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Research article

On some properties of degenerate $\mathfrak{q}$-derangement numbers and polynomials

Waseem Ahmad Khan[1], **Oğuz Yağcı**[2], **Khidir Shaib Mohamed**[3,*], **Mona A. Mohamed**[3] **and Naglaa Mohammed**[3]

[1] Department of Electrical Engineering, Prince Mohammad Bin Fahd University, P.O. Box 1664, Al Khobar 31952, Saudi Arabia

[2] Department of Mathematics, Kırıkkale University, Kırıkkale 71450, Türkiye

[3] Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia

* **Correspondence:** Email: k.idris@qu.edu.sa.

Abstract: Using a Carlitz-type degenerate $\mathfrak{q}$-exponential kernel together with the λ-falling factorial $(\mu)_{\zeta,\lambda}$, we introduce a λ-degenerate $\mathfrak{q}$-analog of the derangement family. The associated exponential generating function defines the degenerate $\mathfrak{q}$-derangement polynomials $\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)$ and yields explicit coefficient formulas, recurrence relations, convolution identities, and determinant representations. The main structural point is that these polynomials are governed by a lower triangular transform in the $\mathfrak{q}$-factorial basis; this transform has a two-term inverse and organizes the connections with degenerate $\mathfrak{q}$-Stirling, $\mathfrak{q}$-Bell, and $\mathfrak{q}$-Fubini polynomials. We also show that the same mechanism is stable under higher-order kernels and under a degenerate $(\mathfrak{p},\mathfrak{q})$-extension. The limiting regimes $\lambda \to 0$ and $\mathfrak{q} \to 1$ recover, respectively, the standard $\mathfrak{q}$-derangements and the classical derangement polynomials.

Keywords: degenerate $\mathfrak{q}$-derangement numbers and polynomials; degenerate $(\mathfrak{p},\mathfrak{q})$-derangement polynomials; degenerate $\mathfrak{q}$-Stirling numbers; degenerate $\mathfrak{q}$-Bell and Fubini polynomials; generating functions; numerical methods; $(\mathfrak{p},\mathfrak{q})$-calculus
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1. Introduction

A *derangement* is a permutation of $\{1,\dots,\zeta\}$ having no fixed points. The classical derangement numbers are obtained by inclusion–exclusion and admit the well-known representation $D_\zeta = \zeta!\sum_{\kappa=0}^{\zeta}(-1)^\kappa/\kappa!$; see, for example, [1, 2]. In many applications, it is natural to deform this family so that additional parameters record combinatorial statistics or so that the corresponding generating functions interact with analytic kernels.

Two complementary deformation schemes are prevalent in the recent literature. First, analogs incorporate $\mathfrak{q}$-calculus, basic hypergeometric structures, and operational methods, and $\mathfrak{q}$-derangement numbers and polynomials have been studied from several viewpoints; see [3, 4] and the references therein. Second, *λ-degenerate* variants are built from Carlitz-type degenerate exponentials and the associated λ-falling factorials, leading to degenerate derangement polynomials, their higher-order extensions, degenerate Pochhammer-type objects, and probabilistic interpretations [1, 2, 5–8]. Degenerate Sheffer sequences supply a unifying framework for many such families [9], and their λ-$\mathfrak{q}$-extensions have proved effective in deriving identities for $\mathfrak{q}$-special polynomials [10]. Related q-Laplace-transform and modified-degenerate operational formulations appear in [11, 12], and analytic variants based on truncated degenerate exponentials and degenerate Mittag-Leffler functions also appear in [13, 14]. Similar generating-function and umbral viewpoints also occur in recent work on probabilistic degenerate Jindalrae and Jindalrae–Stirling polynomial families [15], Hermite-based Fibonacci and Lucas-type polynomial identities [16], and q-Mittag-Leffler-based Bessel and Tricomi functions [17].

The purpose of this paper is to combine these two directions by introducing a λ-degenerate $\mathfrak{q}$-analog of the derangement family. Our construction is driven by an exponential generating function in which the degenerate falling factorial $(\mu)_{\zeta,\lambda}$ is coupled to a degenerate $\mathfrak{q}$-exponential kernel. The resulting polynomials $\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)$ (and their specialization at $\mu = 0$) interpolate between the standard $\mathfrak{q}$-derangement families (as $\lambda \to 0$) and the classical derangement polynomials (as $\mathfrak{q} \to 1$).

The contribution is not merely a formal replacement of the classical exponential by a degenerate $\mathfrak{q}$-exponential. The factor $(1-\xi)^{-1}$, the $\mathfrak{q}$-factorial basis, and the degenerate factorial $(\mu-1)_{\zeta,\lambda}$ together define a lower triangular transform whose inverse has only two nonzero diagonals. This observation gives a compact structural explanation for the recurrence, determinant, and connection-coefficient identities obtained below. It also clarifies why the same family interacts naturally with the degenerate $\mathfrak{q}$-Stirling transform and therefore with the corresponding Bell and Fubini families. Thus, the paper emphasizes the transform structure behind the identities rather than treating them as isolated coefficient extractions. This viewpoint is consistent with recent generating-function approaches to generalized polynomial systems involving auxiliary sequences and hyperharmonic-type numbers [18].

For ease of reference, we summarize the main outcomes of the paper as follows:

- We define the degenerate $\mathfrak{q}$-derangement polynomials and numbers through a two-parameter generating kernel that simultaneously contains the known $\mathfrak{q}$-derangement and degenerate derangement cases.
- We identify the lower triangular transform that sends the degenerate falling-factorial sequence to the degenerate $\mathfrak{q}$-derangement sequence together with its two-term inverse, inhomogeneous recurrence, and determinant representation.
- We express $\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)$ in the basis of degenerate $\mathfrak{q}$-Stirling numbers of the second kind, giving connections with associated Bell- and Fubini-type constructions in the spirit of related degenerate Stirling theory [10, 19, 20].
- We use higher-order kernels and a degenerate $(\mathfrak{p},\mathfrak{q})$-extension to show that the transform mechanism is stable under natural deformations, and we verify the limiting regimes $\lambda \to 0$ and $\mathfrak{q} \to 1$.

The remainder of the paper is organized as follows: Section 2 fixes notation and introduces $\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)$

together with its basic properties. Section 3 develops further identities, higher-order families, and the $(\mathfrak{p},\mathfrak{q})$-extension. We close in Section 4 with remarks and open questions.

2. Degenerate $\mathfrak{q}$-derangement numbers and polynomials

We begin by fixing our $\mathfrak{q}$-calculus conventions and by recalling the degenerate $\mathfrak{q}$-Stirling numbers of the second kind, together with the associated degenerate $\mathfrak{q}$-Bell and $\mathfrak{q}$-Fubini families, as these sequences will serve as connection coefficients in several later expansions. The construction is motivated by the degenerate Stirling framework in [10, 19, 20] and by recent degenerate operational-transform techniques [4, 12]. All generating-function arguments are carried out in the formal power series ring $\mathbb{C}[[t]]$, so coefficient extraction is purely algebraic. After setting notation (see [21, 22] for standard $\mathfrak{q}$-notation and [23] for formal-series manipulations), the degenerate $\mathfrak{q}$-derangement polynomials are introduced by means of an exponential generating function. Their basic properties are then derived (cf. [1–3]).

We denote

$$[\zeta]_{\mathfrak{q}}! = [\zeta]_{\mathfrak{q}}[\zeta-1]_{\mathfrak{q}}\cdots[1]_{\mathfrak{q}}, \qquad \begin{bmatrix}\zeta\\ \kappa\end{bmatrix}_{\mathfrak{q}} = \frac{[\zeta]_{\mathfrak{q}}!}{[\kappa]_{\mathfrak{q}}![\zeta-\kappa]_{\mathfrak{q}}!} \qquad (0\le\kappa\le\zeta).$$

For $\lambda\in\mathbb{C}$ and $\zeta\ge 0$, the *degenerate falling factorial* is

$$(\mu)_{0,\lambda}=1, \qquad (\mu)_{\zeta,\lambda}=\mu(\mu-\lambda)\cdots(\mu-(\zeta-1)\lambda) \quad (\zeta\ge 1).$$

We also use the degenerate $\mathfrak{q}$-exponential functions (cf. [24])

$$\mathfrak{e}_{\mathfrak{q},\lambda}(\xi)=\sum_{\zeta=0}^{\infty}\frac{(1)_{\zeta,\lambda}}{[\zeta]_{\mathfrak{q}}!}\xi^{\zeta}, \qquad \mathfrak{E}_{\mathfrak{q},\lambda}(\xi)=\sum_{\zeta=0}^{\infty}\mathfrak{q}^{\binom{\zeta}{2}}\frac{(1)_{\zeta,\lambda}}{[\zeta]_{\mathfrak{q}}!}\xi^{\zeta},$$

which satisfy the fundamental relation $\mathfrak{e}_{\mathfrak{q},\lambda}(\xi)\mathfrak{E}_{\mathfrak{q},\lambda}(-\xi)=1$ (cf. [24]).

When $\lambda=0$, these series reduce to the standard $\mathfrak{q}$-exponentials

$$\mathfrak{e}_{\mathfrak{q}}(\xi)=\sum_{\zeta=0}^{\infty}\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{q}}!}, \qquad \mathfrak{E}_{\mathfrak{q}}(\xi)=\sum_{\zeta=0}^{\infty}\mathfrak{q}^{\binom{\zeta}{2}}\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{q}}!},$$

and we use the $\mathfrak{q}$-Pochhammer symbol $(a;\mathfrak{q})_{\zeta}=\prod_{j=0}^{\zeta-1}(1-a\mathfrak{q}^{j})$ when convenient (cf. [21, 22]).

Remark 2.1. If

$$A(\xi)=\sum_{\zeta=0}^{\infty}a_{\zeta}\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{q}}!}, \qquad B(\xi)=\sum_{\zeta=0}^{\infty}b_{\zeta}\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{q}}!},$$

then

$$A(\xi)B(\xi)=\sum_{\zeta=0}^{\infty}\left(\sum_{\kappa=0}^{\zeta}\begin{bmatrix}\zeta\\ \kappa\end{bmatrix}_{\mathfrak{q}}a_{\kappa}\,b_{\zeta-\kappa}\right)\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{q}}!}.$$

Indeed,

$$\frac{\xi^{\kappa}}{[\kappa]_{\mathfrak{q}}!}\frac{\xi^{\zeta-\kappa}}{[\zeta-\kappa]_{\mathfrak{q}}!}=\begin{bmatrix}\zeta\\ \kappa\end{bmatrix}_{\mathfrak{q}}\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{q}}!}.$$

This identity is standard in $\mathfrak{q}$-calculus; see, for example, [21, 23].

Definition 2.2. Motivated by the degenerate Stirling framework in [10,19,20], the degenerate $\mathfrak{q}$-Stirling numbers of the second kind are defined by the following generating-function identity:

$$\frac{1}{[\kappa]_{\mathfrak{q}}!}\left(e_{\mathfrak{q},\lambda}(\xi)-1\right)^{\kappa}=\sum_{\zeta=\kappa}^{\infty}\mathcal{S}_{2,\mathfrak{q}}(\zeta,\kappa;\lambda)\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{q}}!}\qquad(\kappa\geq 0).\tag{2.1}$$

Definition 2.3. The *degenerate* $\mathfrak{q}$-*Bell polynomials* are defined by the exponential generating function (cf. [10, 19, 20])

$$e_{\mathfrak{q},\lambda}\left(\mu\left(e_{\mathfrak{q},\lambda}(\xi)-1\right)\right)=\sum_{\zeta=0}^{\infty}\phi_{\zeta,\mathfrak{q}}(\mu\,;\lambda)\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{q}}!}.\tag{2.2}$$

Evaluating at $\mu=1$ yields the degenerate $\mathfrak{q}$-Bell numbers $\phi_{\zeta,\mathfrak{q}}(\lambda):=\phi_{\zeta,\mathfrak{q}}(1\,;\lambda)$.

Theorem 2.4. *For* $\zeta\geq 0$*,*

$$\phi_{\zeta,\mathfrak{q}}(\mu\,;\lambda)=\sum_{\kappa=0}^{\zeta}(\mu)_{\kappa,\lambda}\,\mathcal{S}_{2,\mathfrak{q}}(\zeta,\kappa\,;\lambda).$$

In particular,

$$\phi_{\zeta,\mathfrak{q}}(\lambda)=\sum_{\kappa=0}^{\zeta}(1)_{\kappa,\lambda}\,\mathcal{S}_{2,\mathfrak{q}}(\zeta,\kappa\,;\lambda).$$

Proof. Beginning with (2.2) and expanding $e_{\mathfrak{q},\lambda}$ in its defining series, we arrive at the formal identity

$$e_{\mathfrak{q},\lambda}\left(\mu(e_{\mathfrak{q},\lambda}(\xi)-1)\right)=\sum_{\kappa=0}^{\infty}\frac{(\mu)_{\kappa,\lambda}}{[\kappa]_{\mathfrak{q}}!}\left(e_{\mathfrak{q},\lambda}(\xi)-1\right)^{\kappa}.$$

By the defining relation (2.1) for $\mathcal{S}_{2,\mathfrak{q}}(\zeta,\kappa\,;\lambda)$, we can express

$$\left(e_{\mathfrak{q},\lambda}(\xi)-1\right)^{\kappa}=[\kappa]_{\mathfrak{q}}!\sum_{\zeta=\kappa}^{\infty}\mathcal{S}_{2,\mathfrak{q}}(\zeta,\kappa\,;\lambda)\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{q}}!}.$$

Inserting this into the preceding identity gives

$$e_{\mathfrak{q},\lambda}\left(\mu(e_{\mathfrak{q},\lambda}(\xi)-1)\right)=\sum_{\zeta=0}^{\infty}\left(\sum_{\kappa=0}^{\zeta}(\mu)_{\kappa,\lambda}\mathcal{S}_{2,\mathfrak{q}}(\zeta,\kappa\,;\lambda)\right)\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{q}}!}.\tag{2.3}$$

Extracting the coefficient of $\xi^{\zeta}/[\zeta]_{\mathfrak{q}}!$ from both sides of (2.2) and (2.3) establishes the stated formula for $\phi_{\zeta,\mathfrak{q}}(\mu\,;\lambda)$. Taking $\mu=1$ gives the corresponding identity for the numbers $\phi_{\zeta,\mathfrak{q}}(\lambda)$. □

Definition 2.5. The *degenerate* $\mathfrak{q}$-*Fubini polynomials* $\mathcal{F}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)$ are defined via the following generating function (cf. [10, 20, 25]):

$$\frac{1}{1-\mu\left(e_{\mathfrak{q},\lambda}(\xi)-1\right)}=\sum_{\zeta=0}^{\infty}\mathcal{F}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{q}}!}.\tag{2.4}$$

We denote $\mathcal{F}_{\zeta,\mathfrak{q}}(\lambda):=\mathcal{F}_{\zeta,\mathfrak{q}}(1\,;\lambda)$ for the degenerate $\mathfrak{q}$-Fubini numbers.

Theorem 2.6. *For $\zeta \geq 0$,*

$$\mathcal{F}_{\zeta,\mathfrak{q}}(\mu\,;\lambda) = \sum_{\kappa=0}^{\zeta} \mu^{\kappa}[\kappa]_{\mathfrak{q}}!\,\mathcal{S}_{2,\mathfrak{q}}(\zeta,\kappa\,;\lambda).$$

Proof. From (2.4) and the geometric series expansion, we have

$$\frac{1}{1-\mu(\mathrm{e}_{\mathfrak{q},\lambda}(\xi)-1)} = \sum_{\kappa=0}^{\infty} \mu^{\kappa}(\mathrm{e}_{\mathfrak{q},\lambda}(\xi)-1)^{\kappa}.$$

Apply (2.1) to each power:

$$(\mathrm{e}_{\mathfrak{q},\lambda}(\xi)-1)^{\kappa} = [\kappa]_{\mathfrak{q}}!\sum_{\zeta=\kappa}^{\infty} \mathcal{S}_{2,\mathfrak{q}}(\zeta,\kappa\,;\lambda)\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{q}}!}.$$

Then,

$$\frac{1}{1-\mu(\mathrm{e}_{\mathfrak{q},\lambda}(\xi)-1)} = \sum_{\zeta=0}^{\infty}\left(\sum_{\kappa=0}^{\zeta} \mu^{\kappa}[\kappa]_{\mathfrak{q}}!\,\mathcal{S}_{2,\mathfrak{q}}(\zeta,\kappa\,;\lambda)\right)\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{q}}!}. \tag{2.5}$$

Extracting the coefficient of $\xi^{\zeta}/[\zeta]_{\mathfrak{q}}!$ from both sides of (2.4) and (2.5) gives the stated formula. □

Definition 2.7. The *degenerate* $\mathfrak{q}$*-derangement polynomials* $\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)$ are defined via the following generating function, motivated by the degenerate derangement and $\mathfrak{q}$-derangement frameworks in [1–3]:

$$\frac{1}{1-\xi}\,\mathrm{e}_{\mathfrak{q},\lambda}((\mu-1)\xi) = \sum_{\zeta=0}^{\infty} \mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{q}}!}. \tag{2.6}$$

Setting $\mu = 0$ in $\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)$ yields the numbers $\mathfrak{d}_{\zeta,\mathfrak{q}}(\lambda) := \mathfrak{d}_{\zeta,\mathfrak{q}}(0\,;\lambda)$, which we call the *degenerate* $\mathfrak{q}$-*derangement numbers*.

Theorem 2.8. *For $\zeta \geq 0$,*

$$\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda) = [\zeta]_{\mathfrak{q}}!\sum_{m=0}^{\zeta}\frac{(\mu-1)_{m,\lambda}}{[m]_{\mathfrak{q}}!}.$$

In particular,

$$\mathfrak{d}_{\zeta,\mathfrak{q}}(\lambda) = [\zeta]_{\mathfrak{q}}!\sum_{m=0}^{\zeta}\frac{(-1)_{m,\lambda}}{[m]_{\mathfrak{q}}!}.$$

Proof. Expand both factors in (2.6):

$$\frac{1}{1-\xi} = \sum_{l=0}^{\infty}\xi^{l}, \qquad \mathrm{e}_{\mathfrak{q},\lambda}((\mu-1)\xi) = \sum_{m=0}^{\infty}\frac{(\mu-1)_{m,\lambda}}{[m]_{\mathfrak{q}}!}\,\xi^{m}.$$

Multiply and collect powers of ξ:

$$\frac{1}{1-\xi}\mathrm{e}_{\mathfrak{q},\lambda}((\mu-1)\xi) = \sum_{\zeta=0}^{\infty}\left(\sum_{m=0}^{\zeta}\frac{(\mu-1)_{m,\lambda}}{[m]_{\mathfrak{q}}!}\right)\xi^{\zeta}.$$

Rewriting $\xi^\zeta = [\zeta]_{\mathfrak{q}}!\,\frac{\xi^\zeta}{[\zeta]_{\mathfrak{q}}!}$ yields

$$\frac{1}{1-\xi}\mathfrak{e}_{\mathfrak{q},\lambda}((\mu-1)\xi) = \sum_{\zeta=0}^{\infty}\left([\zeta]_{\mathfrak{q}}!\sum_{m=0}^{\zeta}\frac{(\mu-1)_{m,\lambda}}{[m]_{\mathfrak{q}}!}\right)\frac{\xi^\zeta}{[\zeta]_{\mathfrak{q}}!}. \tag{2.7}$$

Comparing the coefficients in (2.6) and (2.7) gives the desired identity. Setting $\mu = 0$ gives the formula for $\mathfrak{d}_{\zeta,\mathfrak{q}}(\lambda)$. □

Example 2.9. The explicit formula in Theorem 2.8 gives the first values without any further recurrence. Writing $a = \mu - 1$, one obtains

$$\mathfrak{d}_{0,\mathfrak{q}}(\mu\,;\lambda) = 1, \qquad \mathfrak{d}_{1,\mathfrak{q}}(\mu\,;\lambda) = \mu,$$

$$\mathfrak{d}_{2,\mathfrak{q}}(\mu\,;\lambda) = [2]_{\mathfrak{q}}!\,\mu + a(a-\lambda),$$

$$\mathfrak{d}_{3,\mathfrak{q}}(\mu\,;\lambda) = [3]_{\mathfrak{q}}!\,\mu + [3]_{\mathfrak{q}}a(a-\lambda) + a(a-\lambda)(a-2\lambda).$$

Thus, the polynomial parameter μ shifts the input from the ordinary degenerate factorial basis, and the parameter $\mathfrak{q}$ enters through the row factors $[\zeta]_{\mathfrak{q}}!/[m]_{\mathfrak{q}}!$. For the associated numbers obtained by setting $\mu = 0$, the same formulas become

$$\mathfrak{d}_{0,\mathfrak{q}}(\lambda) = 1, \qquad \mathfrak{d}_{1,\mathfrak{q}}(\lambda) = 0,$$

$$\mathfrak{d}_{2,\mathfrak{q}}(\lambda) = 1 + \lambda,$$

$$\mathfrak{d}_{3,\mathfrak{q}}(\lambda) = (1+\lambda)(\mathfrak{q} + \mathfrak{q}^2 - 2\lambda),$$

$$\mathfrak{d}_{4,\mathfrak{q}}(\lambda) = (1+\lambda)\Big([4]_{\mathfrak{q}}[3]_{\mathfrak{q}} - [4]_{\mathfrak{q}}(1+2\lambda) + (1+2\lambda)(1+3\lambda)\Big).$$

As a concrete check, for $\mathfrak{q} = 1/2$ and $\lambda = 0.3$, these formulas yield

$$\mathfrak{d}_{0,\mathfrak{q}}(\lambda) = 1, \quad \mathfrak{d}_{1,\mathfrak{q}}(\lambda) = 0, \quad \mathfrak{d}_{2,\mathfrak{q}}(\lambda) = 1.3, \quad \mathfrak{d}_{3,\mathfrak{q}}(\lambda) = 0.195, \quad \mathfrak{d}_{4,\mathfrak{q}}(\lambda) = 4.317625.$$

These values agree with the tabulated values in Example 3.8 below and give a direct way to verify the two-term inverse relation in Theorem 2.10 for small degrees.

Theorem 2.10. *For $\zeta \geq 1$,*

$$(\mu-1)_{\zeta,\lambda} = \mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda) - [\zeta]_{\mathfrak{q}}\,\mathfrak{d}_{\zeta-1,\mathfrak{q}}(\mu\,;\lambda).$$

In particular,

$$(-1)_{\zeta,\lambda} = \mathfrak{d}_{\zeta,\mathfrak{q}}(\lambda) - [\zeta]_{\mathfrak{q}}\,\mathfrak{d}_{\zeta-1,\mathfrak{q}}(\lambda).$$

Proof. Multiply both sides of (2.6) by $(1-\xi)$:

$$\mathfrak{e}_{\mathfrak{q},\lambda}((\mu-1)\xi) = (1-\xi)\sum_{\zeta=0}^{\infty}\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)\frac{\xi^\zeta}{[\zeta]_{\mathfrak{q}}!}.$$

Expand the right-hand side, and shift indices:

$$(1-\xi)\sum_{\zeta\geq 0}\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{q}}!}=1+\sum_{\zeta=1}^{\infty}\Big(\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)-[\zeta]_{\mathfrak{q}}\,\mathfrak{d}_{\zeta-1,\mathfrak{q}}(\mu\,;\lambda)\Big)\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{q}}!}.$$

Thus, we have the coefficient identity

$$\sum_{\zeta=0}^{\infty}(\mu-1)_{\zeta,\lambda}\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{q}}!}=1+\sum_{\zeta=1}^{\infty}\Big(\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)-[\zeta]_{\mathfrak{q}}\,\mathfrak{d}_{\zeta-1,\mathfrak{q}}(\mu\,;\lambda)\Big)\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{q}}!}. \tag{2.8}$$

Equating the coefficients of $\xi^{\zeta}/[\zeta]_{\mathfrak{q}}!$ for $\zeta\geq 1$ in (2.8) gives the stated recurrence. □

Remark 2.11. Theorems 2.8 and 2.10 show that the sequence $\{\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)\}_{\zeta\geq 0}$ is not just another list of coefficients extracted from a generating function. It is obtained from the degenerate falling-factorial sequence $\{(\mu-1)_{\zeta,\lambda}\}_{\zeta\geq 0}$ by the lower triangular transform

$$\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)=\sum_{m=0}^{\zeta}\frac{[\zeta]_{\mathfrak{q}}!}{[m]_{\mathfrak{q}}!}(\mu-1)_{m,\lambda}.$$

The inverse transform is the finite-band relation

$$(\mu-1)_{\zeta,\lambda}=\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)-[\zeta]_{\mathfrak{q}}\mathfrak{d}_{\zeta-1,\mathfrak{q}}(\mu\,;\lambda),\qquad \zeta\geq 1.$$

Consequently, the identities below may be viewed as compositions of this triangular transform with the degenerate $\mathfrak{q}$-Stirling, Bell, and Fubini transforms. This gives a structural interpretation of the formulas and explains the persistence of the same pattern in the higher-order and $(\mathfrak{p},\mathfrak{q})$ settings.

Theorem 2.12. *For $\zeta\geq 0$,*

$$\sum_{j=0}^{\zeta}\sum_{l=0}^{j}\begin{bmatrix}\zeta\\ j\end{bmatrix}_{\mathfrak{q}}(1)_{\zeta-j,\lambda}\,\mathfrak{d}_{l,\mathfrak{q}}(\mu\,;\lambda)(-1)^{l}\mathcal{S}_{2,\mathfrak{q}}(j,l\,;\lambda)=\sum_{\kappa=0}^{\zeta}(\mu-1)_{\kappa,\lambda}(-1)^{\kappa}\mathcal{S}_{2,\mathfrak{q}}(\zeta,\kappa\,;\lambda).$$

In particular, setting $\mu=0$ yields

$$\sum_{j=0}^{\zeta}\sum_{l=0}^{j}\begin{bmatrix}\zeta\\ j\end{bmatrix}_{\mathfrak{q}}(1)_{\zeta-j,\lambda}\,\mathfrak{d}_{l,\mathfrak{q}}(\lambda)(-1)^{l}\mathcal{S}_{2,\mathfrak{q}}(j,l\,;\lambda)=\sum_{\kappa=0}^{\zeta}(-1)_{\kappa,\lambda}(-1)^{\kappa}\mathcal{S}_{2,\mathfrak{q}}(\zeta,\kappa\,;\lambda).$$

Proof. Beginning with (2.6), substitute $\xi\mapsto 1-\mathrm{e}_{\mathfrak{q},\lambda}(\xi)$. Because $1-(1-\mathrm{e}_{\mathfrak{q},\lambda}(\xi))=\mathrm{e}_{\mathfrak{q},\lambda}(\xi)$, we obtain

$$\frac{1}{\mathrm{e}_{\mathfrak{q},\lambda}(\xi)}\,\mathrm{e}_{\mathfrak{q},\lambda}((\mu-1)(1-\mathrm{e}_{\mathfrak{q},\lambda}(\xi)))=\sum_{l=0}^{\infty}\mathfrak{d}_{l,\mathfrak{q}}(\mu\,;\lambda)\frac{(1-\mathrm{e}_{\mathfrak{q},\lambda}(\xi))^{l}}{[l]_{\mathfrak{q}}!}.$$

Multiplying both sides by $\mathrm{e}_{\mathfrak{q},\lambda}(\xi)$ yields

$$\mathrm{e}_{\mathfrak{q},\lambda}((\mu-1)(1-\mathrm{e}_{\mathfrak{q},\lambda}(\xi)))=\mathrm{e}_{\mathfrak{q},\lambda}(\xi)\sum_{l=0}^{\infty}\mathfrak{d}_{l,\mathfrak{q}}(\mu\,;\lambda)\frac{(1-\mathrm{e}_{\mathfrak{q},\lambda}(\xi))^{l}}{[l]_{\mathfrak{q}}!}. \tag{2.9}$$

We now expand the right-hand side of (2.9). First,

$$\mathrm{e}_{\mathrm{q},\lambda}(\xi)=\sum_{m=0}^{\infty}(1)_{m,\lambda}\frac{\xi^m}{[m]_{\mathrm{q}}!}.$$

Next, $(1-\mathrm{e}_{\mathrm{q},\lambda}(\xi))^l=(-1)^l(\mathrm{e}_{\mathrm{q},\lambda}(\xi)-1)^l$, and by (2.1),

$$\frac{1}{[l]_{\mathrm{q}}!}(\mathrm{e}_{\mathrm{q},\lambda}(\xi)-1)^l=\sum_{j=l}^{\infty}\mathcal{S}_{2,\mathrm{q}}(j,l\,;\lambda)\frac{\xi^j}{[j]_{\mathrm{q}}!}.$$

Hence,

$$\sum_{l\geq 0}\mathfrak{d}_{l,\mathrm{q}}(\mu\,;\lambda)\frac{(1-\mathrm{e}_{\mathrm{q},\lambda}(\xi))^l}{[l]_{\mathrm{q}}!}=\sum_{j=0}^{\infty}\left(\sum_{l=0}^{j}\mathfrak{d}_{l,\mathrm{q}}(\mu\,;\lambda)(-1)^l\mathcal{S}_{2,\mathrm{q}}(j,l\,;\lambda)\right)\frac{\xi^j}{[j]_{\mathrm{q}}!}.$$

Applying Remark 2.1 to the product with $\mathrm{e}_{\mathrm{q},\lambda}(\xi)$ gives

$$\mathrm{e}_{\mathrm{q},\lambda}(\xi)\sum_{l=0}^{\infty}\frac{\mathfrak{d}_{l,\mathrm{q}}(\mu\,;\lambda)}{[l]_{\mathrm{q}}!}(1-\mathrm{e}_{\mathrm{q},\lambda}(\xi))^l=\sum_{\zeta=0}^{\infty}\left(\sum_{j=0}^{\zeta}\sum_{l=0}^{j}\begin{bmatrix}\zeta\\ j\end{bmatrix}_{\mathrm{q}}(1)_{\zeta-j,\lambda}\,\mathfrak{d}_{l,\mathrm{q}}(\mu\,;\lambda)(-1)^l\right.$$
$$\left.\cdot\mathcal{S}_{2,\mathrm{q}}(j,l\,;\lambda)\right)\cdot\frac{\xi^\zeta}{[\zeta]_{\mathrm{q}}!}.$$

On the other hand, expand the left-hand side of (2.9) directly:

$$\begin{aligned}\mathrm{e}_{\mathrm{q},\lambda}((\mu-1)(1-\mathrm{e}_{\mathrm{q},\lambda}(\xi)))&=\sum_{\kappa=0}^{\infty}\frac{(\mu-1)_{\kappa,\lambda}}{[\kappa]_{\mathrm{q}}!}(1-\mathrm{e}_{\mathrm{q},\lambda}(\xi))^\kappa\\&=\sum_{\kappa=0}^{\infty}(\mu-1)_{\kappa,\lambda}(-1)^\kappa\frac{(\mathrm{e}_{\mathrm{q},\lambda}(\xi)-1)^\kappa}{[\kappa]_{\mathrm{q}}!}.\end{aligned}$$

Using (2.1) again, we obtain

$$\mathrm{e}_{\mathrm{q},\lambda}((\mu-1)(1-\mathrm{e}_{\mathrm{q},\lambda}(\xi)))=\sum_{\zeta=0}^{\infty}\left(\sum_{\kappa=0}^{\zeta}(\mu-1)_{\kappa,\lambda}(-1)^\kappa\mathcal{S}_{2,\mathrm{q}}(\zeta,\kappa\,;\lambda)\right)\frac{\xi^\zeta}{[\zeta]_{\mathrm{q}}!}.\tag{2.10}$$

Finally, combining (2.9) with the expansions above and (2.10) and then comparing coefficients of $\xi^\zeta/[\zeta]_{\mathrm{q}}!$ gives the stated identity. The case $\mu=0$ follows by direct substitution. □

Theorem 2.13. *For* $\zeta\geq 0$*,*

$$\sum_{\kappa=0}^{\zeta}\begin{bmatrix}\zeta\\ \kappa\end{bmatrix}_{\mathrm{q}}\mathrm{q}^{\binom{\zeta-\kappa}{2}}(-1)_{\zeta-\kappa,\lambda}\,\phi_{\kappa,\mathrm{q}}(1-\mu\,;\lambda)=\sum_{\kappa=0}^{\zeta}\mathfrak{d}_{\kappa,\mathrm{q}}(\mu\,;\lambda)(-1)^\kappa\mathcal{S}_{2,\mathrm{q}}(\zeta,\kappa\,;\lambda).$$

In particular, for $\mu=0$*,*

$$\sum_{\kappa=0}^{\zeta}\begin{bmatrix}\zeta\\ \kappa\end{bmatrix}_{\mathrm{q}}\mathrm{q}^{\binom{\zeta-\kappa}{2}}(-1)_{\zeta-\kappa,\lambda}\,\phi_{\kappa,\mathrm{q}}(\lambda)=\sum_{\kappa=0}^{\zeta}\mathfrak{d}_{\kappa,\mathrm{q}}(\lambda)(-1)^\kappa\mathcal{S}_{2,\mathrm{q}}(\zeta,\kappa\,;\lambda).$$

Proof. Rewrite (2.9) by using $(\mu-1)(1-\mathrm{e}_{\mathrm{q},\lambda}(\xi)) = (1-\mu)(\mathrm{e}_{\mathrm{q},\lambda}(\xi)-1)$:

$$\mathrm{e}_{\mathrm{q},\lambda}((1-\mu)(\mathrm{e}_{\mathrm{q},\lambda}(\xi)-1)) = \mathrm{e}_{\mathrm{q},\lambda}(\xi)\sum_{\kappa=0}^{\infty}\mathfrak{d}_{\kappa,\mathrm{q}}(\mu\,;\lambda)\frac{(1-\mathrm{e}_{\mathrm{q},\lambda}(\xi))^{\kappa}}{[\kappa]_{\mathrm{q}}!}.$$

Multiply by $\mathfrak{E}_{\mathrm{q},\lambda}(-\xi)$, and use $\mathfrak{E}_{\mathrm{q},\lambda}(-\xi)\mathrm{e}_{\mathrm{q},\lambda}(\xi) = 1$ to obtain

$$\mathfrak{E}_{\mathrm{q},\lambda}(-\xi)\,\mathrm{e}_{\mathrm{q},\lambda}((1-\mu)(\mathrm{e}_{\mathrm{q},\lambda}(\xi)-1)) = \sum_{\kappa=0}^{\infty}\mathfrak{d}_{\kappa,\mathrm{q}}(\mu\,;\lambda)\frac{(1-\mathrm{e}_{\mathrm{q},\lambda}(\xi))^{\kappa}}{[\kappa]_{\mathrm{q}}!}. \tag{2.11}$$

We now expand both sides in the basis $\xi^{\zeta}/[\zeta]_{\mathrm{q}}!$.

By definition,

$$\mathfrak{E}_{\mathrm{q},\lambda}(-\xi) = \sum_{l=0}^{\infty}\mathrm{q}^{\binom{l}{2}}\frac{(1)_{l,\lambda}}{[l]_{\mathrm{q}}!}(-\xi)^{l} = \sum_{l=0}^{\infty}\mathrm{q}^{\binom{l}{2}}\frac{(-1)_{l,\lambda}}{[l]_{\mathrm{q}}!}\xi^{l},$$

and by (2.2),

$$\mathrm{e}_{\mathrm{q},\lambda}((1-\mu)(\mathrm{e}_{\mathrm{q},\lambda}(\xi)-1)) = \sum_{\kappa=0}^{\infty}\phi_{\kappa,\mathrm{q}}(1-\mu\,;\lambda)\frac{\xi^{\kappa}}{[\kappa]_{\mathrm{q}}!}.$$

Using Remark 2.1 for the product, we obtain

$$\begin{aligned}&\mathfrak{E}_{\mathrm{q},\lambda}(-\xi)\,\mathrm{e}_{\mathrm{q},\lambda}((1-\mu)(\mathrm{e}_{\mathrm{q},\lambda}(\xi)-1))\\ &\quad=\sum_{\zeta=0}^{\infty}\left(\sum_{\kappa=0}^{\zeta}\begin{bmatrix}\zeta\\ \kappa\end{bmatrix}_{\mathrm{q}}\mathrm{q}^{\binom{\zeta-\kappa}{2}}(-1)_{\zeta-\kappa,\lambda}\phi_{\kappa,\mathrm{q}}(1-\mu\,;\lambda)\right)\cdot\frac{\xi^{\zeta}}{[\zeta]_{\mathrm{q}}!}.\end{aligned} \tag{2.12}$$

Because $(1-\mathrm{e}_{\mathrm{q},\lambda}(\xi))^{\kappa} = (-1)^{\kappa}(\mathrm{e}_{\mathrm{q},\lambda}(\xi)-1)^{\kappa}$, we get from (2.1)

$$\frac{(1-\mathrm{e}_{\mathrm{q},\lambda}(\xi))^{\kappa}}{[\kappa]_{\mathrm{q}}!} = (-1)^{\kappa}\sum_{\zeta=\kappa}^{\infty}\mathcal{S}_{2,\mathrm{q}}(\zeta,\kappa\,;\lambda)\frac{\xi^{\zeta}}{[\zeta]_{\mathrm{q}}!}.$$

Therefore,

$$\sum_{\kappa=0}^{\infty}\mathfrak{d}_{\kappa,\mathrm{q}}(\mu\,;\lambda)\frac{(1-\mathrm{e}_{\mathrm{q},\lambda}(\xi))^{\kappa}}{[\kappa]_{\mathrm{q}}!} = \sum_{\zeta=0}^{\infty}\left(\sum_{\kappa=0}^{\zeta}\mathfrak{d}_{\kappa,\mathrm{q}}(\mu\,;\lambda)(-1)^{\kappa}\mathcal{S}_{2,\mathrm{q}}(\zeta,\kappa\,;\lambda)\right)\frac{\xi^{\zeta}}{[\zeta]_{\mathrm{q}}!}. \tag{2.13}$$

Comparing the coefficients in (2.11) using (2.12) and (2.13) gives the stated identity. The case $\mu = 0$ is obtained by specialization. □

Theorem 2.14. *For $\zeta \geq 0$,*

$$\sum_{\kappa=0}^{\zeta}\mathfrak{d}_{\kappa,\mathrm{q}}(\mu\,;\lambda)\,\mathcal{S}_{2,\mathrm{q}}(\zeta,\kappa\,;\lambda) = \sum_{l=0}^{\zeta}\sum_{m=0}^{\zeta-l}\begin{bmatrix}\zeta\\ l\end{bmatrix}_{\mathrm{q}}\mathcal{F}_{l,\mathrm{q}}(\lambda)\,(\mu-1)_{m,\lambda}\,\mathcal{S}_{2,\mathrm{q}}(\zeta-l,m\,;\lambda).$$

Proof. Substitute $\xi \mapsto \mathrm{e}_{\mathrm{q},\lambda}(\xi)-1$ in (2.6). Because $1-(\mathrm{e}_{\mathrm{q},\lambda}(\xi)-1) = 2-\mathrm{e}_{\mathrm{q},\lambda}(\xi)$, we obtain

$$\frac{1}{2-\mathrm{e}_{\mathrm{q},\lambda}(\xi)}\,\mathrm{e}_{\mathrm{q},\lambda}((\mu-1)(\mathrm{e}_{\mathrm{q},\lambda}(\xi)-1)) = \sum_{\kappa=0}^{\infty}\mathfrak{d}_{\kappa,\mathrm{q}}(\mu\,;\lambda)\frac{(\mathrm{e}_{\mathrm{q},\lambda}(\xi)-1)^{\kappa}}{[\kappa]_{\mathrm{q}}!}.$$

Using (2.1) to expand each $(\mathrm{e}_{\mathrm{q},\lambda}(\xi)-1)^{\kappa}/[\kappa]_{\mathrm{q}}!$ yields

$$\frac{1}{2-\mathrm{e}_{\mathrm{q},\lambda}(\xi)}\,\mathrm{e}_{\mathrm{q},\lambda}\big((\mu-1)(\mathrm{e}_{\mathrm{q},\lambda}(\xi)-1)\big)=\sum_{\zeta=0}^{\infty}\left(\sum_{\kappa=0}^{\zeta}\mathfrak{d}_{\kappa,\mathrm{q}}(\mu\,;\lambda)\,\mathcal{S}_{2,\mathrm{q}}(\zeta,\kappa\,;\lambda)\right)\frac{\xi^{\zeta}}{[\zeta]_{\mathrm{q}}!}. \tag{2.14}$$

On the other hand, by (2.4) with $\mu=1$, we have

$$\frac{1}{2-\mathrm{e}_{\mathrm{q},\lambda}(\xi)}=\sum_{l=0}^{\infty}\mathcal{F}_{l,\mathrm{q}}(\lambda)\frac{\xi^{l}}{[l]_{\mathrm{q}}!},$$

and by Theorem 2.4,

$$\begin{aligned}\mathrm{e}_{\mathrm{q},\lambda}\big((\mu-1)(\mathrm{e}_{\mathrm{q},\lambda}(\xi)-1)\big)&=\sum_{j=0}^{\infty}\phi_{j,\mathrm{q}}(\mu-1\,;\lambda)\frac{\xi^{j}}{[j]_{\mathrm{q}}!}\\&=\sum_{j=0}^{\infty}\left(\sum_{m=0}^{j}(\mu-1)_{m,\lambda}\,\mathcal{S}_{2,\mathrm{q}}(j,m\,;\lambda)\right)\frac{\xi^{j}}{[j]_{\mathrm{q}}!}.\end{aligned}$$

Applying Remark 2.1 to the product yields

$$\begin{aligned}&\frac{1}{2-\mathrm{e}_{\mathrm{q},\lambda}(\xi)}\,\mathrm{e}_{\mathrm{q},\lambda}\big((\mu-1)(\mathrm{e}_{\mathrm{q},\lambda}(\xi)-1)\big)\\&\quad=\sum_{\zeta=0}^{\infty}\left(\sum_{l=0}^{\zeta}\sum_{m=0}^{\zeta-l}\begin{bmatrix}\zeta\\ l\end{bmatrix}_{\mathrm{q}}\mathcal{F}_{l,\mathrm{q}}(\lambda)\,(\mu-1)_{m,\lambda}\mathcal{S}_{2,\mathrm{q}}(\zeta-l,m\,;\lambda)\right)\cdot\frac{\xi^{\zeta}}{[\zeta]_{\mathrm{q}}!}.\end{aligned} \tag{2.15}$$

Comparing the coefficients in (2.14) and (2.15) establishes the theorem. □

Theorem 2.15. *For $\zeta\geq 0$,*

$$[\zeta]_{\mathrm{q}}!=\sum_{l=0}^{\zeta}\begin{bmatrix}\zeta\\ l\end{bmatrix}_{\mathrm{q}}\mathfrak{d}_{l,\mathrm{q}}(\lambda)\,\mathrm{q}^{\binom{\zeta-l}{2}}(1)_{\zeta-l,\lambda}.$$

Proof. Set $\mu=0$ in (2.6) to get

$$\frac{1}{1-\xi}\,\mathrm{e}_{\mathrm{q},\lambda}(-\xi)=\sum_{l=0}^{\infty}\mathfrak{d}_{l,\mathrm{q}}(\lambda)\frac{\xi^{l}}{[l]_{\mathrm{q}}!}.$$

Multiply both sides by $\mathfrak{E}_{\mathrm{q},\lambda}(\xi)$. Using $\mathrm{e}_{\mathrm{q},\lambda}(-\xi)\mathfrak{E}_{\mathrm{q},\lambda}(\xi)=1$, we obtain

$$\frac{1}{1-\xi}=\left(\sum_{l=0}^{\infty}\mathfrak{d}_{l,\mathrm{q}}(\lambda)\frac{\xi^{l}}{[l]_{\mathrm{q}}!}\right)\left(\sum_{m=0}^{\infty}\mathrm{q}^{\binom{m}{2}}(1)_{m,\lambda}\frac{\xi^{m}}{[m]_{\mathrm{q}}!}\right).$$

By Remark 2.1, this becomes

$$\frac{1}{1-\xi}=\sum_{\zeta=0}^{\infty}\left(\sum_{l=0}^{\zeta}\begin{bmatrix}\zeta\\ l\end{bmatrix}_{\mathrm{q}}\mathfrak{d}_{l,\mathrm{q}}(\lambda)\,\mathrm{q}^{\binom{\zeta-l}{2}}(1)_{\zeta-l,\lambda}\right)\frac{\xi^{\zeta}}{[\zeta]_{\mathrm{q}}!}. \tag{2.16}$$

On the other hand,

$$\frac{1}{1-\xi}=\sum_{\zeta=0}^{\infty}\xi^{\zeta}=\sum_{\zeta=0}^{\infty}[\zeta]_{\mathrm{q}}!\,\frac{\xi^{\zeta}}{[\zeta]_{\mathrm{q}}!}. \tag{2.17}$$

Equating coefficients of $\xi^{\zeta}/[\zeta]_{\mathrm{q}}!$ in (2.16) and (2.17) yields the statement. □

Definition 2.16. The *higher-order degenerate* $\mathfrak{q}$*-derangement polynomials* are introduced via the following generating function for all $r \in \mathbb{N}$.

$$\frac{1}{(1-\xi)^r}\,\mathfrak{e}_{\mathfrak{q},\lambda}((\mu-1)\xi) = \sum_{\zeta=0}^{\infty} \mathfrak{d}^{(r)}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)\frac{\xi^\zeta}{[\zeta]_\mathfrak{q}!}. \tag{2.18}$$

We denote $\mathfrak{d}^{(r)}_{\zeta,\mathfrak{q}}(\lambda) := \mathfrak{d}^{(r)}_{\zeta,\mathfrak{q}}(0\,;\lambda)$.

Theorem 2.17. *For* $\zeta \geq 0$*,*

$$\mathfrak{d}^{(r)}_{\zeta,\mathfrak{q}}(\mu\,;\lambda) = [\zeta]_\mathfrak{q}!\sum_{l=0}^{\zeta}\frac{(\mu-1)_{l,\lambda}}{[l]_\mathfrak{q}!}\binom{\zeta-l+r-1}{\zeta-l}.$$

In particular,

$$\mathfrak{d}^{(r)}_{\zeta,\mathfrak{q}}(\lambda) = [\zeta]_\mathfrak{q}!\sum_{l=0}^{\zeta}\frac{(-1)_{l,\lambda}}{[l]_\mathfrak{q}!}\binom{\zeta-l+r-1}{\zeta-l}.$$

Proof. Expand $(1-\xi)^{-r} = \sum_{m=0}^{\infty}\binom{m+r-1}{m}\xi^m$ and $\mathfrak{e}_{\mathfrak{q},\lambda}((\mu-1)\xi) = \sum_{l=0}^{\infty}(\mu-1)_{l,\lambda}\xi^l/[l]_\mathfrak{q}!$. Then,

$$\frac{1}{(1-\xi)^r}\mathfrak{e}_{\mathfrak{q},\lambda}((\mu-1)\xi) = \sum_{\zeta=0}^{\infty}\left(\sum_{l=0}^{\zeta}\binom{\zeta-l+r-1}{\zeta-l}\frac{(\mu-1)_{l,\lambda}}{[l]_\mathfrak{q}!}\right)\xi^\zeta.$$

Rewriting $\xi^\zeta = [\zeta]_\mathfrak{q}!\,\xi^\zeta/[\zeta]_\mathfrak{q}!$ gives

$$\frac{1}{(1-\xi)^r}\mathfrak{e}_{\mathfrak{q},\lambda}((\mu-1)\xi) = \sum_{\zeta=0}^{\infty}\left([\zeta]_\mathfrak{q}!\sum_{l=0}^{\zeta}\binom{\zeta-l+r-1}{\zeta-l}\frac{(\mu-1)_{l,\lambda}}{[l]_\mathfrak{q}!}\right)\frac{\xi^\zeta}{[\zeta]_\mathfrak{q}!}. \tag{2.19}$$

Comparing the coefficients with (2.18) yields the result. The case $\mu = 0$ follows by direct substitution. □

For later use, we also write $Bel_{\zeta,\mathfrak{q}}(\lambda) := \phi_{\zeta,\mathfrak{q}}(\lambda)$ for the degenerate $\mathfrak{q}$-Bell numbers.

Theorem 2.18. *For* $\zeta \geq 0$*,*

$$\sum_{m=0}^{\zeta}\mathfrak{d}_{m,\mathfrak{q}}(\lambda)(-1)^m\mathcal{S}_{2,\mathfrak{q}}(\zeta,m\mid-\lambda) = \sum_{m=0}^{\zeta}\begin{bmatrix}\zeta\\m\end{bmatrix}_\mathfrak{q} Bel_{m,\mathfrak{q}}(-\lambda)\,(-1)_{\zeta-m,-\lambda}.$$

Proof. Replacing ξ with $1-\mathfrak{e}_{-\lambda,\mathfrak{q}}(\xi)$ in (2.6) with $\mu = 0$ gives

$$\mathfrak{e}^{-1}_{-\lambda,\mathfrak{q}}(\xi)\,\mathfrak{e}^{-1}_{\lambda,\mathfrak{q}}(1-\mathfrak{e}^{-1}_{-\lambda,\mathfrak{q}}(\xi)) = \sum_{\zeta=0}^{\infty}\left(\sum_{m=0}^{\zeta}\mathfrak{d}_{m,\mathfrak{q}}(\lambda)(-1)^m\mathcal{S}_{2,\mathfrak{q}}(\zeta,m\mid-\lambda)\right)\frac{\xi^\zeta}{[\zeta]_\mathfrak{q}!}. \tag{2.20}$$

In contrast, the expansion of the left-hand side using the generating function of $Bel_{m,\mathfrak{q}}(-\lambda)$ and applying Remark 2.1 results in the following equation:

$$\mathfrak{e}^{-1}_{-\lambda,\mathfrak{q}}(\xi)\,\mathfrak{e}^{-1}_{\lambda,\mathfrak{q}}(1-\mathfrak{e}^{-1}_{-\lambda,\mathfrak{q}}(\xi)) = \sum_{\zeta=0}^{\infty}\left(\sum_{m=0}^{\zeta}\begin{bmatrix}\zeta\\m\end{bmatrix}_\mathfrak{q} Bel_{m,\mathfrak{q}}(-\lambda)\,(-1)_{\zeta-m,-\lambda}\right)\frac{\xi^\zeta}{[\zeta]_\mathfrak{q}!}. \tag{2.21}$$

Comparing the coefficients in (2.20) and (2.21) establishes the theorem. □

Theorem 2.19. *For $\zeta \geq 0$,*

$$Bel_{\zeta,\mathfrak{q}}(-\lambda) = \sum_{j=0}^{\zeta}\sum_{m=0}^{j}\begin{bmatrix}\zeta\\ j\end{bmatrix}_{\mathfrak{q}}(1)_{\zeta-j,-\lambda}(-1)^m\mathfrak{d}_{m,\mathfrak{q}}(\lambda)\,\mathcal{S}_{2,\mathfrak{q}}(j,m\mid-\lambda).$$

Proof. From (2.6) with $\mu = 0$, one obtains

$$\frac{1}{1+\xi}\,\mathfrak{e}_{\lambda,\mathfrak{q}}^{-1}(-\xi) = \sum_{m=0}^{\infty}\mathfrak{d}_{m,\mathfrak{q}}(\lambda)(-1)^m\frac{\xi^m}{[m]_{\mathfrak{q}}!}. \tag{2.22}$$

Replace ξ by $\mathfrak{e}_{-\lambda,\mathfrak{q}}(\xi)-1$ in (2.22), and expand $(\mathfrak{e}_{-\lambda,\mathfrak{q}}(\xi)-1)^m$ via (2.1) to obtain

$$\begin{aligned}&\mathfrak{e}_{\lambda,\mathfrak{q}}^{-1}(1-\mathfrak{e}_{-\lambda,\mathfrak{q}}(\xi))\\ &\quad=\sum_{\zeta=0}^{\infty}\left(\sum_{j=0}^{\zeta}\sum_{m=0}^{j}\begin{bmatrix}\zeta\\ j\end{bmatrix}_{\mathfrak{q}}(1)_{\zeta-j,-\lambda}(-1)^m\mathfrak{d}_{m,\mathfrak{q}}(\lambda)\,\mathcal{S}_{2,\mathfrak{q}}(j,m\mid-\lambda)\right)\frac{\xi^\zeta}{[\zeta]_{\mathfrak{q}}!}.\end{aligned} \tag{2.23}$$

Conversely,

$$\mathfrak{e}_{\lambda,\mathfrak{q}}^{-1}(1-\mathfrak{e}_{-\lambda,\mathfrak{q}}(\xi)) = \mathfrak{e}_{-\lambda,\mathfrak{q}}(\mathfrak{e}_{-\lambda,\mathfrak{q}}(\xi)-1) = \sum_{\zeta=0}^{\infty}Bel_{\zeta,\mathfrak{q}}(-\lambda)\frac{\xi^\zeta}{[\zeta]_{\mathfrak{q}}!}. \tag{2.24}$$

Comparing the coefficients in (2.23) and (2.24) gives the stated identity. □

3. Further results and extensions

In this section, we collect further consequences of the constructions from Section 2. The aim is to show that the degenerate $\mathfrak{q}$-derangement family is stable under three natural operations: passage to a $(\mathfrak{p},\mathfrak{q})$-factorial basis, multiplication by higher powers of $(1-\xi)^{-1}$, and specialization to known limiting cases. This organization makes the later formulas part of one transform mechanism rather than independent formal variants. For background on $(\mathfrak{p},\mathfrak{q})$-calculus, we refer to [26, 27]. For $\lambda \in \mathbb{C}$, we use the degenerate falling factorial given by

$$(\mu)_{0,\lambda} = 1, \qquad (\mu)_{\zeta,\lambda} = \mu(\mu-\lambda)(\mu-2\lambda)\cdots(\mu-(\zeta-1)\lambda), \quad (\zeta \geq 1). \tag{3.1}$$

Moreover, for $0 < |\mathfrak{q}| < |\mathfrak{p}| \leq 1$, the $(\mathfrak{p},\mathfrak{q})$-factorial is given (see [26, 27]) by

$$[\zeta]_{\mathfrak{p},\mathfrak{q}}! = \prod_{j=1}^{\zeta}[j]_{\mathfrak{p},\mathfrak{q}}, \qquad [j]_{\mathfrak{p},\mathfrak{q}} = \frac{\mathfrak{p}^j-\mathfrak{q}^j}{\mathfrak{p}-\mathfrak{q}}, \qquad (\zeta \geq 1), \tag{3.2}$$

and we set $[0]_{\mathfrak{p},\mathfrak{q}}! = 1$. We also set

$$\begin{bmatrix}\zeta\\ \kappa\end{bmatrix}_{\mathfrak{p},\mathfrak{q}} = \frac{[\zeta]_{\mathfrak{p},\mathfrak{q}}!}{[\kappa]_{\mathfrak{p},\mathfrak{q}}![\zeta-\kappa]_{\mathfrak{p},\mathfrak{q}}!} \qquad (0 \leq \kappa \leq \zeta). \tag{3.3}$$

We define the degenerate $(\mathfrak{p},\mathfrak{q})$-exponential function by (cf. [28])

$$\mathfrak{e}_{\mathfrak{p},\mathfrak{q},\lambda}(\xi) = \sum_{\zeta=0}^{\infty}(1)_{\zeta,\lambda}\frac{\xi^\zeta}{[\zeta]_{\mathfrak{p},\mathfrak{q}}!}. \tag{3.4}$$

We will also use the $(\mathfrak{p},\mathfrak{q})$-exponential function

$$\mathfrak{e}_{\mathfrak{p},\mathfrak{q}}(\xi)=\sum_{\zeta=0}^{\infty}\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{p},\mathfrak{q}}!},$$

which is standard in $(\mathfrak{p},\mathfrak{q})$-calculus; see [26, 27].

Definition 3.1. The degenerate $(\mathfrak{p},\mathfrak{q})$-derangement polynomials $\mathfrak{d}_{\zeta,\mathfrak{p},\mathfrak{q}}(\mu;\lambda)$ are characterized by the generating function

$$\frac{1}{1-\xi}\,\mathfrak{e}_{\mathfrak{p},\mathfrak{q},\lambda}((\mu-1)\xi)=\sum_{\zeta=0}^{\infty}\mathfrak{d}_{\zeta,\mathfrak{p},\mathfrak{q}}(\mu;\lambda)\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{p},\mathfrak{q}}!}. \tag{3.5}$$

In particular, $\mathfrak{d}_{\zeta,\mathfrak{p},\mathfrak{q}}(\lambda):=\mathfrak{d}_{\zeta,\mathfrak{p},\mathfrak{q}}(0;\lambda)$ are called the degenerate $(\mathfrak{p},\mathfrak{q})$-derangement numbers.

Theorem 3.2. *For $\zeta\geq 0$, we have*

$$\mathfrak{d}_{\zeta,\mathfrak{p},\mathfrak{q}}(\mu;\lambda)=[\zeta]_{\mathfrak{p},\mathfrak{q}}!\sum_{m=0}^{\zeta}\frac{(\mu-1)_{m,\lambda}}{[m]_{\mathfrak{p},\mathfrak{q}}!}.$$

Proof. From (3.4) and (3.5), we obtain

$$\sum_{\zeta=0}^{\infty}\mathfrak{d}_{\zeta,\mathfrak{p},\mathfrak{q}}(\mu;\lambda)\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{p},\mathfrak{q}}!}=\frac{1}{1-\xi}\sum_{m=0}^{\infty}(\mu-1)_{m,\lambda}\frac{\xi^{m}}{[m]_{\mathfrak{p},\mathfrak{q}}!}.$$

Because $1/1-\xi=\sum_{l=0}^{\infty}\xi^{l}$, we have

$$\begin{aligned}\frac{1}{1-\xi}\sum_{m=0}^{\infty}(\mu-1)_{m,\lambda}\frac{\xi^{m}}{[m]_{\mathfrak{p},\mathfrak{q}}!}&=\sum_{l=0}^{\infty}\xi^{l}\sum_{m=0}^{\infty}(\mu-1)_{m,\lambda}\frac{\xi^{m}}{[m]_{\mathfrak{p},\mathfrak{q}}!}\\&=\sum_{\zeta=0}^{\infty}\left(\sum_{m=0}^{\zeta}\frac{(\mu-1)_{m,\lambda}}{[m]_{\mathfrak{p},\mathfrak{q}}!}\right)\xi^{\zeta}.\end{aligned}$$

Multiplying the coefficient of ξ^{ζ} by $[\zeta]_{\mathfrak{p},\mathfrak{q}}!/[\zeta]_{\mathfrak{p},\mathfrak{q}}!=1$, we obtain

$$\sum_{\zeta=0}^{\infty}\mathfrak{d}_{\zeta,\mathfrak{p},\mathfrak{q}}(\mu;\lambda)\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{p},\mathfrak{q}}!}=\sum_{\zeta=0}^{\infty}\left([\zeta]_{\mathfrak{p},\mathfrak{q}}!\sum_{m=0}^{\zeta}\frac{(\mu-1)_{m,\lambda}}{[m]_{\mathfrak{p},\mathfrak{q}}!}\right)\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{p},\mathfrak{q}}!},$$

which yields the desired identity by comparing coefficients. □

Theorem 3.3. *For $\zeta\geq 1$, we have the recurrence relation*

$$(\mu-1)_{\zeta,\lambda}=\mathfrak{d}_{\zeta,\mathfrak{p},\mathfrak{q}}(\mu;\lambda)-[\zeta]_{\mathfrak{p},\mathfrak{q}}\mathfrak{d}_{\zeta-1,\mathfrak{p},\mathfrak{q}}(\mu;\lambda).$$

Proof. Multiplying both sides of (3.5) by $(1-\xi)$ gives

$$\mathfrak{e}_{\mathfrak{p},\mathfrak{q},\lambda}((\mu-1)\xi)=(1-\xi)\sum_{\zeta=0}^{\infty}\mathfrak{d}_{\zeta,\mathfrak{p},\mathfrak{q}}(\mu;\lambda)\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{p},\mathfrak{q}}!}.$$

The right-hand side can be written as

$$\sum_{\zeta=0}^{\infty}\mathfrak{d}_{\zeta,\mathfrak{p},\mathfrak{q}}(\mu\,;\lambda)\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{p},\mathfrak{q}}!}-\sum_{\zeta=0}^{\infty}\mathfrak{d}_{\zeta,\mathfrak{p},\mathfrak{q}}(\mu\,;\lambda)\frac{\xi^{\zeta+1}}{[\zeta]_{\mathfrak{p},\mathfrak{q}}!}$$

$$=\mathfrak{d}_{0,\mathfrak{p},\mathfrak{q}}(\mu\,;\lambda)+\sum_{\zeta=1}^{\infty}\left(\mathfrak{d}_{\zeta,\mathfrak{p},\mathfrak{q}}(\mu\,;\lambda)-\frac{[\zeta]_{\mathfrak{p},\mathfrak{q}}!}{[\zeta-1]_{\mathfrak{p},\mathfrak{q}}!}\mathfrak{d}_{\zeta-1,\mathfrak{p},\mathfrak{q}}(\mu\,;\lambda)\right)\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{p},\mathfrak{q}}!}.$$

Because $[\zeta]_{\mathfrak{p},\mathfrak{q}}!/[\zeta-1]_{\mathfrak{p},\mathfrak{q}}!=[\zeta]_{\mathfrak{p},\mathfrak{q}}$, and

$$\mathfrak{e}_{\mathfrak{p},\mathfrak{q},\lambda}((\mu-1)\xi)=\sum_{\zeta=0}^{\infty}(\mu-1)_{\zeta,\lambda}\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{p},\mathfrak{q}}!},$$

a comparison of coefficients gives the stated recurrence. □

Corollary 3.4. *(i) If* $\mathfrak{p}\to 1$, *then* $[\zeta]_{\mathfrak{p},\mathfrak{q}}!\to[\zeta]_{\mathfrak{q}}!$, *and* $\mathfrak{d}_{\zeta,\mathfrak{p},\mathfrak{q}}(\mu\,;\lambda)\to\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)$.
(ii) If $\mathfrak{q}\to 1$, *and* $\mathfrak{p}\to 1$, *then* $[\zeta]_{\mathfrak{p},\mathfrak{q}}!\to\zeta!$, *and* $\mathfrak{d}_{\zeta,\mathfrak{p},\mathfrak{q}}(\mu\,;\lambda)\to\mathfrak{d}_{\zeta}(\mu\,;\lambda)$, *where*

$$\frac{1}{1-\xi}\,\mathfrak{e}_{\lambda}((\mu-1)\xi)=\sum_{\zeta=0}^{\infty}\mathfrak{d}_{\zeta}(\mu\,;\lambda)\frac{\xi^{\zeta}}{\zeta!},$$

and $\mathfrak{e}_{\lambda}(\xi)=\sum_{\zeta=0}^{\infty}(1)_{\zeta,\lambda}\,\xi^{\zeta}/\zeta!$ *is the usual degenerate exponential function.*

Recall that the degenerate $\mathfrak{q}$-derangement polynomials of order r are defined by (2.18). Let

$$A_{\zeta}^{(s)}=[\zeta]_{\mathfrak{q}}!\binom{\zeta+s-1}{\zeta}\qquad(\zeta\geq 0,\ s\in\mathbb{N}).\tag{3.6}$$

Then,

$$\frac{1}{(1-\xi)^{s}}=\sum_{\zeta=0}^{\infty}\binom{\zeta+s-1}{\zeta}\xi^{\zeta}=\sum_{\zeta=0}^{\infty}A_{\zeta}^{(s)}\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{q}}!}.\tag{3.7}$$

Theorem 3.5. *For* $r,s\in\mathbb{N}$ *and* $\zeta\geq 0$, *we have*

$$\mathfrak{d}_{\zeta,\mathfrak{q}}^{(r+s)}(\mu\,;\lambda)=\sum_{\kappa=0}^{\zeta}\binom{\zeta}{\kappa}_{\mathfrak{q}}\mathfrak{d}_{\kappa,\mathfrak{q}}^{(r)}(\mu\,;\lambda)\,A_{\zeta-\kappa}^{(s)}.$$

Proof. Using (2.18) and (3.7), we have

$$\begin{aligned}\sum_{\zeta=0}^{\infty}\mathfrak{d}_{\zeta,\mathfrak{q}}^{(r+s)}(\mu\,;\lambda)\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{q}}!}&=\frac{1}{(1-\xi)^{r+s}}\mathfrak{e}_{\mathfrak{q},\lambda}((\mu-1)\xi)\\&=\left(\frac{1}{(1-\xi)^{r}}\mathfrak{e}_{\mathfrak{q},\lambda}((\mu-1)\xi)\right)\left(\frac{1}{(1-\xi)^{s}}\right)\\&=\left(\sum_{\zeta=0}^{\infty}\mathfrak{d}_{\zeta,\mathfrak{q}}^{(r)}(\mu\,;\lambda)\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{q}}!}\right)\left(\sum_{\zeta=0}^{\infty}A_{\zeta}^{(s)}\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{q}}!}\right).\end{aligned}$$

By the Cauchy product in the $\mathfrak{q}$-factorial basis, the coefficient of $\xi^\zeta/[\zeta]_{\mathfrak{q}}!$ is

$$\sum_{\kappa=0}^{\zeta}\binom{\zeta}{\kappa}_{\mathfrak{q}}\mathfrak{d}^{(r)}_{\kappa,\mathfrak{q}}(\mu\,;\lambda)\,A^{(s)}_{\zeta-\kappa},$$

which establishes the theorem. □

Corollary 3.6. *For $r\in\mathbb{N}$ and $\zeta\geq 0$, we have*

$$\mathfrak{d}^{(r+1)}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)=\sum_{\kappa=0}^{\zeta}\binom{\zeta}{\kappa}_{\mathfrak{q}}\mathfrak{d}^{(r)}_{\kappa,\mathfrak{q}}(\mu\,;\lambda)\,[\zeta-\kappa]_{\mathfrak{q}}!.$$

Proof. This follows from Theorem 3.5 by taking $s=1$, for which $A^{(1)}_m=[m]_{\mathfrak{q}}!$. □

The following limiting relations show that the families considered in this paper unify the known $\mathfrak{q}$-derangement and classical derangement polynomials.

Theorem 3.7. *For each fixed $\zeta\geq 0$, the following limits hold:*

(i) $\lim_{\lambda\to 0}\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)=\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu)$, *where $\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu)$ is introduced by*

$$\frac{1}{1-\xi}\,\mathfrak{e}_{\mathfrak{q}}((\mu-1)\xi)=\sum_{\zeta=0}^{\infty}\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu)\frac{\xi^\zeta}{[\zeta]_{\mathfrak{q}}!}.$$

(ii) $\lim_{\mathfrak{q}\to 1}\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)=\mathfrak{d}_{\zeta}(\mu\,;\lambda)$, *where $\mathfrak{d}_{\zeta}(\mu\,;\lambda)$ is introduced by*

$$\frac{1}{1-\xi}\,\mathfrak{e}_{\lambda}((\mu-1)\xi)=\sum_{\zeta=0}^{\infty}\mathfrak{d}_{\zeta}(\mu\,;\lambda)\frac{\xi^\zeta}{\zeta!}.$$

(iii) $\lim_{\mathfrak{q}\to 1,\,\lambda\to 0}\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)=\mathfrak{d}_{\zeta}(\mu)$, *where $\mathfrak{d}_{\zeta}(\mu)$ is the classical derangement polynomial determined by*

$$\frac{e^{(\mu-1)\xi}}{1-\xi}=\sum_{\zeta=0}^{\infty}\mathfrak{d}_{\zeta}(\mu)\frac{\xi^\zeta}{\zeta!}.$$

Proof. (i) By (3.1), we have $(\mu-1)_{m,\lambda}\to(\mu-1)^m$ as $\lambda\to 0$ for each fixed m. Hence,

$$\mathfrak{e}_{\mathfrak{q},\lambda}((\mu-1)\xi)=\sum_{m=0}^{\infty}(\mu-1)_{m,\lambda}\frac{\xi^m}{[m]_{\mathfrak{q}}!}\longrightarrow\sum_{m=0}^{\infty}(\mu-1)^m\frac{\xi^m}{[m]_{\mathfrak{q}}!}=\mathfrak{e}_{\mathfrak{q}}((\mu-1)\xi)$$

coefficientwise. Multiplying by $(1-\xi)^{-1}$ and comparing coefficients yields the desired limit for $\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)$.

(ii) For each fixed $m\geq 0$, we have $[m]_{\mathfrak{q}}!\to m!$ as $\mathfrak{q}\to 1$, and hence,

$$\mathfrak{e}_{\mathfrak{q},\lambda}((\mu-1)\xi)=\sum_{m=0}^{\infty}(\mu-1)_{m,\lambda}\frac{\xi^m}{[m]_{\mathfrak{q}}!}\longrightarrow\sum_{m=0}^{\infty}(\mu-1)_{m,\lambda}\frac{\xi^m}{m!}=\mathfrak{e}_{\lambda}((\mu-1)\xi).$$

Therefore, (2.6) reduces to the defining generating function of $\mathfrak{d}_{\zeta}(\mu\,;\lambda)$ in the limit $\mathfrak{q}\to 1$.

(iii) This follows from (ii) by letting $\lambda\to 0$ and using $\mathfrak{e}_{\lambda}((\mu-1)\xi)\to e^{(\mu-1)\xi}$. □

The next identity records the inclusion–exclusion content of the construction in a form that is independent of the particular coefficient extraction used above. The degenerate $\mathfrak{q}$-derangement numbers naturally admit a weighted inclusion–exclusion description. Indeed, the factorial expansion in Theorem 2.15 can be rewritten as

$$[\zeta]_{\mathfrak{q}}! = \sum_{\kappa=0}^{\zeta} \binom{\zeta}{\kappa}_{\mathfrak{q}} \mathfrak{d}_{\kappa,\mathfrak{q}}(\lambda)\, \mathfrak{q}^{\binom{\zeta-\kappa}{2}} (1)_{\zeta-\kappa,\lambda}, \tag{3.8}$$

which shows that the "defect" between $[\zeta]_{\mathfrak{q}}!$ and the degenerate weight $(1)_{\zeta,\lambda}$ is measured by the derangement numbers. This is consistent with the usual inclusion–exclusion principle for derangements, where the parameter $\mathfrak{q}$ encodes a $\mathfrak{q}$-weight and λ produces a degenerate deformation.

Example 3.8. For numerical verification, we list the first few values of $\mathfrak{d}_{\zeta,\mathfrak{q}}(\lambda)$ obtained from Theorem 2.8 for $\mathfrak{q} = 1/2$ and $\lambda = 0.3$:

ζ	0	1	2	3	4	5
$\mathfrak{d}_{\zeta,\mathfrak{q}}(\lambda)$	1	0	1.3	0.195	4.317625	−0.3290015625

These values can be used to check the recurrences in Theorem 2.10 and the convolution identity in Theorem 3.5 in concrete cases. The explicit sums obtained from the generating functions lead to a very simple inhomogeneous recurrence. This recurrence is useful for computational purposes and yields a convenient determinant representation.

Figures 1–3 visualize the values of $\mathfrak{d}_{\zeta,\mathfrak{q}}(\lambda)$ given by Theorem 2.8 for $\zeta = 0, \ldots, 15$ and $\mathfrak{q} \in \{0.25, 0.5, 0.75, 0.9\}$ for three representative choices of λ. The symmetric logarithmic scale is used so that both the wide range of magnitudes and the possible sign changes are visible on the same plot.

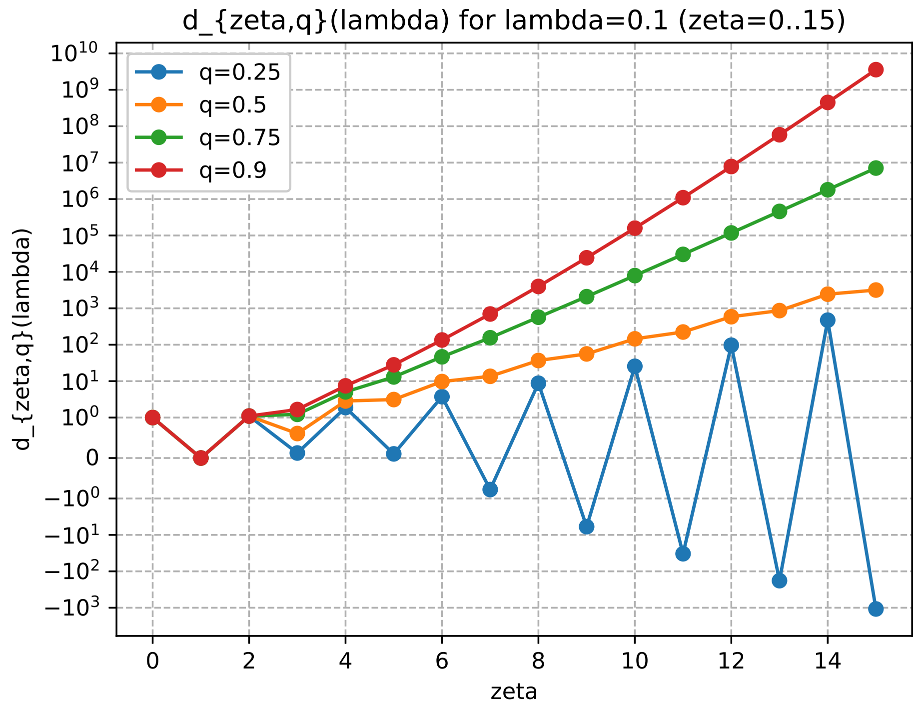

Figure 1. Values of $\mathfrak{d}_{\zeta,\mathfrak{q}}(\lambda)$ (from Theorem 2.8) for $\lambda = 0.1$ and $\mathfrak{q} \in \{0.25, 0.5, 0.75, 0.9\}$, plotted for $\zeta = 0, \ldots, 15$ on a symmetric logarithmic scale.

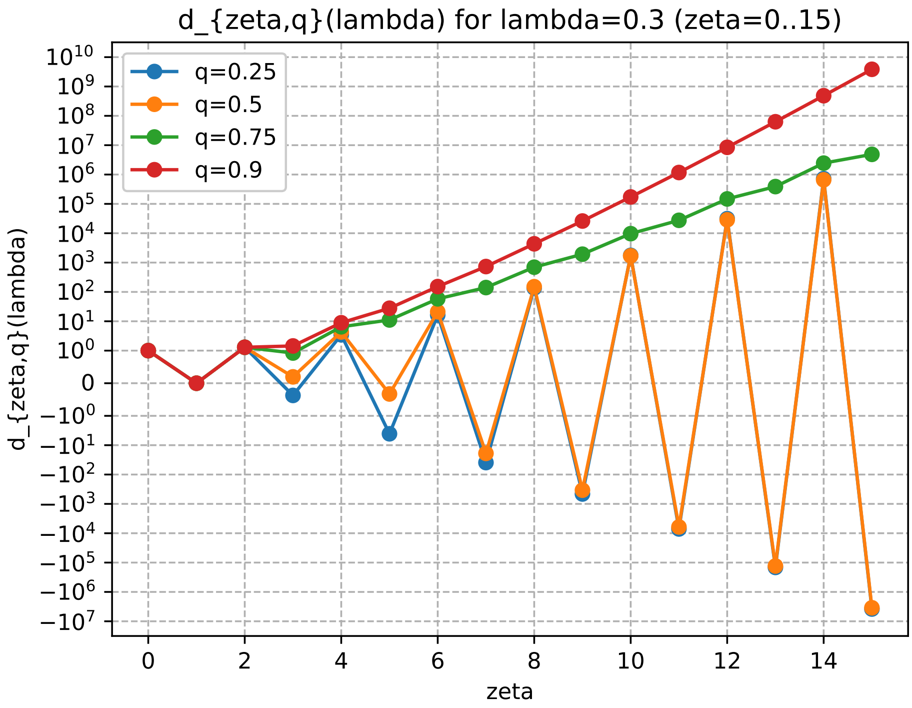

Figure 2. Values of $\mathfrak{d}_{\zeta,\mathfrak{q}}(\lambda)$ (from Theorem 2.8) for $\lambda = 0.3$ and $\mathfrak{q} \in \{0.25, 0.5, 0.75, 0.9\}$, plotted for $\zeta = 0, \ldots, 15$ on a symmetric logarithmic scale.

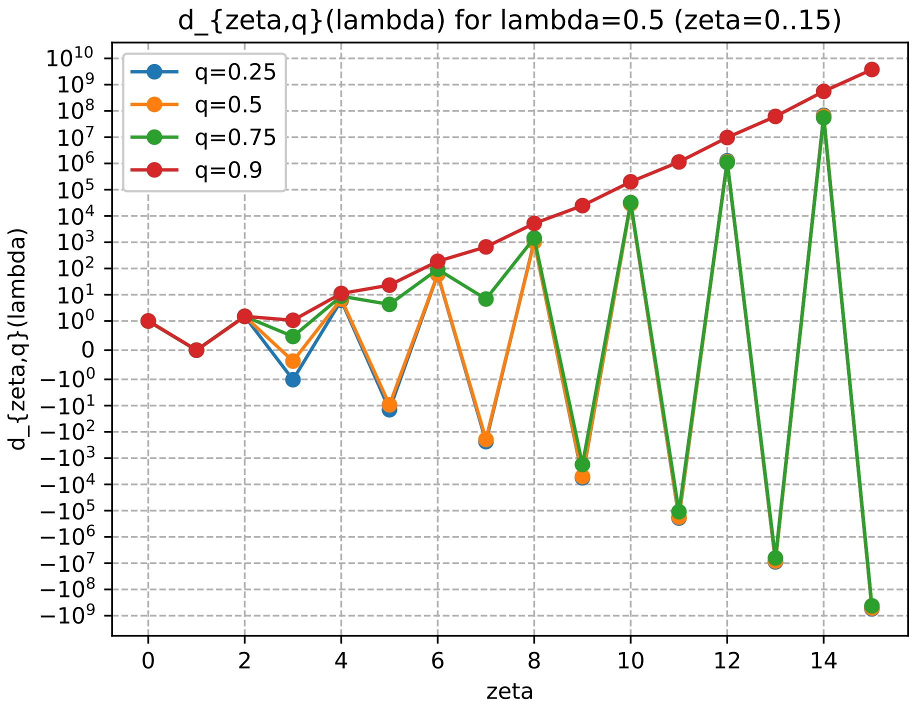

Figure 3. Values of $\mathfrak{d}_{\zeta,\mathfrak{q}}(\lambda)$ (from Theorem 2.8) for $\lambda = 0.5$ and $\mathfrak{q} \in \{0.25, 0.5, 0.75, 0.9\}$, plotted for $\zeta = 0, \ldots, 15$ on a symmetric logarithmic scale.

The plots in Figures 1–3 suggest a clear qualitative dependence on the deformation parameters. All cases satisfy the expected initial values $\mathfrak{d}_{0,\mathfrak{q}}(\lambda) = 1$ and $\mathfrak{d}_{1,\mathfrak{q}}(\lambda) = 0$. For $\lambda = 0.1$, the sequences for $\mathfrak{q} \geq 0.5$ stay positive and increase rapidly with ζ, whereas for $\mathfrak{q} = 0.25$, the values already alternate in sign, and their magnitudes grow with ζ. As λ increases to 0.3 and 0.5, this oscillatory behavior becomes more pronounced and extends to larger values of $\mathfrak{q}$, whereas for $\mathfrak{q}$ close to 1, the values remain positive and essentially monotone over the displayed range.

Proposition 3.9. *For $\zeta \geq 1$, we have*

$$\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda) = [\zeta]_{\mathfrak{q}}\, \mathfrak{d}_{\zeta-1,\mathfrak{q}}(\mu\,;\lambda) + (\mu-1)_{\zeta,\lambda}. \tag{3.9}$$

Moreover, for $0 < |\mathfrak{q}| < |\mathfrak{p}| \leq 1$,

$$\mathfrak{d}_{\zeta,\mathfrak{p},\mathfrak{q}}(\mu\,;\lambda) = [\zeta]_{\mathfrak{p},\mathfrak{q}}\, \mathfrak{d}_{\zeta-1,\mathfrak{p},\mathfrak{q}}(\mu\,;\lambda) + (\mu-1)_{\zeta,\lambda}. \tag{3.10}$$

Proof. By Theorem 2.8, we have

$$\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda) = [\zeta]_{\mathfrak{q}}! \sum_{m=0}^{\zeta} \frac{(\mu-1)_{m,\lambda}}{[m]_{\mathfrak{q}}!}.$$

Multiplying the corresponding identity for $\zeta - 1$ by $[\zeta]_{\mathfrak{q}}$ gives

$$[\zeta]_{\mathfrak{q}}\, \mathfrak{d}_{\zeta-1,\mathfrak{q}}(\mu\,;\lambda) = [\zeta]_{\mathfrak{q}}! \sum_{m=0}^{\zeta-1} \frac{(\mu-1)_{m,\lambda}}{[m]_{\mathfrak{q}}!}.$$

Subtracting the last two equations yields

$$\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda) - [\zeta]_{\mathfrak{q}}\, \mathfrak{d}_{\zeta-1,\mathfrak{q}}(\mu\,;\lambda) = (\mu-1)_{\zeta,\lambda},$$

which proves (3.9). The proof of (3.10) is the same, using Theorem 3.2 instead of Theorem 2.8. □

Corollary 3.10. *For $\zeta \geq 1$,*

$$\mathfrak{d}_{\zeta,\mathfrak{q}}(\lambda) = [\zeta]_{\mathfrak{q}}\, \mathfrak{d}_{\zeta-1,\mathfrak{q}}(\lambda) + (-1)_{\zeta,\lambda}, \qquad \mathfrak{d}_{\zeta,\mathfrak{p},\mathfrak{q}}(\lambda) = [\zeta]_{\mathfrak{p},\mathfrak{q}}\, \mathfrak{d}_{\zeta-1,\mathfrak{p},\mathfrak{q}}(\lambda) + (-1)_{\zeta,\lambda}.$$

Proof. Set $\mu = 0$ in Proposition 3.9. □

Theorem 3.11. *Let $\alpha_\kappa = [\kappa]_{\mathfrak{q}}$ and $\beta_\kappa = (\mu-1)_{\kappa,\lambda}$ $(\kappa \geq 1)$. For $\zeta \geq 1$, set*

$$\Delta_\zeta = \begin{vmatrix} \beta_1 & -1 & 0 & \cdots & 0 \\ \beta_2 & \alpha_2 & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \beta_{\zeta-1} & 0 & \cdots & \alpha_{\zeta-1} & -1 \\ \beta_\zeta & 0 & \cdots & 0 & \alpha_\zeta \end{vmatrix}. \tag{3.11}$$

Then,

$$\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda) = [\zeta]_{\mathfrak{q}}! + \Delta_\zeta. \tag{3.12}$$

Similarly, with $\alpha_\kappa = [\kappa]_{\mathfrak{p},\mathfrak{q}}$ and the same β_κ,

$$\mathfrak{d}_{\zeta,\mathfrak{p},\mathfrak{q}}(\mu\,;\lambda) = [\zeta]_{\mathfrak{p},\mathfrak{q}}! + \Delta_\zeta^{(\mathfrak{p},\mathfrak{q})}, \tag{3.13}$$

where $\Delta_\zeta^{(\mathfrak{p},\mathfrak{q})}$ is introduced by (3.11) with $\alpha_\kappa = [\kappa]_{\mathfrak{p},\mathfrak{q}}$.

Proof. Expanding Δ_ζ along the last row gives

$$\Delta_\zeta = \alpha_\zeta \Delta_{\zeta-1} + \beta_\zeta, \qquad (\zeta \geq 2), \qquad \Delta_1 = \beta_1.$$

Because $[\zeta]_{\mathfrak{q}}! = \alpha_\zeta [\zeta - 1]_{\mathfrak{q}}!$, and $\alpha_1 = [1]_{\mathfrak{q}} = 1$, the sequence $u_\zeta = [\zeta]_{\mathfrak{q}}! + \Delta_\zeta$ satisfies

$$u_\zeta = \alpha_\zeta u_{\zeta-1} + \beta_\zeta, \qquad u_0 = 1.$$

By Proposition 3.9, $\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)$ satisfies the same recurrence with the same initial value, and hence, $u_\zeta = \mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)$ for all $\zeta \geq 0$. This proves (3.12). The proof of (3.13) is identical. □

The following normalized form makes the triangular mechanism explicit and provides the structural viewpoint used throughout the paper. Define the normalized polynomials

$$\widetilde{\mathfrak{d}}_{\zeta,\mathfrak{q}}(\mu\,;\lambda) = \frac{\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)}{[\zeta]_{\mathfrak{q}}!} \qquad (\zeta \geq 0). \tag{3.14}$$

By Theorem 2.8,

$$\widetilde{\mathfrak{d}}_{\zeta,\mathfrak{q}}(\mu\,;\lambda) = \sum_{m=0}^{\zeta} \frac{(\mu-1)_{m,\lambda}}{[m]_{\mathfrak{q}}!}, \qquad (\zeta \geq 0). \tag{3.15}$$

Lemma 3.12. *For* $\zeta \geq 1$*,*

$$\widetilde{\mathfrak{d}}_{\zeta,\mathfrak{q}}(\mu\,;\lambda) = \widetilde{\mathfrak{d}}_{\zeta-1,\mathfrak{q}}(\mu\,;\lambda) + \frac{(\mu-1)_{\zeta,\lambda}}{[\zeta]_{\mathfrak{q}}!}. \tag{3.16}$$

Equivalently,

$$(\mu-1)_{\zeta,\lambda} = \mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda) - [\zeta]_{\mathfrak{q}}\, \mathfrak{d}_{\zeta-1,\mathfrak{q}}(\mu\,;\lambda). \tag{3.17}$$

Proof. Equation (3.16) follows immediately from (3.15) by separating the last term $m = \zeta$. Multiplying (3.16) by $[\zeta]_{\mathfrak{q}}!$ yields (3.17). □

Equations (3.15)–(3.17) show that the passage from the degenerate falling factorials $(\mu - 1)_{\zeta,\lambda}$ to the degenerate $\mathfrak{q}$-derangement polynomials $\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)$ is a triangular transform generated by the factor $1/1-\xi$. In the language of Riordan arrays, multiplication by $1/1-\xi$ corresponds to the classical Riordan pair $(1/1-\xi, \xi)$. See [29, 30] for further details. The presence of the $\mathfrak{q}$-factorials rescales the rows and columns, and the parameter λ changes the input basis from powers to degenerate falling factorials. Thus, the recurrence, determinant, and convolution identities are different manifestations of the same lower triangular operator. This interpretation is useful because it separates the universal transform from the deformation-dependent input sequence. When $\lambda = 0$, the degenerate factorial reduces to an ordinary power, and the $\mathfrak{q}$-exponential becomes analytic. In this case, one can obtain a clean limiting relation as $\zeta \to \infty$.

Proposition 3.13. *Assume* $0 < \mathfrak{q} < 1$ *and* $\lambda = 0$*. If* $|\mu - 1| < 1/1 - \mathfrak{q}$*, then*

$$\lim_{\zeta\to\infty} \frac{\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu \mid 0)}{[\zeta]_{\mathfrak{q}}!} = \mathfrak{e}_{\mathfrak{q}}((\mu-1)) := \sum_{m=0}^{\infty} \frac{(\mu-1)^m}{[m]_{\mathfrak{q}}!}. \tag{3.18}$$

In particular,

$$\lim_{\zeta\to\infty} \frac{\mathfrak{d}_{\zeta,\mathfrak{q}}(0 \mid 0)}{[\zeta]_{\mathfrak{q}}!} = \mathfrak{e}_{\mathfrak{q}}(-1).$$

Proof. By Theorem 2.8 with $\lambda = 0$,

$$\frac{\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu \mid 0)}{[\zeta]_{\mathfrak{q}}!} = \sum_{m=0}^{\zeta} \frac{(\mu - 1)^m}{[m]_{\mathfrak{q}}!}.$$

For $0 < \mathfrak{q} < 1$, we have $[m]_{\mathfrak{q}}! = (\mathfrak{q};\mathfrak{q})_m/(1-\mathfrak{q})^m$ (see [21, 22]), and hence,

$$\left|\frac{(\mu - 1)^{m+1}}{[m+1]_{\mathfrak{q}}!}\right| \Big/ \left|\frac{(\mu - 1)^m}{[m]_{\mathfrak{q}}!}\right| = \frac{|\mu - 1|}{[m+1]_{\mathfrak{q}}} \leq (1 - \mathfrak{q})\, |\mu - 1|.$$

Therefore, the series defining $\mathfrak{e}_{\mathfrak{q}}(\mu - 1)$ converges absolutely whenever $(1 - \mathfrak{q})|\mu - 1| < 1$. Taking the limit $\zeta \to \infty$ in the partial sums proves (3.18). □

For completeness, we briefly record the natural degenerate $(\mathfrak{p},\mathfrak{q})$-extensions of the Stirling numbers and Bell polynomials. These objects provide a convenient interface between generating functions and coefficient identities.

Definition 3.14. For $\kappa \geq 0$, the degenerate $(\mathfrak{p},\mathfrak{q})$-Stirling numbers of the second kind $\mathcal{S}_{2,\mathfrak{p},\mathfrak{q}}(\zeta,\kappa\,;\lambda)$ are given by the generating function

$$\frac{1}{[\kappa]_{\mathfrak{p},\mathfrak{q}}!}(\mathfrak{e}_{\mathfrak{p},\mathfrak{q},\lambda}(\xi) - 1)^{\kappa} = \sum_{\zeta=\kappa}^{\infty} \mathcal{S}_{2,\mathfrak{p},\mathfrak{q}}(\zeta,\kappa\,;\lambda)\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{p},\mathfrak{q}}!}. \tag{3.19}$$

The associated degenerate $(\mathfrak{p},\mathfrak{q})$-Bell polynomials are

$$B_{\zeta,\mathfrak{p},\mathfrak{q}}(\mu\,;\lambda) = \sum_{\kappa=0}^{\zeta} \mathcal{S}_{2,\mathfrak{p},\mathfrak{q}}(\zeta,\kappa\,;\lambda)\,\mu^{\kappa}. \tag{3.20}$$

Theorem 3.15. *The exponential generating function of $B_{\zeta,\mathfrak{p},\mathfrak{q}}(\mu\,;\lambda)$ is given by*

$$\mathfrak{e}_{\mathfrak{p},\mathfrak{q}}(\mu(\mathfrak{e}_{\mathfrak{p},\mathfrak{q},\lambda}(\xi) - 1)) = \sum_{\zeta=0}^{\infty} B_{\zeta,\mathfrak{p},\mathfrak{q}}(\mu\,;\lambda)\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{p},\mathfrak{q}}!}. \tag{3.21}$$

Proof. Using the series definition of $\mathfrak{e}_{\mathfrak{p},\mathfrak{q}}$ and the Cauchy product, we have (cf. [23, 26, 27])

$$\begin{aligned}
\mathfrak{e}_{\mathfrak{p},\mathfrak{q}}(\mu(\mathfrak{e}_{\mathfrak{p},\mathfrak{q},\lambda}(\xi) - 1)) &= \sum_{\kappa=0}^{\infty} \frac{\mu^{\kappa}}{[\kappa]_{\mathfrak{p},\mathfrak{q}}!}(\mathfrak{e}_{\mathfrak{p},\mathfrak{q},\lambda}(\xi) - 1)^{\kappa} \\
&= \sum_{\kappa=0}^{\infty} \mu^{\kappa} \sum_{\zeta=\kappa}^{\infty} \mathcal{S}_{2,\mathfrak{p},\mathfrak{q}}(\zeta,\kappa\,;\lambda)\frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{p},\mathfrak{q}}!} \\
&= \sum_{\zeta=0}^{\infty} \left(\sum_{\kappa=0}^{\zeta} \mathcal{S}_{2,\mathfrak{p},\mathfrak{q}}(\zeta,\kappa\,;\lambda)\mu^{\kappa} \right) \frac{\xi^{\zeta}}{[\zeta]_{\mathfrak{p},\mathfrak{q}}!},
\end{aligned}$$

which gives (3.21) by (3.20). □

It is also convenient to expand the degenerate falling factorials in the power basis. Define the degenerate Stirling numbers of the first kind $\mathfrak{f}_{1,\lambda}(\zeta,\kappa)$ by

$$(\mu)_{\zeta,\lambda} = \sum_{\kappa=0}^{\zeta} \mathfrak{f}_{1,\lambda}(\zeta,\kappa)\,\mu^{\kappa}. \tag{3.22}$$

Then, combining (3.22) with Theorem 3.2 yields an explicit coefficient expansion for $\mathfrak{d}_{\zeta,\mathfrak{p},\mathfrak{q}}(\mu\,;\lambda)$ as a polynomial in $(\mu-1)$.

Proposition 3.16. *For $\zeta \geq 0$,*

$$\mathfrak{d}_{\zeta,\mathfrak{p},\mathfrak{q}}(\mu\,;\lambda) = [\zeta]_{\mathfrak{p},\mathfrak{q}}! \sum_{m=0}^{\zeta} \frac{1}{[m]_{\mathfrak{p},\mathfrak{q}}!} \sum_{\kappa=0}^{m} \mathfrak{f}_{1,\lambda}(m,\kappa)\,(\mu-1)^{\kappa}. \tag{3.23}$$

Proof. Substitute $(\mu-1)_{m,\lambda} = \sum_{\kappa=0}^{m} \mathfrak{f}_{1,\lambda}(m,\kappa)\,(\mu-1)^{\kappa}$ into Theorem 3.2, and interchange the finite sums. □

The preceding identities are algebraic, but the form of the generating functions also indicates several natural contexts in which the family may be useful. First, the specialization $\mu = 0$ gives a degenerate $\mathfrak{q}$-weighted analog of the classical fixed-point-free permutation count. In enumerative models where $\mathfrak{q}$ records a Mahonian-type statistic such as inversions or major index, the parameter λ can be interpreted as a deformation of the weight assigned to the excluded fixed-point structure. The explicit formulas above therefore provide a tractable test family for studying how derangement-type statistics behave under simultaneous $\mathfrak{q}$-weighting and degeneration; compare the nondegenerate $\mathfrak{q}$-derangement setting in [3] with the probabilistic degenerate derangement framework in [6].

Second, the expansion through the degenerate $\mathfrak{q}$-Stirling numbers connects the present polynomials with partition-type objects, Bell polynomials, and ordered partition/Fubini-type polynomials. Consequently, identities such as Theorems 2.12–2.15 can be read as transfer rules between derangement-type structures and set-partition structures. This is relevant whenever one needs to move between fixed-point restrictions and block-decomposition models, for instance, in symbolic enumeration, normal ordering problems, or $\mathfrak{q}$-special polynomial expansions built from Stirling transforms [10,19,20]; related generating-function constructions for generalized Fibonacci–Lucas and Legendre–polylogarithm polynomial families are developed in [18].

Third, the lower triangular transform described in Remark 2.11 and in (3.15)–(3.17) gives computational advantages. Once the degenerate falling factorials are known, the values of $\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)$ can be generated by stable first-order recurrence (3.9), and the determinant formula in Theorem 3.11 supplies an alternative representation suitable for studying Hankel-type determinants and continued fractions. The $(\mathfrak{p},\mathfrak{q})$-extension suggests that the same mechanism can be transported to two-parameter $\mathfrak{q}$-calculus, degenerate Laplace/Sumudu-type transforms, and related families of special functions [4,8,11,26,27].

4. Conclusions

We introduced a λ-degenerate $\mathfrak{q}$-derangement family through a generating kernel that combines the degenerate falling factorial with a Carlitz-type degenerate $\mathfrak{q}$-exponential. This definition produces the polynomials $\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)$ and the associated numbers $\mathfrak{d}_{\zeta,\mathfrak{q}}(\lambda)$, for which we derived explicit coefficient formulas, recurrence relations, convolution identities, numerical illustrations, and determinant representations. The central structural feature is the lower triangular transform

$$(\mu-1)_{\zeta,\lambda} \longmapsto \mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda),$$

whose inverse is the two-term relation in Theorem 2.10. This transform perspective explains the expansion of $\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)$ in terms of the degenerate $\mathfrak{q}$-Stirling numbers of the second kind and links the family to corresponding Bell- and Fubini-type constructions in the spirit of related degenerate Stirling theory [10, 19, 20]. Higher-order analogs and the degenerate $(\mathfrak{p},\mathfrak{q})$-extension preserve this mechanism, rather than merely reproducing the same coefficient extraction in a different notation. Finally, the limiting regimes $\lambda \to 0$ and $\mathfrak{q} \to 1$ recover, respectively, the non-degenerate $\mathfrak{q}$-derangement families [3] and the classical derangement polynomials.

The added low-degree computations, numerical table, and figures illustrate how the formulas may be used in practice: The first values already show where the $\mathfrak{q}$-weight enters and how increasing λ can change the sign pattern of the associated numbers. These observations are not a substitute for a complete combinatorial model, but they help identify the parameter ranges in which positivity, oscillation, or rapid growth should be expected.

Several further developments appear especially promising. First, it would be useful to place the sequence $\{\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)\}_{\zeta\geq 0}$ into a degenerate $\mathfrak{q}$-Sheffer setting, which may yield systematic operational rules, inversion principles, transform formulas, and connection coefficients in the spirit of [9–13]. Second, combinatorial and probabilistic models (cf. [6, 7]) could clarify which statistics are encoded by the parameters $\mathfrak{q}$ and λ, especially in relation to fixed-point restrictions, block decompositions, and weighted random permutations. Third, richer $(\mathfrak{p},\mathfrak{q})$-extensions may reveal further links with other $\mathfrak{q}$-special functions, approximation operators, degenerate Mittag-Leffler-type functions, and Stirling-type families; see [10, 14, 20, 26, 27].

We close with a short list of questions suggested by the present results.

- *Real-rootedness and unimodality.* For fixed $\mathfrak{q} \in (0,1)$ and sufficiently small $|\lambda|$, investigate the zero distribution of $\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)$ and the log-concavity of its coefficients.
- *Hankel determinants and continued fractions.* The determinant representation in Theorem 3.11 points toward Hankel determinant techniques and continued-fraction expansions for suitable normalizations of $\{\mathfrak{d}_{\zeta,\mathfrak{q}}(\mu\,;\lambda)\}_{\zeta\geq 0}$.
- *Full* $(\mathfrak{p},\mathfrak{q})$*-extensions.* Develop a comprehensive theory of degenerate $(\mathfrak{p},\mathfrak{q})$-Stirling and $(\mathfrak{p},\mathfrak{q})$-Bell polynomials, and formulate a complete $(\mathfrak{p},\mathfrak{q})$-analog of the inversion identities obtained in Section 2.

Author contributions

Waseem Ahmad Khan: Methodology, formal analysis, validation, writing–review & editing; Oğuz Yağcı: Conceptualization, methodology, formal analysis, investigation, writing–original draft preparation; Khidir Shaib Mohamed: Conceptualization, supervision, project administration, funding acquisition, validation, writing–review & editing; Mona A. Mohamed: Investigation, formal analysis, visualization, writing–review & editing; Naglaa Mohammed: Investigation, validation, visualization, writing–review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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