widetilde{a}_i|^{n-2}}}_{\widetilde{\delta}_{a_i,\lambda_i}(x)} + O\Big(\underbrace{\frac{1}{\lambda_i^{(n-2)/2} d_i^{n-3}} + \frac{1}{\lambda_i^{(n+2)/2} d_i^n}}_{\chi_0(\lambda_i,d_i)}\Big). \end{aligned} \tag{5.14}$$

Recall that by using (5.13), $\widetilde{\delta}_{a_i,\lambda_i}$ is an even function with respect to the variable $(x-a_i)_k$ for each $k \geq 2$; therefore, the function $\widetilde{\delta}_{a_i,\lambda_i}^{p-\varepsilon} \phi_{i,k}^o$ is an odd function with respect to $(x-a_i)_k$. Hence, we obtain

$$\begin{aligned} \int_{B_i} (P\delta_{a_i,\lambda_i})^{p-\varepsilon} \phi_{i,k}^o &= \int_{B_i} \Big(\widetilde{\delta}_{a_i,\lambda_i} + O(\chi_0(\lambda_i,d_i))\Big)^{p-\varepsilon} \phi_{i,k}^o \\ &= \int_{B_i} \widetilde{\delta}_{a_i,\lambda_i}^{p-\varepsilon} \phi_{i,k}^o + O\Big(\int_{B_i} \widetilde{\delta}_{a_i,\lambda_i}^{p-1} \chi_0(\lambda_i,d_i)) |\phi_{i,k}^o|\Big) \\ &= O\Big(\chi_0(\lambda_i,d_i) \|\phi_i^o\|_{H_0^1(B_i)} \Big(\int_{B_i} \delta_{a_i,\lambda_i}^{\frac{8n}{n^2-4}}\Big)^{\frac{n+2}{2n}}\Big). \end{aligned}$$

However, from (2.9) and (2.11), we deduce that $d_i \leq c\varepsilon$ and $\frac{1}{\lambda_i d_i} \leq c\varepsilon^{1/(n-2)}$. Now, using (1.1), we derive that

$$\begin{aligned} \chi_0(\lambda_i,d_i) \Big(\int_{B_i} \delta_{a_i,\lambda_i}^{\frac{8n}{n^2-4}}\Big)^{\frac{n+2}{2n}} \leq c\chi_0(\lambda_i,d_i)\chi_1(\lambda_i) &\leq c \begin{cases} \frac{d_i}{(\lambda_i d_i)^{n-2}} + \frac{1}{(\lambda_i d_i)^n} & \text{if } n = 4,5, \\ \frac{(\ln \lambda_i)^{2/3} d_i}{(\lambda_i d_i)^4} + \frac{(\ln \lambda_i)^{2/3}}{(\lambda_i d_i)^6}, & \text{if } n = 6 \end{cases} \\ &\leq c\varepsilon^{1+\frac{1}{n-2}}. \end{aligned}$$

Thus, the previous integral becomes

$$\int_{B_i} (P\delta_{a_i,\lambda_i})^{p-\varepsilon} \phi_{i,k}^o = O\Big(\varepsilon^{1+\frac{1}{n-2}} \|\phi_i^o\|_{H_0^1(B_i)}\Big). \tag{5.15}$$

Combining (5.10), (5.11), and (5.15), the proof of the lemma as follows.

Lemma 5.4. *Under the assumption of* (5.8)*, for each $k \in \{2, \cdots, n\}$, it holds that*

$$\int_{B_i} K(P\delta_{a_i,\lambda_i})^{p-\varepsilon-1}\overline{\phi}\phi^o_{i,k} = \int_{B_i} \delta^{p-1}_{a_i,\lambda_i}(\phi^o_{i,k})^2 + o\Big(\|\phi^o_{i,k}\|^2_{H^1_0(B_i)}\Big) + O\Big(\varepsilon^{1+\frac{1}{n-2}}|\ln\varepsilon|\|\phi^o_{i,k}\|_{H^1_0(B_i)}\Big).$$

Proof. Expanding K around a_i and using (2.5) and (2.19), we get

$$\int_{B_i} K(P\delta_{a_i,\lambda_i})^{p-\varepsilon-1}\overline{\phi}\phi^o_{i,k} = K(a_i)\int_{B_i}(P\delta_{a_i,\lambda_i})^{p-\varepsilon-1}\overline{\phi}\phi^o_{i,k} + O\Big(\int_{B_i}|x-a_i|\delta^{p-1}_{a_i,\lambda_i}|\overline{\phi}||\phi^o_{i,k}|\Big). \tag{5.16}$$

Observe that in B_i, $|x-a_i| < d_i/2$; therefore, using Holder's inequality and (3.3), we obtain

$$\int_{B_i}|x-a_i|\delta^{p-1}_{a_i,\lambda_i}|\overline{\phi}||\phi^o_{i,k}| \le cd_i\|\overline{\phi}\|\Big(\int_{B_i}|\nabla\phi^o_{i,k}|^2\Big)^{\frac{1}{2}} \le c\varepsilon^2|\ln\varepsilon|\|\phi^o_{i,k}\|_{H^1_0(B_i)}.$$

Regarding the other integral, using (5.14), we get

$$\begin{aligned}\int_{B_i}(P\delta_{a_i,\lambda_i})^{p-\varepsilon-1}\overline{\phi}\phi^o_{i,k} &= \int_{B_i}\Big(\widetilde{\delta}_{a_i,\lambda_i} + O(\chi_0(\lambda_i,d_i))\Big)^{p-\varepsilon-1}\overline{\phi}\phi^o_{i,k}\\ &= \int_{B_i}(\widetilde{\delta}_{a_i,\lambda_i})^{p-\varepsilon-1}\overline{\phi}\phi^o_{i,k} + O\Big(\chi_0(\lambda_i,d_i)\int_{B_i}\delta^{p-2}_{a_i,\lambda_i}|\overline{\phi}||\phi^o_{i,k}|\Big).\end{aligned} \tag{5.17}$$

Notice that for $n = 6$, we have $p-2 = 0$; therefore, using (3.3), (2.9), and (2.11), we get

$$\begin{aligned}\chi_0(\lambda_i,d_i)\int_{B_i}\delta^{p-2}_{a_i,\lambda_i}|\overline{\phi}||\phi^o_{i,k}| &= \chi_0(\lambda_i,d_i)\int_{B_i}|\overline{\phi}||\phi^o_{i,k}|\\ &\le c\Big(\frac{1}{\lambda_i^2 d_i^3} + \frac{1}{\lambda_i^4 d_i^6}\Big)d_i^2\|\overline{\phi}\|\Big(\int_{B_i}|\nabla\phi^o_{i,k}|^2\Big)^{\frac{1}{2}}\\ &\le c\varepsilon^2|\ln\varepsilon|\|\phi^o_{i,k}\|_{H^1_0(B_i)}.\end{aligned}$$

However, for $n \in \{4,5\}$, we have $p-2 > 0$; therefore

$$\chi_0(\lambda_i,d_i)\int_{B_i}\delta^{p-2}_{a_i,\lambda_i}|\overline{\phi}||\phi^o_{i,k}| \le c\Big(\frac{1}{\lambda_i^{\frac{n-2}{2}}d_i^{n-3}} + \frac{1}{\lambda_i^{\frac{n+2}{2}}d_i^n}\Big)\|\overline{\phi}\|\|\phi^o_{i,k}\|_{H^1_0(B_i)}\Big(\int_{B_i}\delta^{(p-2)\frac{n}{2}}_{a_i,\lambda_i}\Big)^{\frac{2}{n}}.$$

Moreover, using the change of variables $y := \lambda_i(x-a_i)$ and passing to the polar coordinates, easy computations imply that

$$\begin{aligned}\Big(\int_{B_i}\delta^{(p-2)\frac{n}{2}}_{a_i,\lambda_i}\Big)^{\frac{2}{n}} &\le c\Big(\int_{B_i}\frac{\lambda_i^{n(6-n)/4}}{(1+\lambda_i|x-a_i|^2)^{n(6-n)/4}}dx\Big)^{\frac{2}{n}}\\ &\le \frac{c}{\lambda_i^{(n-2)/2}}\Big(\int_{B(0,\lambda_i d_i/2)}\frac{1}{(1+|y|^2)^{n(6-n)/4}}dy\Big)^{\frac{2}{n}}\\ &\le \frac{c}{\lambda_i^{(n-2)/2}}\Big(\int_0^{\lambda_i d_i/2}\frac{r^{n-1}}{(1+r^2)^{n(6-n)/4}}dr\Big)^{\frac{2}{n}} \le c\begin{cases}\ln^{1/2}(\lambda_i d_i)/\lambda_i, & \text{if } n=4,\\ \lambda_i^{-1/2}d_i, & \text{if } n=5.\end{cases}\end{aligned}$$

Thus, for $n \in \{4,5,6\}$, we obtain

$$\chi_0(\lambda_i,d_i)\int_{B_i}\delta^{p-2}_{a_i,\lambda_i}|\overline{\phi}||\phi^o_{i,k}| \le c\varepsilon^2|\ln\varepsilon|\|\phi^o_{i,k}\|_{H^1_0(B_i)}. \tag{5.18}$$

This completes the estimate of the remainder term of (5.17).

Now, we will focus on the first integral of (5.17). We write

$$\int_{B_i}(\widetilde{\delta}_{a_i,\lambda_i})^{p-\varepsilon-1}\overline{\phi}\phi^o_{i,k} = \int_{B_i}(\widetilde{\delta}_{a_i,\lambda_i})^{p-\varepsilon-1}\phi_i\phi^o_{i,k} + \int_{B_i}(\widetilde{\delta}_{a_i,\lambda_i})^{p-\varepsilon-1}(\overline{\phi}-\phi_i)\phi^o_{i,k}. \tag{5.19}$$

Observe that since $\widetilde{\delta}_{a_i,\lambda_i}$ is an even function and $\phi^o_{i,k}$ is the odd part of ϕ_i (with respect to $(x-a_i)_k$; see (5.9)), we find that

$$\int_{B_i}(\widetilde{\delta}_{a_i,\lambda_i})^{p-\varepsilon-1}\phi_i\phi^o_{i,k} = \int_{B_i}(\widetilde{\delta}_{a_i,\lambda_i})^{p-\varepsilon-1}(\phi^o_{i,k})^2 = \int_{B_i}\delta^{p-1}_{a_i,\lambda_i}(\phi^o_{i,k})^2 + o\Big(\|\phi^o_{i,k}\|^2_{H^1_0(B_i)}\Big).$$

However, the second integral of (5.19) needs some precise computations. We decompose it into two parts. The first one is

$$\begin{aligned}\Big|\int_{B_i\setminus B(a_i,d_i/4)}(\widetilde{\delta}_{a_i,\lambda_i})^{p-\varepsilon-1}(\overline{\phi}-\phi_i)\phi^o_{i,k}\Big| &\le c\Big(\int_{B_i\setminus B(a_i,d_i/4)}(\widetilde{\delta}_{a_i,\lambda_i})^{p+1}\Big)^{\frac{2}{n}}\|\overline{\phi}-\phi_i\|\|\phi^o_{i,k}\|_{H^1_0(B_i)}\\ &\le \frac{c}{(\lambda_i d_i)^2}\varepsilon|\ln\varepsilon|\|\phi^o_{i,k}\|_{H^1_0(B_i)}\\ &\le c\varepsilon^{1+\frac{2}{n-2}}|\ln\varepsilon|\|\phi^o_{i,k}\|_{H^1_0(B_i)},\end{aligned}$$

where we have used the fact that $\widetilde{\delta}_{a_i,\lambda_i}\le\delta_{a_i,\lambda_i}$ in B_i, the fact that $\frac{1}{\lambda_i d_i}\le c\varepsilon^{1/(n-2)}$ (which follows from (2.9) and (2.11)), (5.4), (3.3), and (1.3).

Concerning the integral over $B(a_i,d_i/4)$, using (5.5) and (1.1), we have

$$\begin{aligned}\Big|\int_{B(a_i,d_i/4)}(\widetilde{\delta}_{a_i,\lambda_i})^{p-\varepsilon-1}(\overline{\phi}-\phi_i)\phi^o_{i,k}\Big| &\le c\varepsilon^{-\frac{n-4}{2}}|\ln\varepsilon|\|\phi^o_{i,k}\|_{H^1_0(B_i)}\Big(\int_{B(a_i,d_i/4)}\delta_{a_i,\lambda_i}^{\frac{8n}{n^2-4}}\Big)^{\frac{n+2}{2n}}\\ &\le c\varepsilon^{-\frac{n-4}{2}}|\ln\varepsilon|\|\phi^o_{i,k}\|_{H^1_0(B_i)}\chi_1(\lambda_i).\end{aligned}$$

Combining the previous estimates and using (2.9), the proof of the lemma follows.

Lemma 5.5. *The following holds:*

$$\int_{B_i}|\nabla\phi^o_{i,k}|^2 - p\int_{B_i}\delta^{p-1}_{a_i,\lambda_i}(\phi^o_{i,k})^2 \ge c\int_{B_i}|\nabla\phi^o_{i,k}|^2 + O(\varepsilon^3|\ln\varepsilon|^2).$$

Proof. Recall that $B_i\subset\Omega$ and $\phi^o_{i,k}\in H^1_0(B_i)$. Therefore, we will extend it by zero and we define

$$\widetilde{\phi}^o_{i,k} := \begin{cases}\phi^o_{i,k}, & \text{in } B_i,\\ 0, & \text{in } \Omega\setminus B_i.\end{cases}$$

This function $\widetilde{\phi}^o_{i,k}\in H^1_0(\Omega)$, and we have

$$\int_{\Omega}|\nabla\widetilde{\phi}^o_{i,k}|^2 = \int_{B_i}|\nabla\phi^o_{i,k}|^2 \quad , \qquad \int_{\Omega}\delta^{4/(n-2)}_{a_i,\lambda_i}(\widetilde{\phi}^o_{i,k})^2 = \int_{B_i}\delta^{4/(n-2)}_{a_i,\lambda_i}(\phi^o_{i,k})^2, \tag{5.20}$$

which implies that

$$Q_i(\widetilde{\phi}^o_{i,k}) = \int_{B_i} |\nabla \phi^o_{i,k}|^2 - p \int_{B_i} \delta^{p-1}_{a_i,\lambda_i} (\phi^o_{i,k})^2 \quad \text{where} \quad Q_i(v) := \int_\Omega |\nabla v|^2 - p \int_\Omega \delta^{p-1}_{a_i,\lambda_i} v^2. \tag{5.21}$$

Notice that Q_i is a positive quadratic form on $E^\perp_{a_i,\lambda_i}$ (defined in (2.7)). However, the function $\widetilde{\phi}^o_{i,k}$ does not necessarily belong to $E^\perp_{a_i,\lambda_i}$. For this, we will decompose it as

$$\widetilde{\phi}^o_{i,k} = \beta_1 P\delta_{a_i,\lambda_i} + \beta_2 \lambda_i \frac{\partial P\delta_{a_i,\lambda_i}}{\partial \lambda_i} + \sum_{j=1}^n \beta_{j+2} \frac{1}{\lambda_i} \frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,j}} + \widetilde{\phi}^{o,\perp}_{i,k} \quad \text{with} \quad \widetilde{\phi}^{o,\perp}_{i,k} \in E^\perp_{a_i,\lambda_i}, \tag{5.22}$$

where $a_{i,j}$ is the j^{th}-component of the point a_i. Now, we will estimate the coefficients β_i. Taking the scalar product of (5.22) with $P\delta_{a_i,\lambda_i}$, $\lambda_i \partial P\delta_{a_i,\lambda_i}/\partial\lambda_i$, and $\lambda_i^{-1}\partial P\delta_{a_i,\lambda_i}/\partial a_{i,j}$, we obtain

$$\begin{aligned}
c\beta_1 + o\Big(\sum |\beta_j|\Big) &= \langle \widetilde{\phi}^o_{i,k}, P\delta_{a_i,\lambda_i}\rangle = \int_\Omega \delta^p_{a_i,\lambda_i} \widetilde{\phi}^o_{i,k} = \int_{B_i} \delta^p_{a_i,\lambda_i} \phi^o_{i,k} = 0,\\
c\beta_2 + o\Big(\sum |\beta_j|\Big) &= \langle \widetilde{\phi}^o_{i,k}, \lambda_i \frac{\partial P\delta_{a_i,\lambda_i}}{\partial \lambda_i}\rangle\\
&= p\int_\Omega \delta^{p-1}_{a_i,\lambda_i} \lambda_i \frac{\partial \delta_{a_i,\lambda_i}}{\partial\lambda_i} \widetilde{\phi}^o_{i,k} = p\int_{B_i} \delta^{p-1}_{a_i,\lambda_i} \lambda_i \frac{\partial \delta_{a_i,\lambda_i}}{\partial\lambda_i} \phi^o_{i,k} = 0,\\
c\beta_{\ell+2} + o\Big(\sum |\beta_j|\Big) &= \langle \widetilde{\phi}^o_{i,k}, \frac{1}{\lambda_i} \frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,\ell}}\rangle\\
&= p\int_\Omega \delta^{p-1}_{a_i,\lambda_i} \frac{1}{\lambda_i} \frac{\partial \delta_{a_i,\lambda_i}}{\partial a_{i,\ell}} \widetilde{\phi}^o_{i,k} = p\int_{B_i} \delta^{p-1}_{a_i,\lambda_i} \frac{1}{\lambda_i} \frac{\partial \delta_{a_i,\lambda_i}}{\partial a_{i,\ell}} \phi^o_{i,k} = 0 \text{ for } \ell \neq k.
\end{aligned} \tag{5.23}$$

However, for $\ell = k$, the last integral is not zero. In fact, we have

$$\begin{aligned}
c\beta_{k+2} + o\Big(\sum |\beta_j|\Big) &= \langle \widetilde{\phi}^o_{i,k}, \frac{1}{\lambda_i} \frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\rangle\\
&= p\int_\Omega \delta^{p-1}_{a_i,\lambda_i} \frac{1}{\lambda_i} \frac{\partial \delta_{a_i,\lambda_i}}{\partial a_{i,k}} \widetilde{\phi}^o_{i,k} = p\int_{B_i} \delta^{p-1}_{a_i,\lambda_i} \frac{1}{\lambda_i} \frac{\partial \delta_{a_i,\lambda_i}}{\partial a_{i,k}} \phi^o_{i,k}.
\end{aligned} \tag{5.24}$$

Notice that, by using (5.9), we have $\phi^o_{i,k} = \phi_i - \phi^e_{i,k}$, and $\phi^e_{i,k}$ is even. Thus

$$\begin{aligned}
\int_{B_i} \delta^{p-1}_{a_i,\lambda_i} \frac{1}{\lambda_i} \frac{\partial \delta_{a_i,\lambda_i}}{\partial a_{i,k}} \phi^o_{i,k} &= \int_{B_i} \delta^{p-1}_{a_i,\lambda_i} \frac{1}{\lambda_i} \frac{\partial \delta_{a_i,\lambda_i}}{\partial a_{i,k}} \phi_i\\
&= \int_{B_i} \delta^{p-1}_{a_i,\lambda_i} \frac{1}{\lambda_i} \frac{\partial \delta_{a_i,\lambda_i}}{\partial a_{i,k}} \overline{\phi} - \int_{B_i} \delta^{p-1}_{a_i,\lambda_i} \frac{1}{\lambda_i} \frac{\partial \delta_{a_i,\lambda_i}}{\partial a_{i,k}} (\overline{\phi} - \phi_i).
\end{aligned} \tag{5.25}$$

For the second integral in (5.25), using (5.5), (3.3), (1.3), and (5.4), we get

$$\begin{aligned}
&\Big| \int_{B_i} \delta^{p-1}_{a_i,\lambda_i} \frac{1}{\lambda_i} \frac{\partial \delta_{a_i,\lambda_i}}{\partial a_{i,k}} (\overline{\phi} - \phi_i) \Big|\\
&\qquad \leq c\,\varepsilon^{\frac{4-n}{2}} |\ln \varepsilon| \int_{B(a_i,d_i/4)} \delta^p_{a_i,\lambda_i} + c\,(\|\overline{\phi}\| + \|\phi_i\|) \Big(\int_{B_i \setminus B(a_i,d_i/4)} \delta^{p+1}_{a_i,\lambda_i} \Big)^{\frac{n+2}{2n}}
\end{aligned}$$

$$\leq c\,\varepsilon^{\frac{4-n}{2}}|\ln\varepsilon|\frac{1}{\lambda_i^{(n-2)/2}} + c\,\varepsilon|\ln\varepsilon|\frac{1}{(\lambda_i d_i)^{(n+2)/2}}$$
$$\leq c\,\varepsilon^{3/2}|\ln\varepsilon|,$$

where we have used the fact that $\frac{1}{\lambda_i d_i} \leq c\varepsilon^{1/(n-2)}$ and $\frac{1}{\lambda_i} \leq c\varepsilon^{(n-1)/(n-2)}$ (see (2.9) and (2.11)). Concerning the first integral in (5.25), using (3.3), (1.3) , and the fact that $\overline{\phi} \in \mathcal{F}_{a,\lambda}$, we get

$$\int_{B_i} p\,\delta_{a_i,\lambda_i}^{p-1}\frac{1}{\lambda_i}\frac{\partial\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\overline{\phi} = \int_\Omega \cdots - \int_{\Omega\setminus B_i} = \langle\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}, \overline{\phi}\rangle + O\Big(\|\overline{\phi}\|\Big(\int_{\Omega\setminus B_i}\delta_{a_i,\lambda_i}^{p+1}\Big)^{\frac{n+2}{2n}}\Big)$$
$$= O\Big(\varepsilon|\ln\varepsilon|\frac{1}{(\lambda_i d_i)^{(n+2)/2}}\Big)$$
$$= O\Big(\varepsilon^{\frac{3}{2}+\frac{2}{n-2}}|\ln\varepsilon|\Big).$$

Thus, Equation (5.24) becomes

$$c\beta_{k+2} + o\Big(\sum|\beta_j|\Big) = O(\varepsilon^{3/2}|\ln\varepsilon|). \tag{5.26}$$

Combining (5.26) with the system defined in (5.23), we deduce that

$$|\beta_j| \leq c\,\varepsilon^{3/2}|\ln\varepsilon| \qquad \forall\ j \in \{1,\cdots,n+2\},$$

which implies that

$$\widetilde{\phi}_{i,k}^{o} = \widetilde{\phi}_{i,k}^{o,\perp} + O(\varepsilon^{3/2}|\ln\varepsilon|\delta_{a_i,\lambda_i}) \quad \text{ and } \quad \|\widetilde{\phi}_{i,k}^{o}\|^2 = \|\widetilde{\phi}_{i,k}^{o,\perp}\|^2 + O(\varepsilon^3|\ln\varepsilon|^2). \tag{5.27}$$

Now, using (5.22) and (5.27), we deduce that

$$Q_i(\widetilde{\phi}_{i,k}^{o}) = Q_i(\widetilde{\phi}_{i,k}^{o,\perp}) + o(\|\widetilde{\phi}_{i,k}^{o,\perp}\|^2) + O(\varepsilon^3|\ln\varepsilon|^2). \tag{5.28}$$

Recall that $\widetilde{\phi}_{i,k}^{o,\perp} \in E_{a_i,\lambda_i}^{\perp}$; thus, using the fact that Q_i is positive definite on $E_{a_i,\lambda_i}^{\perp}$, we deduce the existence of a positive constant γ such that

$$Q_i(\widetilde{\phi}_{i,k}^{o,\perp}) \geq \gamma\|\widetilde{\phi}_{i,k}^{o,\perp}\|^2. \tag{5.29}$$

Combining (5.28), (5.29), (5.21), and (5.20), we find that

$$\int_{B_i}|\nabla\phi_{i,k}^{o}|^2 - p\int_{B_i}\delta_{a_i,\lambda_i}^{p-1}(\phi_{i,k}^{o})^2 = Q_i(\widetilde{\phi}_{i,k}^{o}) = Q_i(\widetilde{\phi}_{i,k}^{o,\perp}) + o(\|\widetilde{\phi}_{i,k}^{o,\perp}\|^2) + O(\varepsilon^3|\ln\varepsilon|^2)$$
$$\geq \frac{\gamma}{2}\|\widetilde{\phi}_{i,k}^{o,\perp}\|^2 + O(\varepsilon^3|\ln\varepsilon|^2)$$
$$\geq \frac{\gamma}{2}\|\widetilde{\phi}_{i,k}^{o}\|^2 + O(\varepsilon^3|\ln\varepsilon|^2)$$
$$\geq \frac{\gamma}{2}\|\phi_{i,k}^{o}\|_{H_0^1(B_i)}^2 + O(\varepsilon^3|\ln\varepsilon|^2).$$

This achieves the proof of the lemma.

Lemma 5.6. *We have*

$$\|\phi_{i,k}^{o}\|_{H_0^1(B_i)} \leq c\varepsilon^{1+\frac{1}{n-2}}|\ln\varepsilon| \qquad \forall\ k \in \{2,\cdots,n\}. \tag{5.30}$$

Proof. Multiplying the first equation of (5.3) by $\phi^o_{i,k}$, integrating over B_i, and using (3.15), we get

$$\int_{B_i}\nabla\phi_i\nabla\phi^o_{i,k} = \int_{B_i}K|\omega+\widetilde{u}+\overline{\phi}|^{p-\varepsilon-1}(\omega+\widetilde{u}+\overline{\phi})\phi^o_{i,k} - \int_{B_i}K\omega^p\phi^o_{i,k} - \sum_j\alpha_j\int_{B_i}\delta^p_{a_j,\lambda_j}\phi^o_{i,k}$$
$$+\sum_{j=1}^N\Big(A_j\int_{B_i}\delta^p_{a_j,\lambda_j}\phi^o_{i,k} + pB_j\int_{B_i}\delta^{p-1}_{a_j,\lambda_j}\lambda_j\frac{\partial\delta_{a_j,\lambda_j}}{\partial\lambda_j}\phi^o_{i,k} + p\sum_{j=1}^n C_{j,\ell}\int_{B_i}\delta^{p-1}_j\frac{1}{\lambda_j}\frac{\partial\delta_{a_j,\lambda_j}}{\partial a_{j,\ell}}\phi^o_{i,k}\Big).$$

First, by oddness, for $j=i$, since the function δ_{a_i,λ_i} is even and $\phi^o_{i,k}$ is odd with respect to $(x-a_i)_k$, we have

$$\int_{B_i}\delta^p_{a_i,\lambda_i}\phi^o_{i,k}=0,\qquad \int_{B_i}\delta^{p-1}_{a_i,\lambda_i}\lambda_i\frac{\partial\delta_{a_i,\lambda_i}}{\partial\lambda_i}\phi^o_{i,k}=0,\qquad \int_{B_i}\delta^{p-1}_{a_i,\lambda_i}\frac{1}{\lambda_i}\frac{\partial\delta_{a_i,\lambda_i}}{\partial a_{i,\ell}}\phi^o_{i,k}=0\quad\forall\,\ell\neq k.$$

However, using (5.1), we have

$$C_{i,k}\int_{B_i}\delta^{p-1}_{a_i,\lambda_i}\frac{1}{\lambda_i}\frac{\partial\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\phi^o_{i,k} = O\Big(|C_{i,k}|\|\phi^o_{i,k}\|_{H^1_0(B_i)}\Big) = O\Big(\varepsilon^{1+\frac{1}{n-2}}\|\phi^o_{i,k}\|_{H^1_0(B_i)}\Big).$$

Second, by using (5.9), observe that

$$\int_{B_i}\nabla\phi_i\nabla\phi^o_{i,k} = \int_{B_i}|\nabla\phi^o_{i,k}|^2 = \|\phi^o_{i,k}\|^2_{H^1_0(B_i)}. \tag{5.31}$$

Using (2.25), it follows that

$$\Big|\int_{B_i}K\omega^p\phi^o_{i,k}\Big| \le c\varepsilon^p\int_{B_i}|\phi^o_{i,k}| \le c\varepsilon^p d_i^{(n+2)/2}\Big(\int_{B_i}|\nabla\phi^o_{i,k}|^2\Big)^{1/2} \le c\varepsilon^{p+\frac{n+2}{2}}\|\phi^o_{i,k}\|_{H^1_0(B_i)}.$$

In addition, by using (2.11) and (2.14), for $j\neq i$, it holds that

$$\int_{B_i}\delta^p_{a_j,\lambda_j}|\phi^o_{i,k}| \le c\,\varepsilon^{p(\frac{5-n}{2}-\frac{4}{n+2})}\int_{B_i}|\phi^o_{i,k}| \le cd_i^{(n+2)/2}\varepsilon^{p(\frac{5-n}{2}-\frac{4}{n+2})}\|\phi^o_{i,k}\|_{H^1_0(B_i)}$$
$$\le c\varepsilon^{\frac{3}{2}+\frac{2}{n-2}}\|\phi^o_{i,k}\|_{H^1_0(B_i)}.$$

Concerning the first integral, using (2.23), (2.25), and the following analytical formula

$$|t_1+t_2|^\beta(t_1+t_2) = |t_1|^\beta t_1 + (\beta+1)|t_1|^\beta t_2 + O(|t_1|^{\beta-1}t_2^2+|t_2|^{\beta+1})\quad\forall\ t_1,t_2\in\mathbb{R}\text{ and }\ \beta>0,$$

we get

$$\int_{B_i}K\Big|\omega+\widetilde{u}+\overline{\phi}\Big|^{p-\varepsilon-1}\Big(\omega+\widetilde{u}+\overline{\phi}\Big)\phi^o_{i,k}$$
$$=\int_{B_i}K\Big|\alpha_iP\delta_{a_i,\lambda_i}+O(\varepsilon^{-\gamma_1})+\overline{\phi}\Big|^{p-\varepsilon-1}\Big(\alpha_iP\delta_{a_i,\lambda_i}+O(\varepsilon^{-\gamma_1})+\overline{\phi}\Big)\phi^o_{i,k}$$
$$=\int_{B_i}K(\alpha_iP\delta_{a_i,\lambda_i})^{p-\varepsilon}\phi^o_{i,k} + (p-\varepsilon)\int_{B_i}K(\alpha_iP\delta_{a_i,\lambda_i})^{p-\varepsilon-1}(O(\varepsilon^{-\gamma_1})+\overline{\phi})\phi^o_{i,k}$$
$$+O\Big(\int_{B_i}|\overline{\phi}|^{p-\varepsilon}|\phi^o_{i,k}| + \varepsilon^{-\gamma_1 p}\int_{B_i}|\phi^o_{i,k}| + \int_{B_i}\delta^{p-\varepsilon-2}_{a_i,\lambda_i}(\varepsilon^{-2\gamma_1}+|\overline{\phi}|^2)|\phi^o_{i,k}|\Big), \tag{5.32}$$

where γ_1 is defined in (2.15). Using (1.1), (3.3), and the fact that $\varepsilon^{-\gamma_1} \leq \delta_{a_i,\lambda_i}$ in B_i (see (2.15) and (2.16)), observe that

$$\begin{aligned}\varepsilon^{-2\gamma_1}\int_{B_i}\delta_{a_i,\lambda_i}^{p-2}|\phi_{i,k}^o|+\varepsilon^{-\gamma_1}\int_{B_i}\delta_{a_i,\lambda_i}^{p-1}|\phi_{i,k}^o| &\leq c\varepsilon^{-\gamma_1}\int_{B_i}\delta_{a_i,\lambda_i}^{p-1}|\phi_{i,k}^o|\\ &\leq c\varepsilon^{-\gamma_1}\|\phi_{i,k}^o\|_{H_0^1(B_i)}\Big(\int_{B_i}\delta_{a_i,\lambda_i}^{\frac{8n}{n^2-4}}\Big)^{\frac{n+2}{2n}}\\ &\leq c\varepsilon^{-\gamma_1}\|\phi_{i,k}^o\|_{H_0^1(B_i)}\chi_1(\lambda_i)\\ &\leq c\,\varepsilon^{5/3}\|\phi_{i,k}^o\|_{H_0^1(B_i)},\end{aligned}$$

where χ_1 is defined in (1.2) and we have used the fact that $\frac{1}{\lambda_i} \leq c\varepsilon^{(n-1)/(n-2)}$ (see (2.9)).

In the same way, using (3.3), (2.11), and Holder's inequality, we get

$$\int_{B_i}|\overline{\phi}|^{p-\varepsilon}|\phi_{i,k}^o| \leq c\|\overline{\phi}\|^{p-\varepsilon}\|\phi_{i,k}^o\|_{H_0^1(B_i)} \leq c\,\varepsilon^p|\ln\varepsilon|^p\|\phi_{i,k}^o\|_{H_0^1(B_i)},$$

$$\varepsilon^{-\gamma_1 p}\int_{B_i}|\phi_{i,k}^o| \leq c\varepsilon^{-\gamma_1 p}d_i^{\frac{n+2}{2}}\|\phi_{i,k}^o\|_{H_0^1(B_i)} \leq c\,\varepsilon^{\frac{3}{2}+\frac{4}{n-2}}\|\phi_{i,k}^o\|_{H_0^1(B_i)},$$

$$\int_{B_i}\delta_i^{p-\varepsilon-2}|\overline{\phi}|^2|\phi_{i,k}^o| \leq c\|\overline{\phi}\|^2\|\phi_{i,k}^o\|_{H_0^1(B_i)} \leq c\,\varepsilon^2|\ln\varepsilon|^2\|\phi_{i,k}^o\|_{H_0^1(B_i)},$$

$$\int_{B_i}K(\alpha_i P\delta_{a_i,\lambda_i})^{p-\varepsilon-1}|\overline{\phi}||\phi_{i,k}^o| \leq c\|\overline{\phi}\|\|\phi_{i,k}^o\|_{H_0^1(B_i)} \leq c\,\varepsilon|\ln\varepsilon|\|\phi_{i,k}^o\|_{H_0^1(B_i)}.$$

Putting these estimates in (5.32), we obtain

$$\begin{aligned}&\int_{B_i}K\Big|\omega+\widetilde{u}+\overline{\phi}\Big|^{p-\varepsilon-1}\Big(\omega+\widetilde{u}+\overline{\phi}\Big)\phi_{i,k}^o\\ &\quad=\int_{B_i}K(\alpha_i P\delta_{a_i,\lambda_i})^{p-\varepsilon}\phi_{i,k}^o+p\int_{B_i}K(\alpha_i P\delta_{a_i,\lambda_i})^{p-\varepsilon-1}\overline{\phi}\phi_{i,k}^o+O\Big(\varepsilon^{5/3}\|\phi_{i,k}^o\|_{H_0^1(B_i)}\Big).\end{aligned}$$

Combining the previous estimates with Lemmas 5.3 and 5.4, we obtain

$$\int_{B_i}|\nabla\phi_{i,k}^o|^2-p\int_{B_i}\delta_{a_i,\lambda_i}^{p-1}(\phi_{i,k}^o)^2+o\Big(\int_{B_i}|\nabla\phi_{i,k}^o|^2\Big)=O\Big(\varepsilon^{1+\frac{1}{n-2}}|\ln\varepsilon|\|\phi_{i,k}^o\|_{H_0^1(B_i)}\Big).$$

Using Lemma 5.5, we deduce that

$$\|\phi_{i,k}^o\|_{H_0^1(B_i)}^2 \leq c\varepsilon^{1+\frac{1}{n-2}}|\ln\varepsilon|\|\phi_{i,k}^o\|_{H_0^1(B_i)}+c\,\varepsilon^3|\ln\varepsilon|^2.$$

Therefore, we obtain

$$\|\phi_{i,k}^o\|_{H_0^1(B_i)} \leq c\varepsilon^{1+\frac{1}{n-2}}|\ln\varepsilon|,$$

which completes the proof of the lemma.

6. Improvement of Proposition 4.3

The aim of this section is to improve Proposition 4.3. Notice that the norm $\|\overline{\phi}\|^2$ is not negligible with respect to the principal term when we take the scalar product with a tangential vector. However, we are able to improve the estimate of some integrals involving $\overline{\phi}$ by remarking that the odd part is involved in the required integrals. For this aim, in the previous section, we have introduced the projection of this function in $H_0^1(B_i)$ where $B_i := B(a_i, d_i/2)$ (see (5.3)), and we have decomposed the projection into the odd part and the even part. In the following, we will use this decomposition to improve some integrals. We start with the following lemma.

Lemma 6.1. *Under the assumption of* (5.8)*, for each $k \in \{2, \cdots, n\}$, it holds that*

$$\int_\Omega K(P\delta_{a_j,\lambda_j})^{p-\varepsilon}\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}} = K(a_j)\langle\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}, P\delta_{a_j,\lambda_j}\rangle + O(\varepsilon^{\sigma+\sigma_0}), \tag{6.1}$$

$$p\int_\Omega K(P\delta_{a_i,\lambda_i})^{p-\varepsilon-1}\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}P\delta_{a_j,\lambda_j} = K(a_i)\langle\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}, P\delta_{a_j,\lambda_j}\rangle + O(\varepsilon^{\sigma+\sigma_0}), \tag{6.2}$$

$$\int_\Omega K(P\delta_{a_i,\lambda_i})^{p-\varepsilon}\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}} = \frac{c_3}{\lambda_i}\frac{\partial K}{\partial x_k}(a_i) + O(\varepsilon^{\sigma+\sigma_0}), \tag{6.3}$$

where c_3 is defined in (1.4).

Proof. We start by proving the first assertion. Observe that if $\varepsilon\ln\lambda$ is small, using (2.18), we deduce that $\delta_{a,\lambda}^{-\varepsilon} = 1 + O(\varepsilon\ln\lambda)$. In addition, expanding K around a_j and using (2.5), we get

$$\int_\Omega K(P\delta_{a_j,\lambda_j})^{p-\varepsilon}\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}} = K(a_j)\int_\Omega (P\delta_{a_j,\lambda_j})^{p-\varepsilon}\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}} + O\Big(\int_\Omega |x-a_j|\delta_{a_j,\lambda_j}^p\delta_{a_i,\lambda_i}\Big).$$

Using Holder's inequality, Estimate 2 of [6], (2.9), and (2.13), we obtain

$$\begin{aligned}\int_\Omega |x-a_j|\delta_{a_j,\lambda_j}^p\delta_{a_i,\lambda_i} &\le \Big(\int_\Omega |x-a_j|^{\frac{n}{2}}\delta_{a_j,\lambda_j}^{\frac{2n}{n-2}}\Big)^{\frac{2}{n}}\Big(\int_\Omega[\delta_{a_j,\lambda_j}\delta_{a_i,\lambda_i}]^{\frac{n}{n-2}}\Big)^{\frac{n-2}{n}}\\ &\le \frac{c}{\lambda_j}\varepsilon_{ij}\ln(\varepsilon_{ij}^{-1})^{\frac{n-2}{n}} = O(\varepsilon^{\sigma+\sigma_0}).\end{aligned} \tag{6.4}$$

Now, using the fact that

$$(s+t)^\gamma = s^\gamma + O(s^{\gamma-1}t) \qquad \text{for } \gamma > 0 \text{ and } 0 \le t \le s,$$

we get

$$\begin{aligned}&\int_\Omega (P\delta_{a_j,\lambda_j})^{p-\varepsilon}\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\\ &= \int_\Omega \delta_{a_j,\lambda_j}^{p-\varepsilon}\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}} + O\Big(\int_\Omega \delta_{a_j,\lambda_j}^{p-1}\theta_{a_j,\lambda_j}\frac{1}{\lambda_i}\Big|\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\Big|\Big)\\ &= \int_\Omega \delta_{a_j,\lambda_j}^{p}\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}} + O\Big(\varepsilon\ln(\lambda_j)\int_\Omega \delta_{a_j,\lambda_j}^{p}\frac{1}{\lambda_i}\Big|\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\Big| + \int_\Omega \delta_{a_j,\lambda_j}^{p-1}\theta_{a_j,\lambda_j}\frac{1}{\lambda_i}\Big|\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\Big|\Big).\end{aligned}$$

Notice that the first integral is the required scalar product. However, the second and the third integrals are introduced in [9] and they are denoted by (IV) and (VI), respectively. These terms are estimated in [9] (see pages 54–55) and they satisfy

$$\varepsilon \ln(\lambda_j) \int_\Omega \delta^p_{a_j,\lambda_j} \frac{1}{\lambda_i}\Big|\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\Big| + \int_\Omega \delta^{p-1}_{a_j,\lambda_j}\theta_{a_i,\lambda_i}\frac{1}{\lambda_i}\Big|\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\Big| = O(\varepsilon^{\sigma+\sigma_0}),$$

where σ is introduced in (2.20), and σ_0 is a positive constant. This achieves the proof of (6.1).

Regarding Assertion (6.2), following the previous proof, we have

$$\begin{aligned}\int_\Omega K(P\delta_{a_i,\lambda_i})^{p-\varepsilon-1}\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}P\delta_{a_j,\lambda_j} =& K(a_i)\int_\Omega (P\delta_{a_i,\lambda_i})^{p-\varepsilon-1}\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}P\delta_{a_j,\lambda_j}\\ &+ O\Big(\int_\Omega |x-a_i|\delta^p_{a_i,\lambda_i}\delta_{a_j,\lambda_j}\Big).\end{aligned}$$

The last integral is computed in (6.4) by permuting the indices i and j. Concerning the first integral, let $\mu := (m_1/2)\varepsilon^{(n+1)/(n+2)}$ and $\widetilde{B}_k := B(a_k,\mu)$, where m_1 is introduced in (2.10). We remark that by using (2.10), it is easy to see that $\widetilde{B}_k \cap \widetilde{B}_j = \emptyset$ for each $k \neq j$. Now, using (2.19), it holds that

$$\begin{aligned}\int_\Omega (P\delta_{a_i,\lambda_i})^{p-\varepsilon-1}\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}P\delta_{a_j,\lambda_j} &= \int_\Omega \delta^{p-\varepsilon-1}_{a_i,\lambda_i}\frac{1}{\lambda_i}\frac{\partial \delta_{a_i,\lambda_i}}{\partial a_{i,k}}P\delta_{a_j,\lambda_j}\\ + O\Big(\int_{\widetilde{B}_i}\delta^{p-2}_{a_i,\lambda_i}\theta_{a_i,\lambda_i}\frac{1}{\lambda_i}\Big|\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\Big|P\delta_{a_j,\lambda_j} &+ \int_{\widetilde{B}_i}\delta^{p-1}_{a_i,\lambda_i}\frac{1}{\lambda_i}\Big|\frac{\partial\theta_{a_i,\lambda_i}}{\partial a_{i,k}}\Big|P\delta_{a_j,\lambda_j} + \int_{\Omega\setminus\widetilde{B}_i}\delta^p_{a_i,\lambda_i}\delta_{a_j,\lambda_j}\Big).\end{aligned} \tag{6.5}$$

For the first integral in (6.5), using (2.18), it follows that $\delta^{-\varepsilon}_{a_i,\lambda_i} = 1 + O(\varepsilon\ln\lambda_i)$, and therefore

$$p\int_\Omega \delta^{p-\varepsilon-1}_{a_i,\lambda_i}\frac{1}{\lambda_i}\frac{\partial\delta_{a_i,\lambda_i}}{\partial a_{i,k}}P\delta_{a_j,\lambda_j} = p\int_\Omega \delta^{p-1}_{a_i,\lambda_i}\frac{1}{\lambda_i}\frac{\partial\delta_{a_i,\lambda_i}}{\partial a_{i,k}}P\delta_{a_j,\lambda_j} + O\Big(\varepsilon\ln\lambda_i\int_\Omega\delta^p_{a_i,\lambda_i}P\delta_{a_j,\lambda_j}\Big). \tag{6.6}$$

Notice that the first term in (6.6) is the required scalar product. For the second term in (6.6), as before, we write $B_k := B(a_k, d_k/2)$. Using (2.14), (2.15), (2.9), and (2.11), it follows that

$$\begin{aligned}\varepsilon\ln\lambda_i\int_\Omega \delta^p_{a_i,\lambda_i}P\delta_{a_j,\lambda_j} &\le c\varepsilon|\ln\varepsilon|\varepsilon^{-\gamma_1}\int_{B_i}\delta^p_{a_i,\lambda_i} + c\varepsilon|\ln\varepsilon|\varepsilon^{p(\frac{5-n}{2}-\frac{4}{n+2})}\int_{B_j}\delta_{a_j,\lambda_j}\\ &\quad + c\varepsilon|\ln\varepsilon|\Big(\int_{\Omega\setminus B_i}\delta^{p+1}_{a_i,\lambda_i}\Big)^{\frac{p}{p+1}}\Big(\int_{\Omega\setminus B_j}\delta^{p+1}_{a_j,\lambda_j}\Big)^{\frac{1}{p+1}}\\ &\le \frac{c\varepsilon^{1-\gamma_1}|\ln\varepsilon|}{\lambda_i^{(n-2)/2}} + c\varepsilon|\ln\varepsilon|\varepsilon^{p(\frac{5-n}{2}-\frac{4}{n+2})}\frac{d_j^2}{\lambda_j^{\frac{n-2}{2}}} + \frac{c\varepsilon|\ln\varepsilon|}{(\lambda_i d_i)^{\frac{n+2}{2}}(\lambda_j d_j)^{\frac{n-2}{2}}}\\ &\le c\varepsilon^{\sigma+\sigma_0}.\end{aligned}$$

This completes the estimate of (6.6); therefore, the estimate of the first term of (6.5) is completed. Now, observe that

$$\frac{1}{\lambda_i}\Big|\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\Big| \le \frac{1}{\lambda_i}\Big|\frac{\partial\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\Big| + \frac{1}{\lambda_i}\Big|\frac{\partial\theta_{a_i,\lambda_i}}{\partial a_{i,k}}\Big| \le \frac{c}{\sqrt{1+\lambda_i^2|x-a_i|^2}}\delta_{a_i,\lambda_i} + \frac{1}{\lambda_i}\Big|\frac{\partial\theta_{a_i,\lambda_i}}{\partial a_{i,k}}\Big|.$$

Therefore, we get

$$\int_{\widetilde{B}_i} \delta_{a_i,\lambda_i}^{p-2}\theta_{a_i,\lambda_i}\frac{1}{\lambda_i}\Big|\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\Big|P\delta_{a_j,\lambda_j} + \int_{\widetilde{B}_i} \delta_{a_i,\lambda_i}^{p-1}\frac{1}{\lambda_i}\Big|\frac{\partial \theta_{a_i,\lambda_i}}{\partial a_{i,k}}\Big|P\delta_{a_j,\lambda_j}$$
$$\leq c\int_{\widetilde{B}_i} \delta_{a_i,\lambda_i}^{p-1}\Big(\frac{\theta_{a_i,\lambda_i}}{\sqrt{1+\lambda_i^2|x-a_i|^2}} + \frac{1}{\lambda_i}\Big|\frac{\partial \theta_{a_i,\lambda_i}}{\partial a_{i,k}}\Big|\Big)P\delta_{a_j,\lambda_j},$$

which is denoted by (III) in [9] (see page 53), and we derive

$$\int_{\widetilde{B}_i} \delta_{a_i,\lambda_i}^{p-2}\theta_{a_i,\lambda_i}\frac{1}{\lambda_i}\Big|\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\Big|P\delta_{a_j,\lambda_j} + \int_{\widetilde{B}_i} \delta_{a_i,\lambda_i}^{p-1}\frac{1}{\lambda_i}\Big|\frac{\partial \theta_{a_i,\lambda_i}}{\partial a_{i,k}}\Big|P\delta_{a_j,\lambda_j} \leq c\varepsilon^{\sigma+\sigma_0}.$$

In addition, the last integral in (6.5) is computed in Eq (8.3) of [9], and we have

$$\int_{\Omega\setminus\widetilde{B}_i} \delta_{a_i,\lambda_i}^{p}\delta_{a_j,\lambda_j} \leq c\varepsilon^{\sigma+\sigma_0}.$$

This completes the estimate of (6.5). Therefore, the proof of (6.2) is completed.
Finally, taking $\overline{v}=0$ in Proposition 3.3 of [9], the estimate (6.3) follows.

Proposition 6.1. *Let $n\in\{4,5,6\}$ and $\overline{a}_i$ be such that $\overline{a}_i\in\partial\Omega$ and $|a_i-\overline{a}_i| = d(a,\partial\Omega)$. Let $v\in T_{\overline{a}_i}(\partial\Omega)$, where $T_{\overline{a}_i}(\partial\Omega)$ is the tangent vector space at the point $\overline{a}_i$ to the boundary $\partial\Omega$. The following holds:*

$$\int_\Omega K(\omega+\widetilde{u})^{p-\varepsilon-1}\overline{\phi}\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_i}\cdot v = O(\varepsilon^{\sigma+\sigma_0}),$$

for some positive constant σ_0 and where σ is defined in (2.20).
Proof. Without loss of generality, we can assume that (5.8) holds. Hence, we need to prove the lemma for $\partial P\delta_{a_i,\lambda_i}/\partial a_{i,k}$ for each $k\geq 2$ where $a_{i,k}$ is the k^{th}-component of a_i.

First, let $D_i := B(a_i, d_i/4)$. Using Lemma A.2, we see that

$$\frac{1}{\lambda_i}\Big|\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_i}\Big| \leq \frac{1}{\lambda_i}\Big|\frac{\partial \delta_{a_i,\lambda_i}}{\partial a_i}\Big| + \frac{1}{\lambda_i}\Big|\frac{\partial \theta_{a_i,\lambda_i}}{\partial a_i}\Big| \leq \begin{cases} c\delta_{a_i,\lambda_i}, & \text{in } \Omega, \\ \frac{c}{\lambda_i d_i}\delta_{a_i,\lambda_i}, & \text{in } \Omega\setminus D_i. \end{cases} \tag{6.7}$$

Furthermore, using (2.23) and (2.25), we know that

$$\omega+\widetilde{u} = \alpha_i P\delta_{a_i,\lambda_i} + O(\varepsilon^{-\gamma_1}) \qquad \text{in } B_i := B(a_i, d_i/2). \tag{6.8}$$

Thus, using (6.7), (2.19), (1.1), (3.3), (2.11), and (2.9), we deduce that

$$\begin{aligned}\Big|\int_{\Omega\setminus D_i}\cdots\Big| &\leq c\int_{\Omega\setminus D_i}\Big(\sum\delta_{a_j,\lambda_j}^{p-1-\varepsilon} + \omega^{p-1-\varepsilon}\Big)|\overline{\phi}|\frac{1}{\lambda_i d_i}\delta_{a_i,\lambda_i} \\ &\leq \frac{c}{\lambda_i d_i}\int_\Omega|\overline{\phi}|\delta_{a_i,\lambda_i} + \frac{c}{\lambda_i^{n/2}d_i^{n-1}}\sum\int_\Omega\delta_{a_j,\lambda_j}^{p-1}|\overline{\phi}| \\ &\leq c\,\varepsilon^{1/(n-2)}\varepsilon|\ln\varepsilon|\chi_1(\lambda_i) + \frac{c\,\varepsilon|\ln\varepsilon|}{\lambda_i^{n/2}d_i^{n-1}}\sum\chi_1(\lambda_j) \\ &\leq c\varepsilon^{\sigma+\sigma_0} \qquad (\text{for } \sigma_0>0).\end{aligned} \tag{6.9}$$

The integral over D_i remains. Using (6.8), (6.7), and (2.19), we have

$$
\begin{aligned}
\int_{D_i} \cdots &= \int_{D_i} K(\alpha_i P\delta_{a_i,\lambda_i} + O(\varepsilon^{-\gamma_1}))^{p-\varepsilon-1}\overline{\phi}\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}} \\
&= \int_{D_i} K(\alpha_i P\delta_{a_i,\lambda_i})^{p-\varepsilon-1}\overline{\phi}\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}} + O\Big(\varepsilon^{-\gamma_1}\int_{D_i}\delta^{p-1}_{a_i,\lambda_i}|\overline{\phi}|\Big) \\
&= K(a_i)\int_{D_i} (\alpha_i P\delta_{a_i,\lambda_i})^{p-\varepsilon-1}\overline{\phi}\frac{1}{\lambda_i}\frac{\partial \delta_{a_i,\lambda_i}}{\partial a_{i,k}} + O\Big(\int_{D_i}|x-a_i|\delta^{p}_{a_i,\lambda_i}|\overline{\phi}|\Big) \\
&\quad + O\Big(\int_{D_i}\delta^{p-1}_{a_i,\lambda_i}|\overline{\phi}|\frac{1}{\lambda_i}\Big|\frac{\partial\theta_{a_i,\lambda_i}}{\partial a_{i,k}}\Big| + \varepsilon^{-\gamma_1}\int_{D_i}\delta^{p-1}_{a_i,\lambda_i}|\overline{\phi}|\Big).
\end{aligned}
\tag{6.10}
$$

For the last integral in (6.10), using (1.1), the following holds:

$$
\varepsilon^{-\gamma_1}\int_{D_i}\delta^{p-1}_{a_i,\lambda_i}|\overline{\phi}| \leq c\,\varepsilon^{-\gamma_1}\|\overline{\phi}\|\Big(\int_\Omega \delta^{\frac{8n}{n^2-4}}_{a_i,\lambda_i}\Big)^{\frac{n+2}{2n}} \leq c\,\varepsilon^{1-\gamma_1}|\ln\varepsilon|\chi_1(\lambda_i) \leq c\,\varepsilon^{\sigma+\sigma_0}. \tag{6.11}
$$

Concerning the third integral in (6.10), again using (1.1) and (2.4), we have

$$
\int_{D_i}\delta^{p-1}_{a_i,\lambda_i}|\overline{\phi}|\frac{1}{\lambda_i}\Big|\frac{\partial\theta_{a_i,\lambda_i}}{\partial a_i}\Big| \leq \frac{c}{\lambda_i}\Big|\frac{\partial\theta_{a_i,\lambda_i}}{\partial a_i}\Big|_\infty\|\overline{\phi}\|\Big(\int_\Omega \delta^{\frac{8n}{n^2-4}}_{a_i,\lambda_i}\Big)^{\frac{n+2}{2n}} \leq \frac{c\,\varepsilon|\ln\varepsilon|}{\lambda_i^{n/2}d_i^{n-1}}\chi_1(\lambda_i) \leq c\,\varepsilon^{\sigma+\sigma_0}. \tag{6.12}
$$

For the second integral in (6.10), we have

$$
\int_{D_i}|x-a_i|\delta^{p}_{a_i,\lambda_i}|\overline{\phi}| \leq c\,\|\overline{\phi}\|\frac{1}{\lambda_i} \leq c\,\varepsilon^{\sigma+\sigma_0}. \tag{6.13}
$$

We now focus on the first integral of (6.10). Observe that

$$
\begin{aligned}
\int_{D_i}(P\delta_{a_i,\lambda_i})^{p-\varepsilon-1}\overline{\phi}\frac{1}{\lambda_i}\frac{\partial\delta_{a_i,\lambda_i}}{\partial a_{i,k}} &= \int_{D_i}(\delta_{a_i,\lambda_i}-\theta_{a_i,\lambda_i})^{p-\varepsilon-1}\overline{\phi}\frac{1}{\lambda_i}\frac{\partial\delta_{a_i,\lambda_i}}{\partial a_{i,k}} \\
&= \int_{D_i}\delta^{p-\varepsilon-1}_{a_i,\lambda_i}\overline{\phi}\frac{1}{\lambda_i}\frac{\partial\delta_{a_i,\lambda_i}}{\partial a_{i,k}} - (p-\varepsilon-1)\int_{D_i}\delta^{p-\varepsilon-2}_{a_i,\lambda_i}\theta_{a_i,\lambda_i}\overline{\phi}\frac{1}{\lambda_i}\frac{\partial\delta_{a_i,\lambda_i}}{\partial a_{i,k}} \\
&\quad + O\Big(\int_{D_i}\frac{1}{\lambda_i|x-a_i|}\delta^{p-2}_{a_i,\lambda_i}\theta^2_{a_i,\lambda_i}|\overline{\phi}|\Big).
\end{aligned}
\tag{6.14}
$$

Using (3.3), (2.3), and (2.4), the third integral in (6.14) can be estimated as

$$
\begin{aligned}
\int_{D_i}\frac{1}{\lambda_i|x-a_i|}\delta^{p-2}_{a_i,\lambda_i}\theta^2_{a_i,\lambda_i}|\overline{\phi}| &\leq c\|\theta_{a_i,\lambda_i}\|^2_\infty\|\overline{\phi}\|\Big(\int_{D_i}\frac{\delta^{(p-2)\frac{2n}{n+2}}_{a_i,\lambda_i}}{(\lambda_i|x-a_i|)^{2n/(n+2)}}\Big)^{\frac{n+2}{2n}} \\
&\leq \frac{c\,\varepsilon|\ln\varepsilon|}{\lambda_i^{n-2}d_i^{2n-4}}\frac{1}{\lambda_i^{n-2}}\Big(\int_0^{\lambda_i d_i/4}\frac{r^{n-1-2n/(n+2)}}{(1+r^2)^{(6-n)n/(n+2)}}\Big)^{\frac{n+2}{2n}} \\
&\leq c\varepsilon^{\sigma+\sigma_0} \qquad (\text{for } \sigma_0>0).
\end{aligned}
\tag{6.15}
$$

Regarding the second integral in (6.14), it will divided into two parts as follows

$$
\int_{D_i}\cdots = \int_{D_i}\delta^{p-\varepsilon-2}_{a_i,\lambda_i}\theta_{a_i,\lambda_i}(\overline{\phi}-\phi_i)\frac{1}{\lambda_i}\frac{\partial\delta_{a_i,\lambda_i}}{\partial a_{i,k}} + \int_{D_i}\delta^{p-\varepsilon-2}_{a_i,\lambda_i}\theta_{a_i,\lambda_i}\phi_i\frac{1}{\lambda_i}\frac{\partial\delta_{a_i,\lambda_i}}{\partial a_{i,k}}, \tag{6.16}
$$

where ϕ_i is defined in (5.3). Using (5.5) and (2.19), we have

$$\begin{aligned}\Big|\int_{D_i}\delta_{a_i,\lambda_i}^{p-\varepsilon-2}\theta_{a_i,\lambda_i}(\overline{\phi}-\phi_i)\frac{1}{\lambda_i}\frac{\partial\delta_{a_i,\lambda_i}}{\partial a_i}\Big| &\leq c\,\varepsilon^{-\frac{n-4}{2}}|\ln\varepsilon|\|\theta_{a_i,\lambda_i}\|_\infty\int_{D_i}\delta_{a_i,\lambda_i}^{p-1}\\ &\leq \frac{c\,\varepsilon^{-\frac{n-4}{2}}|\ln\varepsilon|}{\lambda_i^{(n-2)/2}d_i^{n-2}}\begin{cases}\ln(\lambda_i d_i)/\lambda_i^2, & \text{if } n=4,\\ d_i^{n-4}/\lambda_i^2, & \text{if } n=5,6\end{cases}\\ &\leq c\,\varepsilon^{\sigma+\sigma_0}\qquad(\text{for }\ \sigma_0>0).\end{aligned}$$

For the second integral of (6.16), using Lemma 7.3 of [9], we know that

$$\theta_{a_i,\lambda_i}=\widetilde{\theta}_{a_i,\lambda_i}+O\Big(\frac{1}{\lambda_i^{(n-2)/2}d_i^{n-2}}\Big(d+\frac{1}{(\lambda_i d_i)^{n+1}}\Big)\Big)=\widetilde{\theta}_{a_i,\lambda_i}+O(\varepsilon^{\frac{5-n}{2}}),\tag{6.17}$$

where $\widetilde{\theta}_{a_i,\lambda_i}$ is an even function. Furthermore, Eq (7.7) of [9] tells us that

$$\|\widetilde{\theta}_{a_i,\lambda_i}\|_\infty\leq\frac{c}{\lambda_i^{(n-2)/2}d_i^{n-2}}\leq c\varepsilon^{(3-n)/2},$$

by using (2.9) and (2.11) (which imply that $\frac{1}{\lambda_i}\leq c\varepsilon^{(n-1)/(n-2)}$ and $c'\varepsilon\leq d_i\leq c\varepsilon$ for each i).
Thus, using (6.17), (1.1), (5.30), and Eq (7.7) of [9] , we get

$$\begin{aligned}\int_{D_i}\delta_{a_i,\lambda_i}^{p-\varepsilon-2}\theta_{a_i,\lambda_i}\phi_i\frac{1}{\lambda_i}\frac{\partial\delta_{a_i,\lambda_i}}{\partial a_{i,k}} &= \int_{D_i}\delta_{a_i,\lambda_i}^{p-\varepsilon-2}\widetilde{\theta}_{a_i,\lambda_i}\phi_i\frac{1}{\lambda_i}\frac{\partial\delta_{a_i,\lambda_i}}{\partial a_{i,k}}+O\Big(\varepsilon^{\frac{5-n}{2}}\int_{D_i}\delta_{a_i,\lambda_i}^{p-1}|\phi_i|\Big)\\ &= \int_{D_i}\delta_{a_i,\lambda_i}^{p-\varepsilon-2}\widetilde{\theta}_{a_i,\lambda_i}\phi_{i,k}^{o}\frac{1}{\lambda_i}\frac{\partial\delta_{a_i,\lambda_i}}{\partial a_{i,k}}+O(\varepsilon^{\sigma+\sigma_0})\\ &\leq c\|\widetilde{\theta}_{a_i,\lambda_i}\|_\infty\|\phi_{i,k}^{o}\|\Big(\int_{D_i}\delta_{a_i,\lambda_i}^{\frac{8n}{n^2-4}}\Big)^{\frac{n+2}{2n}}+c\,\varepsilon^{\sigma+\sigma_0}\\ &\leq\frac{c}{\lambda_i^{(n-2)/2}d_i^{n-2}}\varepsilon^{1+\frac{1}{n-2}}|\ln\varepsilon|\chi_1(\lambda_i)+c\,\varepsilon^{\sigma+\sigma_0}\leq c\,\varepsilon^{\sigma+\sigma_0}.\end{aligned}$$

These complete the estimate of (6.16); therefore, the estimate of the second integral in (6.14) is given by

$$\int_{D_i}\delta_{a_i,\lambda_i}^{p-\varepsilon-2}\theta_{a_i,\lambda_i}\overline{\phi}\frac{1}{\lambda_i}\frac{\partial\delta_{a_i,\lambda_i}}{\partial a_{i,k}}=O(\varepsilon^{\sigma+\sigma_0})\qquad(\text{with }\ \sigma_0>0).\tag{6.18}$$

We now focus on estimating the first integral of (6.14). Let $k\geq 2$, by using (2.18), this integral will be divided into three parts.

$$\begin{aligned}\int_{D_i}\delta_{a_i,\lambda_i}^{p-\varepsilon-1}\overline{\phi}\frac{1}{\lambda_i}\frac{\partial\delta_{a_i,\lambda_i}}{\partial a_{i,k}} =& c_0^{-\varepsilon}\lambda_i^{-\varepsilon\frac{n-2}{2}}\int_{D_i}\delta_{a_i,\lambda_i}^{p-1}\overline{\phi}\frac{1}{\lambda_i}\frac{\partial\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\\ &+c_0^{-\varepsilon}\lambda_i^{-\varepsilon\frac{n-2}{2}}\frac{n-2}{2}\,\varepsilon\int_{D_i}\delta_{a_i,\lambda_i}^{p-1}\overline{\phi}\frac{1}{\lambda_i}\frac{\partial\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\ln(1+\lambda_i^2|x-a_i|^2)\\ &+O\Big(\varepsilon^2\int_{D_i}\delta_{a_i,\lambda_i}^{p}\ln^2(1+\lambda_i^2|x-a_i|^2)|\overline{\phi}|\Big).\end{aligned}\tag{6.19}$$

It is easy to get the estimate of the third integral by using (3.3) as follows:

$$\varepsilon^2 \int_{D_i} \delta^p_{a_i,\lambda_i} \ln^2(1+\lambda_i^2|x-a_i|^2)|\overline{\phi}| \le c\,\varepsilon^2 \|\overline{\phi}\| \Big(\int_{D_i} \delta^{p+1}_{a_i,\lambda_i} (\ln(1+\lambda_i^2|x-a_i|^2))^{\frac{4n}{n+2}} \Big)^{\frac{n+2}{2n}}$$
$$\le c\,\varepsilon^{\sigma+\sigma_0} \qquad (\text{for } \sigma_0 > 0).$$

For the second integral of (6.19), using (5.5), (5.30), (2.9), and the fact that $\phi^o_{i,k}$ is the odd part of ϕ_i, we get

$$\begin{aligned}
\varepsilon \int_{D_i} \delta^{p-1}_{a_i,\lambda_i} \overline{\phi} \frac{1}{\lambda_i} \frac{\partial \delta_{a_i,\lambda_i}}{\partial a_{i,k}} \ln(1+\lambda_i^2|x-a_i|^2) &= \varepsilon \int_{D_i} \delta^{p-1}_{a_i,\lambda_i} \phi^o_{i,k} \frac{1}{\lambda_i} \frac{\partial \delta_{a_i,\lambda_i}}{\partial a_{i,k}} \ln(1+\lambda_i^2|x-a_i|^2) \\
&\quad + O\Big(\varepsilon \|\overline{\phi} - \phi_i\|_\infty \int_{D_i} \delta^p_{a_i,\lambda_i} \ln(1+\lambda_i^2|x-a_i|^2)\Big) \\
&= O\Big(\varepsilon \|\phi^o_{i,k}\| + \varepsilon^{1-\frac{n-4}{2}} |\ln\varepsilon| \lambda_i^{-(n-2)/2}\Big) \\
&= O(\varepsilon^{\sigma+\sigma_0}) \qquad (\text{for } \sigma_0 > 0).
\end{aligned}$$

Concerning the first integral of (6.19), using (1.1), (2.9), and (3.3), it holds that

$$\begin{aligned}
\Big| \int_{\Omega \setminus D_i} \delta^{p-1}_{a_i,\lambda_i} \overline{\phi} \frac{1}{\lambda_i} \frac{\partial \delta_{a_i,\lambda_i}}{\partial a_i} \Big| &\le c \int_{\Omega \setminus D_i} \frac{1}{\lambda_i |x-a_i|} \delta^p_{a_i,\lambda_i} |\overline{\phi}| \\
&\le \frac{c}{\lambda_i d_i} \frac{1}{(\lambda_i d_i^2)^{(n-2)/2}} \int_{\Omega \setminus D_i} \delta^{p-1}_{a_i,\lambda_i} |\overline{\phi}| \\
&\le \frac{c}{\lambda_i d_i} \frac{1}{(\lambda_i d_i^2)^{(n-2)/2}} \|\overline{\phi}\| \chi_1(\lambda_i) \le c\,\varepsilon^{\sigma+\sigma_0} \quad (\text{with } \sigma_0 > 0).
\end{aligned}$$

Hence, using the fact that $\overline{\phi} \perp \partial P\delta_{a_i,\lambda_i}/\partial a_{i,k}$, the first integral of (6.19) becomes

$$\int_{D_i} \delta^{p-1}_{a_i,\lambda_i} \overline{\phi} \frac{1}{\lambda_i} \frac{\partial \delta_{a_i,\lambda_i}}{\partial a_{i,k}} = \int_{\Omega} \delta^{p-1}_{a_i,\lambda_i} \frac{1}{\lambda_i} \frac{\partial \delta_{a_i,\lambda_i}}{\partial a_{i,k}} \overline{\phi} + O(\varepsilon^{\sigma+\sigma_0}) = O(\varepsilon^{\sigma+\sigma_0}) \qquad (\text{for } \sigma_0 > 0).$$

Thus, if we combine the previous estimates, (6.19) becomes

$$\int_{D_i} \delta^{p-\varepsilon-1}_{a_i,\lambda_i} \overline{\phi} \frac{1}{\lambda_i} \frac{\partial \delta_{a_i,\lambda_i}}{\partial a_{i,k}} = O(\varepsilon^{\sigma+\sigma_0}) \qquad (\text{for } \sigma_0 > 0). \tag{6.20}$$

Furthermore, if we combine (6.20), (6.18), and (6.15), the estimate of (6.14) becomes

$$\int_{D_i} (P\delta_{a_i,\lambda_i})^{p-\varepsilon-1} \overline{\phi} \frac{1}{\lambda_i} \frac{\partial \delta_{a_i,\lambda_i}}{\partial a_{i,k}} = O(\varepsilon^{\sigma+\sigma_0}) \qquad (\text{for } \sigma_0 > 0). \tag{6.21}$$

Finally, if we put (6.11), (6.12), (6.13), and (6.21) in Eq (6.10), we deduce that

$$\int_{D_i} K(\omega + \widetilde{u})^{p-\varepsilon-1} \overline{\phi} \frac{1}{\lambda_i} \frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}} = O(\varepsilon^{\sigma+\sigma_0}) \qquad (\text{for } \sigma_0 > 0). \tag{6.22}$$

If we combine (6.22) and (6.9), the proof of the lemma follows.

Proposition 6.2. *Let $\overline{a}_i$ be such that $\overline{a}_i \in \partial\Omega$ and $|a_i - \overline{a}_i| = d(a, \partial\Omega)$. Let $v \in T_{\overline{a}_i}(\partial\Omega)$ where $T_{\overline{a}_i}(\partial\Omega)$ is the tangent vector space at the point $\overline{a}_i$ to the boundary $\partial\Omega$. The following holds:*

$$\int_\Omega K(\omega + \widetilde{u})^{p-\varepsilon-2}\overline{\phi}^2 \frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_i} \cdot v = O(\varepsilon^{\sigma+\sigma_0})$$

for some positive constant σ_0 and where σ is defined in (2.20).

Proof. As in the proof of Proposition 6.1, without loss of generality, we can assume that (5.8) holds. Hence, we need to prove the lemma for $\partial P\delta_{a_i,\lambda_i}/\partial a_{i,k}$ for each $k \geq 2$, where $a_{i,k}$ is the k^{th}-component of a_i. Observe that, by using (2.5), we get

$$\begin{aligned}\Big|\int_{\Omega\setminus D_i} K(\omega + \widetilde{u})^{p-\varepsilon-2}\overline{\phi}^2 \frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_i}\Big| &\leq c\int_{\Omega\setminus D_i}(\omega + \widetilde{u})^{p-2}\overline{\phi}^2\delta_{a_i,\lambda_i}\\ &\leq c\|\overline{\phi}\|^2\Big(\int_{\Omega\setminus D_i}\delta_{a_i,\lambda_i}^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{2n}}\\ &\leq c\|\overline{\phi}\|^2\frac{1}{(\lambda_i d_i)^{(n-2)/2}} \leq c\,\varepsilon^{\sigma+\sigma_0}. \end{aligned} \tag{6.23}$$

Concerning the integral over D_i, observe that

$$\Big|\int_{D_i} K(\omega + \widetilde{u})^{p-\varepsilon-2}\overline{\phi}^2 \frac{1}{\lambda_i}\frac{\partial \theta_{a_i,\lambda_i}}{\partial a_i}\Big| \leq c\|\overline{\phi}\|^2\Big\|\frac{1}{\lambda_i}\frac{\partial\theta_{a_i,\lambda_i}}{\partial a_i}\Big\| \leq \frac{c\|\overline{\phi}\|^2}{(\lambda_i d_i)^{n/2}} \leq c\,\varepsilon^{\sigma+\sigma_0}. \tag{6.24}$$

Furthermore, as in the computations in (6.10) and (6.14), by using (2.23), it follows that

$$\begin{aligned}\int_{D_i} K(\omega + \widetilde{u})^{p-\varepsilon-2}\overline{\phi}^2 \frac{1}{\lambda_i}\frac{\partial \delta_{a_i,\lambda_i}}{\partial a_{i,k}} &= K(a_i)\int_{D_i}(\alpha_i\delta_{a_i,\lambda_i})^{p-\varepsilon-2}\overline{\phi}^2\frac{1}{\lambda_i}\frac{\partial\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\\ &\quad + O\Big(\int_{D_i}|x-a_i|\delta_{a_i,\lambda_i}^{p-1}\overline{\phi}^2 + \int_{D_i}\delta_{a_i,\lambda_i}^{p-2}(\theta_{a_i,\lambda_i} + \varepsilon^{-\gamma_1})\overline{\phi}^2\Big)\\ &= K(a_i)\int_{D_i}(\alpha_i\delta_{a_i,\lambda_i})^{p-\varepsilon-2}\overline{\phi}^2\frac{1}{\lambda_i}\frac{\partial\delta_{a_i,\lambda_i}}{\partial a_{i,k}} + O(\varepsilon^{\sigma+\sigma_0}).\end{aligned}$$

Write $\overline{\phi} = (\overline{\phi} - \phi_i) + \phi_i$ and recall that $\phi_i = \phi_{i,k}^o + \phi_{i,k}^e$, where $\phi_{i,k}^o$ and $\phi_{i,k}^e$ are the odd and the even parts of ϕ_i. Using (5.5) and (5.30), we get

$$\begin{aligned}&\int_{D_i}\delta_{a_i,\lambda_i}^{p-\varepsilon-2}\overline{\phi}^2\frac{1}{\lambda_i}\frac{\partial\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\\ &= \int_{D_i}\delta_{a_i,\lambda_i}^{p-\varepsilon-2}\phi_i^2\frac{1}{\lambda_i}\frac{\partial\delta_{a_i,\lambda_i}}{\partial a_{i,k}} + O\Big(\int_{D_i}\delta_{a_i,\lambda_i}^{p-1}|\overline{\phi} - \phi_i||\phi_i| + \int_{D_i}\delta_{a_i,\lambda_i}^{p-1}|\overline{\phi} - \phi_i|^2\Big)\\ &= 2\int_{D_i}\delta_{a_i,\lambda_i}^{p-\varepsilon-2}\phi_{i,k}^o\phi_{i,k}^e\frac{1}{\lambda_i}\frac{\partial\delta_{a_i,\lambda_i}}{\partial a_{i,k}} + O\Big(\|\overline{\phi} - \phi_i\|_\infty(\|\phi_i\| + \|\overline{\phi} - \phi_i\|)\chi_1(\lambda_i)\Big)\\ &= O\Big(\|\phi_{i,k}^o\|\,\|\phi_{i,k}^e\| + \|\overline{\phi} - \phi_i\|_\infty(\|\phi_i\| + \|\overline{\phi} - \phi_i\|)\chi_1(\lambda_i)\Big)\\ &= O(\varepsilon^{\sigma+\sigma_0}) \qquad (\text{with } \sigma_0 > 0).\end{aligned}$$

Combining the previous estimates, the proof of the proposition follows.

Proposition 6.3. *Let $n \in \{4,5,6\}$. For $i \in \{1,\cdots,N\}$, let $\overline{a}_i$ be the projection of a_i on the boundary (that is, $\overline{a}_i \in \partial\Omega$ and $|a_i - \overline{a}_i| = d(a_i,\partial\Omega)$), the following holds:*

$$\begin{aligned}\langle \nabla I_\varepsilon(\omega+\widetilde{u}+\overline{\phi}), \frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_i}\rangle = &-\sum_{k=1,k\neq i}^{N} \frac{c_1}{\lambda_i}\frac{\alpha_k}{(\lambda_i\lambda_k)^{\frac{n-2}{2}}}\frac{\partial G(a_i,a_k)}{\partial a_i} - \alpha_i^p c_3 \frac{\nabla K(a_i)}{\lambda_i}\\ &+ O(\varepsilon^{\sigma+\sigma_0}) + \left(\frac{\alpha_i}{2}\frac{c_1}{\lambda_i^{n-1}}\frac{\partial H}{\partial \nu}(a_i,a_i) + O(\varepsilon^{\min(1+\frac{2}{n-2};1+\frac{n}{n+2})})\right)\cdot \nu_{\overline{a}_i},\end{aligned}$$

where σ is defined in (2.20), σ_0 is a positive constant, $G(.,.)$ is the Green function with a Dirichlet boundary condition, and $H(.,.)$ is its regular part.

Proof. Without loss of generality, we can assume that (5.8) holds. Hence, $\partial P\delta_{a_i,\lambda_i}/\partial a_{i,1} = \partial P\delta_{a_i,\lambda_i}/\partial a_i \cdot \nu_{\overline{a}_i}$. Hence, using Proposition 4.3, we need only to prove the proposition for $\partial P\delta_{a_i,\lambda_i}/\partial a_{i,k}$ for each $k \geq 2$, where $a_{i,k}$ is the k^{th}-component of a_i.

Let $k \geq 2$, using (2.2), we have

$$\begin{aligned}\langle \nabla I_\varepsilon(\omega+\widetilde{u}+\overline{\phi}), \frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\rangle =&\langle \omega+\widetilde{u}+\overline{\phi}, \frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\rangle\\ &- \int_\Omega K|\omega+\widetilde{u}+\overline{\phi}|^{p-\varepsilon-1}(\omega+\widetilde{u}+\overline{\phi})\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}. \qquad (6.25)\end{aligned}$$

The first term follows from Lemmas 3.4, 8.1, 7.1 of [9], Equation (4.2), and the fact that $\overline{\phi} \in \mathcal{F}_{a,\lambda}$. In fact, Lemma 8.1 of [9] tells us that for $j \neq i$, we have

$$\langle P\delta_{a_j,\lambda_j}, \frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\rangle = \frac{n-2}{\lambda_i^{n/2}\lambda_j^{(n-2)/2}}c_1\left(\frac{(a_j-a_i)_k}{|a_j-a_i|^n} - \frac{(a_j-a_i)_k}{|a_j-\widetilde{a}_i|^n}\right) + O\left(\varepsilon^{\sigma+\sigma_0}\right),$$

for some $\sigma_0 > 0$ and $\widetilde{a}_i = a_i + 2d_i\nu_i$ (this is the symmetry of a_i with respect to $\overline{a}_i$). However, Assertion (c) of Lemma 7.1 of [9] tells us that, for $j \neq i$, we have

$$\frac{\partial G}{\partial a_{i,k}}(a_i,a_j) = (n-2)\left(\frac{a_{j,k}-a_{i,k}}{|a_i-a_j|^n} - \frac{a_{j,k}-a_{i,k}}{|a_j-\widetilde{a}_i|^n}\right) + O\left(\frac{1}{d_i^{n-2}}\right).$$

Concerning Lemma 3.4 of [9], it consists to estimate

$$\langle P\delta_{a_i,\lambda_i}, \frac{1}{\lambda}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\rangle = O(\varepsilon^{\sigma+\sigma_0}), \qquad \text{for some } \sigma_0 > 0.$$

Hence, since $\overline{\phi} \in \mathcal{F}_{a,\lambda}$ (which implies that $\overline{\phi} \perp \partial P\delta_{a_i,\lambda_i}/\partial a_{i,k}$), using (4.2), we deduce that

$$\langle \omega+\widetilde{u}+\overline{\phi}, \frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\rangle = \sum_{j\neq i}\frac{c_1\alpha_j}{\lambda_i^{n/2}\lambda_j^{(n-2)/2}}\frac{\partial G}{\partial a_{i,k}}(a_j,a_i) + O(\varepsilon^{\sigma+\sigma_0}). \qquad (6.26)$$

Concerning the integral in (6.25), let

$$\Omega_1 := \{x \in \Omega : |\overline{\phi}(x)| \leq (1/2)(\omega(x)+\widetilde{u}(x))\} \qquad \text{and} \qquad \Omega_2 := \Omega \setminus \Omega_1.$$

Since $\lambda_i^{-1}|\partial P\delta_{a_i,\lambda_i}/\partial a_i| \le cP\delta_{a_i,\lambda_i} \le c(\omega+\widetilde{u})$, the following holds:

$$\Big|\int_{\Omega_2} K|\omega+\widetilde{u}+\overline{\phi}|^{p-\varepsilon-1}(\omega+\widetilde{u}+\overline{\phi})\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\Big| \le c\int_{\Omega_2}|\overline{\phi}|^{p+1-\varepsilon} \le c\,\varepsilon^{\sigma+\sigma_0}. \tag{6.27}$$

However, in Ω_1, we have

$$\begin{aligned}
&\int_{\Omega_1} K|\omega+\widetilde{u}+\overline{\phi}|^{p-\varepsilon-1}(\omega+\widetilde{u}+\overline{\phi})\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\\
&= \int_{\Omega_1} K(\omega+\widetilde{u})^{p-\varepsilon}\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}} + (p-\varepsilon)\int_{\Omega_1} K(\omega+\widetilde{u})^{p-\varepsilon-1}\overline{\phi}\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\\
&+\frac{1}{2}(p-\varepsilon)(p-\varepsilon-1)\int_{\Omega_1} K(\omega+\widetilde{u})^{p-\varepsilon-2}\overline{\phi}^2\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}} + O\Big(\int_{\Omega_1}(\omega+\widetilde{u})^{p-\varepsilon-2}|\overline{\phi}|^3\Big)\\
&= \int_{\Omega} K(\omega+\widetilde{u})^{p-\varepsilon}\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}} + O(\varepsilon^{\sigma+\sigma_0}),
\end{aligned} \tag{6.28}$$

by using Propositions 6.1 and 6.2. In addition, the last integral can be computed as

$$\begin{aligned}
&\int_{\Omega} K(\omega+\widetilde{u})^{p-\varepsilon}\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\\
&\quad= \int_{\Omega} K\widetilde{u}^{p-\varepsilon}\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}} + O\Big(\int_{\Omega}\widetilde{u}^{p-\varepsilon-1}\omega\frac{1}{\lambda_i}\Big|\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\Big| + \int_{\Omega}\omega^{p-\varepsilon}\frac{1}{\lambda_i}\Big|\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\Big|\Big).
\end{aligned} \tag{6.29}$$

Using (6.7), (2.23), and (2.25), it follows that

$$\begin{aligned}
\int_{\Omega}\omega^{p-\varepsilon}\frac{1}{\lambda_i}\Big|\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\Big| &\le c\int_{D_i}\varepsilon^{p-\varepsilon}\delta_{a_i,\lambda_i} + c\int_{\Omega\setminus D_i}\frac{1}{\lambda_i|x-a_i|}\delta_{a_i,\lambda_i}\\
&\le \frac{c\,\varepsilon^p}{\lambda_i^{(n-2)/2}} + \frac{c}{\lambda_i^{n/2}} \le c\,\varepsilon^{\sigma+\sigma_0},\\
\int_{\Omega}\widetilde{u}^{p-\varepsilon-1}\omega\frac{1}{\lambda_i}\Big|\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}\Big| &\le c\,\varepsilon\int_{D_i}\delta^p_{a_i,\lambda_i} + \frac{c}{\lambda_i^{n/2}d_i^{n-1}}\sum\int_{\Omega\setminus D_i}\delta^{p-1}_{a_j,\lambda_j}\\
&\le \frac{c\,\varepsilon}{\lambda_i^{(n-2)/2}} + \frac{c}{\lambda_i^{n/2}d_i^{n-1}}\begin{Bmatrix}\frac{\ln\lambda_j}{\lambda_j^2} & \text{if } n=4,\\ \frac{1}{\lambda_j^2} & \text{if } n\ge 5\end{Bmatrix} \le c\,\varepsilon^{\sigma+\sigma_0}.
\end{aligned}$$

Hence, we deduce that

$$\int_{\Omega_1} K|\omega+\widetilde{u}+\overline{\phi}|^{p-\varepsilon-1}(\omega+\widetilde{u}+\overline{\phi})\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}} = \int_{\Omega} K\widetilde{u}^{p-\varepsilon}\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}} + O(\varepsilon^{\sigma+\sigma_0}). \tag{6.30}$$

Observe that for $\gamma\in(1,3)$, $t_j\ge 0$ for $j\in\{1,\cdots,N\}$ and $z_i\in\mathbb{R}$ with $|z_i|\le ct_i$, the following holds:

$$\Big(\sum t_j\Big)^{\gamma}z_i = \sum t_j^{\gamma}z_i + \gamma t_i^{\gamma-1}\Big(\sum_{j\ne i}t_j\Big)z_i + O\Big(\sum_{j\ne\ell}(t_jt_\ell)^{(\gamma+1)/2}\Big).$$

Thus, we deduce that

$$\int_{\Omega} K\widetilde{u}^{p-\varepsilon}\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}} = \sum_{j=1}^{N}\int_{\Omega} K(\alpha_jP\delta_{a_j,\lambda_j})^{p-\varepsilon}\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}$$

$$+ (p-\varepsilon)\int_\Omega K(\alpha_i P\delta_{a_i,\lambda_i})^{p-\varepsilon-1}\Big(\sum_{j\neq i}\alpha_j P\delta_{a_j,\lambda_j}\Big)\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}} + O\Big(\sum_{j\neq \ell}\int_\Omega (\delta_{a_j,\lambda_j}\delta_{a_\ell,\lambda_\ell})^{\frac{n}{n-2}}\Big).$$

Notice that the last integral is computed in Estimate (E2) of [6], and we have (by using (2.13))

$$\int_\Omega (\delta_j\delta_\ell)^{n/(n-2)} \leq c\,\varepsilon_{j\ell}^{n/(n-2)}\ln(\varepsilon_{j\ell}^{-1}) \leq c\,\varepsilon^{\sigma+\sigma_0}.$$

Thus, by using Lemma 6.1, we derive

$$\begin{aligned}\int_\Omega K\widetilde{u}^{p-\varepsilon}\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_{i,k}} &= \alpha_i^{p-\varepsilon}\frac{c_3}{\lambda_i}\frac{\partial K}{\partial x_k}(a_i)\\ &+ \sum_{j\neq i}\alpha_j(\alpha_i^{p-\varepsilon-1}K(a_i)+\alpha_j^{p-\varepsilon-1}K(a_j))\langle\frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_{i,k}}, P\delta_{a_j,\lambda_j}\rangle + O(\varepsilon^{\sigma+\sigma_0}).\end{aligned} \tag{6.31}$$

Thus, by using (6.26), (6.27), (6.30), (6.31), and Lemmas 7.1 and 8.1 of [9], the proof of the proposition follows.

7. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. The proof of this kind of constructed solutions becomes standard (following [8]). It relies on the finite dimensional reduction; that is, given $(\alpha, a, \lambda) \in \mathbb{A}_{N,\varepsilon,\xi^*}$, there is a function $\overline{\phi}_{\varepsilon,\alpha,a,\lambda} \in \mathcal{F}_{a,\lambda}$ which extremizes the functional I_ε in the space $\mathcal{F}_{a,\lambda}$. This step is proved, in this paper, in Proposition 3.1. We then need to prove the existence of $(\alpha_\varepsilon, a_\varepsilon, \lambda_\varepsilon) \in \mathbb{A}_{N,\varepsilon,\xi^*}$ such that the associated function $u_\varepsilon := \omega + \sum_{i=1}^N \alpha_{i,\varepsilon}P\delta_{a_{i,\varepsilon},\lambda_{i,\varepsilon}} + \overline{\phi}_\varepsilon$ is a critical point of I_ε. This step relies on the expansions of the scalar product of the gradient ∇I_ε with the functions $P\delta_{a_i,\lambda_i}$ and its derivatives with respect to λ_i and $a_{i,k}$. This expansions are proved in Propositions 4.1, 4.2, and 6.3. These expansions are exactly the same as those found in [16] and [9]; therefore, the proof given in these references still works in our case. □

Proof of Theorem 1.2. Let $a_1 \in \Omega$ be close to ξ_1^* and $a_2 \in \Omega$ be close to ξ_2^* with $\xi_1^* \neq \xi_2^*$; that is, $|a_1 - \xi_1^*| << 1$ and $|a_2 - \xi_2^*| << 1$, it holds that $|a_1 - a_2| \geq c > 0$. Therefore, the interaction between the bubbles $P\delta_{a_1,\lambda_1}$ and $P\delta_{a_2,\lambda_2}$ is of the order of $\frac{1}{(\lambda_1\lambda_2)^{(n-2)/2}} = O(\varepsilon^{n-1})$ (by using (2.9)), which is very small with respect to ε^σ (where $\sigma := 2 + \frac{1}{n-2} - \frac{1}{n+2}$). Thus, this interaction will be considered as a remainder term. Therefore, by taking $((\alpha_1, a_1, \lambda_1), \cdots, (\alpha_m, a_m, \lambda_m)) \in \mathbb{A}_{N_1,\varepsilon,\xi_1^*} \times \cdots \times \mathbb{A}_{N_m,\varepsilon,\xi_m^*}$, Propositions 4.1, 4.2, and 6.3 contain only the parameters in the same set $\mathbb{A}_{N_k,\varepsilon,\xi_k^*}$; that is, we are able to separate each pack alone. In the sense that we will obtain m systems $(S_1, \cdots, S_m)$, each system S_k contains only the parameters $(\alpha_k, a_k, \lambda_k) \in \mathbb{A}_{N_k,\varepsilon,\xi_k^*}$. Thus, the proof of Theorem 1.1 works again. □

8. Conclusions

Taking $n \in \{4, 5, 6\}$, by giving a precise expansion of the associated variational functional, we constructed some positive solutions of the problem (P_ε) which converge weakly to a positive solution of (P_0) and which blow up at a precise point $\xi^* \in \partial\Omega$ (or several points in $\partial\Omega$). This construction shows that for low dimensions, the boundary provides a setting where blow-up and residual mass can coexist.

For this construction, the point ξ^* is not an isolated simple blow-up, in the sense that many bubbles are very close and the concentration points converge to the same point ξ^*.

We note that in our construction, we used the fact that $\frac{\partial K}{\partial \nu}(\xi^*) > 0$. The following natural questions arise:

- What happens if $\frac{\partial K}{\partial \nu}(\xi^*) = 0$ and the function K satisfies some flatness assumptions?
- What is about the changing sign's solutions?

Another natural avenue for future research consists of extending the analysis to the weakly supercritical regime ($\varepsilon < 0$, $|\varepsilon|$ small).

Author contributions

Dalel Almutairi and Mohamed Ben Ayed: Conceptualization, methodology, investigation, writing-original draft. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University for financial support (QU-APC-2026).

Conflicts of interest

The authors declare no conflict of interest.

References

1. M. Ahmedou, M. Ben Ayed, Non simple blow ups for the Nirenberg problem on high dimensional half spheres, *Discrete Cont. Dyn.-A*, **42** (2022), 5967–6005. http://doi.org/10.3934/dcds.2022134
2. M. Ahmedou, M. Ben Ayed, K. El Mehdi, Nirenberg problem on high dimensional spheres: blow up with residual mass phenomenon, *Nonlinear Differ. Equ. Appl.* **32** (2025), 2. https://doi.org/10.1007/s00030-024-01004-8
3. K. Alfaleh, K. El Mehdi, Interior bubbling solutions with residual mass for a neumann problem with slightly subcritical exponent, *Mathematics*, **13** (2025), 1324. https://doi.org/10.3390/math13081324
4. A. Almtuairi, K. El Mehdi, Interior multipeaked solutions for slightly subcritical elliptic problems, *Symmetry*, **17** (2025), 579. https://doi.org/10.3390/sym17040579
5. T. Aubin, *Some nonlinear problem in differential geometry*, Berlin: Springer, 1998. https://doi.org/10.1007/978-3-662-13006-3

6. A. Bahri, *Critical point at infinity in some variational problems*, Harlow: Longman Scientific and Technical, 1989.

7. A. Bahri, J. M. Coron, On a nonlinear elliptic equation involving the critical Sobolev exponent: The effect of the topology on the domain, *Commun. Pur. Appl. Math.*, **41** (1988), 253–294. http://doi.org/10.1002/cpa.3160410302

8. A. Bahri, Y. Y. Li, O. Rey, On a variational problem with lack of compactness: the topological effect of the critical points at infinity, *Calc. Var.*, **3** (1995), 67–93. http://doi.org/10.1007/BF01190892

9. M. Ben Ayed, Concentrating boundary blow up solutions of a slightly subcritical Hénon type problem for all dimensions, *J. Math. Anal. Appl.*, **493** (2021), 124494. http://doi.org/10.1016/j.jmaa.2020.124494

10. M. Ben Ayed, Finite dimensional reduction of a supercritical exponent equation, *Tunisian Journal of Mathematics*, **2** (2020), 379–397. http://doi.org/10.2140/tunis.2020.2.379

11. M. Ben Ayed, H. Chtioui, M. Hammami, A Morse lemma at infinity for Yamabe type problem on domains, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **20** (2003), 543–577. http://doi.org/10.1016/S0294-1449(02)00020-3

12. H. Berestycki, P.-L. Lions, Nonlinear scalar field equations, I existence of a ground state, *Arch. Rational Mech. Anal.*, **82** (1983), 313–345. http://doi.org/10.1007/BF00250555

13. D. M. Cao, S. J. Peng, The asymptotic behaviour of the ground state solutions for Hénon equation, *J. Math. Anal. Appl.*, **278** (2003), 1–17. http://doi.org/10.1016/S0022-247X(02)00292-5

14. J. Davila, J. Faya, F. Mahmoudi, New type solutions to a slightly subcritical Hénon type problem on general domains, *J. Differ. Equations*, **263** (2017), 7221–7249. http://doi.org/10.1016/j.jde.2017.08.005

15. O. Druet, E. Hebey, Elliptic equations of Yamabe type, *International Mathematics Research Surveys*, **2005** (2005), 1–113. http://doi.org/10.1155/IMRS.2005.1

16. H. Fourti, R. Ghoudi, New type of solutions to a slightly subcritical Hénon type problem in dimensions 7, 8 and 9, *J. Differ. Equations*, **268** (2020), 3081–3128. http://doi.org/10.1016/j.jde.2019.09.050

17. M. Hammami, M. Ben Ayed, On a variational problem involving critical Sobolev crowth in dimension four, *Adv. Differential Equations*, **9** (2004), 415–446. http://doi.org/10.57262/ade/1355867950

18. Y. S. Kivshar, G. P. Agrawal, *Optical solitons: from fibers to photonic crystals*, New Yourk: Academic Press, 2003. http://doi.org/ 10.1016/B978-0-12-410590-4.X5000-1

19. Y. Y. Li, Prescribing scalar curvature on S^n and related topics, Part I., *J. Differ. Equations*, **120** (1995), 319–410. http://doi.org/10.1006/jdeq.1995.1115

20. S. J. Peng, Multiple boundary concentrating solutions to Dirichlet problem of Hénon equation, *Acta Math. Appl. Sin.-E.*, **22** (2006), 137–162. http://doi.org/10.1007/s10255-005-0293-0

21. A. Pistoia, G. Vaira, Clustering phenomena for linear perturbation of the Yamabe equation, In: *Partial differential equations arising from physics and geometry*, Cambridge: Cambridge University Press, 2019, 311–331. https://doi.org/10.1017/9781108367639.009

22. P.-D. Thizy, J. Vétois, Positive clusters for smooth perturbations of a critical elliptic equation in dimensions four and five, *J. Funct. Anal.*, **275** (2018), 170–195. http://doi.org/10.1016/j.jfa.2018.02.002

23. O. Rey, The topological impact of critical points at infinity in a variational problem with lack of compactness: the dimension 3, *Adv. Differential Equations*, **4** (1999), 581–616. http://doi.org/10.57262/ade/1411134018

24. O. Rey, The role of the Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent, *J. Funct. Anal.*, **89** (1990), 1–52. http://doi.org/10.1016/0022-1236(90)90002-3

25. R. Schoen, Topics in differential geometry, Graduate course at Stanford University, 1988. Aavailable from: `https://sites.math.washington.edu/~pollack/research/Schoen-1988-notes.html`.

26. C. Zhong, H. Lin, Y. Cheng, J. H. He, Variational approach to (4 + 1)-dimensional Boiti–Leon–Manna–Pempinelli equation, *J. Comput. Appl. Mech.*, **57** (2026), 362–368. http://doi.org/10.22059/jcamech.2026.410601.1779

A. Appendix

For the reader's convenience, we gather two technical lemmas in this appendix. The first lemma is an immediate consequence of the elementary computations, and its proof is therefore omitted. In contrast, a full proof is provided for the second lemma.

Lemma A.1. *Let $a \in \Omega$ and $\lambda > 0$ be such that $\lambda d := \lambda d(a, \partial\Omega)$ is large. The following holds:*

$$\Big(\int_\Omega \delta_{a,\lambda}^{\frac{2n}{n+2}}\Big)^{\frac{n+2}{2n}} \leq c\chi_1(\lambda) \qquad and \qquad \Big(\int_\Omega \delta_{a,\lambda}^{\frac{8n}{(n-2)(n+2)}}\Big)^{\frac{n+2}{2n}} \leq c\chi_1(\lambda), \tag{1.1}$$

where

$$\chi_1(\lambda) := \Big(\lambda^{(2-n)/2} \ \text{ if } 3 \leq n \leq 5; \quad \ln(\lambda)^{2/3}\lambda^{-2} \ \text{ if } n = 6; \quad \lambda^{-2} \ \text{ if } n \geq 7\Big). \tag{1.2}$$

Furthermore, for each fixed $\gamma > 0$, we have

$$\int_{\mathbb{R}^n \setminus B(a,\gamma d)} \delta_{a,\lambda}^{2n/(n-2)} \leq \frac{c}{(\lambda d)^n}. \tag{1.3}$$

Lemma A.2. *Let $a \in \Omega$ and $\lambda > 0$ be such that $\lambda d := \lambda d(a, \partial\Omega)$ is large. The following holds:*

$$\Big|\frac{1}{\lambda}\frac{\partial \theta_{a,\lambda}}{\partial a_j}\Big| \leq \frac{c}{\lambda d}\theta_{a,\lambda} \leq \frac{c}{\lambda d}\delta_{a,\lambda} \qquad in \ \Omega,$$

where a_j is the j^{th}-component of the point a.
Proof. Since $\partial\theta_{a,\lambda}/\partial a_j$ satisfies

$$\Delta\Big(\frac{\partial \theta_{a,\lambda}}{\partial a_j}\Big) = 0 \ \text{ in } \Omega \quad \text{ and } \quad \frac{\partial \theta_{a,\lambda}}{\partial a_j} = \frac{\partial \delta_{a,\lambda}}{\partial a_j} \text{ on } \partial\Omega,$$

for each $x \in \Omega$, it holds that

$$\frac{1}{\lambda}\frac{\partial \theta_{a,\lambda}}{\partial a_j}(x) = -\int_{\partial\Omega} \frac{\partial G}{\partial \nu}(x,y)\frac{1}{\lambda}\frac{\partial \delta_{a,\lambda}}{\partial a_j}(y)dy.$$

However, we know that

$$\left|\frac{1}{\lambda}\frac{\partial \delta_{a,\lambda}}{\partial a_j}(y)\right| \leq \frac{c}{\lambda|y-a|}\delta_{a,\lambda}(y) \quad \forall\, y \in \Omega \setminus \{a\} \qquad \text{and} \qquad \frac{\partial G}{\partial \nu}(x,y) \leq 0 \quad \forall\, y \in \partial\Omega.$$

Thus, it holds that

$$\left|\frac{1}{\lambda}\frac{\partial \theta_{a,\lambda}}{\partial a_j}(x)\right| \leq \frac{c}{\lambda d}\int_{\partial\Omega} -\frac{\partial G}{\partial \nu}(x,y)\delta_{a,\lambda}(y)dy \leq \frac{c}{\lambda d}\theta_{a,\lambda}(x).$$

Hence, the proof is completed.