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Research article

Higher-order dual extensions of dual-type octonions with explicit inverses and block matrix representations

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Abstract: A third-order nilpotent augmentation of the dual-type octonion framework was obtained by replacing the first-order dual layer with a higher-order infinitesimal structure. Algebraic closure and coefficient-wise product expansions were established, enabling all subsequent derivations to be performed in a finite form. A comprehensive invertibility condition was established by limiting the problem to the leading component, and a closed-form inverse was constructed directly up to the second order, producing a computation-ready formula for reciprocal evaluation. The structure of zero divisors and nilpotent elements was characterized, showing how degeneracy and non-invertibility spread via higher-order perturbation layers. The nilpotent constraint was used to establish finite exponential and power expansions, resulting in identities of the Euler and De Moivre types that were obtained for the third-order scenario and explicit formulations for repeated products. A block triangular left-multiplication matrix representation was constructed, from which the determinant and spectral consequences were obtained with the inversion formulas. The resulting framework provides an algebraic tool for explicit computation within dual-type octonionic models and enables higher-order perturbation encoding.

Keywords: third-order dual numbers, dual-type octonions, higher-order duality truncated exponential; De Moivre–type power formulas
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Abbreviations:

O: dual type octonion; $S(O)$: scalar part of dual type octonion; $V(O)$: vector part of dual type octonion; e_0: identity basis element in the scalar part of dual-type octonion algebra; $e_1, e_2, \cdots, e_7$: basis elements spanning the vector part; V: square-zero vector ideal span $\{e_1, e_2 \cdots e_7\}$; η: central nilpotent unit satisfying $\eta^3 = 0$; $\mathbb{D}^{(3)}$: third-order dual coefficient algebra $\mathbb{R}[\eta]/(\eta^3)$; $\mathbb{O}_D$: base non-isotopic dual-type octonion algebras; $\mathbb{O}_D^{(3)}$: third order extension of the non-isotropic dual-type octonion algebra; O_k: coefficients of an element $X = O_0 + \eta O_1 + \eta^2 O_2$ in $\mathbb{O}_D^{(3)}$; a_k: scalar coefficient in the decomposition $O_k = a_k e_0 + v_k$; v_k: vector coefficient in the decomposition $O_k = a_k e_0 + v_k$ with $v_k \in V$; $\sigma(X)$: scalar jet of X; $\nu(X)$: vector jet of X

1. Introduction

Hypercomplex algebras play an important role in modern algebra, geometry, computational mathematics, and mathematical physics. In particular, quaternion-based methods, which are four-dimensional associative division algebras, have been widely used in rigid-body kinematics, robotics, computer graphics, signal processing, image analysis, 3D spatial modeling, aerospace control sequences, the active resolution of complex matrix equations, and the diagonalization of hypercomplex tensors [1–4].

The limitations of quaternionic dimensionality motivated the development of higher-dimensional hypercomplex systems, particularly the eight-dimensional octonionic algebra. As the largest normed division algebra, octonions possess a rich algebraic structure and have attracted considerable attention in non-associative algebra. Octonion-based frameworks have also been investigated as geometric foundations for advanced physical theories, including high-dimensional spatial transformations, relativistic quantum mechanics, and octonion electrodynamics [5–7].

Parallel to the development of hypercomplex algebras, the formulation of dual numbers introduced a powerful tool for modeling spatial mechanics. A dual number incorporates an infinitesimal unit ϵ, defined by the nilpotent property $\epsilon^2 = 0$. This square-zero characteristic geometrically encodes spatial displacement. When applied to pure geometry and software architecture, dual numbers translate standard vectors into dual vectors, forming the absolute basis for computational line geometry [8], instantaneous spatial kinematics, and algorithmic mechanism design [9,10]. Furthermore, the formalizations of these dual-number matrices established the early mathematical groundwork for spatial transformation in robotics [11].

The incorporation of infinitesimal dual-number structures into hypercomplex algebras has greatly advanced the study of rigid-body kinematics and geometric transformations. The combination of dual numbers and quaternions led to the development of dual quaternions, which are now widely used for representing rigid body transformations. They facilitate artifact-free interpolation, simultaneous rotation, translation mapping, and sophisticated control pipelines in robotics and biomechanics [12–14]. The combination of dual numbers and octonions led to the development of dual octonions, extending kinematic capabilities into higher-dimensional spaces and modeling hypercomplex physical phenomena, such as dual octonions electrodynamics [15]. Motivated by these developments, pure algebraic research has produced robust sequence-based constructions, notably the Fibonacci and Horadam dual octonions [16,17]. Moreover, dual-type octonions were introduced as computationally tractable non-isotropic octonionic systems possessing explicit algebraic identities and matrix representations that permit explicit algebraic computations [18].

Furthermore, modern matrix operations have been significantly advanced by algebraic state-space methods, which utilize semi-tensor products to encode complex logical systems and finite-state machines into computationally tractable matrix forms [19].

Despite these advances, most available dual and dual-type octonion frameworks remain fundamentally restricted to first-order infinitesimal structures. In particular, the first-order dual unit successfully encodes the position and velocity and is mathematically incapable of propagating higher-order perturbations, such as acceleration-level corrections or higher spatial derivatives, without exiting the algebra. Advanced analytical models require robust higher-order kinematics embedded within dual Lie algebras [20]. Furthermore, explicit third-order extensions of dual-type octonion algebras, together with closed-form inversion formulas, exact polynomial-functional calculus, finite Euler-De-Moivre types, and computation-ready block matrix realizations, remain largely undeveloped in the literature.

To address this critical gap, we construct a third-order nilpotent extension of the dual-type octonion framework. Here, the designation third-order denotes the nilpotency index of the coefficient algebra (wherein the central unit satisfies $\eta^3 = 0$), conceptually mirroring the exact encoding of Taylor jets up to order 2. By explicitly defining the interaction between this nilpotent scalar layer and the square-zero vector ideal of the base algebra, we establish finite truncation laws that do not exist in the base structure.

The primary theoretical contributions of this study encompass the derivation of a comprehensive invertibility condition driven by the leading component, the construction of closed-form inverses truncated precisely at the second order, and the establishment of exact, finite Euler–De Moivre-type identities. Finally, we construct a 24×24 block lower-triangular matrix. This block-matrix lift translates complex, non-associative hypercomplex operations directly into standard computation-ready linear algebra.

The remainder of this paper is organized as follows: In Section 2, we recall the structure of the underlying dual-type octonion algebra and construct a third-order extension. In Section 3, we develop the principal algebraic properties, including power laws, polynomial-functional calculus, locality, nilpotency bounds, and Euler–De Moivre-type identities, of the operator. In Section 4, we establish the block-matrix representation and its spectral and computational consequences. In Section 5, we present explicit computational examples that illustrate the developed formulas.

2. Preliminaries

2.1. *Dual-type octonions and the square-zero vector ideal*

A dual-type octonion is written as $O = \sum_{s=0}^{7} x_s\, e_s$ and decomposed as $O = S(O) + V(O)$, where $S(O) = x_0 e_0$ and $V(O) = \sum_{s=1}^{7} x_s\, e_s$. The classification is determined by the scalar part: $S(O) = 0$ defines the isotropic class, whereas $S(O) \neq 0$ defines the non-isotropic class. For the non-isotropic multiplication (denoted by ×), the units satisfy $e_1^2 = \cdots = e_7^2 = 0$ and $e_i \times e_j = 0$ for $i \neq j$, and the product of two non-isotropic dual-type octonions takes the coefficient-wise form

$$O_1 \times O_2 = S(O_1)S(O_2) + S(O_1)V(O_2) + S(O_2)V(O_1). \tag{2.1}$$

Consequently, the vector part $V := \operatorname{span}_{\mathbb{R}}\{e_1, \dots, e_7\}$ forms a square-zero ideal in the sense that $V \times V = \{0\}$ under ×. This square-zero property is the key algebraic input behind the explicit inverse formulas used later.

Lemma 2.1. Let $O = x_0 e_0 + v$ and $P = y_0 e_0 + w$ with $x_0, y_0 \in \mathbb{R}$ and $v, w \in V$. Then

$$O \times P = x_0 y_0 e_0 + x_0 w + y_0 v \quad \text{and} \quad v \times w = 0. \tag{2.2}$$

Theorem 2.2. If $O = x_0 e_0 + v$ is non-isotropic with $x_0 \neq 0$, then O is invertible and

$$O^{-1} = \frac{x_0 e_0 - v}{x_0^2}. \tag{2.3}$$

2.2. *The third-order dual coefficient algebra*

Let η be a central nilpotent unit with $\eta^3 = 0$ and $\eta^2 \neq 0$. Define

$$\mathbb{D}^{(3)} := \mathbb{R}[\eta]/(\eta^3) = \{a + b\eta + c\eta^2 : a, b, c \in \mathbb{R}\}. \tag{2.4}$$

For $s = a + b\eta + c\eta^2$ and $t = d + e\eta + f\eta^2 \in \mathbb{D}^{(3)}$, multiplication is induced uniquely by $\eta^3 = 0$:

$$st = ad + (ae + bd)\eta + (af + be + cd)\eta^2. \tag{2.5}$$

The nilpotent filtration is

$$\mathbb{D}^{(3)} \supset (\eta) \supset (\eta^2) \supset \{0\}, \qquad (\eta)^3 = \{0\}. \tag{2.6}$$

Lemma 2.3. An element $s = a + b\eta + c\eta^2 \in \mathbb{D}^{(3)}$ is a unit if and only if $a \neq 0$. In that case,

$$s^{-1} = \frac{1}{a} - \frac{b}{a^2}\eta + \left(\frac{b^2}{a^3} - \frac{c}{a^2}\right)\eta^2. \tag{2.7}$$

2.3. *Construction of the third-order dual-type extension*

The third-order extension of the non-isotropic dual-type octonion algebra is defined by a scalar extension to $\mathbb{D}^{(3)}$:

$$\mathbb{O}_D^{(3)} := \mathbb{O}_D \otimes_{\mathbb{R}} \mathbb{D}^{(3)} \cong \{X = O_0 + \eta O_1 + \eta^2 O_2 : O_0, O_1, O_2 \in \mathbb{O}_D\}. \tag{2.8}$$

For $X = O_0 + \eta O_1 + \eta^2 O_2$ and $Y = P_0 + \eta P_1 + \eta^2 P_2$ in $\mathbb{O}_D^{(3)}$,

$$XY = (O_0 \times P_0) + \eta(O_0 \times P_1 + O_1 \times P_0) + \eta^2(O_0 \times P_2 + O_1 \times P_1 + O_2 \times P_0). \tag{2.9}$$

The subsets $\eta\mathbb{O}_D^{(3)}$ and $\eta^2\mathbb{O}_D^{(3)}$ are two-sided ideals, and

$$\mathbb{O}_D^{(3)} \supset \eta\mathbb{O}_D^{(3)} \supset \eta^2\mathbb{O}_D^{(3)} \supset \{0\}, \quad (\eta\mathbb{O}_D^{(3)})^3 = \{0\}. \tag{2.10}$$

Define $\pi: \mathbb{O}_D^{(3)} \to \mathbb{O}_D$ by $\pi(O_0 + \eta O_1 + \eta^2 O_2) = O_0$. Then π is a surjective homomorphism with $\ker(\pi) = \eta\mathbb{O}_D^{(3)}$. Moreover, $X \in \mathbb{O}_D^{(3)}$ is invertible if and only if O_0 is invertible in $(\mathbb{O}_D, \times)$. In the non-isotropic dual-type algebra, O_0 is invertible if and only if its scalar coefficient $x_0 \neq 0$; hence, the invertibility of X is determined by the leading scalar part.

Theorem 2.4. Let $X = O_0 + \eta O_1 + \eta^2 O_2 \in \mathbb{O}_D^{(3)}$ with O_0 non-isotropic and $S(O_0) = x_0 e_0$ satisfying $x_0 \neq 0$. Then X is invertible and admits

$$X^{-1} = Q_0 + \eta Q_1 + \eta^2 Q_2, \tag{2.11}$$

where

$$Q_0 = O_0^{-1}, \tag{2.12}$$

$$Q_1 = -O_0^{-1} \times O_1 \times O_0^{-1}, \tag{2.13}$$

$$Q_2 = O_0^{-1} \times O_1 \times O_0^{-1} \times O_1 \times O_0^{-1} - O_0^{-1} \times O_2 \times O_0^{-1}. \tag{2.14}$$

Here, O_0^{-1} is explicitly given by (2.3) in the non-isotropic dual-type algebra. Equations (2.12)–(2.14) are the truncated nilpotent-inversion identities adapted to the $\eta^3 = 0$ extension, and the truncation is finite because all η^3-terms vanish.

Corollary 2.5. If $O_0 = 0$, then $X = \eta O_1 + \eta^2 O_2$ is nilpotent of index at most 3. Hence, X is a zero divisor and cannot be invertible.

3. Major results

Let η be a central nilpotent unit satisfying $\eta^3 = 0$ and $\eta^2 \neq 0$. For each coefficient O_k, write

$$O_k = a_k e_0 + v_k, \quad a_k \in \mathbb{R},\ v_k \in V.$$

For $\mathrm{X} = O_0 + \eta O_1 + \eta^2 O_2 \in \mathbb{O}_D$, define the scalar jet and vector jet

$$\sigma(X) := a_0 + a_1\eta + a_2\eta^2 \in \mathbb{D}^{(3)}, \nu(X) := v_0 + \eta v_1 + \eta^2 v_2 \in V^{(3)} := V \otimes_{\mathbb{R}} \mathbb{D}^{(3)}.$$

Then $X = \sigma(X)e_0 + \nu(X)$.

3.1. *Square-zero ideal model and the induced product law*

Theorem 3.1. For all $X, Y \in \mathbb{O}_D^{(3)}$,

$$\nu(X)\nu(Y) = 0, \quad XY = \sigma(X)\sigma(Y)e_0 + \sigma(X)\nu(Y) + \sigma(Y)\nu(X). \tag{3.1}$$

Hence, $V^{(3)}$ is a square-zero ideal in $\mathbb{O}_D^{(3)}$.

Proof. Write $X = \sigma e_0 + \nu$ and $Y = \tau e_0 + \omega$, where $\sigma = \sigma(X)$, $\tau = \sigma(Y)$, $\nu = \nu(X)$, $\omega = \nu(Y)$. Expand:

$$XY = (\sigma e_0 + \nu)(\tau e_0 + \omega) = \sigma\tau(e_0 e_0) + \sigma(e_0\omega) + \tau(\nu e_0) + \nu\omega. \tag{3.2}$$

Because e_0 is the identity, then $e_0\omega = \omega$ and $\nu e_0 = \nu$, so the first three terms become $\sigma\tau e_0 + \sigma\omega + \tau\nu$. It remains to prove that $\nu\omega = 0$.

Now, use bilinearity over $\mathbb{D}^{(3)}$. Write

$$\nu = v_0 + \eta v_1 + \eta^2 v_2, \quad \omega = w_0 + \eta w_1 + \eta^2 w_2,$$

with $v_i, w_i \in V$. Then

$$\nu\omega = (v_0 + \eta v_1 + \eta^2 v_2)(w_0 + \eta w_1 + \eta^2 w_2) = \textstyle\sum_{i,j\in\{0,1,2\}} \eta^{i+j} \left(v_i \times w_j\right). \tag{3.3}$$

Each $v_i \times w_j = 0$ because $v_i, w_j \in V$ and $V \times V = \{0\}$ in the non-isotropic dual-type multiplication. Note that the nilpotency of the vector multiplication follows strictly from the square-zero ideal property of the base algebra, which guarantees that all cross terms vanish. Hence, every summand vanishes and $\nu\omega = 0$. Substituting into (3.2) yields (3.1). Therefore, $V^{(3)}$ is square-zero.

3.2. *Exact power laws and explicit scalar expansions*

Theorem 3.2. Let $X \in \mathbb{O}_D^{(3)}$ and $n \geq 1$. Then

$$\mathrm{X}^{\mathrm{n}} = \sigma(\mathrm{X})^{\mathrm{n}} \mathrm{e}_0 + \mathrm{n}\sigma(\mathrm{X})^{\mathrm{n}-1}\nu(\mathrm{X}). \tag{3.4}$$

Proof. Write $X = \sigma e_0 + \nu$ with $\nu^2 = 0$ by Theorem 3.1.

We prove this theorem using the induction method. The claim is immediate for n=1.

Assume (3.4) holds for some $n \geq 1$. Then

$$X^{n+1} = X^n X = (\sigma^n e_0 + n\sigma^{n-1}\nu)(\sigma e_0 + \nu). \tag{3.5}$$

Using Theorem 3.1,

$$(\sigma^n e_0)(\sigma e_0) = \sigma^{n+1} e_0, \quad (\sigma^n e_0)\nu = \sigma^n \nu, \quad (n\sigma^{n-1}\nu)(\sigma e_0) = n\sigma^n \nu, \tag{3.6}$$

and $(n\sigma^{n-1}\nu)\nu = n\sigma^{n-1}\nu^2 = 0$ because $\nu^2 = 0$. Hence

$$X^{n+1} = \sigma^{n+1} e_0 + \sigma^n \nu + n\sigma^n \nu = \sigma^{n+1} e_0 + (n+1)\sigma^n \nu,$$

This proves the result by induction.

Proposition 3.3. If $\sigma(X) = a_0 + a_1\eta + a_2\eta^2$, then for every integer $n \geq 1$,

$$\sigma(X)^n = a_0^n + n a_0^{n-1} a_1 \eta + \left(n a_0^{n-1} a_2 + \tbinom{n}{2} a_0^{n-2} a_1^2\right)\eta^2. \tag{3.7}$$

Proof. Write $\sigma(X) = a_0 + u$ where $u := a_1\eta + a_2\eta^2$. Since $\eta^3 = 0$, one has $u^3 = 0$ and $u^2 = a_1^2\eta^2$ (the terms involving η^3 vanish). Now expand using the binomial theorem in the commutative ring $\mathbb{D}^{(3)}$:

$$(a_0 + u)^n = \sum_{k=0}^{n} \binom{n}{k} a_0^{n-k} u^k = a_0^n + n a_0^{n-1} u + \binom{n}{2} a_0^{n-2} u^2,$$

because $u^k = 0$ for $k \geq 3$. Substitute $u = a_1\eta + a_2\eta^2$ and $u^2 = a_1^2\eta^2$, then collect coefficients of η and η^2. This yields Eq (3.7).

3.3. *Polynomial and analytic functional calculus*

Theorem 3.4. For any polynomial $p(t) \in \mathbb{R}[t]$ and any $X \in \mathbb{O}_D^{(3)}$,

$$p(X) = p(\sigma(X))e_0 + p'(\sigma(X))\nu(X). \tag{3.8}$$

Proof. Let $p(t) = \sum_{k=0}^{m} c_k t^k$. Then

$$p(X) = \sum_{k=0}^{m} c_k X^k.$$

Apply Theorem 3.2 to each X^k: $X^k = \sigma^k e_0 + k\sigma^{k-1}\nu$. Therefore,

$$p(X) = \sum_{k=0}^{m} c_k \sigma^k e_0 + \sum_{k=1}^{m} c_k k\sigma^{k-1}\nu = p(\sigma)e_0 + p'(\sigma)\nu,$$

where $p'(\sigma) = \sum_{k=1}^{m} c_k k\sigma^{k-1}$ is the formal derivative evaluated in $\mathbb{D}^{(3)}$.

Corollary 3.5. Let $\sigma = a_0 + a_1\eta + a_2\eta^2$. Then

$$p(\sigma) = p(a_0) + p'(a_0)(a_1\eta + a_2\eta^2) + \tfrac{1}{2}p''(a_0)a_1^2\eta^2. \tag{3.9}$$

Proof. Set $u = a_1\eta + a_2\eta^2$. Since $u^3 = 0$ and $u^2 = a_1^2\eta^2$. Expand $p(a_0 + u)$ by Taylor's theorem in a nilpotent direction:

$$p(a_0 + u) = p(a_0) + p'(a_0)u + \frac{1}{2}p''(a_0)u^2,$$

since higher terms involve $u^3 = 0$. Substituting u and u^2 gives (3.9).

Theorem 3.6. Let f be real-analytic in a neighborhood of a_0. Then for $X = \sigma e_0 + v$,

$$f(X) = f(\sigma)e_0 + f'(\sigma)v, \tag{3.10}$$

where $f(\sigma)$ and $f'(\sigma)$ are evaluated in $\mathbb{D}^{(3)}$ and hence truncate exactly at η^2.

Proof. Since f is analytic in a neighborhood of a_0, it admits a convergent Taylor series expansion centered at a_0:

$$f(t) = \textstyle\sum_{k\geq 0} c_k\,(t - a_0)^k.$$

We can rewrite $X = \sigma e_0 + v = a_0e_0 + K$, where $K = (\sigma - a_0)e_0 + v$. Since the scalar part of K is nilpotent in $\mathbb{D}^{(3)}$ and $v^2 = 0$, K is nilpotent, guaranteeing that the series evaluation terminates.

Then,

$$f(X) = \sum_{k\geq 0} c_k X^k.$$

Using Theorem 3.2 on the shifted variable, we have $X^k = (\sigma - a_0)^k e_0 + k(\sigma - a_0)^{k-1}v$. Therefore,

$$f(X) = \sum_{k\geq 0} c_k\,(\sigma - a_0)^k e_0 + \sum_{k\geq 1} c_k\,k(\sigma - a_0)^{k-1}v = f(\sigma)e_0 + f'(\sigma)v.$$

No convergence issue arises from v because all higher vector powers vanish ($v^2 = 0$), and $f(\sigma)$ in $\mathbb{D}^{(3)}$ truncates exactly at η^2 because $\eta^3 = 0$.

3.4. *Conjugation, norm polynomial, and compact inverse*

Define the coefficientwise conjugation by

$$\overline{ae_0 + v} := ae_0 - v, \quad \overline{X} := \overline{O_0} + \eta\overline{O_1} + \eta^2\overline{O_2}.$$

Theorem 3.7. For every $X \in \mathbb{O}_D^{(3)}$,

$$X\bar{X} = \sigma(X)^2 e_0. \tag{3.11}$$

If $a_0 \neq 0$, then X is invertible and

$$X^{-1} = \frac{\bar{X}}{\sigma(X)^2}. \tag{3.12}$$

Proof. Write $X = \sigma e_0 + v$. Since conjugation fixes the scalar jet and negates the vector jet, $\bar{X} = \sigma e_0 - v$.

Multiply using Theorem 3.1,

$$X\bar{X} = (\sigma e_0 + \nu)(\sigma e_0 - \nu) = \sigma^2 e_0 + \sigma(-\nu) + \sigma(\nu) + \nu(-\nu). \tag{3.13}$$

The middle terms cancel each other out. The last term equals $-\nu^2 = 0$ because $\nu^2 = 0$. Hence, $X\bar{X} = \sigma^2 e_0$, proving (3.11).

If $a_0 \neq 0$, then $\sigma = a_0 + a_1\eta + a_2\eta^2$ is a unit in $\mathbb{D}^{(3)}$ by Lemma 2.3. Therefore, σ^2 is also a unit. Multiply

$$X \cdot \frac{\bar{X}}{\sigma^2} = \frac{X\bar{X}}{\sigma^2} = \frac{\sigma^2 e_0}{\sigma^2} = e_0,$$

so $\bar{X}/\sigma^2$ is the two-sided inverse, giving (3.12).

3.5. *Locality, idempotent rigidity, and structural consequences*

Theorem 3.8. The algebra $\mathbb{O}_D^{(3)}$ is local. Its unique maximal ideal is given by

$$\mathfrak{J} = \{X \in \mathbb{O}_D^{(3)} : a_0 = 0\}. \tag{3.14}$$

Proof. By Theorem 3.7, X is invertible if and only if its scalar jet $\sigma(X)$ is invertible in $\mathbb{D}^{(3)}$. By Lemma 2.3, $\sigma(X)$ is invertible if and only if its constant term $a_0 \neq 0$. Hence, the set of all non-units in $\mathbb{O}_D^{(3)}$ is precisely the set $\mathfrak{J}$.

It remains to show that $\mathfrak{J}$ is an ideal. Take $X \in \mathfrak{J}$ so $a_0 = 0$, and take arbitrary $Y \in \mathbb{O}_D^{(3)}$. Write $X = \sigma_X e_0 + \nu_X$ and $Y = \sigma_Y e_0 + \nu_Y$.

$$XY = \sigma_X \sigma_Y e_0 + \sigma_X \nu_Y + \sigma_Y \nu_X. \tag{3.15}$$

Since the constant term of σ_X is 0, the constant term of $\sigma_X\sigma_Y$ is also 0. Moreover, $\sigma_X \nu_Y$ has no constant scalar part because it lies in $V^{(3)}$, and $\sigma_Y \nu_X$ also contributes no scalar constant. Hence, the constant scalar coefficient of XY is 0, i.e., $XY \in \mathfrak{J}$. Thus, $\mathfrak{J}$ is a (two-sided) ideal. Because the complement of $\mathfrak{J}$ is precisely the unit group, $\mathfrak{J}$ is the unique maximal ideal, so the ring is local.

Corollary 3.9. If $X^2 = X$, then $X = 0$ or $X = e_0$.

Proof. In any local ring, the idempotents are trivial. For completeness: If $X^2 = X$, then $X(1 - X) = 0$. Moreover, if X is a unit, then $1 - X = 0$ so $X = 1 = e_0$. Additionally, if X is not a unit, then $X \in \mathfrak{J}$ and $1 - X$ must be a unit, which forces $X = 0$.

3.6. *Polynomial equations: jet-level splitting and freedom of nilpotent layers*

Theorem 3.10. For $p(t) \in \mathbb{R}[t]$ and $X = \sigma e_0 + \nu$,

$$p(X) = 0 \quad \Leftrightarrow \quad p(\sigma) = 0 \text{ in } \mathbb{D}^{(3)} \text{ and } p'(\sigma)\nu = 0 \text{ in } V^{(3)}. \tag{3.16}$$

Proof. By Theorem 3.4, $p(X) = p(\sigma)e_0 + p'(\sigma)\nu$. The scalar part $p(\sigma)e_0$ lies in $\mathbb{D}^{(3)}e_0$, while the vector part $p'(\sigma)\nu$ lies in $V^{(3)}$. These two components are in complementary $\mathbb{D}^{(3)}$-submodules, so their sum is zero if and only if each component is zero. This yields (3.16).

Theorem 3.11. Assume $p(\sigma) = 0$ in $\mathbb{D}^{(3)}$. Write $p'(\sigma) = \eta^r u$, where u is a unit in $\mathbb{D}^{(3)}$ and $r \in \{0,1,2\}$, or $p'(\sigma) = 0$. Then the constraint $p'(\sigma)\nu = 0$ is equivalent to

$$\nu \in \eta^{3-r}V^{(3)}\ (r = 0,1,2), \text{ and is void if } p'(\sigma) = 0. \tag{3.17}$$

Proof. If $p'(\sigma) = 0$, then $p'(\sigma)\nu = 0$ for all ν, hence no restriction.

Assume $p'(\sigma) = \eta^r u$ with u a unit. Then $p'(\sigma)\nu = 0$ is equivalent to $\eta^r(u\nu) = 0$. Because u is a unit, the map $\nu \mapsto u\nu$ is bijective on $V^{(3)}$, so this is equivalent to $\eta^r\nu = 0$. Now in $\mathbb{D}^{(3)}$, the annihilator ideal of η^r is (η^{3-r}). Indeed, $\eta^r\eta^{3-r} = \eta^3 = 0$, and if $\eta^r z = 0$ then z must be divisible by η^{3-r} because the basis $\{1,\eta,\eta^2\}$ makes the divisibility check direct. Hence, $\eta^r\nu = 0$ holds exactly when each coefficient of ν is divisible by η^{3-r}, i.e., $\nu \in \eta^{3-r}V^{(3)}$.

This theorem gives a high-resolution parameterization of solution families: The order of vanishing of $p'(\sigma)$ determines how much of the vector jet can be freely chosen.

3.7. *Block-matrix lift and spectral consequences*

Let L_O be the real 8×8 left-multiplication matrix for $O \in \mathbb{O}_D$. Define L_X on $\mathbb{O}_D^{(3)}$ by $L_X(Y) = XY$.

Theorem 3.12. (Block-lower-triangular matrix representation) Relative to the decomposition $Y = P_0 + \eta P_1 + \eta^2 P_2$, the left-multiplication operator L_X has the block form

$$[L_X] = \begin{pmatrix} L_{O_0} & 0 & 0 \\ L_{O_1} & L_{O_0} & 0 \\ L_{O_2} & L_{O_1} & L_{O_0} \end{pmatrix}. \tag{3.18}$$

Consequently, its determinant and characteristic polynomial are

$$\det(L_X) = \det\left(L_{O_0}\right)^3, \quad \chi_{L_X}(\lambda) = \chi_{L_{O_0}}(\lambda)^3. \tag{3.19}$$

Proof. Take an arbitrary element $Y = P_0 + \eta P_1 + \eta^2 P_2$. Using the coefficient expansion:

$$XY = (O_0 \times P_0) + \eta(O_0 \times P_1 + O_1 \times P_0) + \eta^2(O_0 \times P_2 + O_1 \times P_1 + O_2 \times P_0). \tag{3.20}$$

Thus, the output coefficient of 1 equals $O_0 \times P_0$, the output coefficient of η equals $O_1 \times P_0 + O_0 \times P_1$, and the output coefficient of η^2 equals $O_2 \times P_0 + O_1 \times P_1 + O_0 \times P_2$. Written as an operator on the column vector $(P_0, P_1, P_2)^\top$, this is exactly the block action in (3.18), with each block being a left-multiplication map $P \mapsto O \times P$, i.e., L_{O_i}.

For (3.19), note that (3.18) is block lower -triangular with diagonal blocks L_{O_0}, so the determinant is the product of diagonal block determinants: $\det(L_X) = \det(L_{O_0})^3$. The characteristic polynomial factorization follows similarly because $\lambda I - [L_X]$ remains block lower-triangular with diagonal blocks $\lambda I - L_{O_0}$, yielding $\chi_{L_X}(\lambda) = \chi_{L_{O_0}}(\lambda)^3$.

3.8. *Sharp nilpotency bound: an index-4 signature*

Define $\rho(X) := a_0 \in \mathbb{R}$ and set $N := X - \rho(X)e_0$.

Then N has scalar jet $\sigma(N) = a_1\eta + a_2\eta^2$ and vector jet $\nu(N) = v_0 + \eta v_1 + \eta^2 v_2$.

Theorem 3.13. For every $X \in \mathbb{O}_D^{(3)}$,

$$N^4 = 0. \tag{3.21}$$

Moreover,

$$N^3 = 3a_1^2\eta^2 v_0, \tag{3.22}$$

so the nilpotency index is exactly 4 whenever $a_1 \neq 0$ and $v_0 \neq 0$.

Proof. Write $N = se_0 + \nu$ with

$$\mathrm{s} := a_1\eta + a_2\eta^2, \qquad \nu := v_0 + \eta v_1 + \eta^2 v_2.$$

First compute the powers of s in $\mathbb{D}^{(3)}$. Then one has

$$\mathrm{s}^2 = (a_1\eta + a_2\eta^2)^2 = a_1^2\eta^2 + 2a_1a_2\eta^3 + a_2^2\eta^4 = a_1^2\eta^2,$$

because $\eta^3 = 0$ implies $\eta^4 = 0$. Additionally,

$$\mathrm{s}^3 = \mathrm{s}^2\mathrm{s} = (a_1^2\eta^2)(a_1\eta + a_2\eta^2) = a_1^3\eta^3 + a_1^2a_2\eta^4 = 0.$$

Now, apply the exact power law (Theorem 3.2) to N:

$$N^n = \mathrm{s}^n\mathrm{e}_0 + \mathrm{n}\mathrm{s}^{n-1}\mathrm{\nu}. \tag{3.23}$$

For $n = 4$,

$$N^4 = \mathrm{s}^4\mathrm{e}_0 + 4\mathrm{s}^3\mathrm{\nu} = 0 + 0 = 0,$$

since $\mathrm{s}^3 = 0$, proving the nilpotency bound.

For $n = 3$,

$$N^3 = \mathrm{s}^3\mathrm{e}_0 + 3\mathrm{s}^2\mathrm{\nu} = 3\mathrm{s}^2\mathrm{\nu} = 3(a_1^2\eta^2)\mathrm{\nu}.$$

Now multiply η^2 into $\mathrm{\nu} = v_0 + \eta v_1 + \eta^2 v_2$:

$$\eta^2\nu = \eta^2 v_0 + \eta^3 v_1 + \eta^4 v_2 = \eta^2 v_0.$$

Hence, $N^3 = 3a_1^2\eta^2 v_0$, giving (3.22). This is nonzero exactly when $a_1 \neq 0$ and $v_0 \neq 0$, while $N^4 = 0$, so the nilpotency index is exactly 4 in that case.

3.9. *Euler and De Moivre-type identities for* $\mathbb{O}_{\boldsymbol{D}}^{(\mathbf{3})}$

In the third order extension $\mathbb{O}_{\boldsymbol{D}}^{(\mathbf{3})}$, the combination of the central nilpotent scalar layer $(\eta^3 = 0)$ and the square-zero vector ideal $(v^2 = 0)$ yields exact, finite exponential and polar-power expansions.

Theorem 3.14. (Euler-type identity) For any $X = \sigma e_0 + v \in \mathbb{O}_{\boldsymbol{D}}^{(\mathbf{3})}$ with $\sigma = a_0 + a_1\eta + a_2\eta^2$ the exponential mp yields the exact finite expansion:

$$e^X = e^{a_0}\left[1 + a_1\eta + \left(a_2 + \tfrac{1}{2}a_1^2\right)\eta^2\right](e_0 + v).$$

Proof. Let $u = a_1\eta + a_2\eta^2$. Since $\eta^3 = 0$, we have $u^2 = a_1^2\eta^2$ and $u^3 = 0$. The Taylor expansion of e^u truncates exactly at the second order.

$$e^\sigma = e^{a_0 + u} = e^{a_0}e^u = e^{a_0}\left(1 + u + \tfrac{1}{2}u^2\right).$$

Substituting u and u^2 yields:

$$e^{\sigma} = e^{a_0}\left[1 + a_1\eta + \left(a_2 + \tfrac{1}{2}a_1^2\right)\eta^2\right]. \tag{3.24}$$

From Theorem 3.6, $\mathrm{e}^{\mathrm{X}} = e^{\sigma}e_0 + (e^{\sigma})'v = e^{\sigma}(e_0 + v)$, substituting (3.24) here, we get the required results.

Theorem 3.15. (De Moivre-type power formula) Let $X = \sigma e_0 + v \in \mathbb{O}_{\boldsymbol{D}}^{(\mathbf{3})}$ be invertible. Define the normalized vector jet $\mathrm{w} = \sigma^{-1}v \in V^{(3)}$.Then X admits the generalized polar form $\mathrm{X} = \sigma(\mathrm{e}_0 + w)$, and for any integer $\mathrm{n} \geq 1$, the following De Moivre-type identity holds:

$$\mathrm{X}^{\mathrm{n}} = \sigma^{n}(e_0 + nw).$$

Proof. Since $\mathrm{a}_0 \neq 0$, the scalar jet σ is a unit in $\mathbb{D}^{(3)}$. We can therefore factor σ out of X to obtain $\mathrm{X} = \sigma\mathrm{e}_0 + \sigma\sigma^{-1}v = \sigma(e_0 + w)$.

Applying the binomial expansion to $(e_0 + w)^n$ yields:

$$(e_0 + w)^n = e_0^n + ne_0^{n-1}w + \tfrac{n(n-1)}{2}e_0^{n-2}w^2 + \cdots.$$

Since $\mathrm{w}^2 = (\sigma^{-1}v)(\sigma^{-1}v) = 0$, all higher-order terms vanish identically, leaving $(e_0 + w)^n = e_0 + nw$. Multiplying by the scalar factor σ^{n} gives $\mathrm{X}^{\mathrm{n}} = \sigma^{n}(e_0 + nw)$, proving the generalized De Moivre formula for the third-order extension.

4. Block-Matrix representation and computation for $\mathbb{O}_{\boldsymbol{D}}^{(\mathbf{3})}$

Throughout this section, let $X = O_0 + \eta O_1 + \eta^2 O_2 \in \mathbb{O}_D^{(3)}$ with $O_k = a_k e_0 + v_k$ ($a_k \in \mathbb{R}, v_k \in V = \operatorname{span}\{e_1, \ldots, e_7\}$). The scalar jet is $\sigma(X) = a_0 + a_1\eta + a_2\eta^2$, and the vector jet is $v(X) = v_0 + \eta v_1 + \eta^2 v_2$.

4.1. The lifted $\mathbf{24} \times \mathbf{24}$ *real representation*

Let $L_O: \mathbb{O}_D \to \mathbb{O}_D$ denote left multiplication by $O \in \mathbb{O}_D$. Under the non-isotropic dual-type product, $V \times V = \{0\}$; hence, for $O = ae_0 + v$ and $P = be_0 + w$,

$$OP = (ab)e_0 + aw + bv. \tag{4.1}$$

Lemma 4.1. Relative to the decomposition $\mathbb{O}_D = \mathbb{R}e_0 \oplus V$ and the coordinate vector $(b, w)^{\mathsf{T}}$ with $w \in \mathbb{R}^7$, the matrix of L_O is

$$[L_O] = \begin{pmatrix} a & 0 \\ v & aI_7 \end{pmatrix}. \tag{4.2}$$

Consequently, $\det(L_O) = a^8$ and $\chi_{L_O}(\lambda) = (\lambda - a)^8$.

Proof. From the product rule, $L_O(b, w) = (ab, aw + bv)$. The scalar component is ab, giving the top row $(a, 0)$. The vector component is $aw + bv$, giving the bottom block row (v, aI_7), which yields (4.2). The determinant and characteristic polynomial follow because (4.2) is lower block triangular with diagonal blocks a and aI_7.

Now define $L_X: \mathbb{O}_D^{(3)} \to \mathbb{O}_D^{(3)}$ by $L_X(Y) = XY$. Write $Y = P_0 + \eta P_1 + \eta^2 P_2$ and identify $\mathbb{O}_D^{(3)} \cong \mathbb{O}_D^{\oplus 3}$ via $Y \leftrightarrow (P_0, P_1, P_2)^{\mathsf{T}}$.

Theorem 4.2. The matrix of L_X relative to $(P_0, P_1, P_2)^\top$ is

$$[L_X] = \begin{pmatrix} A & 0 & 0 \\ B & A & 0 \\ C & B & A \end{pmatrix}, \quad A = [L_{O_0}],\ B = [L_{O_1}],\ C = [L_{O_2}], \tag{4.3}$$

where each block is 8×8. In particular,

$$\det(L_X) = \det(A)^3 = a_0^{24}, \quad \chi_{L_X}(\lambda) = \chi_A(\lambda)^3 = (\lambda - a_0)^{24}. \tag{4.4}$$

Proof. Using distributivity, centrality of η, and $\eta^3 = 0$, one obtains

$$XY = (O_0P_0) + \eta(O_0P_1 + O_1P_0) + \eta^2(O_0P_2 + O_1P_1 + O_2P_0). \tag{4.5}$$

Thus, the coefficient of 1 depends only on P_0 via L_{O_0}; the coefficient of η depends on (P_0, P_1) via (L_{O_1}, L_{O_0}); and the coefficient of η^2 depends on (P_0, P_1, P_2) via $(L_{O_2}, L_{O_1}, L_{O_0})$. This yields the block matrix (4.3). Since (4.3) is block lower triangular with diagonal blocks A,, the determinant and characteristic polynomial factorization in (4.4) follow from standard block triangular properties.

Corollary 4.3. The map $X \mapsto L_X$ is injective. In particular, $L_X = 0$ implies $X = 0$.

Proof. If $L_X = 0$, then $0 = L_X(e_0) = Xe_0 = X$.

Corollary 4.4. All eigenvalues of L_X are equal to a_0, and

$$\operatorname{tr}(L_X) = 24a_0, \quad \det(L_X) = a_0^{24}. \tag{4.6}$$

Proof. The eigenvalue statement and determinant follow from (4.2). The trace equals the sum of eigenvalues counted with algebraic multiplicity; hence, $24a_0$.

4.2. *Minimal polynomial and a sharp nilpotent shift bound*

Set $a_0 = \rho(X)$ and $N := X - a_0e_0$. From Section 3, $N^4 = 0$ and $N^3 = 3a_1^2\eta^2v_0$.

Theorem 4.4.

$$(L_X - a_0I)^4 = 0. \tag{4.7}$$

Hence, the minimal polynomial of L_X divides $(\lambda - a_0)^4$. Moreover, if $a_1 \neq 0$ and $v_0 \neq 0$, then $(L_X - a_0I)^3 \neq 0$, so the minimal polynomial is exactly $(\lambda - a_0)^4$.

Proof. Since $L_X - a_0I == L_N$, it suffices to show $L_N^4 = 0$. For any Y,

$$L_N^4(Y) = N^4Y.$$

Because $N^4 = 0$ in the algebra, $N^4Y = 0$ for all Y, so $L_N^4 = 0$, proving (4. 7). For sharpness, if $a_1 \neq 0$ and $v_0 \neq 0$, then $N^3 = 3a_1^2\eta^2v_0 \neq 0$. Therefore,

$$(L_X - a_0I)^3(e_0) = L_N^3(e_0) = N^3 \neq 0,$$

so $(L_X - a_0I)^3 \neq 0$, forcing the minimal polynomial to have exponent 4.

4.3. *Closed-form inverse of the lifted matrix*

Assume $a_0 \neq 0$, so $A = [L_{O_0}]$ is invertible by Lemma 4.1.

Theorem 4.6. If $a_0 \neq 0$, then $[L_X]$ is invertible and

$$[L_X]^{-1} = \begin{pmatrix} A^{-1} & 0 & 0 \\ -A^{-1}BA^{-1} & A^{-1} & 0 \\ A^{-1}BA^{-1}BA^{-1} - A^{-1}CA^{-1} & -A^{-1}BA^{-1} & A^{-1} \end{pmatrix}. \tag{4.8}$$

Proof. Let the right-hand side be M. Compute $[L_X]M$ blockwise. The $(1,1)$ block is $AA^{-1} = I$, and the $(1,2)$ and $(1,3)$ blocks are 0.

For the $(2,1)$ block:

$$BA^{-1} + A(-A^{-1}BA^{-1}) = BA^{-1} - BA^{-1} = 0.$$

The $(2,2)$ block is $AA^{-1} = I$, and $(2,3) = 0$.

For the $(3,1)$ block:

$$CA^{-1} + B(-A^{-1}BA^{-1}) + A(A^{-1}BA^{-1}BA^{-1} - A^{-1}CA^{-1}) \\ = CA^{-1} - BA^{-1}BA^{-1} + BA^{-1}BA^{-1} - CA^{-1} = 0.$$

The $(3,2)$ block is

$$BA^{-1} + A(-A^{-1}BA^{-1}) = 0,$$

and $(3,3) = AA^{-1} = I$. Thus, $[L_X]M = I$. A symmetric check gives $M[L_X] = I$; hence (4.8).

Corollary 4.7. If $a_0 \neq 0$, then $L_{X^{-1}} = [L_X]^{-1}$, where $X^{-1} = \overline{X}/\sigma(X)^2$.

Proof. Associativity of the induced product implies $L_{XY} = L_X L_Y$ for all X, Y. With $Y = X^{-1}$, it follows that $L_X L_{X^{-1}} = L_{e_0} = I$, so $L_{X^{-1}} = L_X^{-1}$. Although the underlying octonion algebra is non-associative, the identity $L_{X^{-1}} = [L_X]^{-1}$ holds because the closed- form inverse is constructed purely via scalar central element in $\mathbb{D}^{(3)}$ and the square-zero vector ideal, meaning the operators compose associativity in this specific matrix representation. The explicit block form gives the corresponding real matrix.

4.4. *Exact matrix functions by finite truncation*

Because $(L_X - a_0 I)^4 = 0$, every analytic matrix function reduces to a finite polynomial in $(L_X - a_0 I)$.

Theorem 4.8. Let $K \coloneqq L_X - a_0 I$. Then

$$e^{tL_X} = e^{ta_0}\left(I + tK + \frac{t^2}{2}K^2 + \frac{t^3}{6}K^3\right). \tag{4.9}$$

In particular,

$$\cos(L_X) = \cos(a_0)I - \sin(a_0)K - \frac{\cos(a_0)}{2}K^2 + \frac{\sin(a_0)}{6}K^3, \tag{4.10}$$

$$\sin(L_X) = \sin(a_0)I + \cos(a_0)K - \frac{\sin(a_0)}{2}K^2 - \frac{\cos(a_0)}{6}K^3. \tag{4.11}$$

Proof. Since $K^4 = 0$, the series $e^{tK} = \sum_{m\geq 0} \frac{t^m}{m!}K^m$ truncates after $m = 3$. Additionally, $L_X = a_0 I + K$ implies $e^{tL_X} = e^{ta_0}e^{tK}$, yielding (4.9).

The cosine and sine series are polynomials in K of degree at most 3 for the same reason; using $\cos(a_0 I + K) = \cos(a_0)\cos(K) - \sin(a_0)\sin(K)$ and truncations of $\cos(K), \sin(K)$ (which truncate because $K^4 = 0$) yields (4.10) and (4.11) by direct series expansion.

4.5. A computable invariant detecting the third-order activity of $\boldsymbol{X}$

The cube $N^3 = 3a_1^2\eta^2 v_0$ controls whether the extension genuinely uses the third-order layer in a nontrivial way.

Theorem 4.9. Let $K = L_X - a_0 I$. Then

$$K^3 = L_{N^3} = L_{3a_1^2\eta^2 v_0}. \tag{4.12}$$

Moreover,

$$\operatorname{rank}(K^3) = \begin{cases} 1, & \text{if } a_1 \neq 0 \; and \; v_0 \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{4.13}$$

Consequently, K has a Jordan block of size 4 if and only if $a_1 \neq 0$ and $v_0 \neq 0$; in that case, exactly one such block occurs.

Proof. First, $K = L_{X - a_0 e_0} = L_N$, hence $K^3 = L_N^3 = L_{N^3}$, giving (4.12).

If $a_1 = 0$ or $v_0 = 0$, then $N^3 = 0$, so $K^3 = L_{N^3} = 0$ and the rank is 0. Assume $a_1 \neq 0$ and $v_0 \neq 0$. Then $N^3 = 3a_1^2\eta^2 v_0 \neq 0$. For any Y,

$$L_{\eta^2 v_0}(Y) = (\sigma(Y))(\eta^2 v_0) \quad \text{by} v(\eta^2 v_0)^2 = 0 \; and \; \sigma(\eta^2 v_0) = 0,$$

so only the constant term of $\sigma(Y)$ contributes because $(b_0 + b_1\eta + b_2\eta^2)\eta^2 = b_0\eta^2$. Hence, every output is a real multiple of $\eta^2 v_0$, so $\operatorname{Im}(L_{\eta^2 v_0}) = \operatorname{span}_{\mathbb{R}}\{\eta^2 v_0\}$, which is 1-dimensional. Therefore, $\operatorname{rank}(L_{\eta^2 v_0}) = 1$, and scaling by $3a_1^2$ preserves rank, giving $\operatorname{rank}(K^3) = 1$. This proves (4.13).

Finally, for a nilpotent operator, the rank of the third power equals the number of Jordan blocks of size at least 4. Since $\operatorname{rank}(K^3) \leq 1$, at most one such block occurs; it occurs exactly when the rank is 1.

5. Computational applications and examples

5.1. Closed-form computational rules in $\mathbb{O}_{\boldsymbol{D}}^{(\mathbf{3})}$

Having established the core algebraic properties of $\mathbb{O}_{\boldsymbol{D}}^{(\mathbf{3})}$, we now demonstrate its practical utility. The exact product, power law, and closed-form inverse provide a complete calculus for symbolic implementation. Rather than restating these fundamental operations, in this section, we show how the nilpotent constraints $(\eta^3 = 0, v^2 = 0)$ elegantly resolve polynomial equations and enable higher-order kinematic perturbation modeling without requiring infinite series expansions.

5.2. Polynomial constraints and solution families controlled by root multiplicity

Polynomial equations $p(X) = 0$ are resolved exactly by the jet-splitting principle (Theorem 3.10), which is the algebraic core behind Weil-algebra evaluation and Taylor-mode propagation in higher-order differential settings. In this algebra, this splitting becomes especially sharp because the vector jet is square-zero.

Let $p(t) \in \mathbb{R}[t]$, let $X = \sigma e_0 + \nu$, and write $\sigma = a_0 + a_1\eta + a_2\eta^2$. Since $\eta^3 = 0$, the scalar condition $p(\sigma) = 0$ in $\mathbb{D}^{(3)}$ is equivalent to the finite Taylor identity

$$p(\sigma) = p(a_0) + p'(a_0)(a_1\eta + a_2\eta^2) + \frac{1}{2}p''(a_0)a_1^2\eta^2 = 0.$$

Theorem 5.1. Assume $p(a_0) = 0$. Consider the multiplicity of a_0 as a real root of p.
(1) Simple root: If $p'(a_0) \neq 0$, then $p(X) = 0$ holds if and only if

$$a_1 = 0, \quad a_2 = 0, \quad \nu = 0. \tag{5.1}$$

Hence, only the constant scalar lift $X = a_0 e_0$ is a solution.
(2) Double root: If $p'(a_0) = 0$ and $p''(a_0) \neq 0$, then $p(X) = 0$ holds if and only if

$$a_1 = 0, \quad \text{and} \quad p'(\sigma)\nu = 0. \tag{5.2}$$

In this case $\sigma = a_0 + a_2\eta^2$ and $p'(\sigma) = p''(a_0)a_2\eta^2$. Consequently,

$$\text{if } a_2 \neq 0, \text{ then } \nu \in \eta V^{(3)} \text{ (i.e., } v_0 = 0); \quad \text{if } a_2 = 0, \text{ then } \nu \text{ is arbitrary.} \tag{5.3}$$

(3) Triple (or higher) root: If $p'(a_0) = p''(a_0) = 0$, then $p(\sigma) = 0$ imposes no restriction on (a_1, a_2), and the constraint is carried by $p'(\sigma)\nu = 0$. In particular, if $p'''(a_0) \neq 0$, then $p'(\sigma) = \frac{1}{2}p'''(a_0)a_1^2\eta^2$, so

$$\text{if } a_1 \neq 0, \text{ then } \nu \in \eta V^{(3)}; \quad \text{if } a_1 = 0, \text{ then } \nu \text{ is arbitrary.} \tag{5.4}$$

If all derivatives up to order 3 vanish at a_0, then $p'(\sigma) = 0$ for every σ, and ν is unconstrained.

Proof. The equivalence $p(X) = 0 \Leftrightarrow p(\sigma) = 0$ and $p'(\sigma)\nu = 0$ is exactly Theorem 3.10. The scalar condition $p(\sigma) = 0$ reduces to the finite Taylor identity because $(a_1\eta + a_2\eta^2)^3 = 0$.

For the simple root case, the identity forces $p'(a_0)a_1 = 0$ and $p'(a_0)a_2 = 0$, so $a_1 = a_2 = 0$. Then $p'(\sigma) = p'(a_0)$ is a unit in $\mathbb{D}^{(3)}$, so $p'(\sigma)\nu = 0$ implies $\nu = 0$.

For the double root case, the identity reduces to $\frac{1}{2}p''(a_0)a_1^2\eta^2 = 0$, so $a_1 = 0$, while a_2 remains free; then $p'(\sigma) = p''(a_0)a_2\eta^2$, and the annihilator identity $\eta^2 \cdot \nu = 0 \Leftrightarrow \nu \in \eta V^{(3)}$ gives the stated condition.

For the higher multiplicity case, the same computation shows that $p(\sigma) = 0$ imposes no restriction and $p'(\sigma)$ is of order η^2 whenever the first non-vanishing derivative is $p^{(3)}(a_0)$; the same annihilator argument yields the stated condition.

This theorem is practically important: It gives an explicit "degrees of freedom" count for solution families and shows that the nilpotent layers η and η^2 interact with root multiplicity in a controlled, finite way.

5.3. Verification examples that demonstrate sharpness

The following examples are included to make the results checkable directly using only $\eta^3 = 0$ and $V^{(3)} \cdot V^{(3)} = \{0\}$.

Example 5.1. Let $\sigma = 2 + \eta$ (so $a_0 = 2, a_1 = 1, a_2 = 0$) and let $\nu = v_0 + \eta v_1 + \eta^2 v_2$ be arbitrary.

Using the scalar inverse formulas in $\mathbb{D}^{(3)}$:

$$\sigma^{-1} = \frac{1}{2} - \frac{1}{4}\eta + \frac{1}{8}\eta^2, \qquad \sigma^{-2} = \frac{1}{4} - \frac{1}{4}\eta + \frac{3}{16}\eta^2.$$

Hence, the closed- form inverse yields

$$X^{-1} = \left(\frac{1}{2} - \frac{1}{4}\eta + \frac{1}{8}\eta^2\right) e_0 - \left(\frac{1}{4} - \frac{1}{4}\eta + \frac{3}{16}\eta^2\right) \nu.$$

A direct multiplication gives

$$XX^{-1} = \sigma\sigma^{-1} e_0 + \sigma(-\sigma^{-2}\nu) + \sigma^{-1}\nu = e_0 - \sigma^{-1}\nu + \sigma^{-1}\nu = e_0,$$

where $\sigma\sigma^{-1} = 1$ in $\mathbb{D}^{(3)}$ and $\nu \cdot \nu = 0$ are the only facts used. This verifies the inverse formula for a fully explicit instance.

Example 5.2. Let the base algebra be $\mathbb{O}_D = span_{\mathbb{R}}\{e_0, e_1, \cdots, e_7\}$, where e_0 is the identity and $V = span_{\mathbb{R}}\{e_1, e_2, \cdots, e_7\}$ forms the square-zero ideal. We define three simple base elements as follows:

$$O_0 = 2e_0 + e_1; O_1 = e_0 - e_2; O_2 = 3e_3.$$

We construct the third-order element $X \in \mathbb{O}_D^{(3)}$ using the nilpotent unit η (where η^3):

$$X = (2e_0 + e_1) + \eta(e_0 - e_2) + \eta^2(3e_3).$$

The scalar jet is extracted from the real coefficients of e_0: $\sigma(X) = 2 + \eta$.

By the coefficient-wise conjugation definition, the vector parts are negated, whereas the scalar parts remain fixed.

$$\bar{X} = (2e_0 - e_1) + \eta(e_0 + e_2) + \eta^2(-3e_3).$$

To compute the norm, we apply Theorem 3.7, $N(X) = X\bar{X} = \sigma(X)^2 e_0 = (2 + \eta)^2 e_0 = (4 + 4\eta + \eta^2)e_0$. Since the base scalar $a_0 = 2 \neq 0, X$ is invertible. We compute the inverse of the norm polynomial within the coefficient ring $\mathbb{D}^{(3)}$ using the truncated binomial expansion,

$$(\sigma(X)^2)^{-1} = (4 + 4\eta + \eta^2)^{-1} = \tfrac{1}{4}\left(1 + \eta + \tfrac{1}{4}\eta^2\right)^{-1} = \tfrac{1}{4} - \tfrac{1}{4}\eta + \tfrac{3}{16}\eta^2.$$

Applying the compact inverse formula $X^{-1} = \frac{\bar{X}}{\sigma(X)^2}$ from Eq (3.2), we have

$$X^{-1} = (\frac{1}{4} - \frac{1}{4}\eta + \frac{3}{16}\eta^2)((2e_0 - e_1) + \eta(e_0 + e_2) + \eta^2(-3e_3))$$
$$= \left(\tfrac{1}{2}e_0 - \tfrac{1}{4}e_1\right) + \eta\left(-\tfrac{1}{4}e_0 + \tfrac{1}{4}e_1 + \tfrac{1}{4}e_2\right) + \eta^2\left(\tfrac{1}{8}e_0 - \tfrac{3}{16}e_1 - \tfrac{1}{4}e_2 - \tfrac{3}{4}e_3\right).$$

The lower triangular block matrix representation derived in Section 4 bypasses the need for symbolic hypercomplex solvers. This structure is particularly advantageous for applications in higher-order kinematics, where multiparameter perturbations and higher-order spatial derivatives must be computationally operationalized within dual algebraic frameworks.

Now, we encode the Left-multiplication operator $\mathrm{L_X}$ as a real matrix. Let $\mathrm{I_8}$ be the 8×8 identity matrix, and $\mathrm{L_{e_k}}$ denotes the standard 8×8 left multiplication matrix for the basis vector $\mathrm{e_k}$ in the base algebra. The sub-blocks are:

$$\mathrm{L}_{O_0} = 2I_8 + L_{e_1},\ \mathrm{L}_{O_1} = I_8 - L_{e_2}, \mathrm{L}_{O_2} = 3L_{e_3}.$$

From Eq (3.18), the third-order nilpotency forces the exact 24×24 lower-triangular Toeplitz-block structure:

$$[L_X] = \begin{pmatrix} 2I_8 + L_{e_1} & 0 & 0 \\ I_8 - L_{e_2} & 2I_8 + L_{e_1} & 0 \\ 3L_{e_3} & I_8 - L_{e_2} & 2I_8 + L_{e_1} \end{pmatrix}.$$

This matrix representation immediately operationalizes the algebra for computational kinematics and multi-parameter perturbations, bypassing the need for symbolic hypercomplex solvers.

Example 5.3. Let $p(t) = (t-1)^2$, so $a_0 = 1$ is a double root with $p'(1) = 0$ and $p''(1) = 2$. Theorem 5. 1 gives $a_1 = 0$ and $\sigma = 1 + a_2\eta^2$. Additionally, if $a_2 \neq 0$, then $p'(\sigma) = 2a_2\eta^2 \neq 0$, and the constraint is $v \in \eta V^{(3)}$, meaning $v_0 = 0$ but v_1, v_2 remain free. Moreover, if $a_2 = 0$, then $p'(\sigma) = 0$, and the vector jet is completely free. In either case, the solution set is strictly larger than the scalar solution. Explicitly, the calculation $p''(1) = 2 \neq 0$ confirms the double-root condition, which strictly enforces the solution family parameterization restricting $v \in \eta V^3$ and $a_2 \neq 0$.

Example 5.4. Let $p(t) = t(t-2)$. At the root $a_0 = 2$, one has $p'(2) = 2 \neq 0$. Theorem 5.1 forces $a_1 = a_2 = 0$ and $v = 0$, so the only solution in $\mathbb{O}_D^{(3)}$ lying over $a_0 = 2$ is $X = 2e_0$. Thus, the η-layers and all vector layers are eliminated automatically for simple roots.

These examples confirm that the theory is internally consistent and that the sharp dichotomy between simple and multiple roots is not superficial; it is the mechanism that determines whether the higher-order structure survives or collapses.

5.4. *Application to higher-order kinematics perturbations*

To illustrate the concrete value of $\mathbb{O}_D^{(3)}$, consider its applications to kinematic perturbation models. In standard dual quaternion or dual-type octonion kinematics, the real part encodes the orientation, and the first-order dual part encodes the position or velocity. However, standard dual numbers $(\epsilon^2 = 0)$ cannot natively capture acceleration without leaving the algebra.

By utilizing the third-order nilpotent layer $(\eta^3 = 0)$, a motion trajectory jet can be encoded natively as: $X(t) \approx X_0 + \eta X_1 + \eta^2 X_2$.

Here, $X_0 \in \mathbb{O}_D$ represents the base pose, $X_1 \in \mathbb{O}_D$ encodes the first-derivative (velocity/ angular velocity) perturbations, and $X_2 \in \mathbb{O}_D$ captures the second-derivative (acceleration) data. Because the matrix lift $[L_X]$ exactly preserves this block structure, algorithms can compute forward kinematics, inverses, and transformations for accelerated bodies in a single automated matrix operation, avoiding truncation errors associated with finite difference methods.

6. Conclusions

In this paper, we establish $\mathbb{O}_D^{(3)}$, a third-order nilpotent extension of the dual-type octonion algebra. By unifying a nilpotent coefficient layer $(\eta^3 = 0)$ with a square-zero vector ideal $(v^2 = 0)$, we provide a complete, closed-form computational calculus encompassing exact formulas for multiplication, inversion, and finite power expansions. The key theoretical contributions include the jet splitting principle, which parameterizes polynomial solution families strictly via root multiplicity, and a 24×24 block lower triangular matrix representation that reduces complex non-associative operations to stable and exact matrix algorithms.

Furthermore, by exploiting the generalized polar form of the square-zero vector ideal, we explicitly establish Euler and De Moivre-type identities adapted to the third-order scenario. We demonstrate the practical utility of this algebra by showing how it natively accommodates higher-order kinematic multi-derivative perturbations, such as acceleration, without requiring infinite series approximations. In future research, we will explore the application of these newly derived identities in advanced cryptography, along with systematic integer sequence constructions within the dual-type octonion domain.

Author contributions

All authors of this article have been contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no competing interests.

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