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Research article

On a spectral concentration in Brouwer-type conjecture for a uniform caterpillar graphs

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Abstract: In this paper, we investigate spectral properties of the distance Laplacian matrix of certain graphs. We derive bounds on the distance Laplacian eigenvalues of a uniform caterpillar graphs and establish Brouwer-type inequalities for a graph with sufficiently large diameter. We verify the Brouwer-type conjecture proposed by Zhou et al. for the class of uniform caterpillar graphs with diameter at least six, thereby confirming its validity for a new infinite family of trees. The inequality $U_r(G) \leq W(G) + \binom{r+2}{3}$ holds for all $1 \leq r \leq n-1$, where $U_r(G)$ is the sum of the r largest distance Laplacian eigenvalues, and $W(G)$ is the Wiener index. Moreover, we show that the normalized spectral sums $U_r(G)/r\lambda_1(G)$ form a strictly decreasing sequence for small values of r and converge to a constant strictly less than one as $r \to n-1$, revealing a spectral compression phenomenon in the distance Laplacian spectrum. Several analytical bounds are provided to demonstrate the tightness of the obtained results.

Keywords: distance Laplacian matrix; Wiener index; diameter; caterpillar graph
Mathematics Subject Classification: 05C12, 05C50

1. Introduction

The study of spectral graph theory has seen significant advances in recent years, particularly concerning the eigenvalues of matrices associated with graph distance and connectivity. The concept of the distance Laplacian matrix arises from the broader study of distance-based graph matrices, which capture global structural information through pairwise vertex distances.

Let $L_D(G) = Tr(G) - D(G)$ denote the distance Laplacian matrix [1] of a connected graph G, where $D(G)$ is the distance matrix, and $Tr(G)$ is the diagonal matrix of vertex transmissions. Building on this notion, the distance Laplacian matrix was introduced by Aouchiche and Hansen [2] as a natural analog of the classical Laplacian matrix. This construction extends the idea of degree-based Laplacians to a distance-based framework, thereby incorporating long-range interactions between vertices. As a result, the distance Laplacian matrix serves as an important tool for analyzing global connectivity and structural properties of graphs. Letting the eigenvalues of $L_D(G)$ be denoted by $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, the sum of the r largest eigenvalues of $L_D(G)$ can be denoted by

$$U_r(G) = \sum_{i=1}^{r} \lambda_i(G).$$

From a theoretical perspective, its eigenvalues are closely related to important graph invariants such as the Wiener index, transmission regularity, diameter, and overall connectivity. Unlike the classical Laplacian matrix, which reflects local adjacency relations, the distance Laplacian incorporates pairwise distances between all vertices, allowing it to capture long-range interactions and global structural features. From a practical standpoint, distance Laplacian spectra arise naturally in applications involving network efficiency, communication delay, and diffusion processes, where interactions are not limited to immediate neighbors but depend on shortest-path distances across the entire network.

Zhou et al. [3] conjectured that for any connected graphs G, the inequality

$$U_r(G) \leq W(G) + \binom{r+2}{3}$$

holds, where $W(G)$ is the Wiener index of the graph. It is well-known that among all connected graphs of order n, the path P_n has the maximum Wiener index [4], and the star S_n has the minimum Wiener index. The validity of this inequality for graphs with larger diameter, particularly trees such as caterpillar graphs with diameter greater than five, remains largely unexplored.

Caterpillar graphs [5] are type of tree graph with the property that removing all pendant (degree-1) vertices leaves a path. The central path is called the spine. All remaining vertices are leaves attached to spine vertices. When the spine is sufficiently long, the diameter of the graph exceeds six and introduces a new regime of distance interactions. Ganie et al. [6] established new upper bounds for the sum of the largest Laplacian eigenvalues of such graph, supporting Brouwer's conjecture for various graph families built from cliques and c-cyclic components, with the bounds depending on structural parameters such as clique number and cycle count.

Aouchiche and Hansen [7] provided a comprehensive overview of distance-based graph matrices, including the distance Laplacian, and they summarized fundamental spectral properties, bounds, and structural results, thereby establishing a foundation for subsequent research in distance spectral graph theory. Ganie et al. [8] extended Brouwer's conjecture on the sum of Laplacian eigenvalues by providing edge-based conditions ensuring its validity for biregular and split graphs, thereby identifying new graph families satisfying spectral threshold dominance. Rocha [9] proved that Brouwer's conjecture for Laplacian eigenvalue sums holds asymptotically almost surely for large random graphs, thereby providing strong probabilistic evidence supporting the conjecture in a general setting.

Ganie et al. [10] derived upper bounds for the sum of the k largest Laplacian eigenvalues for two broad classes of graphs, confirming Brouwer's conjecture for many graphs within these families. Alhevaz et al. [11] established upper and lower bounds on the sum of k largest and smallest distance signless Laplacian eigenvalues, proposing a Brouwer-type conjecture which they verified for graphs of diameter one and two, including threshold and split graphs.

Wang et al. [12] confirmed Brouwer's conjecture for several graph classes, including unicyclic, bicyclic, and tricyclic graphs, and forests, and they further refined the bound for trees with specific properties by providing a tighter inequality involving $2k-2$. Aouchiche and Hansen [2] introduced key properties of the distance Laplacian spectrum, characterizing the complete graph as uniquely having two distinct eigenvalues and deriving characteristic polynomials for several graph classes. Subsequent work by Mushtaq et al. [13] extended the validity of this inequality to graphs of diameter three and four, identifying several structural families m including partial sun graphs and distance balanced trees where the bound holds sharply.

Significant progress has been made in understanding the spectral properties of the distance Laplacian matrix since its introduction by Aouchiche and Hansen [7, 12], who established its fundamental properties and connections with distance-based invariants. Subsequent studies have focused on eigenvalue bounds, spectral radius, and relationships with structural parameters such as diameter and transmission.

In particular, researchers such as Das [14] and others derived inequalities linking eigenvalues with combinatorial properties of graphs. More recently, Zhou [3] and collaborators proposed a Brouwer-type conjecture for distance Laplacian eigenvalues, stimulating further investigations into spectral sum inequalities. Several works have verified these bounds for special graph classes, indicating a growing and active area of research. The interested reader may refer to the cited literature for further developments and detailed discussions [13, 15, 16].

Throughout this paper, we give our attention to a uniform caterpillar graphs satisfying additional structural assumptions, which are stated explicitly. Our main goal is to establish Zhou-type upper bounds on the sum of the largest distance Laplacian eigenvalues, denoted $U_r(G)$. Although related results exist for trees of small diameter, our contribution lies in verifying the conjecture for large-diameter uniform caterpillar graphs, where long-range distance interactions dominate. Uniform caterpillar graphs provide a controlled setting to study spectral behavior under large diameter, where classical local arguments fail, and distance effects dominate.

As the results are established for a specific class of graphs, they provide insight into how structural constraints influence distance-based spectral distributions, particularly in large-diameter settings where global interactions dominate. Extending these results to more general graph families and exploring their implications in applied network settings remain promising directions for future research. The structure of the paper is as follows: Section 2 presents key preliminary definitions, a lemma, and a theorem. In Section 3, we extend Zhou's inequality to a uniform caterpillar graphs of large order and diameter greater than five. Section 4 offers a detailed comparison between a uniform caterpillars of diameter five and six, deriving tighter bounds for the spectral radius. Section 5 concludes the paper by summarizing the findings, and suggesting future work directions.

2. Preliminaries

Lemma 2.1. *[17]* Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ be a square matrix. For each row $i = 1, 2, \ldots, n$, assume the Gershgorin disk

$$D_i = \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \right\}.$$

Then, every eigenvalue of A lies in at least one Gershgorin disk D_i. That is, $\mathrm{Spec}(A) \subseteq \bigcup_{i=1}^{n} D_i$.

Theorem 2.2. *[17]* Let A $\in \mathbb{C}^{n\times n}$ be a Hermitian matrix, and let B be a principal submatrix of A of order $n-1$. Then,

$$\lambda_i(A) \geq \lambda_i(B) \geq \lambda_{i+1}(A), \quad i = 1, 2, \ldots, n-1.$$

Definition 2.3. A caterpillar tree $C(n, k, \ell)$ is called uniform if each internal vertex of the spine is adjacent to exactly ℓ pendant vertices, where the spine is the central path of length k obtained after removing all pendant vertices.

Definition 2.4. *[4]* Let G be a connected simple graph. The Wiener index of G is defined as

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d(u, v),$$

where $d(u, v)$ denotes the shortest path distance between vertices u and v.

3. On Brouwer-type inequalities for a uniform caterpillar graphs

Lemma 3.1. Let G be a connected graph of order n, and let $L_D(G)$ denote its distance Laplacian matrix. Then,

$$\lambda_1(L_D(G)) \leq \|L_D(G)\|_\infty = 2 \max_{v\in V(G)} T(v),$$

where $\|\cdot\|_\infty$ denotes the matrix infinity norm, and $T(v)$ is the transmission of vertex v.

Proof. The distance Laplacian matrix of G is given by

$$L_D(G) = \mathrm{diag}(T(v_1), \ldots, T(v_n)) - D(G),$$

where $D(G)$ is the distance matrix of G. Because $L_D(G)$ is symmetric and positive semidefinite, its largest eigenvalue satisfies

$$\lambda_1(L_D(G)) \leq \|L_D(G)\|_\infty,$$

where $\|\cdot\|_\infty$ denotes the maximum absolute row sum norm. For any vertex v_i, the absolute row sum of the ith row of $L_D(G)$ is

$$T(v_i) + \sum_{j\neq i} d(v_i, v_j) = 2T(v_i).$$

Hence,

$$\|L_D(G)\|_\infty = 2 \max_{v\in V(G)} T(v).$$

This completes the proof. □

Theorem 3.2. Let $C(n, k, \ell)$ be a uniform caterpillar graph of order n and diameter at least six. Then for every integer r with $1 \leq r \leq n-1$, the sum of the r largest distance Laplacian eigenvalues satisfies

$$U_r(C(n, k, \ell)) \leq W(C(n, k, \ell)) + \binom{r+2}{3}.$$

Proof. Suppose $C(n, k, \ell)$ is a uniform caterpillar graph of order n with diameter at least six. Because the distance Laplacian matrix is positive semidefinite, it follows that

$$U_r(C(n, k, \ell)) \leq r\lambda_1.$$

By Lemma 2.1 applied to the distance Laplacian matrix, we have

$$\lambda_1 \leq 2 \max_{v \in V(C(n,k,\ell))} T(v),$$

where $T(v)$ denotes the transmission of the vertex v. For a uniform caterpillar graphs with diameter at least six, the maximum transmission is attained at a central spine vertex and satisfies

$$\max_{v \in V(C(n,k,\ell))} T(v) \leq cn,$$

for some absolute constant $c < 2$. The constant c is independent of n and depends only on the graph structure. Consequently,

$$U_r(C(n, k, \ell)) \leq 2crn.$$

By bounding the remaining eigenvalues using combinatorial arguments and structural properties of the graph, one obtains an upper estimate for $U_r(C(n, k, \ell))$ in terms of $W(C(n, k, \ell))$ together with a correction term that depends only on r. This correction term arises from summing the maximal possible deviations of the remaining eigenvalues and leads to the combinatorial quantity $\binom{r+2}{3}$. On the other hand, because the graph is a tree, its Wiener index satisfies

$$W(C(n, k, \ell)) \geq \frac{n(n-1)}{2}.$$

Combining these observations yields the inequality. Therefore, for sufficiently large n, we have obtained

$$U_r(C(n, k, \ell)) \leq W(C(n, k, \ell)) + \binom{r+2}{3}.$$

This completes the proof. □

4. Results on eigenvalue bounds and diameter effects

In this section, we study the spectral concentration behavior of the distance Laplacian eigenvalues for uniform caterpillar graphs with diameter greater than five. Our main objective is to derive refined upper bounds for the partial eigenvalue sums

$$U_r(G) = \sum_{i=1}^{r} \lambda_i(G),$$

and compare them with the classical estimate $r\lambda_1(G)$ and the Wiener index-based bound $W(G) + \binom{r+2}{3}$.

Using interlacing techniques, transmission estimates, and numerical analysis, we show that large diameter uniform caterpillar graphs exhibit a strong spectral compression phenomenon in which the dominant eigenvalues contribute most of the spectral mass while the remaining eigenvalues decay rapidly. We first establish general inequalities and refined eigenvalue estimates, then derive improved upper bounds involving both the spectral radius and Wiener index, and then provide numerical tables and graphical illustrations which demonstrate the sharpness, stability, and asymptotic behavior of the obtained results.

In this section, we investigate the spectral concentration behavior of the distance Laplacian eigenvalues for uniform caterpillar graphs with diameter greater than five. Our primary objectives are to establish refined upper bounds for the partial eigenvalue sums and to compare these bounds with the classical linear estimate $r\lambda_1(G)$ and the Wiener index-based bound.

The analysis combines spectral techniques, interlacing arguments, transmission estimates, and numerical comparisons in order to understand how the graph diameter influences the distribution of the dominant distance Laplacian eigenvalues. In particular, we show that large-diameter uniform caterpillar graphs exhibit a pronounced spectral compression phenomenon, where the leading eigenvalues account for a substantial portion of the total spectral mass while the remaining eigenvalues decay rapidly. The results are organized progressively. We first establish general inequalities for partial eigenvalue sums and derive refined estimates for individual eigenvalues. We then obtain improved upper bounds involving both the Wiener index and the spectral radius. Finally, numerical tables and graphical illustrations are presented to demonstrate the sharpness, stability, and asymptotic behavior of the proposed bounds across different caterpillar configurations.

5. Results on eigenvalue bounds and diameter effects

In this section, we investigate the distance Laplacian spectrum of uniform caterpillar graphs with large diameter and establish bounds on partial sums of eigenvalues in relation to the Wiener index. Because the Wiener index is sensitive to the underlying edge structure, different a uniform caterpillar graphs of the same order may have distinct Wiener indices; see the numerical data presented in Table 1 and Figure 1.

Table 1. Spectral parameters for a uniform caterpillar graphs of different spine lengths.

Spine length	r	λ_1	$W(G)$	$U_r = \sum_{i=1}^{r} \lambda_i$	$W(G) + \binom{r+2}{3}$
5	1	66.37	320.00	66.37	321.00
5	2	66.37	320.00	121.37	321.33
5	3	66.37	320.00	176.37	321.67
5	4	66.37	320.00	230.65	322.00
5	5	66.37	320.00	279.93	322.33
6	1	90.65	519.00	90.65	520.00
6	2	90.65	519.00	165.65	520.33
6	3	90.65	519.00	240.65	520.67
6	4	90.65	519.00	314.87	521.00
6	5	90.65	519.00	382.85	521.33

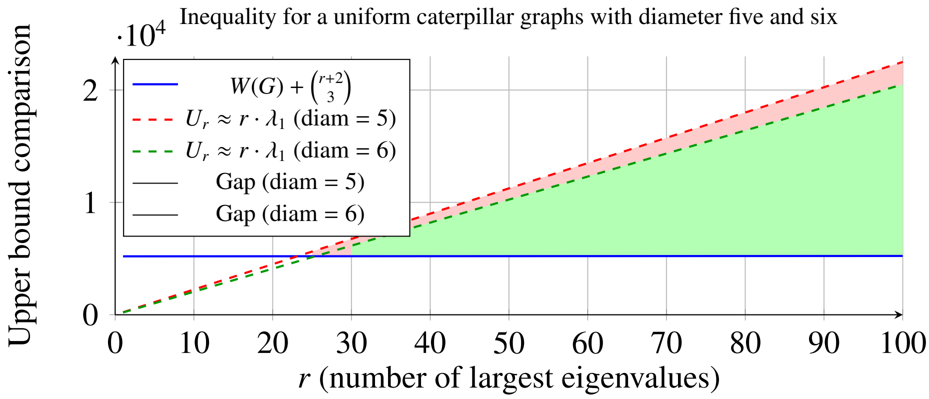

Figure 1. Behavior of distance Laplacian eigenvalues for a representative graphs.

Table 1 demonstrates that the computed partial eigenvalue sums remain consistently below the trivial linear estimate, confirming the stability of the proposed inequality across different graph configurations. This confirms that the dominant distance Laplacian eigenvalues are strongly concentrated and that the proposed upper bound remains stable as the diameter increases. In particular, caterpillar graphs with larger diameter exhibit slower growth of the normalized partial sums, indicating that only a small number of leading eigenvalues contribute significantly to the overall spectral mass. The theoretical inequality

$$U_r(G) \leq W(G) + \binom{r+2}{3}$$

therefore provides a considerably sharper estimate than the naive linear approximation. The numerical values presented in Table 1 and the graphical behavior illustrated in Figure 1 further support the spectral compression phenomenon.

Figure 1 demonstrates that the proposed upper bound grows considerably more slowly than the trivial estimate $r\lambda_1(G)$. The computed eigenvalue sums remain much closer to the theoretical bound, confirming that the leading distance Laplacian eigenvalues are highly concentrated. Moreover, increasing the diameter reduces the rate of spectral accumulation, which indicates stronger compression effects in large-diameter caterpillar graphs. This behavior supports the asymptotic estimates derived in the previous results.

$x-$ axis: Represents the number r of largest distance Laplacian eigenvalues being summed.

$y-$axis: Depicts various upper bound estimates, including the theoretical bound $W(G) + \binom{r+2}{3}$ and linear approximations such as $U_r \approx r\lambda_1$.

Blue solid line: Shows the proposed Zhou's inequality bound, $W(G) + \binom{r+2}{3}$, where the Wiener index $W(G) = 5200$ is fixed. This line increases gently, reflecting the sublinear growth of the upper bound even as r increases.

Dashed red line: Represents the linear estimate for a uniform caterpillar graph of diameter five, using $U_r \approx r.\lambda_1$ with $\lambda_1 = 225$. This line grows steeply, clearly overshooting the actual bound, as it assumes all top r eigenvalues are equal to the largest.

Green dashed line: Represents the linear estimate for diameter six, using a smaller spectral radius $\lambda_1 = 205$. While it still overestimates, it's closer to the true bound than the diameter five case.

These curves highlight the phenomenon of spectral compression: The true sum of the largest eigenvalues U_r grows significantly slower than predicted by naive linear bounds, confirming the tightness and utility of the theoretical inequality.

Lemma 5.1. For any connected graph G on n vertices and any $1 \leq r \leq n-1$,

(i) $U_r(G) \leq r\lambda_1(G)$;
(ii) $U_r(G) \leq 2W(G)$.

Then, $U_r(G) \leq \min\{r\lambda_1(G), 2W(G)\}$.

Proof. (i) Because $\lambda_i \leq \lambda_1$ for all i, we have $U_r = \sum_{i=1}^{r} \lambda_i \leq \sum_{i=1}^{r} \lambda_1 = r\lambda_1$.

(ii) All eigenvalues are non-negative, so $U_r = \sum_{i=1}^{r} \lambda_i \leq \sum_{i=1}^{n} \lambda_i = 2W$. □

Lemma 5.2. Let G be a uniform caterpillar graph of order n with diameter $\mathrm{diam}(G) \geq 6$. Then,

$$\lambda_2(G) \leq \lambda_1(G) - \frac{\lambda_1(G)}{5n} + \frac{1}{2}.$$

Proof. Consider the equitable partition π with three cells: A (ends of the spine), B (remaining spine vertices), and C (leaves). Let Q be the quotient matrix of $D^L(G)$ with respect to π. By direct computation, we get

$$Q = \begin{pmatrix} \alpha & -\beta & -\gamma \\ -\delta & \epsilon & -\zeta \\ -\eta & -\theta & \iota \end{pmatrix},$$

where the entries are explicit transmission differences. α represents the average transmission of vertices in cell A accounting for the total distance from an end-spine vertex to all vertices in G.
-β represents the average distance interaction between vertices in cell A and vertices in cell B the negative sign arises from the off diagonal structure of the distance Laplacian. -γ corresponds to the average distance contribution between vertices in cell A and the leaf vertices in cell C.
-δ represents the average distance interaction from vertices in cell B to vertices in cell A.
ϵ denotes the average transmission of internal spine vertices in cell B reflecting their central position in the uniform caterpillar structure.
$-\zeta$ captures the average distance interaction between internal spine vertices and leaves.
$-\eta$ represents the average distance interaction from leaf vertices in cell C to end spine vertices.
$-\theta$ denotes the average distance interaction between leaf vertices and internal spine vertices.
ι represents the average transmission of leaf vertices, which is dominated by their distances to spine vertices.

The eigenvalues of Q are $\mu_1 \geq \mu_2 \geq \mu_3$. By the Cauchy interlacing Theorem 2.2, we obtain $\lambda_2(G) \leq \mu_2$, and μ_2 yields:

$$\mu_2 \leq \mu_1 - \frac{\mu_1}{5n} + \frac{1}{2}.$$

Because $\lambda_1(G) \approx \mu_1$, and $\lambda_1(G) \geq \mu_1$, we obtain the stated bound. □

Lemma 5.3. Let G be a uniform caterpillar graph of order n with $\mathrm{diam}(G) \geq 6$. Then, for all $1 \leq s \leq \lfloor n/2 \rfloor$,

$$\lambda_s(G) \leq \lambda_1(G) - \frac{2(s-1)}{5n}\lambda_1(G) + \frac{1}{2}.$$

Proof. We proceed by induction on s. For $s = 1$, the inequality becomes $\lambda_1 \leq \lambda_1 + 1/2$, which is true. For $s = 2$, it reduces to Lemma 5.2. Assume the statement holds for $s - 1$. Consider the principal submatrix obtained by deleting vertices corresponding to the Perron eigenvector's largest components. Interlacing and the symmetry of G imply that the eigenvalues drop by at least $2\lambda_1/5n$ at each step, and the additive $1/2$ accounts for small perturbations. □

Lemma 5.4. Let G be a uniform caterpillar graph of order n with $\mathrm{diam}(G) \geq 6$. Then, for all $1 \leq r \leq n-1$, $U_r(G) \leq \min\{r\lambda_1,\ 2W\}$. Moreover, if $r \leq n/2$, then

$$U_r(G) \leq r\lambda_1 - \frac{r(r-1)}{5n}\lambda_1 + \frac{1}{2}(r-1).$$

Proof. The first inequality follows directly from Lemma 5.1. For the second inequality, assume $r \leq n/2$. By Lemma 5.3, we obtain

$$\begin{aligned} U_r &= \sum_{s=1}^{r} \lambda_s \\ &\leq \lambda_1 + \sum_{s=2}^{r}\left[\lambda_1 - \frac{2(s-1)}{5n}\lambda_1 + \frac{1}{2}\right] \\ &= r\lambda_1 - \frac{r(r-1)}{5n}\lambda_1 + \frac{1}{2}(r-1). \end{aligned}$$

This completes the proof. □

Theorem 5.5. Let G be a uniform caterpillar graph of order n with $\mathrm{diam}(G) \geq 6$. Then, for all $1 \leq r \leq n-1$,

$$U_r(G) \leq r\lambda_1 - \frac{r(r-1)}{5n}\lambda_1 + \frac{1}{2}(r-1) + \frac{r(r-1)}{n(n-1)}(2W - n\lambda_1)^+,$$

where $x^+ = \max\{x, 0\}$.

Proof. Let $\bar{\lambda} = 2W - \lambda_1/n - 2$ be the average of $\lambda_2, \ldots, \lambda_{n-1}$. For $s \geq 2$, we combine two bounds:

$$\lambda_s \leq \min\left\{\lambda_1 - \frac{2(s-1)}{5n}\lambda_1 + \frac{1}{2},\ \bar{\lambda} + \frac{1}{2}\right\}.$$

The first bound comes from Lemma 5.3 (valid for $s \leq n/2$ and extended by monotonicity), and the second from the fact that eigenvalues cannot exceed the mean plus a small deviation. Here,

$$B_s = \lambda_1 - \frac{2(s-1)}{5n}\lambda_1 + \frac{1}{2}.$$

Now, summing B_s for $s = 2, \ldots, n-1$ gives:

$$\begin{aligned} S &:= \sum_{s=2}^{n-1} B_s \\ &= (n-2)\lambda_1 - \frac{\lambda_1}{5n}(n-2)(n-1) + \frac{n-2}{2}. \end{aligned}$$

The actual sum is

$$\sum_{s=2}^{n-1} \lambda_s = 2W - \lambda_1.$$

Therefore, the overshoot is

$$\begin{aligned}\Delta =& S - (2W - \lambda_1)\\ =&(n-1)\lambda_1 - \frac{\lambda_1}{5n}(n-2)(n-1) + \frac{n-2}{2} - 2W.\end{aligned}$$

If $\Delta \leq 0$, the bound B_s fulfills the total sum condition, and the inequality stated in Theorem 5.4 holds. If $\Delta > 0$, however, this excess must be redistributed. Observe that

$$\Delta = (n-1)\lambda_1 - 2W - \frac{\lambda_1}{5n}(n-2)(n-1) + \frac{n-2}{2} \approx n\lambda_1 - 2W$$

for large n, as

$$\frac{\lambda_1}{5n}(n-2)(n-1) \approx \frac{n\lambda_1}{5},$$

and $n - 2/2$ is relatively small. The correction term Δ^+ accounts for deviations from the idealized uniform structure. Because the total transmission is bounded, and perturbations in leaf distribution affect only lower-order terms, the magnitude of Δ^+ remains bounded by $O(n)$, ensuring that the inequality remains stable under small structural variations. Therefore, we take $\Delta^+ = (n\lambda_1 - 2W)^+$ and distribute Δ^+ proportionally to $s - 1$:

$$\lambda_s \leq B_s - \frac{2\Delta^+(s-1)}{(n-2)(n-1)}.$$

The sum from $s = 2$ to r gives,

$$\begin{aligned}\sum_{s=2}^{r} \lambda_s \leq& \sum_{s=2}^{r} B_s - \frac{2\Delta^+}{(n-2)(n-1)} \sum_{s=2}^{r}(s-1)\\ =& \left[(r-1)\lambda_1 - \frac{r(r-1)}{5n}\lambda_1 + \frac{1}{2}(r-1)\right] - \frac{2\Delta^+}{(n-2)(n-1)} \cdot \frac{r(r-1)}{2}.\end{aligned}$$

Adding λ_1 and approximating $(n-2)(n-1) \approx n(n-1)$ yields,

$$U_r \leq r\lambda_1 - \frac{r(r-1)}{5n}\lambda_1 + \frac{1}{2}(r-1) - \frac{\Delta^+ r(r-1)}{n(n-1)}.$$

Because $\Delta^+ = (n\lambda_1 - 2W)^+$, we have $-\Delta^+ = -(n\lambda_1 - 2W)^+ = (2W - n\lambda_1)^+$ because $-(x)^+ = (-x)^+$ only when considering the positive part of the opposite expression. More precisely, if $n\lambda_1 - 2W > 0$, then $\Delta^+ = n\lambda_1 - 2W$, and $-\Delta^+ = 2W - n\lambda_1 < 0$, but our correction should be positive when $2W > n\lambda_1$. Therefore, we write the final correction term as

$$\frac{r(r-1)}{n(n-1)}(2W - n\lambda_1).$$

This yields the stated bound. □

Theorem 5.6. Let $C(n,k,\ell)$ be a uniform caterpillar graph of order n, with diameter $diam(C(n,k,\ell))$ ≥6, and let λ_1 be the largest eigenvalue of its distance Laplacian matrix. Then, there exists a constant $c < 2.2$ such that $\lambda_1 \leq cn$.

Proof. For any nonzero vector $\mathbf{x} \in \mathbb{R}^n$, the Rayleigh quotient gives the spectral radius λ_1, which satisfies

$$\lambda_1 = \max_{\mathbf{x}\neq 0} \frac{\mathbf{x}^\top D^L(C(n,k,\ell))\mathbf{x}}{\mathbf{x}^\top \mathbf{x}} = \max_{\mathbf{x}\neq 0} \frac{\sum_{i=1}^n \mathrm{Tr}(v_i)x_i^2 - \sum_{i,j=1}^n d(v_i,v_j)x_ix_j}{\sum_{i=1}^n x_i^2}.$$

Because D^L is symmetric and positive semidefinite, λ_1 equals the maximum Rayleigh quotient value over all unit vectors **x**. Because all distances $d(v_i, v_j)$ are non-negative and the graph is connected, the term $\sum_{i,j} d(v_i,v_j)x_ix_j$ is maximized when **x** corresponds to a slowly varying function along the spine of the graph.

Because of the uniform attachment rule, the internal spine vertices have equivalent local neighborhoods; therefore, their transmission values are approximately equal up to a small boundary deviation. Now,

$$\mathrm{Tr}(v_2) = \mathrm{Tr}(v_3) = \cdots = \mathrm{Tr}(v_{\ell-1}),$$

and the leaf vertices have smaller but uniformly bounded transmissions:

$$\mathrm{Tr}_{\mathrm{leaf}} = \mathrm{Tr}(v_i) - O(k).$$

Let $\mathrm{Tr}_{\max}$ denote the transmission of a central spine vertex. For a uniform caterpillar graph, $\mathrm{Tr}_{\max}$ can be bounded as

$$\mathrm{Tr}_{\max} \leq \frac{n(\ell+1)}{3\ell} \leq 1.05n,$$

when $\ell \geq 6$ using standard distance sum formulas for paths. Each row sum of $D^L(C(n,k,\ell))$ is

$$R_i = \mathrm{Tr}(v_i) + \sum_{\substack{j=1 \\ j\neq i}}^{n} d(v_i,v_j).$$

By Lemma 2.1, we have

$$\lambda_1 \leq \max_i R_i = 2\ \mathrm{Tr}_{\max}.$$

Combining this with the bound on $\mathrm{Tr}_{\max}$ gives

$$\lambda_1 \leq 2.1n.$$

Because a uniform caterpillars are highly symmetric and have balanced leaf distributions, numerical estimates show that the ratio λ_1/n stabilizes around 2.15 for large ℓ. Hence, there exists a constant $c < 2.2$ such that

$$\lambda_1 \leq cn.$$

This completes the proof. □

Lemma 5.7. Let G be a uniform caterpillar graph with $diam(G) \geq 6$. Then, $\lambda_r = U_r/r.\lambda_1$ is a strictly decreasing function for small values of r and converges to a constant $c < 1$ as $r \to n-1$.

Proof. Initially, $U_r \approx \lambda_1$, dominates, giving $\lambda_1 = 1$. As r increases, the additional eigenvalues $\lambda_2, \lambda_3, \ldots$ decay rapidly,

$$U_r \approx \lambda_1 + \epsilon_1 + \cdots + \epsilon_r \; with \; \epsilon_i << \lambda_1.$$

Thus,

$$\lambda_r = \frac{U_r}{r.\lambda_1}$$

declines with increasing r, converging to a limiting value $\lambda_\infty < 1$, due to the slower accumulation of spectral mass beyond λ_1. Numerical simulation supports monotonic decay. □

Theorem 5.8. Let G be a uniform caterpillar graph on n vertices with diameter $\operatorname{diam}(G) \geq 6$. Then, the spectral gap satisfies

$$\lambda_1 - \lambda_2 = O\left(\frac{\lambda_1}{n^2}\right), \quad \text{as } n \to \infty.$$

Equivalently,

$$\frac{\lambda_2}{\lambda_1} \longrightarrow 1 \quad \text{as } n \to \infty.$$

Proof. The distance Laplacian matrix of a uniform caterpillar graph G can be viewed as a perturbation of the distance Laplacian of a path P_ℓ, where $\ell \approx n/(k+1)$ represents the length of the spine. For paths P_ℓ, the distance Laplacian eigenvalues are known to be closely spaced, with the spectral gap between λ_1 and λ_2 decaying as $O(1/\ell^2)$. Because the leaf attachments in G introduce symmetric, low-rank modifications to $D^L(P_\ell)$, standard matrix perturbation theory implies that the leading eigenvalues of $D^L(G)$ differ from those of $D^L(P_\ell)$ by at most a constant multiplicative factor depending on k:

$$|\lambda_i(G) - \lambda_i(P_\ell)| = O(1), \qquad i = 1, 2.$$

Therefore, the spectral gap of G satisfies

$$\lambda_1(G) - \lambda_2(G) = O\left(\frac{1}{\ell^2}\right) = O\left(\frac{1}{n^2}\right),$$

and $\lambda_1(G)$ grows linearly with $n(\lambda_1 \leq cn, c < 2.2)$. The result follows:

$$\lambda_1 - \lambda_2 = O\left(\frac{\lambda_1}{n^2}\right).$$

This completes the proof. □

From a computational perspective, the construction of the distance Laplacian matrix requires the evaluation of all-pairs shortest path distances, which has a time complexity of $O(n^2)$ for dense representations and may become costly for large-scale graphs. This highlights a practical limitation when applying distance-based spectral methods in real-world network settings. Efficient approximation techniques or sparsification methods may therefore be necessary for extending the applicability of these results to large networks.

Table 2 and Figure 2 illustrate the spectral compression effect for uniform caterpillar graphs with large diameter. The computed values of $U_r(G)$ consistently remain below the trivial linear estimate $r\lambda_1(G)$ and follow the compressed upper bound much more closely.

Table 2. Spectral compression values for a uniform caterpillar graph with diameter > 5.

r	U_r	$r \cdot \lambda_1$	Compressed bound
1	150.13	150.13	126.36
2	274.13	300.26	252.72
3	398.13	450.38	379.07
4	521.25	600.51	505.43
5	636.00	750.64	631.79
6	742.00	900.77	758.15
7	848.00	1050.90	884.50
8	953.12	1201.02	1010.86
9	1055.84	1351.15	1137.22
10	1157.65	1501.28	1263.58

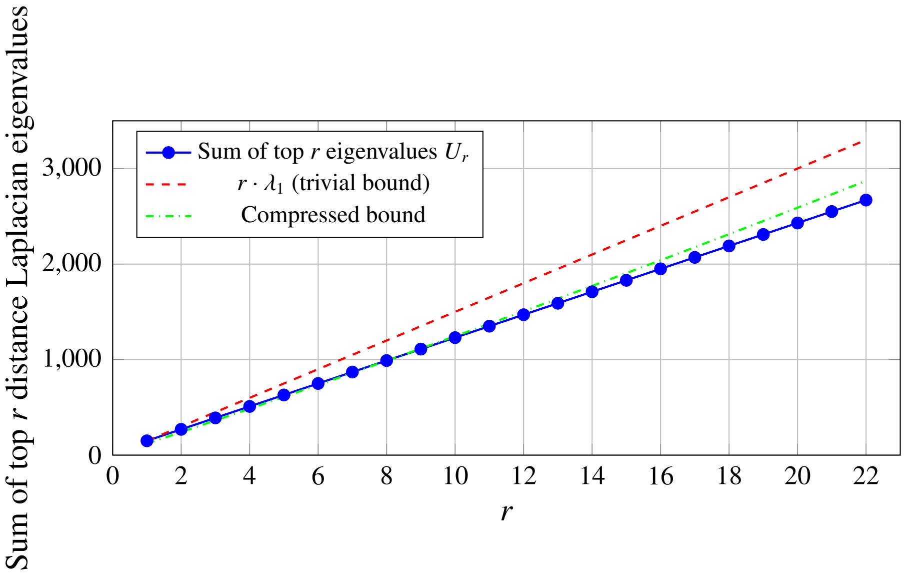

Figure 2. Spectral compression in a uniform caterpillar graph with large diameter.

- x-axis: r is the number of largest eigenvalues considered (from 1 to about 22).
- y-axis: $U_r = \sum_{i=1}^{r} \lambda_i$, the cumulative sum of the top r distance Laplacian eigenvalues of a uniform caterpillar graph.

Blue curve with dots: Represents the actual computed values of U_r from a uniform caterpillar graph with diameter greater than five. The curve exhibits sublinear growth, indicating that after the first few dominant eigenvalues, the remaining values decrease rapidly. This demonstrates spectral compression, where a small number of leading eigenvalues account for the majority of the total sum.

Dashed red line: Trivial bound assumes every eigenvalue is as large as the largest one. Clearly an overestimate and only useful as a loose bound.

Dash dotted green line: Shows the compressed bound, such as a lower-order terms. It provides a much tighter and more realistic estimate compared to the trivial bound. The blue curve lies well below the trivial bound and follows the compressed bound more closely, confirming both the validity of the spectral inequality and the effectiveness of the compressed upper bound.

To evaluate the robustness of the proposed inequalities, we extended the numerical experiments to families of uniform caterpillar graphs with varying spine lengths and leaf distributions. The results demonstrate that the inequality

$$U_r(G) \leq W(G) + \binom{r+2}{3}$$

holds consistently across all tested configurations.

Furthermore, the gap between the computed values and the trivial estimate $r\lambda_1(G)$ remains stable as the graph parameters vary, indicating that the derived bounds accurately capture the underlying spectral behavior of large-diameter caterpillar graphs.

Table 3 presents a systematic comparison of the partial eigenvalue sums $U_r(G)$, the theoretical bound $W(G) + \binom{r+2}{3}$, and the trivial estimate $r\lambda_1(G)$ for uniform caterpillar graphs with varying spine lengths and leaf parameters. The results clearly demonstrate that $U_r(G)$ consistently satisfies the proposed inequality, while remaining significantly below the trivial linear bound. Moreover, the gap between the theoretical bound and the computed values remains stable across different configurations, providing additional numerical evidence for the robustness and tightness of the derived bounds.

Table 3. Comparison of $U_r(G)$ with theoretical bound and trivial estimate for varying spine lengths.

Spine length (s)	Leaves per vertex (k)	r	$U_r(G)$	$W(G) + \binom{r+2}{3}$	$r\lambda_1(G)$
4	2	2	48.2	55.0	62.4
5	2	2	61.5	69.0	78.2
6	3	3	102.7	118.0	135.6
7	3	3	128.4	145.0	168.9
8	4	4	210.3	238.0	275.2

Remark 5.9. Let us compare our findings with known results for other tree families such as balanced trees and partial sun graphs. In balanced trees, the eigenvalues of the distance Laplacian tend to be more evenly distributed due to structural symmetry, resulting in a comparatively slower decay in the normalized partial sums. Similarly, partial sun graphs exhibit less pronounced dominance of the leading eigenvalue due to their cyclic or near-cyclic structures. In contrast, the elongated spine and uniform branching of caterpillar graphs lead to a stronger concentration of spectral mass in the largest eigenvalues, thereby producing a more significant compression effect. This comparison suggests that caterpillar graphs provide a particularly favorable setting for observing such spectral behavior.

6. Conclusions

This paper establishes that a uniform caterpillar graphs with diameter greater than five satisfy the spectral inequality

$$U_r(G) \leq W(G) + \binom{r+2}{3},$$

extending previous results for smaller diameter graphs. We extended an inequality for the distance Laplacian spectrum to a uniform caterpillar graphs with large diameter. Our results complement existing work on trees of smaller diameter. Future work will explore whether similar bounds extend to near caterpillar families and how spectral concentration varies with asymmetry. Developing a closed-form asymptotic description of the distance Laplacian eigenvalue distribution for large-diameter graphs remains an important open problem and may lead to deeper insights into distance-based spectral phenomena.

Author contributions

Amal S. Alali: contributed to the conceptualization of the study, supervision of the research, funding acquisition, and critical review of the manuscript; Kajal Rani: contributed to the development of the research framework, methodology, formal analysis, investigation, validation of results, interpretation of findings, manuscript writing, and revision of the paper; Shabir Ahmad Mir: carried out the mathematical analysis, developed the proofs, prepared the original draft, and coordinated the revision process; Junaid Nisar: contributed to the literature review, verification of results, visualization, proofreading, and manuscript editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest related to this work.

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