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Research article

Complete moment convergence of moving average processes generated by m-widely acceptable sequences under sub-linear expectations

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Abstract: In this article, the complete moment convergence for the partial sum of moving average processes $\{X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}, n \geq 1\}$ is established under some proper conditions, where $\{Y_i, -\infty < i < \infty\}$ is a sequence of m-widely acceptable (m-WA) random variables, which is stochastically dominated by a random variable Y in sub-linear expectations space $(\Omega, \mathcal{H}, \mathbb{E})$, and $\{a_i, -\infty < i < \infty\}$ is an absolutely summable sequence of real numbers. The results extend the relevant results in probability space to those under sub-linear expectations. A Rosenthal-type inequality for m-WA random variables is also eatablished.

Keywords: m-widely acceptable random variables; moving average processes; complete convergence; complete moment convergence; sub-linear expectation
Mathematics Subject Classification: 60F05, 60F15

1. Introduction

In order to study the uncertainty in probability, Peng [1, 2] introduced the concepts of the sub-linear expectations space. Motivated by the works of Peng [1, 2], many people have tried to extend the results of classic probability space to those of the sub-linear expectations space. Zhang [3–5] proved the exponential inequalities, Rosenthal's inequalities, and Donsker's invariance principle under sub-linear expectations. Under sub-linear expectations, Xu and Cheng [6] studied how small the increments of G-Brownian motion are. Xu and Zhang [7, 8] got a three series theorem of independent random variables and a law of logarithm for arrays of row-wise extended negatively dependent (END) random variables under the sub-linear expectations. Zhong and Wu [9] obtained the complete convergence and complete moment convergence for weighted sums of END random variables under sub-linear expectations. For more limit theorems under sub-linear expectations, the interested readers could refer to Chen [10], Zhang [11], Kuczmaszewska [12], Chen and Wu [13], Xu and Cheng [14], Xu et al. [15, 16], and Xu and Kong [17] and references therein.

Guan et al. [18] proved under proper conditions the complete moment convergence for the partial sum of moving average processes produced by m-widely orthant dependent (m-WOD) random variables holds. For more results about complete moment convergence of moving average processes, the interested reader could refer to Zhang and Ding [19] and Hosseini and Nezakati [20, 21] and refercences therein. Recently, under the sub-linear expectations, Xu et al. [15] and Xu and Kong [17] proved complete convergence and complete moment convergence for negatively dependent (ND) random variables, Xu et al. [16] established complete convergence and complete moment convergence of linear processes produced by ND random variables, Xu [22] got complete convergence and complete moment convergence for maximal weighted sums of END random variables, Kuczmaszewska [12] obtained complete convergence for widely acceptable (WA) random variables, Hu and Wu [23] considered the complete convergence theorem for weighted sums of WA random variables, and Wu, Deng, and Wang [24] studied capacity inequalities and strong laws for m-widely acceptable (m-WA) random variables. By the definition of ND (see Definition 1.5 of Zhang [4]), END (see Definition 2.4 of Zhang [25]), and m-WA random variables under sub-linear expectations here, as discussed in Section 5 of Kuczmaszewska [12], ND and END random variables (cf. Definition 2.4 of Zhang [25]) are also m-WA random variables, and m-WA random variables are not necessarily ND or END random variables under sub-linear expectations. The complete convergence and complete moment convergence for maximal weighted sums of m-WA random variables and the complete moment convergence for partial sums and maximal partial sums for moving average processes produced by m-WA random variables remain unkown. It is natural to ask whether or not the relevant results of Guan et al. [18] hold for moving average processes produced by m-WA random variables under sub-linear expectations. As discussed in Example 2.5 of Zhang [25], m-WOD random variables in probability space are m-WA random variables in sub-linear expectation space if the sub-linear expectation $\mathbb{E}$ is reduced to linear expectation E. Why m-WA random variables under sub-linear expectations nontrivially differ from m-WOD random variables in probability space could be seen heuristically by the nontrivial difference between G-normal random variables and normal random variables as pointed out in Remark 3.1 of Xu [26]. The main contributions of this paper are as follows: We prove a Rosenthal-type inequality for m-WA random variables, which is an inevitable and useful tool to investigate the complete moment convergences related to m-WA random variables under sub-linear expectations. Secondly, we exactly establish the complete moment convergence for the partial sum of moving average processes produced by m-WA random variables under sub-linear expectations, carefully extending the relevant results obtained in Guan et al. [18].

We organize the rest of this paper as follows. We give some necessary basic notions, concepts, and corresponding properties and provide necessary lemmas under sub-linear expectations in the next section. In Section 3, we give our main results, Theorems 3.1 and 3.2, the proofs of which are also presented in this section.

2. Preliminary

As in Xu and Cheng [14], we use similar notations as in the work by Peng [2], Chen [10], and Zhang [4]. Suppose that $(\Omega, \mathcal{F})$ is a given measurable space. Assume that $\mathcal{H}$ is a subset of all random variables on $(\Omega, \mathcal{F})$ such that $X_1, \cdots, X_n \in \mathcal{H}$ implies $\varphi(X_1, \cdots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}^n)$, where $C_{l,Lip}(\mathbb{R}^n)$ represents the linear space of (local Lipschitz) function φ fulfilling

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \le C(1 + |\mathbf{x}|^m + |\mathbf{y}|^m)(|\mathbf{x} - \mathbf{y}|), \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

for some $C > 0$, $m \in \mathbb{N}$ depending on φ.

Definition 2.1. *A sub-linear expectation* $\mathbb{E}$ *on* $\mathcal{H}$ *is a functional* $\mathbb{E} : \mathcal{H} \mapsto \bar{\mathbb{R}} := [-\infty, \infty]$ *fulfilling the following properties: For all* $X, Y \in \mathcal{H}$*, we have*

(a) *Monotonicity: If* $X \ge Y$*, then* $\mathbb{E}[X] \ge \mathbb{E}[Y]$*;*

(b) *Constant preserving:* $\mathbb{E}[c] = c$*,* $\forall c \in \mathbb{R}$*;*

(c) *Positive homogeneity:* $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$*,* $\forall \lambda \ge 0$*;*

(d) *Sub-additivity:* $\mathbb{E}[X + Y] \le \mathbb{E}[X] + \mathbb{E}[Y]$ *whenever* $\mathbb{E}[X] + \mathbb{E}[Y]$ *is not of the form* $+\infty - \infty$ *or* $-\infty + \infty$.

A set function $V : \mathcal{F} \mapsto [0, 1]$ is named to be a capacity if

(a) $V(\emptyset) = 0$, $V(\Omega) = 1$;

(b) $V(A) \le V(B)$, $A \subset B$, $A, B \in \mathcal{F}$.

A capacity V is called sub-additive if $V(A \bigcup B) \le V(A) + V(B)$, $A, B \in \mathcal{F}$.

In this sequel, given a sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, set $\mathbb{V}(A) := \inf\{\mathbb{E}[\xi] : I_A \le \xi, \xi \in \mathcal{H}\}$, $\forall A \in \mathcal{F}$ (see (2.3) and the definitions of $\mathbb{V}$ above (2.3) in Zhang [3]). $\mathbb{V}$ is a sub-additive capacity. Set

$$C_{\mathbb{V}}(X) := \int_0^\infty \mathbb{V}(X > x)\mathrm{d}x + \int_{-\infty}^0 (\mathbb{V}(X > x) - 1)\mathrm{d}x.$$

As in 4.3 of Zhang [3], throughout this paper, define an extension of $\mathbb{E}$ on the space of all random variables by

$$\mathbb{E}^*(X) = \inf\{\mathbb{E}[Y] : X \le Y, Y \in \mathcal{H}\}.$$

Then $\mathbb{E}^*$ is a sub-linear expectation on the space of all random variables, $\mathbb{E}[X] = \mathbb{E}^*[X]$, $\forall X \in \mathcal{H}$, and $\mathbb{V}(A) = \mathbb{E}^*(I_A)$, $\forall A \in \mathcal{F}$.

Definition 2.2. *Suppose* $\{Y_i, -\infty < i < \infty\}$ *is a sequence of random variables in sub-linear expectations space* $(\Omega, \mathcal{H}, \mathbb{E})$. $\{Y_i, -\infty < i < \infty\}$ *is called to be WA, if there exists a positive sequence* $\{g(n), n \ge 1\}$ *of dominating coefficients such that for all* $n \in \mathbb{N}$ *and for all* $-\infty < j < \infty$*, we have*

$$\mathbb{E}\exp\left(\sum_{i=1}^n a_{ni} f_i(Y_{j+i})\right) \le g(n) \prod_{i=1}^n \mathbb{E}\exp\left(a_{ni} f_i(Y_{j+i})\right), \quad 0 < g(n) < \infty, \tag{2.1}$$

where $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ *is an array of nonnegative constants and* $f_i(\cdot) \in C_{b,Lip}(\mathbb{R})$*,* $i = 1, 2, \ldots$ *are all non-decreasing (or all non-increasing) real-valued truncation functions.*

Definition 2.3. *Let* $m \ge 1$ *be fixed integer. A sequence of random variables* $\{X_n, n \ge 1\}$ *is called to be m-WA, if, for any* $n \ge 2$ *and* $i_1, i_2, \cdots, i_n$ *fulfilling* $|i_k - i_j| \ge m$ *for all* $1 \le k \ne j \le n$*, we have* $X_{i_1}, X_{i_2}, \cdots, X_{i_n}$ *are WA.*

Definition 2.4. *We say that* $\{Y_n; n \geq 1\}$ *is stochastically dominated by a random variable* Y *in* $(\Omega, \mathcal{H}, \mathbb{E})$ *if there exists a constant* C *such that* $\forall n \geq 1$ *for all nonnegative* $h \in C_{l,Lip}(\mathbb{R})$, $\mathbb{E}(h(Y_n)) \leq C\mathbb{E}(h(Y))$.

Assume that $\mathbf{X}_1$ and $\mathbf{X}_2$ are two n-dimensional random vectors defined, respectively, in sub-linear expectation spaces $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$. They are called identically distributed if, for every function $\psi \in C_{l,Lip}(\mathbb{R})$ such that $\psi(\mathbf{X}_1) \in \mathcal{H}_1, \psi(\mathbf{X}_2) \in \mathcal{H}_2$,

$$\mathbb{E}_1[\psi(\mathbf{X}_1)] = \mathbb{E}_2[\psi(\mathbf{X}_2)]$$

whenever the sub-linear expectations are finite. $\{X_n\}_{n=1}^{\infty}$ is named to be identically distributed if, for each $i \geq 1$, X_i and X_1 are identically distributed.

In the paper, we assume that $\mathbb{E}$ is countably sub-additive, i.e., $\mathbb{E}(X) \leq \sum_{n=1}^{\infty} \mathbb{E}(X_n)$ whenever $X \leq \sum_{n=1}^{\infty} X_n$, $X, X_n \in \mathcal{H}$ and $X \geq 0$, $X_n \geq 0$, $n = 1, 2, \ldots$. Hence, $\mathbb{E}^*$ is also countably sub-additive. Let C stand for a positive constant that may change from place to place. $I(A)$ or I_A represents the indicator function of A. Write $\log(x) = \ln \max\{\mathrm{e}, x\}$, $x > 0$.

We cite the following lemma (cf. Lemma 2.2 of Xu et al. [16]).

Lemma 2.1. *If, for a random variable* X *on* $(\Omega, \mathcal{F})$, $C_{\mathbb{V}}\{|X|\} < \infty$, *then*

$$\mathbb{E}^*[|X|] \leq C_{\mathbb{V}}\{|X|\}.$$

Next, we give some useful lemmas.

Lemma 2.2. *(cf. Wu et al. [24]) Let* $\{X_n, n \geq 1\}$ *be a sequence of m-WA random variables with dominating coefficients* $g(n)$. *If* $\{f_n(\cdot), n \geq 1\}$ *are all non-decreasing (non-increasing), then* $\{f_n(X_n), n \geq 1\}$ *are still m-WA with dominating coefficients* $\{g(n), n \geq 1\}$.

Lemma 2.3. *Let* $0 < t \leq 1$ *or* $t = 2$ *and* $\{X_n, n \geq 1\}$ *be a sequence of WA random variables in sub-linear expectations space* $(\Omega, \mathcal{H}, \mathbb{E})$. *Assume further that* $\mathbb{E}(X_n) \leq 0$ *for each* $n \geq 1$ *when* $t = 2$. *Then for all* $x > 0$ *and* $y > 0$,

$$\mathbb{V}(S_n \geq x) \leq \sum_{i=1}^{n} \mathbb{V}(X_i > y) + g(n) \exp\left(\frac{x}{y} - \frac{x}{y} \ln\left(1 + \frac{xy^{t-1}}{\sum_{i=1}^{n} \mathbb{E}|X_i|^t}\right)\right). \tag{2.2}$$

Proof. If $0 < t \leq 1$, then we can establish (2.2) by the adapted proof of Theorem 2.1 of Shen [27]. In summary, we use the truncation of X_i and the non-decreasing property of the function $\frac{\mathrm{e}^{hu}-1}{u^t}$ on $u \geq 0$ for $h > 0$, and then we use the minimum of $\exp\left\{\frac{\mathrm{e}^{hy}-1}{y^t} \sum_{i=1}^{n} \mathbb{E}\left(|X_i|^t\right) - hx\right\}$ over $y > 0$ to obtain (2.2); the details are postponed in Appendix. If $t = 2$, (2.2) follows immediately from Lemma 2.1 of Wu et al. [24]. □

Lemma 2.4. *For a positive real number* $q \geq 2$, $\{X_n, n \geq 1\}$ *is a sequence of m-WA random variables with dominating coefficients* $\{g(n), n \geq 1\}$. *If* $C_{\mathbb{V}}\{|X_i|^q\} < \infty$ *for every* $i \geq 1$, *then for all* $n \geq 1$, *there exist positive constants* $C_1(m, q)$, $C_2(m, q)$, *and* $C_3(m, q)$ *depending on* q *and* m *such that*

$$\begin{aligned}\mathbb{E}\left(\left|\sum_{i=1}^{n} X_i\right|^q\right) \leq & C_1(m, q) \sum_{i=1}^{n} C_{\mathbb{V}}\{|X_i|^q\} + C_2(m, q) g(n) \left(\sum_{i=1}^{n} \mathbb{E}X_i^2\right)^{q/2} \\ & + C_3(m, q) \left(\sum_{i=1}^{n} [|\mathbb{E}(X_i)| + |\mathbb{E}(-X_i)|]\right)^q.\end{aligned}$$

Proof. For readers' convenience, we give a short sketch of the proof here. First, we prove Lemma 2.4 in the case that $\{X_n, n \geq 1\}$ is a sequence of WA random variables with dominating coefficients $\{g(n), n \geq 1\}$, which can be obtained by multiplying both sides of (4.1) by qx^{q-1} and integrating on the half line, and C_r inequality. Then combining the above result with the similar proof of Corollary 3 of Fang et al. [28] (see the end of this proof) and C_r inequality, it concludes that Lemma 2.4 holds. Note that

$$C_{\mathbb{V}}\{|X^+|^q\} = \int_0^\infty \mathbb{V}\left(|X^+|^q > x\right) \mathrm{d}x = \int_0^\infty qx^{q-1}\mathbb{V}\left\{|X^+| > x\right\} \mathrm{d}x,$$

where $X^+ := \max\{X, 0\}$. We first suppose that $\{X_n, n \geq 1\}$ is a sequence of WA random variables with dominating coefficients $\{g(n), n \geq 1\}$ and $\mathbb{E}(X_n) \leq 0$. Putting $y = x/r$ in (2.2) with $t = 2$ yields

$$\mathbb{V}\left(S_n^+ \geq x\right) \leq \sum_{i=1}^n \mathbb{V}\left(X_i^+ > y\right) + g(n)\mathrm{e}^r \left(\frac{r\sum_{i=1}^n \mathbb{E}\left(|X_i|^2\right)}{r\sum_{i=1}^n \mathbb{E}\left(|X_i|^2\right) + x^2}\right)^r. \tag{2.3}$$

By the similar proof of (3.4) of Zhang [11], $n \geq 1$, multiplying both sides of (2.3) by qx^{q-1}, and integrating on the half line, we have

$$\begin{aligned}\mathbb{E}\left((S_n^+)^q\right) &\leq C_{\mathbb{V}}\left\{(S_n^+)^q\right\} \\ &\leq \sum_{i=1}^n CC_{\mathbb{V}}\left\{|X_i^+|^q\right\} + Cg(n)\left(\sum_{i=1}^n \mathbb{E}X_i^2\right)^{q/2}.\end{aligned}$$

Hence, when $\{X_n, n \geq 1\}$ is a sequence of WA random variables with dominating coefficients $\{g(n), n \geq 1\}$, by C_r inequality, we see that

$$\begin{aligned}\mathbb{E}\left((S_n^+)^q\right) &\leq C\mathbb{E}\left(\left((S_n - \sum_{i=1}^n \mathbb{E}X_i)^+\right)^q\right) + C\left(\sum_{i=1}^n |\mathbb{E}(X_i)|\right)^q \\ &\leq \sum_{i=1}^n CC_{\mathbb{V}}\left\{|(X_i - \mathbb{E}X_i)^+|^q\right\} + Cg(n)\left(\sum_{i=1}^n \mathbb{E}X_i^2\right)^{q/2} + +C\left(\sum_{i=1}^n |\mathbb{E}(X_i)|\right)^q \\ &\leq \sum_{i=1}^n CC_{\mathbb{V}}\left\{|X_i^+|^q\right\} + C\sum_{i=1}^n |\mathbb{E}(X_i)|^q + Cg(n)\left(\sum_{i=1}^n \mathbb{E}X_i^2\right)^{q/2} + +C\left(\sum_{i=1}^n |\mathbb{E}(X_i)|\right)^q \\ &\leq \sum_{i=1}^n CC_{\mathbb{V}}\left\{|X_i^+|^q\right\} + Cg(n)\left(\sum_{i=1}^n \mathbb{E}X_i^2\right)^{q/2} + +C\left(\sum_{i=1}^n |\mathbb{E}(X_i)|\right)^q,\end{aligned}$$

and

$$\mathbb{E}\left(((-S_n)^+)^q\right) \leq \sum_{i=1}^n CC_{\mathbb{V}}\left\{|X_i^-|^q\right\} + Cg(n)\left(\sum_{i=1}^n \mathbb{E}X_i^2\right)^{q/2} + +C\left(\sum_{i=1}^n |\mathbb{E}(-X_i)|\right)^q.$$

Therefore, combining the two equations above yields

$$\begin{aligned}\mathbb{E}\left(\left|\sum_{i=1}^n X_i\right|^q\right) \leq& C_1(m,q)\sum_{i=1}^n C_{\mathbb{V}}\left\{|X_i|^q\right\} + C_2(m,q)g(n)\left(\sum_{i=1}^n \mathbb{E}X_i^2\right)^{q/2} \\ &+ C_3(m,q)\left(\sum_{i=1}^n \left[|\mathbb{E}(X_i)| + |\mathbb{E}(-X_i)|\right]\right)^q.\end{aligned}$$

When $\{X_n, n \geq 1\}$ is a sequence of m-WA random variables with dominating coefficients $\{g(n), n \geq 1\}$, we may assume that $n = mj + l$ for some $j \geq 0$ and $1 \leq l \leq m$. Then $\sum_{i=1}^{n} X_i = \sum_{i=0}^{m} \sum_{j_1=1}^{l} X_{im+j_1} + \sum_{i=0}^{m-1} \sum_{j_2=1}^{m} X_{im+j_2}$. By Definition 2.3, $\{X_{im+j_1}, 0 \leq i \leq m\}$ is WA for each $1 \leq j_1 \leq l$, and $\{X_{im+j_2}, 0 \leq i \leq m-1\}$ is WA for each $1 \leq j_2 \leq m$. By the equation above and C_r inequality, we see that

$$\begin{aligned}
&\mathbb{E}\left(\left|\sum_{i=1}^{n} X_i\right|^q\right) \\
&\leq (2m)^{q-1}\left(\sum_{j_1=1}^{l} \mathbb{E}\left(\left|\sum_{i=0}^{m} X_{im+j_1}\right|^q\right) + \sum_{j_2=1}^{m} \mathbb{E}\left(\left|\sum_{i=0}^{m-1} X_{im+j_2}\right|^q\right)\right) \\
&\leq (2m)^{q-1}\left(\sum_{j_1=1}^{l}\left(C_1(m,q)\sum_{i=0}^{m} C_{\mathbb{V}}\left\{|X_{im+j_1}|^q\right\} + C_2(m,q)g(n)\left(\sum_{i=0}^{m} \mathbb{E}X_{im+j_1}^2\right)^{q/2}\right.\right. \\
&\quad \left.\left. +C_3(m,q)\left(\sum_{i=0}^{m}\left[|\mathbb{E}(X_{im+j_1})| + |\mathbb{E}(-X_{im+j_1})|\right]\right)^q\right)\right) \\
&\quad + (2m)^{q-1}\left(\sum_{j_2=1}^{m}\left(C_1(m,q)\sum_{i=0}^{m-1} C_{\mathbb{V}}\left\{|X_{im+j_2}|^q\right\}\right.\right. \\
&\quad \left.\left. +C_2(m,q)g(n)\left(\sum_{i=0}^{m-1} \mathbb{E}X_{im+j_2}^2\right)^{q/2} + C_3(m,q)\left(\sum_{i=0}^{m-1}\left[|\mathbb{E}(X_{im+j_2})| + |\mathbb{E}(-X_{im+j_2})|\right]\right)^q\right)\right) \\
&\leq (2m)^{q-1}\left(C_1(m,q)\sum_{i=1}^{n} C_{\mathbb{V}}\{|X_i|^q\} + C_2(m,q)g(n)\left(\sum_{i=1}^{n} \mathbb{E}X_i^2\right)^{q/2} + C_3(m,q)\left(\sum_{i=1}^{n}[|\mathbb{E}(X_i)| + |\mathbb{E}(-X_i)|]\right)^q\right).
\end{aligned}$$

This finishes the proof. □

Remark 2.1. *Lemma 2.4 differs from Rosenthal's inequality for ND random variables under sub-linear expectations (cf. Theorem 2.1 of Zhang [4]) by noting that here, there is $g(n)$ with $C_{\mathbb{V}}\{|X_i|^q\}$ and m-WA in place of $\mathbb{E}\{|X_i|^q\}$ and NA in Theorem 2.1 of Zhang [4]. In the proof of Theorems 3.1 and 3.2, Lemma 2.4 is indispensable to obtain $I_2 < \infty$ and $J_2 < \infty$.*

3. Main results

Our main results, considered as an extension of Guan et al. [18] in some sense, are as follows.

3.1. Notation and standing assumptions

Hereafter, suppose $l(x)$ is a function slowly varying at infinity. Suppose that $\{a_i, -\infty < i < \infty\}$ is an absolutely summable sequence of real numbers. Assume that $\{X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}, n \geq 1\}$ is a moving average process produced by a sequence $\{Y_i, -\infty < i < \infty\}$ of m-WA random variables with dominating coefficients $g(n)$ and $\{Y_i, -\infty < i < \infty\}$ is stochastically dominated by Y in sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$.

Theorem 3.1. *Suppose $p \geq 1$, $\alpha > \frac{1}{2}$, $\alpha p > 1$, and $g(n) = O(n^\delta)$ for some $\delta \geq 0$. If $\mathbb{E}(Y_i) = \mathbb{E}(-Y_i) = 0$, $i = 1, 2, \cdots$, for $\frac{1}{2} < \alpha \leq 1$, $C_{\mathbb{V}}\left\{|Y|^p l(|Y|^{1/\alpha})\right\} < \infty$ for $p > 1$, and $C_{\mathbb{V}}\left\{|Y|^{1+\lambda}\right\} < \infty$ for $p = 1$ and some*

$\lambda > 0$, *then for any* $\epsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) C_{\mathbb{V}} \left\{ \left(\left| \sum_{j=1}^{n} X_j \right| - \epsilon n^{\alpha} \right)^{+} \right\} < \infty. \tag{3.1}$$

Remark 3.1. *In Theorem 3.1, if $\mathbb{E}$ is reduced to linear expectation E, then* (3.1) *is just* (3.1) *of Guan et al. [18]. The rate $n^{\alpha p-2-\alpha} l(n)$ implies* $\lim_{n\to\infty} n^{\alpha p-1} l(n) \int_{\epsilon}^{\infty} \mathbb{V}\left(\left| \sum_{j=1}^{n} X_j \right| / n^{\alpha} > y \right) \mathrm{d}y = 0$, $\forall \epsilon > 0$. *In Theorems 3.1 and 3.2, we assume that* $\mathbb{E}(Y_i) = \mathbb{E}(-Y_i) = 0$ *for* $-\infty < i < \infty$. *Under these restrictive conditions, the intrinsic difference between probability space and sub-linear expectation space is discussed heuristically in Remark 3.1 of Xu [26].*

Proof of Theorem 3.1. For $2^{-\alpha} < \mu < 1$, define $\tilde{g}_{\mu}(x) \in C_{l,Lip}(\mathbb{R})$ such that $I\{|x| \le \mu\} < \tilde{g}_{\mu}(x) < I\{|x| \le 1\}$. Define $g_j(x) \in C_{l,Lip}(\mathbb{R})$, $j \ge 1$ such that $g_j(x)$ is an even function, and for x, $0 \le g_j(x) \le 1$, $g_j(x/2^{j\alpha}) = 1$ while $2^{(j-1)\alpha} < |x| \le 2^{j\alpha}$, and $g_j(x/2^{j\alpha}) = 0$ while $|x| \le \mu 2^{(j-1)\alpha}$ or $|x| > (1+\mu)2^{j\alpha}$. We see that

$$g_j(|Y|/2^{j\alpha}) \le I\left\{ \mu 2^{(j-1)\alpha} < |Y| \le (1+\mu)2^{j\alpha} \right\}, |Y|^l \tilde{g}_{\mu}\left(\frac{|Y|}{2^{k\alpha}} \right) \le 1 + \sum_{j=1}^{k} |Y|^l g_j\left(\frac{|Y|}{2^{j\alpha}} \right), \tag{3.2}$$

$$1 - \tilde{g}_{\mu}\left(\frac{|Y|}{2^{k\alpha}} \right) \le \sum_{j=k}^{\infty} g_j\left(\frac{|Y|}{2^{j\alpha}} \right). \tag{3.3}$$

Let $f(n) = n^{\alpha p-2-\alpha} l(n)$, $Y_{xj}^{(1)} = -xI\{Y_j < -x\} + Y_j I\{|Y_j| \le x\} + xI\{Y_j > x\}$, $Y_{xj}^{(2)} = Y_j - Y_{xj}^{(1)}$ be the monotone truncations of $\{Y_j, -\infty < j < \infty\}$ for $x > 0$. Write $Y_x^{(1)} = -xI\{Y < -x\} + YI\{|Y| \le x\} + xI\{Y > x\}$, $Y_x^{(2)} = Y - Y_x^{(1)}$. Then by Lemma 2.2, we see that $\{Y_{xj}^{(1)}, -\infty < j < \infty\}$ and $\{Y_{xj}^{(2)}, -\infty < j < \infty\}$ are two sequences of m-WA random variables. We observe that

$$\begin{aligned}
&\sum_{n=1}^{\infty} f(n) C_{\mathbb{V}} \left\{ \left(\left| \sum_{j=1}^{n} X_j \right| - \epsilon n^{\alpha} \right)^{+} \right\} \\
\le &\sum_{n=1}^{\infty} f(n) \int_{\epsilon n^{\alpha}}^{\infty} \mathbb{V}\left\{ \left| \sum_{j=1}^{n} X_j \right| > x \right\} \mathrm{d}x \le C \sum_{n=1}^{\infty} f(n) \int_{n^{\alpha}}^{\infty} \mathbb{V}\left\{ \left| \sum_{j=1}^{n} X_j \right| > \epsilon x \right\} \mathrm{d}x \\
\le &C \sum_{n=1}^{\infty} f(n) \int_{n^{\alpha}}^{\infty} \mathbb{V}\left\{ \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_{xj}^{(2)} \right| > \epsilon x/2 \right\} \mathrm{d}x \\
&+ C \sum_{n=1}^{\infty} f(n) \int_{n^{\alpha}}^{\infty} \mathbb{V}\left\{ \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_{xj}^{(1)} \right| > \epsilon x/2 \right\} \mathrm{d}x =: I_1 + I_2.
\end{aligned} \tag{3.4}$$

First, we establish $I_1 < \infty$. Observe $|Y_{xj}^{(2)}| < |Y_j| \left(1 - \tilde{g}_{\mu}\left(\frac{|Y_j|}{x} \right) \right)$. Then by Markov's inequality under sub-linear expectations, we see that

$$I_1 \le C \sum_{n=1}^{\infty} f(n) \int_{n^{\alpha}}^{\infty} x^{-1} \mathbb{E}^{*} \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_{xj}^{(2)} \right| \mathrm{d}x$$

$$
\begin{aligned}
&\le C\sum_{n=1}^{\infty} f(n)\int_{n^{\alpha}}^{\infty} x^{-1}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}\mathbb{E}^{*}\left|Y_{xj}^{(2)}\right|\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty} f(n)\int_{n^{\alpha}}^{\infty} x^{-1}\mathbb{E}|Y|\left(1-\tilde{g}_{\mu}\left(\frac{|Y|}{x}\right)\right)\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty} f(n)\sum_{m=n}^{\infty}\int_{m^{\alpha}}^{(m+1)^{\alpha}} x^{-1}\mathbb{E}|Y|\left(1-\tilde{g}_{\mu}\left(\frac{|Y|}{m^{\alpha}}\right)\right)\mathrm{d}x\\
&\le C\sum_{m=1}^{\infty} m^{-1}\mathbb{E}|Y|\left(1-\tilde{g}_{\mu}\left(\frac{|Y|}{m^{\alpha}}\right)\right)\sum_{n=1}^{m} f(n).
\end{aligned}
$$

If $p>1$, $\alpha p-1-\alpha>-1$, we conclude that

$$
\begin{aligned}
I_1 &= C\sum_{k=0}^{\infty}\sum_{m=2^k}^{2^{k+1}-1} m^{\alpha p-1-\alpha} l(m)\mathbb{E}|Y|\left(1-\tilde{g}_{\mu}\left(\frac{|Y|}{m^{\alpha}}\right)\right)\\
&\le C\sum_{k=1}^{\infty} 2^{\alpha pk-\alpha k} l(2^k)\mathbb{E}|Y|\left(1-\tilde{g}_{\mu}\left(\frac{|Y|}{2^{k\alpha}}\right)\right)\\
&\le C\sum_{k=1}^{\infty} 2^{\alpha pk-\alpha k} l(2^k)\mathbb{E}^{*}\sum_{j=k}^{\infty}|Y| g_j\left(\frac{|Y|}{2^{j\alpha}}\right)\\
&= C\sum_{j=1}^{\infty}\mathbb{E}^{*}|Y| g_j\left(\frac{|Y|}{2^{j\alpha}}\right)\sum_{k=1}^{j} 2^{\alpha pk-\alpha k} l(2^k)\\
&\le C\sum_{j=1}^{\infty} 2^{\alpha pj} l(2^j)\mathbb{V}\left\{|Y|>\mu 2^{(j-1)\alpha}\right\}\\
&\le C\sum_{m=1}^{\infty} m^{\alpha p-1} l(m)\mathbb{V}\left\{|Y|>\mu 2^{-1} m^{\alpha}\right\}\le C C_{\mathbb{V}}\left\{|Y|^{p} l(|Y|^{1/\alpha})\right\}<\infty.
\end{aligned}
$$

If $p=1$, $C_{\mathbb{V}}\left\{|Y|^{1+\lambda}\right\}<\infty$ yields $C_{\mathbb{V}}\left\{|Y|^{1+\lambda'} l(|Y|^{1/\alpha})\right\}<\infty$ for any $0<\lambda'<\lambda$, then by Lemma 2.3 of Xu [26], we see that

$$
\begin{aligned}
I_1 &\le C\sum_{m=1}^{\infty} m^{-1}\mathbb{E}|Y|\left(1-\tilde{g}_{\mu}\left(\frac{|Y|}{m^{\alpha}}\right)\right)\sum_{n=1}^{m} n^{-1} l(n)\\
&\le C\sum_{m=1}^{\infty} m^{-1}\mathbb{E}|Y|\left(1-\tilde{g}_{\mu}\left(\frac{|Y|}{m^{\alpha}}\right)\right)\sum_{n=1}^{m} n^{-1+\alpha\lambda'} l(n)\\
&\le C\sum_{m=1}^{\infty} m^{\alpha\lambda'-1} l(n)\mathbb{E}|Y|\left(1-\tilde{g}_{\mu}\left(\frac{|Y|}{m^{\alpha}}\right)\right)\\
&= C\sum_{k=0}^{\infty}\sum_{m=2^k}^{2^{k+1}-1} m^{\alpha\lambda'-1} l(m)\mathbb{E}|Y|\left(1-\tilde{g}_{\mu}\left(\frac{|Y|}{m^{\alpha}}\right)\right)\\
&\le C\sum_{k=1}^{\infty} 2^{k(\alpha\lambda')} l(2^k)\mathbb{E}|Y|\left(1-\tilde{g}_{\mu}\left(\frac{|Y|}{2^{k\alpha}}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\le C\sum_{k=1}^{\infty}2^{k(\alpha\lambda')}l(2^k)\mathbb{E}^*\left(|Y|\sum_{l=k}^{\infty}g_l\left(\frac{|Y|}{2^{l\alpha}}\right)\right)\\
&\le C\sum_{l=1}^{\infty}\mathbb{E}^*\left(|Y|g_l\left(\frac{|Y|}{2^{l\alpha}}\right)\right)\sum_{k=1}^{l}2^{k(\alpha\lambda')}l(2^k)\\
&\le C\sum_{l=1}^{\infty}\mathbb{E}\left(|Y|g_l\left(\frac{|Y|}{2^{l\alpha}}\right)\right)2^{l(\alpha\lambda')}l(2^l)\\
&\le C\sum_{l=1}^{\infty}\mathbb{V}\left\{|Y|>\mu 2^{(l-1)\alpha}\right\}2^{l\alpha(\lambda'+1)}l(2^l)<\infty.
\end{aligned}
$$

Hence, we get

$$
I_1<\infty. \tag{3.5}
$$

Next, we establish $I_2 < \infty$. From Markov's inequality under sub-linear expectations, Hölder's inequality, and Lemma 2.4, follows that

$$
\begin{aligned}
I_2 &\le C\sum_{n=1}^{\infty}f(n)\int_{n^\alpha}^{\infty}x^{-r}\mathbb{E}^*\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+n}Y_{xj}^{(1)}\right|^r\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty}f(n)\int_{n^\alpha}^{\infty}x^{-r}\mathbb{E}^*\left|\sum_{i=-\infty}^{\infty}|a_i|^{\frac{r-1}{r}}\left(|a_i|^{\frac{1}{r}}\left|\sum_{j=i+1}^{i+n}Y_{xj}^{(1)}\right|\right)\right|^r\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty}f(n)\int_{n^\alpha}^{\infty}x^{-r}\left(\sum_{i=-\infty}^{\infty}|a_i|\right)^{r-1}\left(\sum_{i=-\infty}^{\infty}|a_i|\mathbb{E}^*\left|\sum_{j=i+1}^{i+n}Y_{xj}^{(1)}\right|^r\right)\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty}f(n)\int_{n^\alpha}^{\infty}x^{-r}\left(\sum_{i=-\infty}^{\infty}|a_i|\mathbb{E}\left|\sum_{j=i+1}^{i+n}Y_{xj}^{(1)}\right|^r\right)\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty}f(n)\int_{n^\alpha}^{\infty}x^{-r}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}C_{\mathbb{V}}\left\{|Y_{xj}^{(1)}|^r\right\}\mathrm{d}x\\
&\quad +C\sum_{n=1}^{\infty}f(n)g(n)\int_{n^\alpha}^{\infty}x^{-r}\sum_{i=-\infty}^{\infty}|a_i|\left(\sum_{j=i+1}^{i+n}\mathbb{E}\left|Y_{xj}^{(1)}\right|^2\right)^{r/2}\mathrm{d}x\\
&\quad +C\sum_{n=1}^{\infty}f(n)\int_{n^\alpha}^{\infty}x^{-r}\sum_{i=-\infty}^{\infty}|a_i|\left(\sum_{j=i+1}^{i+n}\left|\mathbb{E}(Y_{xj}^{(1)})\right|+\left|\mathbb{E}(-Y_{xj}^{(1)})\right|\right)^r\mathrm{d}x\\
&=: I_{21}+I_{22}+I_{23},
\end{aligned} \tag{3.6}
$$

where $r \ge 2$ is given later.

By (4.3)–(4.6) in Appendix, we see that

$$
I_{21}<\infty,\quad I_{22}<\infty. \tag{3.7}
$$

For I_{23}, we take $r > 2$. By $\mathbb{E}(Y_i) = \mathbb{E}(-Y_i) = 0$, Proposition 1.3.7 of Peng [2], and Lemma 4.5 (iii) of Zhang [3], we get

$$
\begin{aligned}
I_{23} &\le C\sum_{n=1}^{\infty} f(n)\sum_{m=n}^{\infty}\int_{m^\alpha}^{(m+1)^\alpha} x^{-r}\left(\sup_i \sum_{j=i+1}^{i+n}\mathbb{E}\left|Y_{xj}^{(1)}-Y_j\right|\right)^r \mathrm{d}x\\
&\le C\sum_{n=1}^{\infty} f(n)\sum_{m=n}^{\infty}\int_{m^\alpha}^{(m+1)^\alpha} x^{-r}n^r\left(\mathbb{E}|Y|\left(1-\tilde{g}_\mu\left(\frac{|Y|}{x}\right)\right)\right)^r \mathrm{d}x\\
&\le C\sum_{n=1}^{\infty} f(n)n^r\sum_{m=n}^{\infty} m^{\alpha(1-r)-1}\left(\mathbb{E}|Y|\left(1-\tilde{g}_\mu\left(\frac{|Y|}{m^\alpha}\right)\right)\right)^r\\
&\le C\sum_{m=1}^{\infty} m^{\alpha(1-r)-1}\frac{\mathbb{E}\left(|Y|^p l(|Y|^{1/\alpha})\right)^r}{m^{\alpha(p-1)r}l^r(m)}\sum_{n=1}^{m} f(n)n^r\\
&\le C\sum_{m=1}^{\infty} m^{-(\alpha p-1)(r-1)-1}/l^r(m)\left(C_{\mathbb{V}}\left\{|Y|^p l(|Y|^{1/\alpha})\right\}\right)^r<\infty. \qquad (3.8)
\end{aligned}
$$

Hence, (3.1) is established by (3.4)–(3.8). □

We next investigate the case $\alpha p = 1$.

Theorem 3.2. *Assume that* $1 \le p < 2$. *Suppose that* $\sum_{i=-\infty}^{\infty}|a_i|^\theta < \infty$, *where* $\theta \in (0,1)$ *if* $p = 1$. *Assume that* $g(n) = O(n^\delta)$ *for some* $0 \le \delta < (2-p)/p$. *While* $p = 1$, *assume that* $0 < \delta < 1$. *If* $\mathbb{E}(Y_i) = \mathbb{E}(-Y_i) = 0$ *and* $C_{\mathbb{V}}\left\{|Y|^{p(1+\delta)}l(|Y|^p)\right\} < \infty$, *then for any* $\varepsilon > 0$,

$$
\sum_{n=1}^{\infty} n^{-1-1/p}l(n)C_{\mathbb{V}}\left\{\left(\left|\sum_{j=1}^{n}X_j\right|-\varepsilon n^{1/p}\right)^+\right\}<\infty. \qquad (3.9)
$$

Remark 3.2. *In Theorem 3.2, if* $\mathbb{E}$ *is reduced to linear expectation* E, *then* (3.9) *is just* (3.9) *of Guan et al. [18]. The rate* $n^{-1-1/p}l(n)$ *implies* $\lim_{n\to\infty} l(n)\int_\epsilon^\infty \mathbb{V}\left(\left|\sum_{j=1}^n X_j\right|/n^{1/p} > y\right)\mathrm{d}y = 0$, $\forall\epsilon > 0$.

Proof of Theorem 3.2. Let $h(x) = n^{-1-1/p}l(n)$. As in the proof of (3.4), we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} h(n)C_{\mathbb{V}}\left\{\left(\left|\sum_{j=1}^{k}X_j\right|-\varepsilon n^{1/p}\right)^+\right\}\\
\le& C\sum_{n=1}^{\infty} h(n)\int_{n^{1/p}}^{\infty}\mathbb{V}\left\{\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+n}Y_{xj}^{(2)}\right|>\varepsilon x/2\right\}\mathrm{d}x\\
&+C\sum_{n=1}^{\infty} h(n)\int_{n^{1/p}}^{\infty}\mathbb{V}\left\{\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+n}Y_{xj}^{(1)}\right|>\varepsilon x/2\right\}\mathrm{d}x\\
=:& J_1+J_2. \qquad (3.10)
\end{aligned}
$$

For J_1, take $\varepsilon' = \delta$. By Markov's inequality under sub-linear expectations, C_r inequality, and Lemma 2.3 of Xu [26], we see that

$$
J_1 \le C\sum_{n=1}^{\infty} h(n)\int_{n^{1/p}}^{\infty} x^{-\theta}\mathbb{E}^*\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+n}Y_{xj}^{(2)}\right|^2\mathrm{d}x
$$

$$
\begin{aligned}
&\le C\sum_{n=1}^{\infty} nh(n)\int_{n^{1/p}}^{\infty} x^{-\theta}\mathbb{E}|Y_x^{(2)}|^{\theta}\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty} nh(n)\sum_{m=n}^{\infty}\int_{m^{1/p}}^{(m+1)^{1/p}} x^{-\theta}\mathbb{E}|Y|^{\theta}\left(1-\tilde{g}_{\mu}\left(\frac{|Y|}{x}\right)\right)\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty} nh(n)\sum_{m=n}^{\infty} m^{(1-\theta)/p-1}\mathbb{E}|Y|^{\theta}\left(1-\tilde{g}_{\mu}\left(\frac{|Y|}{m^{1/p}}\right)\right)\\
&= C\sum_{m=1}^{\infty} m^{(1-\theta)/p-1}\mathbb{E}|Y|^{\theta}\left(1-\tilde{g}_{\mu}\left(\frac{|Y|}{m^{1/p}}\right)\right)\sum_{n=1}^{m} nh(n)\\
&\le \begin{cases} C\sum_{m=1}^{\infty} m^{-1/p}l(m)\mathbb{E}|Y|\left(1-\tilde{g}_{\mu}\left(\frac{|Y|}{m^{1/p}}\right)\right), & 1<p<2\\ C\sum_{m=1}^{\infty} m^{(1-\theta)/p-1}\mathbb{E}|Y|^{\theta}\left(1-\tilde{g}_{\mu}\left(\frac{|Y|}{m^{1/p}}\right)\right)\sum_{n=1}^{m} n^{\varepsilon'-1}l(n), & p=1\end{cases}\\
&\le \begin{cases} C\sum_{k=0}^{\infty}\sum_{m=2^k}^{2^{k+1}-1} m^{-1/p}l(m)\mathbb{E}|Y|\left(1-\tilde{g}_{\mu}\left(\frac{|Y|}{m^{1/p}}\right)\right), & 1<p<2\\ C\sum_{k=0}^{\infty}\sum_{m=2^k}^{2^{k+1}-1} m^{-\theta+\varepsilon'}l(m)\mathbb{E}|Y|^{\theta}\left(1-\tilde{g}_{\mu}\left(\frac{|Y|}{m}\right)\right), & p=1\end{cases}\\
&\le \begin{cases} C\sum_{k=1}^{\infty} 2^{k(-1/p+1)}l(2^k)\mathbb{E}^*\left(|Y|\sum_{j=k}^{\infty} g_j\left(\frac{|Y|}{2^{j/p}}\right)\right), & 1<p<2\\ C\sum_{k=1}^{\infty} 2^{k(-\theta+\varepsilon'+1)}l(2^k)\mathbb{E}^*\left(|Y|^{\theta}\sum_{j=k}^{\infty} g_j\left(\frac{|Y|}{2^{j}}\right)\right), & p=1\end{cases}\\
&\le \begin{cases} C\sum_{j=1}^{\infty} \mathbb{E}^*\left(|Y|g_j\left(\frac{|Y|}{2^{j/p}}\right)\right)\sum_{k=1}^{j} 2^{k(-1/p+1)}l(2^k), & 1<p<2\\ C\sum_{j=1}^{\infty} \mathbb{E}^*\left(|Y|^{\theta}g_j\left(\frac{|Y|}{2^{j}}\right)\right)\sum_{k=1}^{j} 2^{k(-\theta+\varepsilon'+1)}l(2^k), & p=1\end{cases}\\
&\le \begin{cases} C\sum_{j=1}^{\infty} \mathbb{V}\left\{|Y|>\mu 2^{(j-1)/p}\right\}2^{j}l(2^j)<\infty, & 1<p<2\\ C\sum_{j=1}^{\infty} \mathbb{V}\left\{|Y|>\mu 2^{(j-1)}\right\}2^{j(\varepsilon'+1)}l(2^k)<\infty, & p=1,\end{cases}
\end{aligned}
\tag{3.11}
$$

where $\tilde{g}_{\mu}(\cdot)$, $g_j(\cdot)$ are defined as those of (3.2) and (3.3) with only $1/p$ here in place of α there.

For J_2, as in the proof of I_2, observing that $g(n)=O(n^{\delta})$, for some $0\le\delta<(2-p)/p$, taking $q=2$ by Lemma 2.4 and the similar proof of (2.8) of Zhang [25], we get

$$
\begin{aligned}
J_2 &\le C\sum_{n=1}^{\infty} h(n)\int_{n^{1/p}}^{\infty} x^{-2}\mathbb{E}^*\left|\sum_{i=-\infty}^{\infty} a_i\sum_{j=i+1}^{i+n} Y_{xj}^{(1)}\right|^2\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty} nh(n)(1+g(n))\sum_{m=n}^{\infty}\int_{m^{1/p}}^{(m+1)^{1/p}} x^{-2}C_{\mathbb{V}}\left\{|Y_x^{(1)}|^2\right\}\mathrm{d}x\\
&\quad+C\sum_{n=1}^{\infty} h(n)\sum_{m=n}^{\infty}\int_{m^{1/p}}^{(m+1)^{1/p}} x^{-2}\left[\sum_{i=1}^{n}|\mathbb{E}(Y_{xj}^{(1)})|+|\mathbb{E}(-Y_{xj}^{(1)})|\right]^2\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty} nh(n)(1+g(n))\sum_{m=n}^{\infty} m^{-1/p-1}\int_0^{(m+1)^{2/p}}\mathbb{V}\left\{|Y|^2>y\right\}\mathrm{d}y\\
&\quad+C\sum_{n=1}^{\infty} h(n)\sum_{m=n}^{\infty} m^{-1/p-1}\left[\sum_{i=1}^{n}\mathbb{E}(|Y_{m^{1/p}j}^{(2)}|)\right]^2\\
&\le C\sum_{m=1}^{\infty} m^{-1/p-1}\int_0^{(m+1)^{2/p}}\mathbb{V}\left\{|Y|^2>y\right\}\mathrm{d}y\sum_{n=1}^{\infty} n^{-1/p}l(n)(1+g(n))
\end{aligned}
$$

$$
\begin{aligned}
&+ C\sum_{m=1}^{\infty} m^{-1/p-1}\left[\mathbb{E}(|Y^{(2)}_{m^{1/p}}|)\right]^2 \sum_{n=1}^{m} n^{1-1/p} l(n)\\
\leq\ & C\sum_{m=1}^{\infty} m^{-2/p+\delta} l(m)\int_0^{(m+1)^{2/p}} \mathbb{V}\left\{|Y|^2>y\right\}\mathrm{d}y\\
&+ C\sum_{m=1}^{\infty} m^{-2/p+1} l(m)\left[\mathbb{E}(|Y^{(2)}_{m^{1/p}}|)\right]^2\\
\leq\ & C\sum_{m=1}^{\infty} m^{-2/p+\delta} l(m)\sum_{\ell=1}^{m+1}\int_{(\ell-1)^{2/p}}^{(\ell)^{2/p}} \mathbb{V}\left\{|Y|^2>y\right\}\mathrm{d}y\\
&+ C\sum_{m=1}^{\infty} m^{-2/p+1} l(m)\left[\mathbb{E}|Y|\left(1-\tilde{g}_{\mu}\left(\frac{|Y|}{m^{1/p}}\right)\right)\right]^2\\
\leq\ & C\sum_{\ell=1}^{\infty}\int_{(\ell-1)^{2/p}}^{(\ell)^{2/p}} \mathbb{V}\left\{|Y|^2>y\right\}\mathrm{d}y \sum_{m=1\bigvee(\ell-1)}^{\infty} m^{-2/p+\delta} l(m)\\
&+ C\sum_{m=1}^{\infty} m^{-2/p+1} l(m)\frac{\left(\mathbb{E}\left(|Y|^{p(1+\delta)} l(|Y|^p)\right)\right)^2}{\left(m^{(p(1+\delta)-1)/p} l(m)\right)^2}\\
\leq\ & C\sum_{\ell=2}^{\infty}\int_{(\ell-1)^{2/p}}^{(\ell)^{2/p}} \mathbb{V}\left\{|Y|^2>y\right\}\ell^{1+\delta-2/p} l(\ell)\mathrm{d}y + C\\
&+ C\sum_{m=1}^{\infty} m^{-2\delta-1}/l(m)\left(C_{\mathbb{V}}\left\{|Y|^{p(1+\delta)} l(|Y|^p)\right\}\right)^2\\
\leq\ & C\sum_{\ell=2}^{\infty}\int_{(\ell-1)^{2/p}}^{\ell^{2/p}} \mathbb{V}\left\{|Y|^2>y\right\} y^{(1+\delta-2/p)p/2} l(y^{p/2})\mathrm{d}y + C\\
\leq\ & C\int_1^{\infty} \mathbb{V}\left\{|Y|^p>y\right\}\mathrm{d}(l(y)y^{1+\delta}) + C \leq CC_{\mathbb{V}}\left\{|Y|^{p(1+\delta)} l(|Y|^p)\right\} < \infty.
\end{aligned}
\tag{3.12}
$$

Therefore, combining (3.10)–(3.12) results in (3.9). This completes the proof. □

By Theorems 3.1 and 3.2, we conclude that the following corollary holds.

Corollary 3.1. *Under the same conditions of Theorem 3.1, for any $\epsilon > 0$, we have*

$$
\sum_{n=1}^{\infty} n^{\alpha p-2} l(n)\mathbb{V}\left\{\left|\sum_{j=1}^{n} X_j\right| > \epsilon n^{\alpha}\right\} < \infty. \tag{3.13}
$$

Under the conditions of Theorem 3.2, for any $\epsilon > 0$, we have

$$
\sum_{n=1}^{\infty} n^{-1} l(n)\mathbb{V}\left\{\left|\sum_{j=1}^{n} X_j\right| > \epsilon n^{1/p}\right\} < \infty. \tag{3.14}
$$

4. Conclusions

We have obtained new results about the complete moment convergence for the partial sum of moving average processes produced by m-WA random variables under sub-linear expectations. Results obtained here carefully extend the relevant ones for m-WOD random variables in probability space to those under sub-linear expectations, which complement those for ND and END random variables, and serve as a basis for further complete q-order moment convergence for partial sums and maximal partial sums of moving average processes generated by m-WA random variables under sub-linear expectations. In addition, Lemma 2.4 is the Rosenthal-type inequality for m-WA random variables, which is another innovation point of this paper, and could serve as basis for further complete moment convergence for maximal partial sums of m-WA random variables under sub-linear expectations.

Author contributions

Mingzhou Xu: Writing - original draft, writing - review & editing, conceptualization, formal analysis, funding acquisition, methodology, project administration. Xuhang Kong: Writing - review & editing, conceptualization, validation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence tools in the creation of this article.

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Conflict of interest

All authors state no conflicts of interest in this article.

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Appendix

Detailed proof of Lemma 2.3. For readers' convenience, here we give a detailed proof when $0 < t \le 1$.

For $y > 0$, write $\bar{X}_i = \min\{X_i, y\}$, $i = 1, 2, \cdots, n$, and $T_n = \sum_{i=1}^n \bar{X}_i$, $n \ge 1$. We easily see that

$$\{S_n \ge x\} = \{T_n \ne S_n\} \bigcup \{T_n \ge x\},$$

which yields that for any positive h,

$$\mathbb{V}(S_n \ge x) \le \mathbb{V}(T_n \ne S_n) + \mathbb{V}(T_n \ge x) \le \sum_{i=1}^n \mathbb{V}(X_i > y) + \mathrm{e}^{-hx}\mathbb{E}\mathrm{e}^{hT_n}.$$

It follows that

$$\mathbb{V}(S_n \geq x) \leq \sum_{i=1}^{n} \mathbb{V}(X_i > y) + g(n)\mathrm{e}^{-hx} \prod_{i=1}^{n} \mathbb{E}\mathrm{e}^{h\bar{X}_i}. \tag{4.1}$$

For $0 < t \leq 1$, $h > 0$, the function $\frac{\mathrm{e}^{hu}-1}{u^t}$ is increasing on $u > 0$. Hence,

$$\begin{aligned}
\mathbb{E}\mathrm{e}^{h\bar{X}_i} &\leq 1 + \mathbb{E}\left(\frac{\mathrm{e}^{h\bar{X}_i}-1}{|\bar{X}_i|^t}|\bar{X}_i|^t\right) \leq 1 + \mathbb{E}\left(\frac{\mathrm{e}^{hy}-1}{|y|^t}|\bar{X}_i|^t\right) \\
&\leq 1 + \frac{\mathrm{e}^{hy}-1}{|y|^t}\mathbb{E}\left(|\bar{X}_i|^t\right) \leq \exp\left\{\frac{\mathrm{e}^{hy}-1}{|y|^t}\mathbb{E}\left(|\bar{X}_i|^t\right)\right\} \\
&\leq \exp\left\{\frac{\mathrm{e}^{hy}-1}{|y|^t}\mathbb{E}\left(|X_i|^t\right)\right\}.
\end{aligned}$$

Combining the inequality above and (4.1) yields that

$$\mathbb{V}(S_n \geq x) \leq \sum_{i=1}^{n} \mathbb{V}(X_i > y) + g(n)\exp\left\{\frac{\mathrm{e}^{hy}-1}{y^t}\sum_{i=1}^{n}\mathbb{E}\left(|X_i|^t\right) - hx\right\}. \tag{4.2}$$

Taking $h = \frac{1}{y}\log\left(1 + \frac{xy^{t-1}}{\sum_{i=1}^{n}\mathbb{E}(|X_i|^t)}\right)$ in the right-hand side of (4.2), we obtain (2.2). $\square$

Proof of $I_{21} < \infty$ and $I_{22} < \infty$ in the proof of Theorem 3.1 . For I_{21}, if $p > 1$, taking $r > \max\{2, p\}$, then by C_r inequality, the similar proof of (2.8) of Zhang [25], and Lemma 2.3 of Xu [26], we see that

$$\begin{aligned}
I_{21} &\leq C\sum_{n=1}^{\infty} f(n)\int_{n^\alpha}^{\infty} x^{-r}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n} C_{\mathbb{V}}\left\{|Y_x^{(1)}|^r\right\}\mathrm{d}x \\
&\leq C\sum_{n=1}^{\infty} f(n)n\sum_{m=n}^{\infty}\int_{m^\alpha}^{(m+1)^\alpha} x^{-r}C_{\mathbb{V}}\left\{|Y_x^{(1)}|^r\right\}\mathrm{d}x \\
&\leq C\sum_{n=1}^{\infty} f(n)n\sum_{m=n}^{\infty} m^{\alpha(1-r)-1}C_{\mathbb{V}}\left\{|Y_{(m+1)^\alpha}^{(1)}|^r\right\} \\
&\leq C\sum_{m=1}^{\infty} m^{\alpha(1-r)-1}\int_0^{(m+1)^\alpha}\mathbb{V}(|Y| > x)\, x^{r-1}\mathrm{d}x\sum_{n=1}^{m} f(n)n \\
&\leq C\sum_{m=1}^{\infty} m^{\alpha(1-r)-1}\int_0^{(m+1)^\alpha}\mathbb{V}(|Y| > x)\, x^{r-1}\mathrm{d}x m^{\alpha p-\alpha}l(m) \\
&\leq C\sum_{m=1}^{\infty} m^{\alpha(p-r)-1}l(m)\sum_{k=1}^{m+1}\int_{(k-1)^\alpha}^{k^\alpha}\mathbb{V}(|Y| > x)\, x^{r-1}\mathrm{d}x \\
&\leq C\sum_{k=1}^{\infty}\int_{(k-1)^\alpha}^{k^\alpha}\mathbb{V}(|Y| > x)\, x^{r-1}\mathrm{d}x\sum_{m=1\bigvee(k-1)}^{\infty} m^{\alpha(p-r)-1}l(m) \\
&\leq C\sum_{k=2}^{\infty}\mathbb{V}\left(|Y| > (k-1)^\alpha\right)k^{r\alpha-1}k^{\alpha(p-r)}l(k) + C\sum_{m=1}^{\infty} m^{\alpha(p-r)-1}l(m)
\end{aligned}$$

$$\leq C\sum_{k=2}^{\infty}\mathbb{V}\left(|Y|>(1/2)^{\alpha}k^{\alpha}\right)k^{\alpha p-1}l(k)+C<\infty. \tag{4.3}$$

For I_{21}, if $p=1$, taking $r>\max\{1+\lambda',2\}$, where $0<\lambda'<\lambda$, then by the similar discussion as above, we see that

$$\begin{aligned}
I_{21} &\leq C\sum_{m=1}^{\infty}m^{\alpha(1-r)-1}\int_{0}^{(m+1)^{\alpha}}\mathbb{V}\left(|Y|>x\right)x^{r-1}\mathrm{d}x\sum_{n=1}^{m}f(n)n\\
&\leq C\sum_{m=1}^{\infty}m^{\alpha(1-r+\lambda')-1}\int_{0}^{(m+1)^{\alpha}}\mathbb{V}\left(|Y|>x\right)x^{r-1}\mathrm{d}xl(m)\\
&\leq C\sum_{m=1}^{\infty}m^{\alpha(1-r+\lambda')-1}l(m)\sum_{k=1}^{m+1}\int_{(k-1)^{\alpha}}^{k^{\alpha}}\mathbb{V}\left(|Y|>x\right)x^{r-1}\mathrm{d}x\\
&\leq C\sum_{k=1}^{\infty}\int_{(k-1)^{\alpha}}^{k^{\alpha}}\mathbb{V}\left(|Y|>x\right)x^{r-1}\mathrm{d}x\sum_{m=1\bigvee(k-1)}^{\infty}m^{\alpha(1-r+\lambda')-1}l(m)\\
&\leq C\sum_{k=2}^{\infty}\mathbb{V}\left(|Y|>(k-1)^{\alpha}\right)k^{r\alpha-1}k^{\alpha(1-r+\lambda')}l(k)+C\sum_{m=1}^{\infty}m^{\alpha(1-r+\lambda')-1}l(m)\\
&\leq C\sum_{k=2}^{\infty}\mathbb{V}\left(|Y|>(1/2)^{\alpha}k^{\alpha}\right)k^{\alpha(1+\lambda')-1}l(k)+C<\infty.
\end{aligned} \tag{4.4}$$

For I_{22}, if $1\leq p<2$, observing that $g(n)=O(n^{\delta})$ and taking $r>2$ fulfilling that $\alpha p+r/2-\alpha pr/2-1+\delta=(\alpha p-1)(1-r/2)+\delta<0$ and $r/2+\alpha p-2-\alpha+\delta>-1$, then by C_r inequality, we conclude that

$$\begin{aligned}
I_{22} &\leq C\sum_{n=1}^{\infty}n^{r/2}f(n)g(n)\int_{n^{\alpha}}^{\infty}x^{-r}\left(\mathbb{E}\left|Y_x^{(1)}\right|^2\right)^{r/2}\mathrm{d}x\\
&\leq C\sum_{n=1}^{\infty}n^{r/2}f(n)g(n)\sum_{m=n}^{\infty}\int_{m^{\alpha}}^{(m+1)^{\alpha}}\\
&\times\left[x^{-r}\left(\mathbb{E}\left(|Y|^2\tilde{g}_{\mu}\left(\frac{\mu|Y|}{x}\right)\right)\right)^{r/2}+\left(\mathbb{E}\left(1-\tilde{g}_{\mu}\left(\frac{|Y|}{x}\right)\right)\right)^{r/2}\right]\mathrm{d}x\\
&\leq C\sum_{m=1}^{\infty}\left[m^{\alpha(1-r)-1}\mathbb{E}\left(|Y|^2\tilde{g}_{\mu}\left(\frac{\mu|Y|}{(m+1)^{\alpha}}\right)\right)^{r/2}+m^{\alpha-1}\left(\mathbb{E}\left(1-\tilde{g}_{\mu}\left(\frac{|Y|}{m^{\alpha}}\right)\right)\right)^{r/2}\right]\\
&\times\sum_{n=1}^{m}n^{r/2}f(n)g(n)\\
&\leq C\sum_{m=1}^{\infty}m^{\alpha(p-r)+r/2+\delta-2}l(m)\left[\mathbb{E}\left(|Y|^p|Y|^{2-p}\tilde{g}_{\mu}\left(\frac{\mu|Y|}{(m+1)^{\alpha}}\right)\right)\right]^{r/2}\\
&+C\sum_{m=1}^{\infty}m^{\alpha p+r/2+\delta-2}l(m)\left(\mathbb{E}\left(1-\tilde{g}_{\mu}\left(\frac{|Y|}{m^{\alpha}}\right)\right)\right)^{r/2}\\
&\leq C\sum_{m=1}^{\infty}m^{\alpha p+r/2-\alpha pr/2+\delta-2}l(m)\left(\mathbb{E}|Y|^p\right)^{r/2}<\infty.
\end{aligned} \tag{4.5}$$

For I_{22}, if $p \geq 2$, observing that $g(n) = O(n^\delta)$ and taking $r > (\alpha p - 1)/(\alpha - 1/2) \geq p$ satisfying $\alpha(p-r) + r/2 + \delta - 1 < 0$, then by C_r inequality and the similar proof of (4.5), we have

$$\begin{aligned}
I_{22} \leq C \sum_{m=1}^{\infty} &\left[m^{\alpha(1-r)-1} \mathbb{E}\left(|Y|^2 \tilde{g}_\mu\left(\frac{\mu|Y|}{(m+1)^\alpha}\right)\right)^{r/2} + m^{\alpha-1}\left(\mathbb{E}\left(1 - \tilde{g}_\mu\left(\frac{|Y|}{m^\alpha}\right)\right)\right)^{r/2}\right] \\
&\times \sum_{n=1}^{m} n^{r/2} f(n) g(n) \\
&\leq C \sum_{m=1}^{\infty} m^{\alpha(p-r)+r/2+\delta-2} l(m) \left(\mathbb{E}|Y|^2 \tilde{g}_\mu\left(\frac{\mu|Y|}{(m+1)^\alpha}\right)\right)^{r/2} \\
&+ C \sum_{m=1}^{\infty} m^{\alpha p + r/2 + \delta - 2} l(m) \left(\mathbb{E}\left(1 - \tilde{g}_\mu\left(\frac{|Y|}{m^\alpha}\right)\right)\right)^{r/2} \\
&\leq C \sum_{m=1}^{\infty} m^{\alpha(p-r)+r/2+\delta-2} l(m) \left(\mathbb{E}|Y|^2\right)^{r/2} < \infty. \qquad (4.6)
\end{aligned}$$

□