AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(5): 15056–15073.
DOI:10.3934/math.2026620
Received: 18 March 2026
Revised: 28 April 2026
Accepted: 07 May 2026
Published: 29 May 2026

Research article

Qualitative analysis of delay differential equations with non-instantaneous impulses: existence and controllability

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Abstract: In this paper, we mainly studied the non-instantaneous impulsive evolution equations with state-dependent delay under integral boundary conditions, and the evolution process of the non-instantaneous impulsive system was given. First, sufficient conditions were presented for the problem to have at least one solution when the related semigroup is compact. Second, the case when the associated semigroup is equicontinuous was presented by utilizing the theory of a semigroup. Third, the controllability of the considered problem was acquired via fixed point theory. In the end, examples were given to demonstrate the validity of the obtained results.

Keywords: non-instantaneous impulsive equations; state-dependent delay; integral boundary conditions; fixed point theorem; existence result; controllability
Mathematics Subject Classification: 34A12, 34A37, 34B37

1. Introduction

Impulse is a pervasive phenomenon in nature and human social activities, such as pharmacokinetics, secure communication, and population dynamics [1, 2]. Impulsive differential equations are usually employed to probe into impulse phenomenon, and it is not a simple superposition of the continuous system and discrete system, but a synthesis of the characteristics of the continuous system and discrete system. However, it goes beyond the research scope of continuous and discrete systems and forms a new hybrid system, which can effectively delineate the impulse phenomenon, for example, in the stability of impulsive neural networks, the infectious disease model of impulse immunity, the orbit-changing technology of satellite orbit and so on. Generally speaking, the impulsive systems consist of continuous dynamics that are depicted by an ordinary differential equation; discrete jumps at some instances with abrupt changes, showing when a resetting event happens, and the states of the system change instantaneously; and a criterion that resolves when the states of the systems will be reset. The development of impulsive differential equations has furnished a more accurate means of modeling some real-world processes and phenomenon, such as harvesting, disasters, and so on. The existence results and qualitative properties of solutions have been widely explored, see, e.g., [3–5] and the cited references.

Nevertheless, some phenomena in real life cannot be delineated by the action of instantaneous impulses, for instance, earthquakes and tsunamis. In view of the duration of the change process, the impulse can be divided into an instantaneous impulse and non-instantaneous impulse. As the name implies, the instantaneous impulse shows that the time of the sudden change process is very short relative to the whole development process and can be ignored. A non-instantaneous impulse signifies that the process of change is dependent on the state and lasts for a period of time that cannot be ignored. Hernández and O'Regan [6] first introduced the non-instantaneous impulsive equations based on the context of a person injecting drugs. In Banach space, by utilizing the theory of a semigroup, they obtained the existence and uniqueness results. Along this line, non-instantaneous impulse differential equations have received a significant amount of attention, see, e.g., [7–9]. Yang and Wang [8] studied non-instantaneous impulsive equations with boundary value conditions and found the existence results by the fixed point theorem. Wang [9] considered a kind of non-instantaneous impulse equation with state-dependent delay in Banach space. Combining semigroup theory, the existence result of the system was obtained by the fixed point method. In addition, the controllability of differential systems with non-instantaneous impulse has received considerable attention, see, e.g., [10–14] and the references cited therein. In [10], the authors considered the controllability of the nonlinear integro-differential equation and nonlocal problem with non-instantaneous impulse via Rothe's fixed point theory. In [13], the authors considered the fuzzy delay differential system with non-instantaneous impulses and investigated the existence, uniqueness, and total controllability results via nonlinear functional analysis, the Banach fixed point theorem, and fuzzy theory, respectively.

Since the speed limitation and connection between the system internal subsystem takes time, which makes time delay inevitable in almost all dynamic systems. A differential equation is usually utilized as the mathematical model of time delay. For example, the mathematical model put forward in the fields of biomedicine, chemical reactions, and so on have apparent time-delay phenomenon [15, 16]. Recently, delay differential equations have been deeply studied and incorporated into models in different branches of science [17–19]. Nelson [17] applied the delay equation to pharmacokinetics to study the correlation between drug administration and decreased viral load in HIV infections.

To our knowledge, the two-point boundary value condition is common for the first-order impulsive boundary value problem (BVP), which is a special case of the integral BVP, and the works on nonlinear differential equations including non-instantaneous impulses, integral boundary conditions, and state-dependent delay are rare. Motivated by this, in this paper, we study the non-instantaneous impulsive evolution equations with state-dependent delay under integral boundary conditions. The main contributions of this paper are as follows:

- We present the evolution processes of 1D and 2D non-instantaneous impulsive delay systems to improve understanding, which are rarely addressed in existing literature.
- We introduce the C_0-semigroup and infinitesimal generator $-B$, clarify their relation to mild solutions, and use an abstract phase space $\mathcal{D}$ to address delayed systems.

- The integral BVP is studied as an extension and supplement to the two-point BVP.
- Sufficient existence conditions are obtained under compact and equicontinuous semigroups, where the equicontinuous case complements the compact one.

Notations: $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{Z}_+$ signify the sets of real numbers, positive real numbers, and positive integers, respectively. $\mathbb{R}^n$ signifies n-dimensional Euclidean space with norm $\|\cdot\|$. Set $\mathrm{K} := [0,1]$. $C(\mathrm{K}, G)$ is the set of mapping $\zeta : \mathrm{K} \to G$ whose components are continuous functions. It forms a Banach space and $\|\zeta\|_C$ denotes the norm. $PC(\mathrm{K}, G) = \{\zeta : \mathrm{K} \to G : \zeta \in C((t_j, t_{j+1}], G)$ and there exist $\zeta(t_j^-)$ and $\zeta(t_j^+)$ with $\zeta(t_j^-) = \zeta(t_j)\}$, where the norm is denoted by $\|\zeta\|_{PC}$. Denote $M := \sup_{t\in\mathrm{K}} \|U(t)\|$. The noncompact Kuratowski measure is represented by $\mu(\cdot), \mu_C(\cdot), \mu_{PC}(\cdot)$ on the bounded set of $G, C(\mathrm{K}, G), PC(\mathrm{K}, G)$, and we refer readers to [20] and the reference therein for more details.

2. Preliminaries

Consider the existence of solutions of the following delay differential equation with non-instantaneous impulse,

$$\begin{cases} \zeta'(t) + B\zeta(t) = h(t, \zeta_{\varpi(t,\zeta_t)}), & t \in \bigcup_{j=0}^{k}(s_j, t_{j+1}], \\ \zeta(t) = U(t - t_j)I_j(t, \zeta(t)), & t \in \bigcup_{j=1}^{k}(t_j, s_j], \\ \zeta(0) = \displaystyle\int_0^1 \zeta(s)ds, \end{cases} \tag{1}$$

where $B : Z(B) \subseteq G \to G$ is a linear operator and is also closed, $\{U(t), t \geq 0\}$ denotes the C_0-semigroup, the infinitesimal generator is denoted by $-B$ on Hilbert space G, and $0 = s_0 < t_1 < s_1 < t_2 < \cdots < t_k < s_k < t_{k+1} = 1$. $\zeta(t) \in \mathbb{R}^n$ signifies the state vector. $h \in C(\mathrm{K} \times \mathcal{D}, G)$, where $\mathcal{D}$ is a phase space, which will be specified later and $I_j \in C([t_j, s_j] \times G, G)$, $j = 1, 2, \cdots, k$. For $\zeta_t : (-\infty, 0] \to G$, we define $\zeta_t(\theta) = \zeta(t + \theta)$, $\theta \in (-\infty, 0]$, where $\zeta_t(\cdot)$ is the element of $\mathcal{D}$ and it denotes the history of the state from each time θ up to the present time t. For any $t \geq 0$, let $\zeta_{\varpi(t,\zeta_t)} = \zeta(\varpi(t, \zeta(t + \theta)))$, $\theta \in (-\infty, 0]$.

Remark 1. In order to facilitate the understanding of non-instantaneous impulsive systems, the evolution process of a one-dimensional non-instantaneous impulsive system and two-dimensional non-instantaneous impulsive system are taken as examples, see Figure 1.

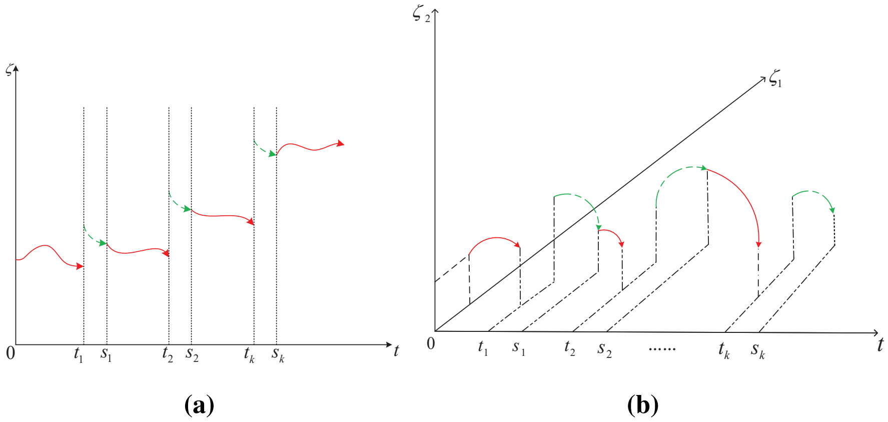

Figure 1. One-dimensional and two-dimensional non-instantaneous impulsive systems.

One may observe from the above two figures that the system evolves continuously on the intervals $(s_j, t_{j+1}]$, $j = 0, 1, 2, \cdots, k$ (red line with arrow); when $t = t_j$, $j = 1, 2, \cdots, k$, the state of the system changes abruptly (impulse occurs) and then the impulse lasts on the intervals $(t_j, s_j]$, $j = 1, 2, \cdots, k$ (green line with arrow), which cannot be ignored during the whole evolution process.

Lemma 1. *([21]) $\mathcal{D}$ is a function mapping $(-\infty, 0]$ into G, which is seminormed linear, endowed with the norm $\|\cdot\|_{\mathcal{D}}$, and satisfies:*

(i). If $\zeta \in PC(K, G)$ and $\zeta_0 \in \mathcal{D}$, then for every $t \in \mathrm{K}$, it holds that

- $\zeta_t \in \mathcal{D}$.
- There is $C_0 > 0$ such that $\|\zeta(t)\| \leq C_0 \|\zeta_t\|_{\mathcal{D}}$, where C_0 is a constant.
- There exist $C_1 : \mathbb{R}_+ \to \mathbb{R}_+$ and $C_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|\zeta_t\|_{\mathcal{D}} \leq C_1(t) \sup_{s \in [0,t]} \|\zeta(s)\| + C_2(t)\|\zeta_0\|_{\mathcal{D}}, \tag{2}$$

where C_1, C_2 are both independent of $\zeta(\cdot)$ with C_1 continuous and C_2 locally bounded.

(ii). For function $\zeta(\cdot)$ defined in (i), ζ_t is a $\mathcal{D}$-valued continuous function on K.

(iii). $\mathcal{D}$ is a complete space.

Lemma 2. *([20]) G is a complete normed vector space, Λ is a bounded subset of G, and there is a countable subset Λ_0 of Λ satisfying $\mu(\Lambda) \leq 2\mu(\Lambda_0)$.*

Lemma 3. *([20]) G is a complete normed vector space, $\Lambda = \{\zeta_n\}$ is a subset of $PC(\mathrm{K}, G)$, which is bounded and countable, and therefore, $\mu(\Lambda(t))$ satisfies*

$$\mu\Big(\Big\{\int_{\mathrm{K}} \zeta_n(t)ds \big| n \in \mathbb{N}\Big\}\Big) \leq 2\int_{\mathrm{K}} \mu(\Lambda(t))dt.$$

In addition, it is Lebesgue integral on K.

Lemma 4. *([20]) G is a complete normed vector space and for each $[t_j, t_{j+1}]$, $j = 0, 1, \cdots, k$, $\Lambda \subset PC(\mathrm{K}, G)$ is bounded and equicontinuous, thus $\mu(\Lambda(t)) \in PC(\mathrm{K}, \mathbb{R}^+)$ and $\mu_{PC}(\Lambda) = \sup_{t \in \mathrm{K}} \mu(\Lambda(t))$.*

Lemma 5. *([20]) G is a complete normed vector space. $S \subset G$ and S is nonempty. $Q : S \to G$ is continuous, which is called the strict μ-set-contraction operator if for every $\Gamma \subset S$ (Γ is bounded), there is a constant $0 \leq \Delta < 1$ such that $\mu(Q(\Gamma)) \leq \Delta\mu(\Gamma)$.*

Lemma 6. *([20]) G is a complete normed vector space. Suppose Γ is a bounded subset on G and is also closed and convex. $\Phi : \Gamma \to \Gamma$ is the μ-set-contraction operator, therefore, Φ has at least one fixed point in Γ.*

Lemma 7. *([8]) $\mathcal{M} \subset G$ is closed, nonempty, and convex, and a mapping $\mathcal{P}\gamma = \mathcal{A}\gamma + \mathcal{B}\gamma$ satisfies*

(i) $\mathcal{A}\sigma + \mathcal{B}\beta \in \mathcal{M}$, $\sigma, \beta \in \mathcal{M}$;

(ii) $\mathcal{A}$ is a continuous mapping with compactness;

(iii) $\mathcal{B}$ is contractive.

Therefore, there exists $\zeta \in \mathcal{M}$ satisfying $\zeta = \mathcal{A}\zeta + \mathcal{B}\zeta$.

Lemma 8. *A function* $\zeta \in PC(\mathrm{K}, G)$ *is a mild solution of problem* (1) *if and only if* ζ *satisfies*

$$\zeta(t) = \begin{cases} \dfrac{Z_1 + Z_2 + Z_3}{1 - t_1} + \displaystyle\int_0^t U(t-s)h(s, \zeta_{\varpi(s,\zeta_s)})ds, & t \in [0, t_1], \\ U(t - t_j)I_j(t, \zeta(t)), & t \in \displaystyle\bigcup_{j=1}^{k}(t_j, s_j], \\ U(t - t_j)I_j(s_j, \zeta(s_j)) + \displaystyle\int_{s_j}^t U(t-s)h(s, \zeta_{\varpi(s,\zeta_s)})ds, & t \in \displaystyle\bigcup_{j=0}^{k}(s_j, t_{j+1}], \end{cases} \tag{3}$$

where

$$\begin{aligned} Z_1 &= \int_0^{t_1}\int_0^s U(s-\tau)h(\tau, \zeta_{\varpi(\tau,\zeta_\tau)})d\tau ds, \\ Z_2 &= \sum_{j=1}^{k}\int_{t_j}^{s_j} U(s-t_j)I_j(s, \zeta(s))ds, \\ Z_3 &= \sum_{j=1}^{k}\int_{s_j}^{t_{j+1}} U(s-t_j)I_j(s_j, \zeta(s_j))ds + \sum_{j=1}^{k}\int_{s_j}^{t_{j+1}}\int_{s_j}^{s} U(s-\tau)h(\tau, \zeta_{\varpi(\tau,\zeta_\tau)})d\tau ds. \end{aligned}$$

Proof: Suppose ζ satisfies (1). Then for $t \in [0, t_1]$, we integrate the first equation in (1) from 0 to t yielding

$$\zeta(t) = \zeta(0) + \int_0^t U(t-s)h(s, \zeta_{\varpi(s,\zeta_s)})ds. \tag{4}$$

For $t \in (s_j, t_{j+1}]$, combining the impulsive condition in (1), one has

$$\zeta(t) = U(t-t_j)I_j(s_j, \zeta(s_j)) + \int_{s_j}^t U(t-s)h(s, \zeta_{\varpi(s,\zeta_s)})ds. \tag{5}$$

According to the integral boundary value condition in (1), we have

$$\begin{aligned} \zeta(0) = \int_0^1 \zeta(s)ds &= \int_0^{t_1}\zeta(s)ds + \int_{t_1}^{s_1}\zeta(s)ds + \int_{s_1}^{t_2}\zeta(s)ds + \cdots + \int_{t_k}^{s_k}\zeta(s)ds + \int_{s_k}^{t_{k+1}}\zeta(s)ds \\ &= \int_0^{t_1}\Big[\zeta(0) + \int_0^s U(s-\tau)h(\tau, \zeta_{\varpi(\tau,\zeta_\tau)})d\tau\Big]ds + \sum_{j=1}^{k}\int_{t_j}^{s_j} U(s-t_j)I_j(s, \zeta(s))ds \\ &\quad + \sum_{j=1}^{k}\int_{s_j}^{t_{j+1}}\Big[U(s-t_j)I_j(s_j, \zeta(s_j)) + \int_{s_j}^{s} U(s-\tau)h(\tau, \zeta_{\varpi(\tau,\zeta_\tau)})d\tau\Big]ds \\ &= \zeta(0)t_1 + Z_1 + Z_2 + Z_3. \end{aligned} \tag{6}$$

Then, from (6), we have

$$\zeta(0) = \frac{Z_1 + Z_2 + Z_3}{1 - t_1}. \tag{7}$$

Thus, combining (4) and (7), one obtains

$$\zeta(t) = \frac{Z_1 + Z_2 + Z_3}{1 - t_1} + \int_0^t U(t-s)h(s, \zeta_{\varpi(s,\zeta_s)})ds, \ t \in [0, t_1]. \tag{8}$$

Hence, from (5) and (8), Eq (3) is satisfied.

If $\zeta \in PC(K, G)$ satisfies (3), then it can be easily verified that the conclusion holds, which completes the proof.

Lemma 9. *Suppose* $\{U(t),\ t \geq 0\}$ *is the operator semigroup and satisfies*

$$\lim_{t \to 0^+} \|U(t) - I\| = 0,$$

where I represents the identity operator. Then, we call $\{U(t),\ t \geq 0\}$ the uniformly continuous operator semigroup.

3. Main results

3.1. Existence of mild solutions

We assume h and I_j satisfy the conditions:

(H_1) $h \in C(\mathrm{K} \times \mathcal{D}, G)$ and $I_j \in C([t_j, s_j] \times G, G)$, $j = 1, 2, \cdots, k$.

(H_2) For $\zeta_1, \zeta_2 \in G$ and each $t \in [s_j, t_{j+1}], j = 0, 1, \cdots, k$, there exists a positive constant L_1 satisfying $\|h(t, \zeta_1) - h(t, \zeta_2)\| \leq L_1 \|\zeta_1 - \zeta_2\|_{\mathcal{D}}$.

(H_3) For $\zeta \in G$ and each $t \in [s_j, t_{j+1}]$, there is $\|h(t, \zeta)\| \leq \phi(t) w(\|\zeta\|_{\mathcal{D}})$, where $\phi : [s_j, t_{j+1}] \to G$ is integrable and $w : G \to G$ is integrable and nondecreasing.

(H_4) For $\zeta_1, \zeta_2 \in G$ and each $t \in [t_j, s_j]$, there is $\|I_j(t, \zeta_1) - I_j(t, \zeta_2)\| \leq L_2 \|\zeta_1 - \zeta_2\|$, where $L_2 > 0$ is a constant, $j = 1, 2, \cdots, k$.

(H_5) For $\zeta \in G$ and each $t \in [t_j, s_j]$, there is $\|I_j(t, \zeta)\| \leq \xi_j(t) \alpha(\|\zeta\|_{PC})$, where $\xi_j : [t_j, s_j] \to G$ is integrable and $\alpha : \mathrm{K} \to G$ is an integrable and nondecreasing function.

(H_6) For any countable subset D of G, there is $L_j > 0$ such that

$$\mu(h(t, D)) \leq L_j \mu(D), \ t \in \bigcup_{j=0}^{k} (s_j, t_{j+1}].$$

Theorem 1. *Suppose conditions* (H_1),(H_3),(H_4), *and* (H_5) *hold,* $\{U(t), t \geq 0\}$ *is compact, and* $ML_2 < 1$. *Then, at least one mild solution of system* (1) *can be obtained.*

Proof: Set $M_j = \int_{t_j}^{s_j} \xi_j(t)dt$, $L = \max\limits_{j=1,2,\cdots,k} M_j$, $\mathcal{K} = \int_0^1 \phi(t)dt$, $B_r = \{\zeta \in PC(\mathrm{K}, G) : \|\zeta\|_{PC} \leq r\}$, where $r \geq \max\{\frac{2kML\alpha(r) + (2-t_1)M\mathcal{K}w(\varsigma)}{1-t_1},\ M\xi_j(s_j)\alpha(r) + M\mathcal{K}w(\varsigma)\}$, $\varsigma = \max\limits_{s \in [0,t]}\{\delta r + C_2(s)r\}$. Define

$$(P\zeta)(t) = \begin{cases} \dfrac{1}{1-t_1}\Big(\displaystyle\sum_{j=1}^{k} \int_{t_j}^{s_j} U(s-t_j) I_j(s, \zeta(s))ds + \sum_{j=1}^{k} \int_{s_j}^{t_{j+1}} U(s-t_j) I_j(s_j, \zeta(s_j))ds\Big), & t \in [0, t_1], \\ U(t-t_j) I_j(t, \zeta(t)), & t \in \displaystyle\bigcup_{j=1}^{k} (t_j, s_j], \\ U(t-t_j) I_j(s_j, \zeta(s_j)), & t \in \displaystyle\bigcup_{j=1}^{k} (s_j, t_{j+1}], \end{cases}$$

and

$$(Q\zeta)(t)=\begin{cases}\displaystyle\int_0^t U(t-s)h(s,\zeta_{\varpi(s,\zeta_s)})ds+\frac{1}{1-t_1}\sum_{j=0}^{k}\int_{s_j}^{t_{j+1}}\int_{s_j}^{s}U(s-\tau)h(\tau,\zeta_{\varpi(\tau,\zeta_\tau)})d\tau ds, & t\in[0,t_1],\\ 0, & t\in\displaystyle\bigcup_{j=1}^{k}(t_j,s_j],\\ \displaystyle\int_{s_j}^{t}U(t-s)h(s,\zeta_{\varpi(s,\zeta_s)})ds, & t\in\displaystyle\bigcup_{j=1}^{k}(s_j,t_{j+1}].\end{cases}$$

(a) For any $\zeta\in B_r$, we will prove $H\zeta=P\zeta+Q\zeta\in B_r$ holds.

For $t\in[0,t_1]$, one gets

$$\begin{aligned}\|H\zeta\|\le&\int_0^t M\phi(s)w(\|\zeta_{\varpi(s,\zeta_s)}\|)ds+\frac{M}{1-t_1}\Big(\sum_{j=1}^{k}\int_{t_j}^{s_j}\xi_j(s)\alpha(\|\zeta\|)ds+\sum_{j=1}^{k}\int_{s_j}^{t_{j+1}}\xi_j(s_j)\alpha(\|\zeta\|)ds\\&+\sum_{j=0}^{k}\int_{s_j}^{t_{j+1}}\int_{s_j}^{s}\phi(\tau)w(\|\zeta_{\varpi(s,\zeta_s)}\|)d\tau ds\Big).\end{aligned}\tag{9}$$

From (2), we have

$$\|\zeta_{\varpi(s,\zeta_s)}\|\le C_1(t)\sup_{s\in[0,t]}\|\zeta(s)\|+C_2(t)\|\zeta_0\|_{\mathcal{D}}\le\delta\|\zeta(s)\|+C_2(t)\|\zeta_0\|_{\mathcal{D}}\le\delta r+C_2(t)r\le\varsigma,\tag{10}$$

where $\delta=\sup_{s\in[0,t]}C_1(s)$. Combining (9), (10), and the fact that w is nondecreasing, we have

$$\|H\zeta\|\le\frac{2kML\alpha(r)+(2-t_1)M\mathcal{K}w(\varsigma)}{1-t_1}.\tag{11}$$

For each $t\in(t_j,s_j]$ and $t\in(s_j,t_{j+1}]$, we have

$$\begin{aligned}&\|H\zeta\|\le\|U(t-t_j)\|\|I_j(s_j,\zeta(s_j))\|\le M\xi_j(s_j)\alpha(r),\\&\|H\zeta\|\le M\xi_j(s_j)\alpha(r)+M\mathcal{K}w(\varsigma).\end{aligned}\tag{12}$$

Therefore, from (11) and (12), for any $\zeta\in B_r$, $H\zeta=P\zeta+Q\zeta\in B_r$ holds.

(b) Next, we will prove P is contractive on B_r.

For each $t\in[0,t_1]$, one gets

$$\|(P\zeta_1)(t)-(P\zeta_2)(t)\|\le\tfrac{ML_2}{1-t_1}\Big(\sum_{j=1}^{k}(s_j-t_j)+\sum_{j=1}^{k}(t_{j+1}-s_j)\Big)\|\zeta_1-\zeta_2\|=ML_2\|\zeta_1-\zeta_2\|.\tag{13}$$

For each $t\in\bigcup_{j=1}^{k}(t_j,s_j]$ and $t\in\bigcup_{j=1}^{k}(s_j,t_{j+1}]$, one can easily verify

$$\|(P\zeta_1)(t)-(P\zeta_2)(t)\|\le ML_2\|\zeta_1-\zeta_2\|,\tag{14}$$

and then, from (13) and (14), one can obtain that P is a contractive mapping.

(c) We are going to prove Q is continuous.

For $t \in [0, t_1]$ and $t \in (s_j, t_{j+1}]$, we have

$$\|(Q\zeta')(t) - (Q\zeta)(t)\| \leq \Big(\frac{M}{2(1-t_1)} \sum_{j=0}^{k} (t_{j+1} - s_j)^2 + Mt_1\Big)\|h(\cdot, \zeta'_{\varpi(\cdot,\zeta')}) - h(\cdot, \zeta_{\varpi(\cdot,\zeta)})\|. \tag{15}$$

$$\|(Q\zeta')(t) - (Q\zeta)(t)\| \leq M(t_{j+1} - s_j)\|h(\cdot, \zeta'_{\varpi(\cdot,\zeta')}) - h(\cdot, \zeta_{\varpi(\cdot,\zeta)})\|. \tag{16}$$

For $t \in (t_j, s_j]$, one has

$$\|(Q\zeta')(t) - (Q\zeta)(t)\| = 0. \tag{17}$$

Therefore, from (15) and (17), we have $\|(Q\zeta')(t) - (Q\zeta)(t)\| \to 0$ as $n \to \infty$ since h is continuous.
(d) We will show that Q is compact.

Since $\|Q\zeta\|_{PC} \leq r$, Q is uniformly bounded on B_r. For $\alpha_1, \alpha_2 \in [0, t_1]$ and $\alpha_1 < \alpha_2$, we have

$$\begin{aligned}
&\|(Q\zeta)(\alpha_2) - (Q\zeta)(\alpha_1)\| \\
&\leq \int_0^{\alpha_1} \|U(\alpha_2 - s) - U(\alpha_1 - s)\|\|h(s, \zeta_{\varpi(s,\zeta_s)})\|ds + \int_{\alpha_1}^{\alpha_2} \|U(\alpha_2 - s)\|\|h(s, \zeta_{\varpi(s,\zeta_s)})\|ds \\
&\leq Mw(\varsigma)\Big(\int_0^{\alpha_1} \|U(\alpha_2 - \alpha_1) - I\|\|\phi(s)\|ds + \int_{\alpha_1}^{\alpha_2} \|\phi(s)\|ds\Big).
\end{aligned} \tag{18}$$

For $\alpha_1, \alpha_2 \in (t_j, s_j]$ and $\alpha_1 < \alpha_2$, we have

$$\|(Q\zeta)(\alpha_2) - (Q\zeta)(\alpha_1)\| = 0. \tag{19}$$

For $\alpha_1, \alpha_2 \in (s_j, t_{j+1}]$ and $\alpha_1 < \alpha_2$, one gets

$$\|(Q\zeta)(\alpha_2) - (Q\zeta)(\alpha_1)\| \leq Mw(\varsigma)\Big(\int_{\alpha_1}^{\alpha_2} \|\phi(s)\|ds + \int_{s_j}^{\alpha_1} \|U(\alpha_2 - \alpha_1) - I\|\|\phi(s)\|ds\Big). \tag{20}$$

Since $\{U(t), t \geq 0\}$ is compact, according to semigroup theory, we can derive that $\{U(t), t \geq 0\}$ is uniformly continuous (uniformly operator topology continuous). Thus, from Lemma 9, we have $\|U(\alpha_2 - \alpha_1) - I\| \to 0$ as $\alpha_2 \to \alpha_1$. Besides, from ϕ being an integral and (18)–(20), we can conclude $\|(Q\zeta)(\alpha_2) - (Q\zeta)(\alpha_1)\| \to 0$ as $\alpha_2 \to \alpha_1$, which means Q is equicontinuous.

We define $E(t) = \{(Q\zeta)(t) : \zeta \in B_r\}$, $t \in$ K. Since $U(\cdot)$ is compact, then $E(0)$ is relatively compact. Set

$$E_\epsilon(t) := Q_\epsilon B_r(t) = \{(Q\zeta)(t - \epsilon) : \zeta \in B_r\},\ 0 < \epsilon < t \leq 1,$$

and it can be easily verified that $E_\epsilon(t)$ is relatively compact for $t \in (\epsilon, 1]$. For $t \in (0, t_1]$ and $t \in (s_j, t_{j+1}]$, we get

$$\|(Q\zeta)(t) - (Q_\epsilon\zeta)(t)\| \leq Mw(\varsigma)\Big(\int_{t-\epsilon}^{t} \|\phi(s)\|ds + \int_0^{t-\epsilon} \|U(\epsilon) - I\|\|\phi(s)\|ds\Big), \tag{21}$$

$$\|(Q\zeta)(t) - (Q_\epsilon\zeta)(t)\| \leq Mw(\varsigma)\Big(\int_{t-\epsilon}^{t} \|\phi(s)\|ds + \int_{s_j}^{t-\epsilon} \|U(\epsilon) - I\|\|\phi(s)\|ds\Big). \tag{22}$$

For $t \in (t_j, s_j]$, we have

$$\|(Q\zeta)(t) - (Q_\epsilon\zeta)(t)\| = 0, \tag{23}$$

which is independent of ϵ. Since $\{U(\cdot), t \geq 0\}$ is compact and ϕ is integrable, according to Lemma 9 and (21)–(23), we have $\|(Q\zeta)(t) - (Q_\epsilon\zeta)(t)\| \to 0$ as $\epsilon \to 0$. By the Arzela-Ascoli theorem, we get that $Q : B_r \to B_r$ is completely continuous. From Lemma 7, one can obtain that ζ is a fixed point of the operator H, which means ζ is a solution of problem (1).

Remark 2. *Compared with [8], we have introduced the concepts of a C_0-semigroup and infinitesimal generator $-B$. Moreover, one can notice that there exists a link between the generator $-B$ and the mild solutions of problem* (1).

Theorem 2. *Suppose (H_1), (H_3), (H_4), and (H_6) hold, $\{U(t), t \geq 0\}$ is equicontinuous, and $M\Big(L_2 + \frac{4(k+2)}{1-t_1}\eta\Big) < 1$, where $\eta := \max\limits_{j=0,1,\cdots,k} L_j(t_{j+1} - s_j)$. Then at least one mild solution of problem* (1) *can be obtained.*

Proof: Combining the proof of Theorem 1, one can easily get that Q is continuous and for any $\zeta \in B_r$, there is $H\zeta \in B_r$, where $B_r = \{\zeta \in PC(\mathrm{K}, G) : \|\zeta\|_{PC} \leq r\}$.

For $t \in [0, t_1]$, $t \in (t_j, s_j]$, and $t \in (s_j, t_{j+1}]$, it holds that

$$\|(P\zeta_1)(t) - (P\zeta_2)(t)\| \leq ML_2\|\zeta_1 - \zeta_2\|, \tag{24}$$

which implies P is Lipschitz continuous. For $e_1, e_2 \in [0, t_1]$ and $e_1 < e_2$, we have

$$\|(Q\zeta)(e_2) - (Q\zeta)(e_1)\| \leq w(\varsigma) \int_0^{e_1} \|U(e_2 - s) - U(e_1 - s)\|\|\phi(s)\|ds + Mw(\varsigma) \int_{e_1}^{e_2} \|\phi(s)\|ds. \tag{25}$$

For $e_1, e_2 \in (t_j, s_j]$ and $e_1 < e_2$, we have

$$\|(Q\zeta)(e_2) - (Q\zeta)(e_1)\| = 0. \tag{26}$$

For $s_j < e_1 < e_2 < t_{j+1}$, one gets

$$\|(Q\zeta)(e_2) - (Q\zeta)(e_1)\| \leq w(\varsigma) \int_{s_j}^{e_1} \|U(e_2 - s) - U(e_1 - s)\|\|\phi(s)\|ds + Mw(\varsigma) \int_{e_1}^{e_2} \|\phi(s)\|ds. \tag{27}$$

Since the semigroup is equicontinuous and ϕ is integrable, combining (25)–(27), we can conclude Q is equicontinuous as $e_2 \to e_1$. For any $D \subset B_r$, according to Lemma 2, there is $D_0 = \{\zeta'\}$, which is a subset of D and countable such that

$$\mu(Q(D))_{PC} \leq 2\mu(Q(D_0))_{PC}. \tag{28}$$

From the boundedness and equicontinuity of $Q(D_0) \subset Q(B_r)$, by Lemma 4, one gets

$$\mu_{PC}(Q(D_0)) = \max_{t \in [t_j, t_{j+1}]} \mu(Q(D_0)(t)). \tag{29}$$

Since

$$\int_{s_j}^{t_{j+1}} \int_{s_j}^{s} U(s - \tau)h(\tau, \zeta'_{\varpi(\tau,\zeta'_\tau)})d\tau ds = \int_{s_j}^{t_{j+1}} \int_{\tau}^{t_{j+1}} U(s - \tau)h(\tau, \zeta'_{\varpi(\tau,\zeta'_\tau)})dsd\tau,$$

for $t \in [0, t_1]$, from condition (H_6), Lemma 3, and $-\infty < \varpi(s, \zeta'_s) \le s$, we have

$$\begin{aligned}
\mu(Q(D_0)(t)) &= \mu\Big(\int_0^t U(t-s)h(s, \zeta'_{\varpi(s,\zeta'_s)})ds + \frac{1}{1-t_1}\sum_{j=0}^{k}\int_{s_j}^{t_{j+1}}\int_{\tau}^{t_{j+1}} U(s-\tau)h(\tau, \zeta'_{\varpi(\tau,\zeta'_\tau)})dsd\tau\Big) \\
&\le M\mu\Big(\int_0^t h(s, \zeta'_{\varpi(s,\zeta'_s)})ds\Big) + \frac{M}{1-t_1}\mu\Big(\sum_{j=0}^{k}\int_{s_j}^{t_{j+1}} h(s, \zeta'_{\varpi(s,\zeta'_s)})ds\Big) \\
&\le 2ML_j\int_0^t \sup_{\theta\in(-\infty,0]}\mu(\zeta'(s+\theta))ds + \frac{2M}{1-t_1}\sum_{j=0}^{k}\int_{s_j}^{t_{j+1}} L_j \sup_{\theta\in(-\infty,0]}\mu(\zeta'(s+\theta))ds \\
&\le 2Mt_1L_j\mu_{PC}(D) + \frac{2M}{1-t_1}\sum_{j=0}^{k}((t_{j+1}-s_j)L_j)\mu_{PC}(D). \qquad (30)
\end{aligned}$$

Therefore, combining (28)–(30), one has

$$\mu(Q(D))_{PC} \le 4M\eta\mu_{PC}(D) + \frac{4M}{1-t_1}(k+1)\eta\mu_{PC}(D) \le \frac{4M(k+2)}{1-t_1}\eta\mu(D)_{PC}. \qquad (31)$$

By (24) and the properties of noncompact Kuratowski measures, for any bounded $D \subset B_r$, we have

$$\mu(P(D))_{PC} \le ML_2\mu(D)_{PC}. \qquad (32)$$

According to (31) and (32), we can obtain

$$\mu(H(D))_{PC} \le \mu(P(D))_{PC} + \mu(Q(D))_{PC} \le M\Big(L_2 + \frac{4(k+2)}{1-t_1}\eta\Big)\mu(D)_{PC}.$$

For $t \in (s_j, t_{j+1}]$, from (28), (H_6), and Lemma 3, we have

$$\mu(Q(D))_{PC} \le 2\mu(Q(D_0))_{PC} \le 4M\Big(\int_{s_j}^t \mu(h(s, \zeta'_{\varpi(s,\zeta'(s))}))ds\Big) \le 4M\eta\mu(D)_{PC}.$$

Combining (32), we get

$$\mu(H(D))_{PC} \le \mu(P(D))_{PC} + \mu(Q(D))_{PC} \le M(L_2 + 4\eta)\mu(D)_{PC}.$$

Then, the operator $H : B_r \to B_r$ is a μ-set-contraction operator. We can conclude from Lemma 6 that the defined operator H has at least one fixed point, which is a PC-mild solution of (1).

Remark 3. *In the previous literature, most authors assume the semigroup is compact. However, in Theorem 2, we obtain the existence result when the associated semigroup is equicontinuous, which can be regarded as a supplement to the fact that the solution semigroup is compact.*

3.2. Controllability

Consider the controllability of the following system:

$$\begin{cases} \zeta'(t) + B\zeta(t) = h(t, \zeta_{\varpi(t,\zeta_t)}) + \mathcal{B}u(t), & t \in \bigcup_{j=0}^{k}(s_j, t_{j+1}], \\ \zeta(t) = U(t-t_j)I_j(t, \zeta(t)), & t \in \bigcup_{j=1}^{k}(t_j, s_j], \\ \zeta(0) = \displaystyle\int_0^1 \zeta(s)ds, \end{cases} \qquad (33)$$

where $\mathcal{B}$ is a bounded linear operator from K into a Hilbert space G, and we denote $M_{\mathcal{B}} := \sup\|\mathcal{B}\|$. $u : K \to G$ is a control function and all other symbols are the same as in system (1). In the following, $\mathcal{B}^*$ and $U^*(t)$ represent the adjoint operator of $\mathcal{B}$ and adjoint semigroup of $U(t)$, respectively.

Definition 1. *A function $\zeta \in PC(K, G)$ is called a mild solution of system* (33)*, if and only if ζ satisfies*

$$\zeta(t) = \begin{cases} \dfrac{Z_1 + Z_2 + Z_3}{1 - t_1} + \displaystyle\int_0^t U(t-s)h(s, \zeta_{\varpi(s,\zeta_s)})ds + \int_0^t U(t-s)\mathcal{B}u(s)ds, & t \in [0, t_1], \\ U(t - t_j)I_j(t, \zeta(t)), & t \in \displaystyle\bigcup_{j=1}^{k}(t_j, s_j], \\ U(t-t_j)I_j(s_j, \zeta(s_j)) + \displaystyle\int_{s_j}^t U(t-s)h(s, \zeta_{\varpi(s,\zeta_s)})ds + \int_{s_j}^t U(t-s)\mathcal{B}u(s)ds, & t \in \displaystyle\bigcup_{j=0}^{k}(s_j, t_{j+1}], \end{cases} \tag{34}$$

where Z_1, Z_2, and Z_3 are the same as in Lemma 8.

{**Definition 2.** *The non-instantaneous delay system* (33) *is said to be controllable on* $[t_0, T]$*, if it is controllable on* $(0, t_1]$ *and* $(s_j, t_{j+1}]$ *for $j = 1, 2, \cdots, n$, i.e., there exists $\zeta_{t_1} \in G$ and $\zeta_{t_{j+1}} \in G$ such that $\zeta(t_1) = \zeta_{t_1}$ and $\zeta(t_{j+1}) = \zeta_{t_{j+1}}$, $j = 0, 1, 2, \cdots, n$. The target states ζ_{t_1} and $\zeta_{t_{j+1}}$ are arbitrary within the state space G, serving as the equation constraints $\zeta(t_1) = \zeta_{t_1}$ and $\zeta(t_{j+1}) = \zeta_{t_{j+1}}$ that the system must satisfy at the terminal time t_1 and t_{j+1}, respectively.*

Lemma 9. *For any $\zeta_{t_1} \in G$, the solutions of system* (33) *on* $[0, t_1]$ *satisfy* $\zeta(t_1) = \zeta_{t_1}$ *with the control function*

$$\begin{aligned} u(t) = &-\frac{\mathcal{B}(s)^* U(t_0 - s)^*}{t_1}\Big(\int_0^{t_1} U(t_1 - s)\mathcal{B}(s)\mathcal{B}(s)^* U^*(t_0 - s)ds\Big)^{-1}\Big[\frac{Z_1 + Z_2 + Z_3}{1 - t_1} - \zeta_{t_1} \\ &+ \int_0^{t_1} U(t_1 - s)h(s, \zeta_{\varpi(s,\zeta_s)})ds\Big]. \end{aligned} \tag{35}$$

Proof: Consider the solution $\zeta(t)$ of system (33) on $[0, t_1]$ defined by (34). For $t = t_1$, we have

$$\begin{aligned} \zeta(t_1) &= \frac{Z_1 + Z_2 + Z_3}{1 - t_1} + \int_0^{t_1} U(t_1 - s)\mathcal{B}u(s)ds \\ &= \frac{Z_1 + Z_2 + Z_3}{1 - t_1} + \int_0^{t_1} U(t_1 - s)h(s, \zeta_{\varpi(s,\zeta_s)})ds - \frac{1}{t_1}\int_0^{t_1} U(t_1 - s)\mathcal{B}\mathcal{B}(s)^* U(t_0 - s)^* \\ &\cdot\Big(\int_0^{t_1} U(t_1 - s)\mathcal{B}(s)\mathcal{B}(s)^* U^*(t_0 - s)ds\Big)^{-1}\Big[\frac{Z_1 + Z_2 + Z_3}{1 - t_1} - \zeta_{t_1} + \int_0^{t_1} U(t_1 - s)h(s, \zeta_{\varpi(s,\zeta_s)})ds\Big]ds \\ &= \zeta_{t_1}, \end{aligned}$$

which completes the proof of Lemma 9.

Lemma 10. *For any $\zeta_{t_{j+1}} \in G$, $j = 1, 2, \cdots, n$, the solutions of system* (33) *on* $(s_j, t_{j+1}]$ *satisfy* $\zeta(t_{j+1}) = \zeta_{t_{j+1}}$ *with the control function*

$$u(t) = -\frac{\mathcal{B}(s)^* U(s_j - s)^*}{t_{j+1} - s_j}\Big(\int_{s_j}^{t_{j+1}} U(t_{j+1} - s)\mathcal{B}(s)\mathcal{B}(s)^* U^*(s_j - s)ds\Big)^{-1}\Big[U(t_{j+1} - t_j)I_j(s_j, \zeta(s_j)) - \zeta_{t_{j+1}}$$

$$+\int_{s_j}^{t_{j+1}} U(t_{j+1}-s)h(s,\zeta_{\varpi(s,\zeta_s)})ds\Big]ds. \tag{36}$$

Proof: Consider the solution $\zeta(t)$ of system (33) on $(s_j, t_{j+1}]$ defined by (34). For $t = t_{j+1}$, we have

$$\begin{aligned}
\zeta(t_{j+1}) &= U(t_{j+1}-t_j)I_j(s_j,\zeta(s_j)) + \int_{s_j}^{t_{j+1}} U(t_{j+1}-s)h(s,\zeta_{\varpi(s,\zeta_s)})ds + \int_{s_j}^{t_{j+1}} U(t_{j+1}-s)\mathcal{B}u(s)ds \\
&= U(t_{j+1}-t_j)I_j(s_j,\zeta(s_j)) + \int_{s_j}^{t_{j+1}} U(t_{j+1}-s)h(s,\zeta_{\varpi(s,\zeta_s)})ds \\
&\quad - \frac{1}{t_{j+1}-s_j}\int_{s_j}^{t_{j+1}} U(t_{j+1}-s)\mathcal{B}\mathcal{B}(s)^*U(s_j-s)^*\Big(\int_{s_j}^{t_{j+1}} U(t_{j+1}-s)\mathcal{B}(s)\mathcal{B}(s)^*U^*(s_j-s)ds\Big)^{-1} \\
&\quad \cdot\Big[U(t_{j+1}-t_j)I_j(s_j,\zeta(s_j)) - \zeta_{t_{j+1}} + \int_{s_j}^{t_{j+1}} U(t_{j+1}-s)h(s,\zeta_{\varpi(s,\zeta_s)})ds\Big]ds \\
&= \zeta_{t_{j+1}},
\end{aligned}$$

which completes the proof of Lemma 10.

Theorem 3. *Suppose (H_1), (H_2), and (H_4) hold. If*

$$\begin{aligned}
\rho := \max\Big\{&\frac{M^2M_{\mathcal{B}}^2+1}{1-t_1}\Big(ML_2(1-t_1) + \frac{\delta ML_1}{2}\sum_{j=0}^{k}(t_{j+1}-s_j)^2\Big) + \delta(1+2Mt_1L_1), \\
&M^2M_{\mathcal{B}}^2(t_{j+1}-s_j)(ML_2 + \delta(ML_1(t_{j+1}-s_j)+1)\Big\} < 1,
\end{aligned} \tag{37}$$

and then, system (33) is controllable.

Proof: Define operator $H : PC(\mathrm{K}, G) \to PC(\mathrm{K}, G)$ by

$$(H\zeta)(t) = \begin{cases}
\dfrac{Z_1+Z_2+Z_3}{1-t_1} + \displaystyle\int_0^t U(t-s)h(s,\zeta_{\varpi(s,\zeta_s)})ds + \int_0^t U(t-s)\mathcal{B}u(s)ds, & t\in[0,t_1], \\
U(t-t_j)I_j(t,\zeta(t)), & t\in\displaystyle\bigcup_{j=1}^{k}(t_j,s_j], \\
U(t-t_j)I_j(s_j,\zeta(s_j)) + \displaystyle\int_{s_j}^t U(t-s)h(s,\zeta_{\varpi(s,\zeta_s)})ds + \int_{s_j}^t U(t-s)\mathcal{B}u(s)ds, & t\in\displaystyle\bigcup_{j=0}^{k}(s_j,t_{j+1}],
\end{cases}$$

where $u(t)$ is defined by (35) and (36) in $(0, t_1]$ and $(s_j, t_{j+1}]$, $j = 1, 2, \cdots, n$, respectively. From Lemmas 9 and 10, we have $\zeta(t_1) = \zeta_{t_1}$ and $\zeta(t_{j+1}) = \zeta_{t_{j+1}}$. In the following, we will prove the defined operator H has a unique fixed point.

Obviously, H is well-defined. For $\zeta \in PC(\mathrm{K}, G)$, $H\zeta \in PC(\mathrm{K}, G)$. For $\zeta' \in PC(\mathrm{K}, G)$, $\zeta'' \in PC(\mathrm{K}, G)$ and $t \in [0, t_1]$, and one gets

$$\|Z_1'-Z_1''\| \le M\int_0^{t_1}\int_0^s \|h(\tau,\zeta'_{\varpi(\tau,\zeta'_\tau)}) - h(\tau,\zeta''_{\varpi(\tau,\zeta''_\tau)})\|d\tau ds \le ML_1\int_0^{t_1}\int_0^s \|\zeta'_{\varpi(\tau,\zeta'_\tau)} - \zeta''_{\varpi(\tau,\zeta''_\tau)}\|_{\mathcal{D}}d\tau ds. \tag{38}$$

From (2), we have

$$\|\zeta'_{\varpi(s,\zeta'_s)} - \zeta''_{\varpi(s,\zeta''_s)}\|_{\mathcal{D}} \le C_1(t) \sup_{s\in[0,t]} \|\zeta'(s) - \zeta''(s)\| \le \delta\|\zeta' - \zeta''\|, \text{ where } \delta = \sup_{s\in[0,t]} C_1(s). \tag{39}$$

Therefore, combining (38) and (39), yields

$$\|Z_1' - Z_1''\| \le \frac{Mt_1^2}{2} L_1 \|\zeta'_{\varpi(s,\zeta'_s)} - \zeta''_{\varpi(s,\zeta''_s)}\|_{\mathcal{D}} \le \frac{\delta M t_1^2}{2} L_1 \|\zeta' - \zeta''\|.$$

Similarly, we have

$$\|Z_2' - Z_2''\| \le ML_2 \sum_{j=1}^{k} (s_j - t_j) \|\zeta' - \zeta''\|,$$

and

$$\|Z_3' - Z_3''\| \le \Big(ML_2 \sum_{j=1}^{k} (t_{j+1} - s_j) + \frac{\delta M L_1}{2} \sum_{j=1}^{k} (t_{j+1} - s_j)^2\Big) \|\zeta' - \zeta''\|.$$

What's more, we have

$$\begin{aligned}
&\|u'(t) - u''(t)\| \\
&= \Big\| \frac{\mathcal{B}(s)^* U(t_0 - s)^*}{t_1} \Big(\int_0^{t_1} U(t_1 - s)\mathcal{B}(s)\mathcal{B}(s)^* U^*(t_0 - s) ds \Big)^{-1} \Big\| \\
&\cdot \Big\| \frac{(Z_1' - Z_1'') + (Z_2' - Z_2'') + (Z_3' - Z_3'')}{1 - t_1} - (\zeta'_{t_1} - \zeta''_{t_1}) + \int_0^{t_1} U(t_1 - s)(h'(s, \zeta_{\varpi(s,\zeta_s)}) - h''(s, \zeta_{\varpi(s,\zeta_s)})) ds \Big\| \\
&\le \frac{MM_{\mathcal{B}}}{t_1} \Big[\frac{1}{1 - t_1} \Big(\|Z_1' - Z_1''\| + \|Z_2' - Z_2''\| + \|Z_3' - Z_3''\| \Big) + \delta \|\zeta' - \zeta''\| + M t_1 L_1 \|\zeta' - \zeta''\| \Big] \\
&\le \frac{MM_{\mathcal{B}}}{t_1} \Big[\frac{1}{1 - t_1} \Big(ML_2(1 - t_1) + \frac{\delta M L_1}{2} \sum_{j=0}^{k} (t_{j+1} - s_j)^2 \Big) + \delta(1 + M t_1 L_1) \Big] \|\zeta' - \zeta''\|.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\|(H\zeta')(t) - (H\zeta'')(t)\| \\
&\le \frac{1}{1 - t_1} \Big(\|Z_1' - Z_1''\| + \|Z_2' - Z_2''\| + \|Z_3' - Z_3''\| \Big) + MM_B t_1 \|u'(t) - u''(t)\| + \delta M t_1 L_1 \|\zeta' - \zeta''\| \\
&\le \Big[\frac{M^2 M_{\mathcal{B}}^2 + 1}{1 - t_1} \Big(ML_2(1 - t_1) + \frac{\delta M L_1}{2} \sum_{j=0}^{k} (t_{j+1} - s_j)^2 \Big) + \delta(1 + 2M t_1 L_1) \Big] \|\zeta' - \zeta''\|.
\end{aligned} \tag{40}$$

For $\zeta' \in PC(\mathrm{K}, G)$, $\zeta'' \in PC(\mathrm{K}, G)$ and $t \in (t_j, s_j]$, and we have

$$\|(H\zeta')(t) - (H\zeta'')(t)\| \le ML_2 \|\zeta' - \zeta''\|. \tag{41}$$

For $\zeta' \in PC(\mathrm{K}, G)$, $\zeta'' \in PC(\mathrm{K}, G)$ and $t \in (s_j, t_{j+1}]$, and one can obtain

$$\|(H\zeta')(t) - (H\zeta'')(t)\| \le M^2 M_{\mathcal{B}}^2 (t_{j+1} - s_j)(ML_2 + \delta(ML_1(t_{j+1} - s_j) + 1) \|\zeta' - \zeta''\|. \tag{42}$$

Then, according to (40)–(42), we can get

$$\|(H\zeta')(t) - (H\zeta'')(t)\| \leq \rho\|\zeta' - \zeta''\|,$$

which implies the operator H is contractive. Therefore, combining with Banach fixed point theory, system (33) has a unique solution, which means system (33) is controllable.

Remark 5. *Compared with [8], in this paper, we introduce time delay into non-instantaneous impulsive systems, and analyze the existence of solutions via the fixed point theorem and semigroup theory respectively, under the conditions that the operator semigroup possesses compactness and equicontinuity. Since [8] only derive existence results by means of the fixed point theorem, the conclusions obtained in this paper can be regarded as a supplement and generalization to the above two references. Besides the existence of solutions, we also discuss the controllability of non-instantaneous impulsive systems with delay, which has not been involved in [8].*

4. Examples

Set $G = L^2([0,1],\mathbb{R})$ as a complete normed vector space equipped with the L^2 norm $\|\cdot\|_2$. Set K = $[0,1]$, $0 = s_0 < t_1 = \frac{1}{4} < s_1 = \frac{1}{2} < t_2 = \frac{3}{4} = s_2 < t_3 = 1$, $k = 2$. Define $B\zeta = -\frac{\partial^2}{\partial x^2}\zeta$ for $\zeta \in \mathcal{D}(B)$ with $\mathcal{D}(B) = \{\zeta \in G : \frac{\partial\zeta}{\partial x}, \frac{\partial^2\zeta}{\partial x^2} \in G, \zeta(0) = \zeta(1) = 0\}$. From [22], $-B$ generates an analytic C_0-semigroup $\{U(t), t \geq 0\}$ on G with $M = \sup\limits_{t \in K}\|U(t)\| \leq 1$.

Example 1. Consider

$$\begin{cases} \dfrac{\partial}{\partial t}\zeta(t,x) - \dfrac{\partial^2}{\partial x^2}\zeta(t,x) = \displaystyle\int_{-\infty}^{t} e^{s-t} \cdot \frac{\zeta(s - \varpi_1(s)\varpi_2\|\zeta(s)\|, x)}{4}ds, & x \in (0,1), t \in (0,\frac{1}{4}]\bigcup(\frac{1}{2},\frac{3}{4}], \\ \zeta(t,x) = \dfrac{1}{7}U(t - \dfrac{1}{4})\zeta(t), & t \in (\frac{1}{4},\frac{1}{2}], \\ \zeta(t,0) = \zeta(t,1) = 0, & t \in (0,\frac{1}{4}]\bigcup(\frac{1}{2},\frac{3}{4}], \\ \zeta(0,x) = \displaystyle\int_0^1 \zeta(s,x)ds, & x \in (0,1). \end{cases} \tag{43}$$

Set $\zeta(t)(x) = \zeta(t,x)$, $\varpi(t,\zeta) = \varpi_1(s)\varpi_2(\|\zeta(0)\|)$ and $I_j(t,\zeta(t)) = \frac{1}{7}|\zeta(t)|$ with $L_2 = \frac{1}{7}$, and then we can rewrite (43) as

$$\begin{cases} \zeta'(t) + B\zeta(t) = \int_{-\infty}^{0} e^{s} \cdot \frac{\zeta}{4}ds, & t \in (0,\frac{1}{4}]\bigcup(\frac{1}{2},\frac{3}{4}], \\ \zeta(t) = U(t - \dfrac{1}{4})I_j(t,\zeta(t)), & t \in (\frac{1}{4},\frac{1}{2}], \\ \zeta(0) = \displaystyle\int_0^1 \zeta(s)ds. \end{cases} \tag{44}$$

Then, for all $\zeta \in G$ and each $t \in [0,\frac{1}{4}]\bigcup(\frac{1}{2},\frac{3}{4}]$, one has $\|h(t,\zeta(t))\| \leq \frac{1}{4}\|\zeta\|_{\mathcal{D}}$, $\phi(t) = \frac{1}{4}$, and $w(t) = t$. For $t \in (\frac{1}{4},\frac{1}{2}]$, we have $\|I(t,\zeta(t))\| \leq \frac{1}{7}\|\zeta\|$ with $\xi(t) = \frac{1}{7}$ and $\alpha(t) = t$. Besides, $L = \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{7}dt = \frac{1}{28}$ and $\mathcal{K} = \int_0^1 \frac{1}{4}dt = \frac{1}{4}$. Obviously, the conditions in Theorem 1 are satisfied, and system (44) has at least one mild solution.

Example 2. Set $I_j(t,\zeta)(x) = \frac{1}{3}\sin\zeta(t,x)$, where $h(t,\zeta)$ and other symbols keep the same as in Example 1. We consider the following control problem:

$$\begin{cases} \zeta'(t) + B\zeta(t) = \int_{-\infty}^{0} e^s \cdot \frac{\zeta}{4} ds + \mathcal{B}u(t), & t \in (0, \frac{1}{4}] \bigcup (\frac{1}{2}, \frac{3}{4}], \\ \zeta(t) = U(t - \dfrac{1}{4}) I(t, \zeta(t)), & t \in (\frac{1}{4}, \frac{1}{2}], \\ \zeta(0) = \displaystyle\int_0^1 \zeta(s) ds. \end{cases} \tag{45}$$

Evidently, $h(t,\zeta)$ and $I_j(t,\zeta)$ are both continuous functions and the condition (H_1) is satisfied. In addition, we have

$$\|h(t,\zeta)\|_2 \le \Big(\int_0^1 \Big(\int_{-\infty}^0 e^s \cdot \frac{\|\zeta\|}{4} ds\Big)^2 dx\Big)^{\frac{1}{2}} \le \Big(\int_0^1 \Big(\frac{1}{4}\int_{-\infty}^0 e^s \cdot \sup\|\zeta\| ds\Big)^2 dx\Big)^{\frac{1}{2}} \le \frac{1}{4}\|\zeta\|_{\mathcal{D}}$$

and

$$\begin{aligned} \|h(t,\zeta_1) - h(t,\zeta_2)\|_2 \le \Big(\int_0^1 \Big(\int_{-\infty}^0 e^s \cdot \frac{\|\zeta_1 - \zeta_2\|}{4} ds\Big)^2 dx\Big)^{\frac{1}{2}} &\le \frac{1}{4}\Big(\int_0^1 \Big(\int_{-\infty}^0 e^s \cdot \sup\|\zeta_1 - \zeta_2\| ds\Big)^2 dx\Big)^{\frac{1}{2}} \\ &\le \frac{1}{4}\|\zeta_1 - \zeta_2\|_{\mathcal{D}}, \end{aligned}$$

which implies that condition (H_2) is satisfied with $L_1 = \frac{1}{4}$. Besides, there exists

$$\|I_j(t,\zeta_1) - I_j(t,\zeta_2)\| \le \frac{1}{3}\|\zeta_1 - \zeta_2\|,$$

which signifies that conditions (H_4) is satisfied with $L_2 = \frac{1}{3}$. Obviously, the condition $\rho < 1$ is also satisfied. Then, combining with Theorem 3, we can conclude that system (45) is controllable.

5. Conclusions

We have mainly considered non-instantaneous impulsive equations with state-dependent delay under integral boundary conditions. The abstract phase space has been brought out to provide the system with delay by means of fulfilling the ensuing axioms put forward by Hale and Kato. By utilizing the operator semigroup theorem and standard fixed point theory, the existence results have been obtained when the related semigroups are compact and equicontinuous, respectively. Besides, the controllability of the non-instantaneous impulsive probem has been taken into consideration. However, compared with the classical instantaneous impulse differential system, the theoretical development of the existing non-instantaneous impulse differential system is still lagging behind and the research results on the properties of solutions are not perfect. Therefore, in later work, we may consider the periodicity and stability of solutions of other different non-instantaneous impulse boundary value equations. Also, we can consider whether the novel step-function method is suitable for the non-instantaneous impulsive problem.

Author contributions

Shengke Xu: Conceptualization, Methodology, Writing—original draft, Validation; Yinuo Wang: Writing—review and editing, Supervision, Methodology, Formal analysis, Conceptualization; Kaibo Shi: Writing—review and editing, Supervision, Validation; Tao Liu: Conceptualization, Methodology. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by Chengdu University's 2025 Undergraduate Innovation Training Program Project (Stability Analysis of Nonlinear Delayed Systems with Impulsive Disturbances, 202511079005).

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

1. Y. Q. Zhang, H. Q. Wu, J. D. Cao, Prescribed-time synchronization for stochastic reaction-diffusion networks with delayed impulse and application in secure communication, *Expert Syst. Appl.*, **301** (2026), 130308. http://dx.doi.org/10.1016/j.eswa.2025.130308
2. M. Hritonenko, N. Kato, Y. Yatsenko, Impulse controls in optimal harvesting of age-structured populations, *Int. J. Biomath.*, **16** (2023), 2250128. http://dx.doi.org/10.1142/S1793524522501285
3. Y. N. Wang, C. D. Li, H. J. Wu, H. Deng, Stabilization of nonlinear delayed systems subject to impulsive disturbance via aperiodic intermittent control, *J. Franklin I.*, **361** (2024), 106675. http://dx.doi.org/10.1016/j.jfranklin.2024.106675
4. Q. Fang, X. D. Li, Event-triggered impulsive control for nonlinear impulsive disturbed systems with applications, *Commun. Nonlinear Sci.*, **152** (2026), 109292. http://dx.doi.org/10.1016/j.cnsns.2025.109292
5. Y. N. Wang, C. D. Li, H. J. Wu, H. Deng, Stability of nonlinear delayed impulsive control systems via step-function method, *Chaos Soliton. Fract.*, **189** (2024), 115631. http://dx.doi.org/10.1016/j.chaos.2024.115631

6. E. Hernández, D. O'Regan, On a new class of abstract impulsive differential equations, *Proc. Amer. Math. Soc.*, **141** (2013), 1641–1649. http://dx.doi.org/10.1090/S0002-9939-2012-11613-2
7. Y. N. Wang, C. D. Li, H. J. Wu, H. Deng, Existence of solutions for fractional instantaneous and non-instantaneous impulsive differential equations with perturbation and Dirichlet boundary value, *Discrete Cont. Dyn.-S*, **15** (2022), 1767–1776. http://dx.doi.org/10.3934/dcdss.2022005
8. D. Yang, J. R. Wang, Integral boundary value problems for nonlinear non-instantaneous impulsive differential equations, *J. Appl. Math. Comput.*, **55** (2017), 59–78. http://dx.doi.org/10.1007/s12190-016-1025-8
9. Y. N. Wang, C. D. Li, H. Deng, H. J. Wu, $\mathcal{S}$-asymptotically ω-periodic mild solutions for noninstantaneous impulsive integro-differential equations with state-dependent delay, *Math. Method. Appl. Sci.*, **46** (2023), 11229–11245. http://dx.doi.org/10.1002/mma.9177
10. M. Malik, R. Dhayal, S. Abbas, A. Kumar, Controllability of non-autonomous nonlinear differential system with non-instantaneous impulses, *RACSAM Rev. R. Acad. A*, **113** (2019), 103–118. http://dx.doi.org/10.1007/s13398-017-0454-z
11. M. Muslim, A. Kumar, Controllability of fractional differential equation of order (1,2] with non-instantaneous impulses, *Asian J. Control*, **20** (2018), 935–942. http://dx.doi.org/10.1002/asjc.1604
12. S. Kumar, S. M. Abdal, Approximate controllability for a class of instantaneous and non-instantaneous impulsive semilinear systems, *J. Dyn. Control Syst.*, **28** (2022), 725–737. http://dx.doi.org/10.1007/s10883-021-09540-7
13. A. Kumar, M. Malik, K. S. Nisar, Existence and total controllability results of fuzzy delay differential equation with non-instantaneous impulses, *Alex. Eng. J.*, **60** (2021), 6001–6012. http://dx.doi.org/10.1016/j.aej.2021.04.017
14. Y. Wang, Y. Y. Liu, Y. S. Liu, Total controllability of non-autonomous second-order measure evolution systems with state-dependent delay and non-instantaneous impulses, *Math. Biosci. Eng.*, **20** (2023), 2061–2080. http://dx.doi.org/10.3934/mbe.2023095
15. W. P. London, J. A. Yorke, Recurrent outbreaks of measles, chickenpox and mumps. I. Seasonal variation in contact rates, *Am. J. Epidemiol.*, **98** (1973), 453–468. http://dx.doi.org/10.1093/oxfordjournals.aje.a121575
16. M. Malek-Zavarei, M. Jamshidi, *Time-delay system: analysis, optimization, and applications*, New York: Elsevier Science Inc., 1987.
17. P. W. Nelson, A. S. Perelson, Mathematical analysis of delay differential equation models of HIV-1 infection, *Math. Biosci.*, **179** (2002), 73–94. http://dx.doi.org/10.1016/S0025-5564(02)00099-8
18. S. Li, Y. Y. Yan, Y. Yu, Q. S. Zhong, L. F. Hua, D. X. Liao, Fast finite-time position tracking control of electro-hydraulic servo systems with parametric uncertainty via dynamic surface and neural adaptive method, *Mathematics*, **14** 2026, 1551. https://dx.doi.org/10.3390/math14091551
19. S. Li, K. Yan, Y. L. Xie, Q. S. Zhong, J. Yang, D. X. Liao, Finite-time neural adaptive control of electro-hydraulic servo systems with minimal input delay and parametric uncertainty via Padé approximation, *Mathematics*, **14** (2026), 1368. https://dx.doi.org/10.3390/math14081368

20. K. Deimling, *Nonlinear functional analysis*, Berlin: Springer 1985. https://dx.doi.org/10.1007/978-3-662-00547-7

21. J. Hale, J. Kato, Phase space for retarded equations with infinite delay, *Funkcial Ekvac*, **21** (1978), 11–41.

22. A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, New York: Springer, 1988. https://doi.org/10.1007/978-1-4612-5561-1

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