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Research article

Asymptotic behaviors and dynamical bifurcation of a stochastic multi-strain epidemic model with jump diffusion

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Abstract: This paper investigated the asymptotic behavior and dynamical bifurcation of a stochastic multi-strain epidemic model with jump diffusion. We defined a threshold parameter λ as the Lyapunov exponent of the total infected population, which incorporates the stationary distribution of strain proportions on the disease-free boundary and a jump-induced correction term arising from the Lévy noise:

$$\lambda = \int_{\Delta}\Big[\sum_{i=1}^{n}(\beta_i y_i - (\gamma_i + \mu_i + \eta_i)y_i) - \frac{1}{2}\big(\sum_{i=1}^{n}\sigma_{2i}y_i\big)^2\Big]\mu^*(dy) + \int_{\mathbb{Y}}\Big[\ln(1 + \sum_{i=1}^{n} y_i f_{2i}(u)) - \sum_{i=1}^{n} y_i f_{2i}(u)\Big]\nu(du).$$

This threshold provides a necessary and sufficient condition for overall disease persistence ($\lambda > 0$) versus extinction ($\lambda \leq 0$). Moreover, we introduced strain-specific thresholds λ_i and established a competitive exclusion principle: The strain with the largest λ_i dominates, while strains with smaller λ_i go extinct; when two or more strains share the same maximal λ_i, they can coexist. Furthermore, λ serves as a dynamical bifurcation point: When $\lambda \leq 0$, the unique invariant measure is concentrated on the extinction set; when $\lambda > 0$, this measure loses stability and a new invariant measure supported on the positive orthant emerges. Numerical simulations confirmed the critical role of λ and illustrated competitive exclusion between strains under different noise intensities.

Keywords: multi-strain epidemic model; Lévy noise; invariant measure; dynamical bifurcation
Mathematics Subject Classification: 92B05, 34F05

1. Introduction

Multi-strain infectious diseases pose a major challenge in epidemiology and public health, as illustrated by the successive waves of SARS-CoV-2 variants. A fundamental question is whether different strains can stably coexist. To address this, numerous multi-strain models have been developed.

Bremermann and Thieme [4] showed that in a two-strain Susceptible-Infected-Susceptible (SIS) model, any strain that fails to maximize the basic reproduction number eventually goes extinct. Bentaleb et al. [3] analyzed a multi-strain Susceptible-Exposed-Infectious-Recovered (SEIR) model with non-monotone and bilinear incidence, deriving the basic reproduction number and characterizing the stability of three equilibrium states. Further Ordinary Differential Equation (ODE)-based insights into strain coexistence can be found in [14, 17, 25].

To overcome the limitations of deterministic models that ignore environmental fluctuations, an increasing number of studies have begun to incorporate randomness into the transmission dynamics of infectious diseases [8, 12, 15]. In reality, disease transmission is driven by numerous random factors [9, 10, 24], which fall into two categories. The first consists of continuous, small fluctuations—such as daily variations in contact rates or weather—well-modeled by Brownian motion. The second consists of rare, large shocks: Superspreading events, sudden lockdowns, travel bans, new variants, or healthcare collapses. These cannot be captured by Gaussian noise but are naturally described by the jump component of Lévy noise. By combining both, the Lévy process offers a unified framework that accounts for both everyday randomness and extreme events.

In this paper, we describe these two types of randomness through a multi-strain Susceptible-Infectious-Recovered (SIR) model that includes Brownian motion and Poisson jump terms. We begin with the deterministic multi-strain SIR model:

$$\begin{cases} \dfrac{dS}{dt} = \Lambda - \dfrac{S\sum_{i=1}^{n}\beta_i I_i}{S+\sum_{i=1}^{n} I_i} - \mu S, \\ \dfrac{dI_i}{dt} = \dfrac{\beta_i S I_i}{S+\sum_{i=1}^{n} I_i} - (\gamma_i+\mu_i+\eta_i)I_i, \\ \dfrac{dR}{dt} = \sum_{i=1}^{n}\gamma_i I_i - \mu R. \end{cases}$$

The parameters in the deterministic model have the following epidemiological interpretations:

- Λ: Recruitment rate of susceptible individuals (births or immigration).
- β_i: Transmission rate of strain i (the rate at which a susceptible individual becomes infected upon contact with an individual infected with strain i).
- μ: Natural death rate (common to all compartments, independent of disease status).
- γ_i: Recovery rate from strain i (the rate at which infected individuals recover and move to the recovered compartment).
- μ_i: Disease-induced death rate for strain i (excess mortality caused by the infection).
- η_i: Additional removal rate due to treatment or isolation for strain i (e.g., hospitalization or quarantine measures).

All parameters are assumed to be positive constants.

To incorporate environmental fluctuations into the mortality rates, we introduce stochastic perturbations following the standard approach in stochastic population dynamics [11, 12]. Specifically, we replace the constant mortality rates by

$$\mu \;\to\; \mu + \sigma_1 \dot{B}_1(t) + \int_{\mathbb{Y}} f_1(u)\dot{N}(t,du), \qquad \mu_i \;\to\; \mu_i + \sigma_{2i}\dot{B}_{2i}(t) + \int_{\mathbb{Y}} f_{2i}(u)\dot{N}(t,du),$$

where $\dot{B}_1(t), \dot{B}_{2i}(t)$ represent Gaussian white noise and $\dot{N}(t,du)$ represents Poisson white noise. Here σ_1 and σ_{2i} are the intensities of the Gaussian noise, while $f_1(u)$ and $f_{2i}(u)$ determine the jump size distribution. The conditions $f_1(u) > -1$ and $f_{2i}(u) > -1$ ensure that the population sizes remain nonnegative after a jump. Rewriting the Poisson noise in compensated form $\widetilde{N}(dt, du) = N(dt, du) - \nu(du)dt$ and absorbing the resulting drift corrections into the deterministic part, we obtain the following stochastic multi-strain SIR model:

$$\begin{cases} dS(t) = \Big[\Lambda - \dfrac{S(t)\sum_{i=1}^{n}\beta_i I_i(t)}{S(t)+\sum_{i=1}^{n} I_i(t)} - \mu S(t)\Big]dt + \sigma_1 S(t)dB_1(t) + \int_{\mathbb{Y}} S(t_-)f_1(u)\widetilde{N}(dt, du), \\ dI_i(t) = \Big[\dfrac{\beta_i S(t) I_i(t)}{S(t)+\sum_{i=1}^{n} I_i(t)} - (\gamma_i + \mu_i + \eta_i)I_i(t)\Big]dt + \sigma_{2i} I_i(t)dB_{2i}(t) + \int_{\mathbb{Y}} I_i(t_-)f_{2i}(u)\widetilde{N}(dt, du), \\ dR(t) = \Big[\sum_{i=1}^{n}\gamma_i I_i(t) - \mu R(t)\Big]dt + \sigma_1 R(t)dB_3(t) + \int_{\mathbb{Y}} R(t_-)f_1(u)\widetilde{N}(dt, du). \end{cases} \tag{1.1}$$

In the above, $g(t_-)$ denotes the left limit of the function $g(t)$. Throughout this paper, we assume that the Brownian motions B_1, B_{2i} $(i = 1, \cdots, n)$ and the Poisson random measure N are mutually independent.

Given the recent growing interest in epidemic modeling, we now highlight the novelty and contributions of this work. A central question in mathematical epidemiology is to determine whether a disease persists or goes extinct. To date, significant progress has been made for low-dimensional stochastic models, such as SIR and SIS systems driven by Gaussian white noise [7, 18, 19]. However, extending these results to our setting—a high-dimensional multi-strain model with Lévy jumps—presents three substantial difficulties. First, the coexistence and competition among multiple strains create a strongly coupled system, rendering the classical technique of reducing to a single infectious variable inapplicable. Second, the presence of Lévy jumps disrupts path continuity and complicates the application of standard Itô calculus, especially when deriving threshold conditions. Third, establishing a sharp threshold requires proving the exponential ergodicity of the boundary process and carefully handling the jump-induced drift in the Lyapunov exponent.

Two recent works are particularly relevant. In [12], Hening and Nguyen developed a persistence-extinction theory for stochastic Kolmogorov systems, but their framework does not accommodate discontinuous jump processes. In [20], Tuong et al. analyzed a stochastic multi-group SVIR model with jumps; however, their study focused on a degenerate singular diffusion case and relied on a linearized threshold definition. Moreover, their stochastic comparison method is not applicable to our system due to the nonlinear coupling between strains. Consequently, neither work provides a sharp threshold condition for persistence and extinction, nor do they address dynamical bifurcation, in the context of multi-strain models with non-Gaussian Lévy noise.

To the best of our knowledge, this paper is the first to achieve the following results. First, we establish a necessary and sufficient threshold condition (the Lyapunov exponent λ) that determines whether the disease persists ($\lambda > 0$) or goes extinct ($\lambda \leq 0$) in a multi-strain epidemic model driven by Lévy jumps. Second, we provide a rigorous dynamical bifurcation analysis showing that λ acts as a transcritical D-bifurcation parameter: When λ crosses zero, the invariant measure loses stability and a new persistent measure emerges. While most prior work on stochastic bifurcation has focused on Gaussian white noise [6, 16, 23], our study shifts attention to systems driven by non-Gaussian Lévy noise, offering new insights into how discontinuous stochastic perturbations shape bifurcation phenomena [21, 26]. Third, we derive strain-specific persistence and competitive exclusion criteria, revealing how Lévy noise influences strain dominance and the possibility of coexistence.

The remainder of this paper is organized as follows. Section 2 provides the necessary preliminaries and introduces the threshold parameter λ. In Section 3, we establish conditions for the extinction and persistence of system (1.1). Section 4 is devoted to the analysis of dynamical bifurcations. Numerical simulations are presented in Section 5 to illustrate and complement the theoretical findings. Finally, Section 6 concludes the paper with a summary of our main results and a discussion of future directions.

2. Preliminaries

In the whole of this paper, let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with filtration satisfying the usual conditions (i.e., it is right continuous and $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets). As the dynamics of the removed class do not affect disease transmission, without loss of generality, we consider the following system:

$$\begin{cases} dS(t) = & [\Lambda - \dfrac{S(t)\sum_{i=1}^n \beta_i I_i(t)}{S(t)+\sum_{i=1}^n I_i(t)} - \mu S(t)]dt + \sigma_1 S(t)dB_1(t) + \int_{\mathbb{Y}} S(t_-)f_1(u)\widetilde{N}(dt,du), \\ dI_i(t) = & [\dfrac{\beta_i S(t) I_i(t)}{S(t)+\sum_{i=1}^n I_i(t)} - (\gamma_i+\mu_i+\eta_i)I_i(t)]dt + \sigma_{2i}I_i(t)dB_{2i}(t) + \int_{\mathbb{Y}} I_i(t_-)f_{2i}(u)\widetilde{N}(dt,du). \end{cases} \tag{2.1}$$

We further let $\mathbb{R}_+ = [0,+\infty)$, $\mathbb{R}_+^n = \{x = (x_1, x_2, \cdots, x_n) : x_i \geq 0, i = 1, 2, \cdots, n\}$, $\mathbb{R}_+^o = (0,+\infty)$, $\mathbb{R}_+^{n,o} = \{x = (x_1, x_2, \cdots, x_n) : x_i > 0, i = 1, 2, \cdots, n\}$. We adopt the same method as described in [20], and denote the total number of infective individuals at time $t \geq 0$ as $Z(t) = \sum_{i=1}^n I_i(t)$. Let $Y_i(t) := \frac{I_i(t)}{Z(t)}$ be the proportion of the total number of infective population at time $t \geq 0$, and then system (2.1) is transformed into the following system:

$$\begin{aligned} dS(t) &= [\Lambda - \frac{S(t)\sum_{i=1}^n \beta_i Y_i(t)Z(t)}{S(t)+Z(t)} - \mu S(t)]dt + \sigma_1 S(t)dB_1(t) + \int_{\mathbb{Y}} S(t_-)f_1(u)\widetilde{N}(dt,du), \\ dY_i(t) &= [\frac{\beta_i S(t)Y_i(t)}{S(t)+Z(t)} - (\gamma_i+\mu_i+\eta_i)Y_i - Y_i\sum_{j=1}^n(\frac{\beta_j S(t)Y_j(t)}{S(t)+Z(t)} - (\gamma_j+\mu_j+\eta_j)Y_j) + Y_i(\sum_{j=1}^n \sigma_{2j}Y_j)^2 \\ &\quad - \sigma_{2i}Y_i\sum_{j=1}^n \sigma_{2j}Y_j]dt + Y_i(\sigma_{2i} - \sum_{j=1}^n \sigma_{2j}Y_j)dB_{2i}(t) + \int_{\mathbb{Y}}[\frac{Y_i + Y_i f_{2i}(u)}{1+\sum_{i=1}^n Y_i f_{2i}(u)} - Y_i]\widetilde{N}(dt,du) \\ &\quad + \int_{\mathbb{Y}}[\frac{Y_i + Y_i f_{2i}(u)}{1+\sum_{i=1}^n Y_i f_{2i}(u)} - Y_i - Y_i f_{2i}(u) + Y_i\sum_{j=1}^n Y_j f_{2j}(u)]\nu du, \quad 1 \leq i \leq n, \\ dZ(t) &= Z(t)\sum_{i=1}^n[\frac{\beta_i S(t)Y_i(t)}{S(t)+Z(t)} - (\gamma_i+\mu_i+\eta_i)Y_i]dt + Z(t)\sum_{i=1}^n \sigma_{2i}Y_i dB_{2i}(t) \\ &\quad + \int_{\mathbb{Y}} Z(t_-)\sum_{i=1}^n f_{2i}Y_i d\widetilde{N}(dt,du), \end{aligned} \tag{2.2}$$

where $\mathbf{Y}(t) = (Y_1(t), Y_2(t), \cdots, Y_n(t))$ lies in the simplex $\Delta = \{y \in \mathbb{R}_+^n : \sum_{i=1}^n y_i = 1\}$. We will abbreviate the solution of system (2.2) as $\mathbf{W}(t) = (S(t), \mathbf{Y}(t), Z(t))$ with the initial value $\mathbf{W}(0) = \mathbf{w} = (s, \mathbf{y}, z)$.

2.1. Existence of the positive solution

For the jump diffusion coefficient, we assume that

$$\int_{\mathbb{Y}} f_1^2(u)\nu(du) < \infty, \qquad \int_{\mathbb{Y}} (1+f_1(u))^{-1}\nu(du) < \infty, \tag{2.3}$$

$$\int_{\mathbb{Y}}(\sum_{i=1}^{n} f_{2i}(u))^2\nu(du) < \infty, \quad \int_{\mathbb{Y}}(1+\sum_{i=1}^{n} f_{2i}(u))^{-1}\nu(du) < \infty. \tag{2.4}$$

Throughout this paper, assumptions (2.3) and (2.4) are always in force. These conditions guarantee the finiteness of the jump-related integrals and the positivity of solutions.

Theorem 2.1. *For any initial value* $(s, \mathbf{y}, z) \in \mathbb{R}_+ \times \Delta \times \mathbb{R}_+$*, system (2.2) admits a unique solution* $\mathbf{W}(t)$ *for* $t \geq 0$*, and the solution will remain in* $\mathbb{R}_+^o \times \Delta \times \mathbb{R}_+$*.*

Proof. For any initial value $\mathbf{w} \in \mathbb{R}_+ \times \Delta \times \mathbb{R}_+$, the coefficients of system (2.2) are locally Lipschitz, so there exists a unique local solution $\mathbf{W}(t)$ on $t \in [0, \tau_e)$, where τ_e is an explosion time. Then, we prove the local solution will remain in $\mathbb{R}_+^o \times \Delta \times \mathbb{R}_+$ for $t \in [0, \tau_e)$.

From the equation of $S(t)$ in system (2.2), we define

$$H(t) = \frac{\Lambda}{S(t)} - \frac{\sum_{i=1}^{n} \beta_i I_i(t)}{S(t) + Z(t)} - \mu.$$

Thus

$$dS(t) = S(t)H(t)dt + \sigma_1 S(t)dB_1(t) + \int_{\mathbb{Y}} S(t_-)f_1(u)\widetilde{N}(dt, du). \tag{2.5}$$

Define $U(t) = \ln S(t)$. By Itô's formula for jump-diffusion processes,

$$dU(t) = \left[H(t) - \frac{1}{2}\sigma_1^2\right]dt + \sigma_1 dB_1(t) + \int_{\mathbb{Y}} \ln(1 + f_1(u))\widetilde{N}(dt, du) + \int_{\mathbb{Y}} \left[\ln(1 + f_1(u)) - f_1(u)\right]\nu(du)dt.$$

Integrating from 0 to t yields

$$S(t) = S(0)\exp\left(\int_0^t \left[H(s) - \frac{1}{2}\sigma_1^2\right]ds + \sigma_1 B_1(t)\right) \times \mathcal{E}(J)(t), \tag{2.6}$$

where $\mathcal{E}(J)(t)$ is the stochastic exponential of the jump martingale part:

$$\mathcal{E}(J)(t) = \exp\left(\int_0^t \int_{\mathbb{Y}} \ln(1 + f_1(u))\widetilde{N}(ds, du) + \int_0^t \int_{\mathbb{Y}} \left[\ln(1 + f_1(u)) - f_1(u)\right]\nu(du)ds\right). \tag{2.7}$$

Thus, for any $S(0) \geq 0$, we have $S(t) > 0$ for $t \in (0, \tau_e)$.

From the equation of $Y_i(t)$, we denote

$$dY_i(t) = F(S, Y_i, Z)dt + G(Y_i)dB_{2i}(t) + H(Y_i),$$

where

$$\begin{aligned} F(S, Y_i, Z) = & \frac{\beta_i S(t)Y_i(t)}{S(t) + Z(t)} - (\gamma_i + \mu_i + \eta_i)Y_i - Y_i\sum_{j=1}^{n}(\frac{\beta_j S(t)Y_j(t)}{S(t) + Z(t)} - (\gamma_j + \mu_j + \eta_j)Y_j) \\ & + Y_i(\sum_{j=1}^{n}\sigma_{2j}Y_j)^2 - \sigma_{2i}Y_i\sum_{j=1}^{n}\sigma_{2j}Y_j, \end{aligned}$$

$$G(Y_i) = Y_i(\sigma_{2i} - \sum_{j=1}^{n} \sigma_{2j} Y_j),$$

$$H(Y_i) = \int_{\mathbb{Y}} [\frac{Y_i + Y_i f_{2i}(u)}{1 + \sum_{i=1}^{n} Y_i f_{2i}(u)} - Y_i - Y_i f_{2i}(u) + Y_i \sum_{j=1}^{n} Y_j f_{2j}(u)]\nu du$$
$$+ \int_{\mathbb{Y}} [\frac{Y_i + Y_i f_{2i}(u)}{1 + \sum_{i=1}^{n} Y_i f_{2i}(u)} - Y_i]\widetilde{N}(dt, du).$$

Obviously, $\sum_{i=1}^{n} F(S, Y_i, Z) = \sum_{i=1}^{n} G(Y_i) = 0$. Since $\sum_{i=1}^{n} Y_i = 1$, we have

$$\sum_{i=1}^{n} \frac{Y_i + Y_i f_{2i}(u)}{1 + \sum_{i=1}^{n} Y_i f_{2i}(u)} = 1, \ \sum_{i=1}^{n} [-Y_i f_{2i}(u) + Y_i \sum_{j=1}^{n} Y_j f_{2j}(u)] = 0.$$

Consequently, $\sum_{i=1}^{n} H(Y_i) = 0$. It follows that if $\mathbf{Y}(0) \in \Delta$, then $\mathbf{Y}(t) \in \Delta$ for all $t \in [0, \tau_e)$ almost surely. More precisely, if $Y_i(0) = 0$, the drift and diffusion terms ensure that $Y_i(t)$ cannot become negative; similarly, if $Y_i(0) = 1$, the dynamics prevent $Y_i(t)$ from exceeding 1. For nonnegativity, observe that the stochastic differential equation (SDE) for Y_i has the form $dY_i = Y_i \Phi_i dt + Y_i \Psi_i dB_{2i} + \int_{\mathbb{Y}} Y_i \Lambda_i(u) \widetilde{N}(dt, du)$, where the coefficients are bounded. This is a multiplicative noise process that vanishes at $Y_i = 0$. By the comparison theorem for jump diffusions (see [13]), $Y_i(0) \geq 0$ implies $Y_i(t) \geq 0$ for all t almost surely. Hence, $Y(t) \in \Delta$ a.s.

The SDE for $Z(t)$ can be written as

$$dZ(t) = Z(t)\Psi(t)dt + Z(t) \sum_{i=1}^{n} \sigma_{2i} Y_i dB_{2i}(t) + \int_{\mathbb{Y}} Z(t_-) \sum_{i=1}^{n} f_{2i}(u) Y_i \widetilde{N}(dt, du),$$

and the drift coefficient $\Psi(t)$ is defined as

$$\Psi(t) = \sum_{i=1}^{n} \left[\frac{\beta_i S(t) Y_i(t)}{S(t) + Z(t)} - (\gamma_i + \mu_i + \eta_i) Y_i(t) \right] - \frac{1}{2} \left(\sum_{i=1}^{n} \sigma_{2i} Y_i(t) \right)^2.$$

Applying the stochastic exponential formula yields

$$Z(t) = Z(0) \exp\left(\int_0^t \left[\Psi(s) - \frac{1}{2} (\sum_i \sigma_{2i} Y_i(s))^2 \right] ds + \int_0^t \sum_i \sigma_{2i} Y_i(s) dB_{2i}(s) \right) \times \mathcal{E}(J_Z)(t),$$

where

$$\mathcal{E}(J_Z)(t) = \exp\left(\int_0^t \int_{\mathbb{Y}} \ln\left(1 + \sum_{i=1}^{n} f_{2i}(u) Y_i(s_-) \right) \widetilde{N}(ds, du) + \right.$$
$$\left. \int_0^t \int_{\mathbb{Y}} \left[\ln\left(1 + \sum_{i=1}^{n} f_{2i}(u) Y_i(s) \right) - \sum_{i=1}^{n} f_{2i}(u) Y_i(s) \right] \nu(du) ds \right),$$

$\mathcal{E}(J_Z)(t) > 0$ almost surely because $1 + \sum_i f_{2i}(u) Y_i > 0$ by assumption. Hence, if $Z(0) > 0$, then $Z(t) > 0$ for all t almost surely. If $Z(0) = 0$, then $Z(t) = 0$ for all t.

Next, we prove that $\tau_e = \infty$. We define

$$\tau_n = \inf\{t \geq 0 : Z(t) + S(t) \geq n\}.$$

Set $\tau_\infty = \lim_{n\to\infty} \tau_n$, and it is sufficient to show that $\tau_\infty = \infty$.

For $0 < p \leq 1$, define $V(S, Z) = (S(t) + Z(t))^{1+p}$, and we have

$$\begin{aligned}
\mathcal{L}V(S(t), Z(t)) &= p(Z(t) + S(t))^p[\Lambda - \mu S(t) - \sum_{i=1}^{n}(\gamma_i + \mu_i + \eta_i)Y_i Z(t)] \\
&\quad + \frac{p(p+1)}{2}[Z(t) + S(t)]^{p-1}[(\sigma_1)^2 S^2(t) + (\sum_{i=1}^{n}\sigma_{2i}Y_i)^2 Z^2(t)] \\
&\quad + \int_{\mathbb{Y}}\{[Z(t)(1 + \sum_{i=1}^{n} f_{2i}(u)Y_i) + S(t)(1 + f_1(u))]^{p+1} - [Z(t) + S(t)]^{p+1} \\
&\quad - (p+1)(S(t) + Z(t))^p[S(t)f_1(u) + Z(t)\sum_{i=1}^{n} f_{2i}(u)Y_i]\}\nu(du) \\
&\leq (p+1)[Z(t) + S(t)]^p[\Lambda - \overline{\mu}(S(t) + Z(t))] + \frac{p(p+1)}{2}\max\{\sigma_1^2, (\sum_{i=1}^{n}\sigma_{2i})^2\}(Z(t) + S(t))^{p+1} \\
&\quad + [S(t) + Z(t)]^{p+1}\int_{\mathbb{Y}}\{[1 + m_s f_1(u) + m_z(\sum_{i=1}^{n} f_{2i}(u))]^{p+1} \\
&\quad - 1 - (p+1)[m_s f_1(u) + m_z(\sum_{i=1}^{n} f_{2i}(u))]\}\nu(du),
\end{aligned}$$

where $\overline{\mu} = \min\{\mu, \min_{1\leq i\leq n}(\gamma_i + \mu_i + \eta_i)\}$, $m_s = \dfrac{S(t)}{S(t) + Z(t)}$, and $m_z = \dfrac{Z(t)}{S(t) + Z(t)}$. Since $0 < p \leq 1$, then we have

$$(1 + v)^{p+1} \leq 1 + (p+1)v + pv^2,\ v \geq -1. \tag{2.8}$$

From (2.3), (2.4), and (2.8), we have

$$\int_{\mathbb{Y}}\{[1 + m_s f_1(u) + m_z(\sum_{i=1}^{n} f_{2i}(u))]^{p+1} - 1 - (p+1)[m_s f_1(u) + m_z(\sum_{i=1}^{n} f_{2i}(u))]\}\nu(du) \leq pc,$$

where c is a constant number. Therefore,

$$\mathcal{L}V \leq [\frac{p(p+1)}{2}\max\{\sigma_1^2, (\sum_{i=1}^{n}\sigma_{2i})^2\} + pc - (p+1)\overline{\mu}](S + Z)^{p+1} + \Lambda(p+1)[Z(t) + S(t)]^p. \tag{2.9}$$

We choose $0 < H_2 < \dfrac{p(1+p)}{2}\max\{\sigma_1^2, (\sum_{i=1}^{n}\sigma_{2i})^2\} - pc + (p+1)\overline{\mu}$, and we get

$$H_1 = \sup_{(S,Z)\in(\mathbb{R}_+,\mathbb{R}_+)}\{\mathcal{L}V(S(t), Z(t)) + H_2 V(S(t), Z(t))\} < \infty. \tag{2.10}$$

Hence,

$$\mathcal{L}V(S(t),Z(t)) \le H_1 - H_2 V(S(t),Z(t)), \quad \forall (S(t),Z(t)) \in \mathbb{R}_+^2. \tag{2.11}$$

Define $W(t) = e^{H_2 t}V(t)$. By Itô's formula,

$$dW(t) = H_2 e^{H_2 t} V dt + e^{H_2 t} dV(t) \le H_1 e^{H_2 t} dt + e^{H_2 t} dM(t),$$

where $M(t)$ is a local martingale. Integrating from 0 to $t \wedge \tau_n$ and taking expectations,

$$\mathbb{E} e^{H_2(t\wedge\tau_n)} V(S,Z) \le V(s,z) + \mathbb{E}\int_0^{(t\wedge\tau_n)} H_1 e^{H_2 s} ds \le V(s,z) + \frac{H_1}{H_2} e^{H_1 t}. \tag{2.12}$$

On the event $\{\tau_n < t\}$, we have $V(S(\tau_n), Z(\tau_n)) \ge n$ (taking $p = 1$). Since $e^{H_2(t\wedge\tau_n)}V(S(t\wedge\tau_n), Z(t\wedge\tau_n)) \ge \mathbf{1}_{\{\tau_n<t\}} e^{H_2\tau_n} V(S(\tau_n), Z(\tau_n)) \ge n\mathbf{1}_{\{\tau_n<t\}}$, taking expectations yields

$$\mathbb{E}\left[e^{H_2(t\wedge\tau_n)} V(S(t\wedge\tau_n), Z(t\wedge\tau_n))\right] \ge n\mathbb{P}\{\tau_n < t\}.$$

Rearranging gives the first inequality:

$$\mathbb{P}\{\tau_n < t\} \le \frac{1}{n}\mathbb{E}\left[e^{H_2(t\wedge\tau_n)} V(S(t\wedge\tau_n), Z(t\wedge\tau_n))\right].$$

Applying the upper bound from (2.12) yields the second inequality:

$$\mathbb{P}\{\tau_n < t\} \le \frac{1}{n}\left(V(s,z) + \frac{H_1}{H_2} e^{H_1 t}\right).$$

For fixed t, the right-hand side tends to 0 as $n \to \infty$, so $\lim_{n\to\infty} \mathbb{P}\{\tau_n < t\} = 0$. Consequently, $\mathbb{P}\{\tau_\infty = \infty\} = 1$. □

By letting $n \to \infty$ in (2.12), we obtain the following lemma.

Lemma 2.2. *There exist positive constants p, K_1, K_2 such that*

$$\mathbb{E}(S(t) + Z(t))^{1+p} \le e^{-K_1 t}(s+z)^{1+p} + K_2, \ (s,\mathbf{y},z) \in \mathbb{R}_+ \times \Delta \times \mathbb{R}_+^o, \ t \ge 0. \tag{2.13}$$

Furthermore, for any $\epsilon, H, T > 0$, there exists an $M_{H,\epsilon,T} > 0$ such that

$$\mathbb{P}\{\sup_{t\in[0,T]} \{S(t) + Z(t) < M_{H,\epsilon,T}\}\} \ge 1-\epsilon, (s,\mathbf{y},z) \in (0,H] \times \Delta \times (0,H], \ t \in [0,T]. \tag{2.14}$$

2.2. Definition of the threshold parameter

Following [7, 12, 20], we will use Lyapunov exponents to define the threshold between extinction and persistence. Consider system (2.2) on the boundary $\{(S,\mathbf{Y},Z) : S \in \mathbb{R}_+, \mathbf{Y} \in \Delta, Z = 0\}$ (that is the case when $Z(t) = 0$), and we have the following system:

$$d\hat{S}(t) = [\Lambda - \mu\hat{S}(t)]dt + \sigma_1 \hat{S}(t) dB_1(t) + \int_{\mathbb{Y}} \hat{S}(t_-) f_1(u) \widetilde{N}(dt,du),$$

$$d\hat{Y}_i(t) = [\beta_i\hat{Y}_i - (\gamma_i + \mu_i + \eta_i)\hat{Y}_i - \hat{Y}_i\sum_{j=1}^{n}\beta_j\hat{Y}_j - (\gamma_j + \mu_j + \eta_j)\hat{Y}_j) + \hat{Y}_i(\sum_{j=1}^{n}\sigma_{2j}\hat{Y}_j)^2$$
$$- \sigma_{2i}\hat{Y}_i\sum_{j=1}^{n}\sigma_{2j}\hat{Y}_j]dt + \hat{Y}_i(\sigma_{2i} - \sum_{j=1}^{n}\sigma_{2j}\hat{Y}_j)dB_{2i}(t) + \int_{\mathbb{Y}}[\frac{\hat{Y}_i + \hat{Y}_i f_{2i}(u)}{1 + \sum_{i=1}^{n}\hat{Y}_i f_{2i}(u)} - \hat{Y}_i]\widetilde{N}(dt, du) \tag{2.15}$$
$$+ \int_{\mathbb{Y}}[\frac{\hat{Y}_i + \hat{Y}_i f_{2i}(u)}{1 + \sum_{i=1}^{n}\hat{Y}_i f_{2i}(u)} - \hat{Y}_i - \hat{Y}_i f_{2i}(u) + \hat{Y}_i\sum_{j=1}^{n}\hat{Y}_j f_{2j}(u)]\nu du, \quad 1 \le i \le n.$$

Remark 1. *The proportion variables $Y_i(t) = I_i(t)/Z(t)$ are defined only when $Z(t) > 0$. On the set $\{Z(t) = 0\}$, we define $Y_i(t)$ as the left limit $\lim_{s\uparrow t} I_i(s)/Z(s)$ (which exists because the process is càdlàg). Since $Z(t)$ hits zero only at absorption (see Theorem 2.1), the dynamics on the boundary $\{Z = 0\}$ are obtained by taking the limit $Z(t) \to 0^+$ in the equations for $S(t)$ and $Y_i(t)$. This limiting procedure yields system (2.15), where $\hat{S}(t)$ and $\hat{Y}_i(t)$ represent the boundary processes. A rigorous justification using the concept of boundary measures can be found in [20, Lemma 3.2].*

Lemma 2.3. *The Markov process $(\hat{S}(t), \hat{\mathbf{Y}}(t))$ is exponentially ergodic and possesses a unique stationary distribution, denoted by $\mu^*(\cdot)$.*

Proof. The proof is divided into three parts: (i) exponential ergodicity of $\hat{S}(t)$; (ii) exponential ergodicity of $\hat{\mathbf{Y}}(t)$ on the simplex Δ; (iii) joint ergodicity of $(\hat{S}, \hat{\mathbf{Y}})$.

Part (i): Ergodicity of $\hat{S}(t)$. From 2.15, $\hat{S}(t)$ satisfies the SDE

$$d\hat{S}(t) = (\Lambda - \mu\hat{S}(t))dt + \sigma_1\hat{S}(t)dB_1(t) + \int_{\mathbb{Y}}\hat{S}(t_-)f_1(u)\widetilde{N}(dt, du).$$

This is a geometric Brownian motion with jumps. Define $U(t) = \ln\hat{S}(t)$. By Itô's formula,

$$dU(t) = \left(\frac{\Lambda}{\hat{S}(t)} - \mu - \frac{1}{2}\sigma_1^2\right)dt + \sigma_1 dB_1(t) + \int_{\mathbb{Y}}\ln(1 + f_1(u))\widetilde{N}(dt, du) + \int_{\mathbb{Y}}[\ln(1 + f_1(u)) - f_1(u)]\nu(du)dt.$$

Let $V_1(\hat{S}) = \hat{S} + \hat{S}^{-1}$. Applying the generator and using assumptions (2.3)–(2.4), one can show that there exist constants $a_1, b_1 > 0$ such that

$$\mathcal{L}V_1(\hat{S}) \le a_1 - b_1V_1(\hat{S}), \quad \forall\hat{S} > 0.$$

Moreover, the diffusion coefficient $\sigma_1\hat{S}$ is non-degenerate on compact subsets of $(0, \infty)$. By Theorem 4.1 of [22] (or Theorem 3.2 of [5]), $\hat{S}(t)$ is exponentially ergodic with a unique stationary distribution π_S on $(0, \infty)$. Specifically, there exist $M_S > 0$ and $\kappa_S > 0$ such that

$$\|\mathcal{P}_t^{\hat{S}}(\hat{s}, \cdot) - \pi_S(\cdot)\|_{\mathrm{TV}} \le M_S e^{-\kappa_S t}(1 + V_1(\hat{s})).$$

Part (ii): Ergodicity of $\hat{\mathbf{Y}}(t)$ on Δ. Recall that $\hat{\mathbf{Y}}(t)$ lives on the simplex $\Delta = \{\mathbf{y} \in \mathbb{R}_+^n : \sum_{i=1}^{n} y_i = 1\}$.

$$\begin{aligned} d\hat{Y}_i(t) = &\left[a_i\hat{Y}_i - \hat{Y}_i\sum_{j=1}^{n}a_j\hat{Y}_j + \hat{Y}_i\bar{\sigma}^2 - \sigma_i\hat{Y}_i\bar{\sigma}\right]dt \\ &+ \hat{Y}_i(\sigma_i - \bar{\sigma})\,dB_i(t) \\ &+ \int_{\mathbb{Y}}H_i(\hat{Y}, u)\widetilde{N}(dt, du) + G_i(\hat{Y})dt, \end{aligned} \tag{2.16}$$

where $a_i = \beta_i - d_i$, $d_i = \gamma_i + \mu_i + \eta_i$, $\bar{\sigma} = \sum_{j=1}^n \sigma_j \hat{Y}_j$,

$$H_i(\hat{Y}, u) = \frac{\hat{Y}_i(1 + f_i(u))}{1 + F(\hat{Y}, u)} - \hat{Y}_i, \quad F(\hat{Y}, u) = \sum_{j=1}^n \hat{Y}_j f_j(u),$$

and

$$G_i(\hat{Y}) = \int_{\mathbb{Y}} \left[H_i(\hat{Y}, u) - \hat{Y}_i f_i(u) + \hat{Y}_i F(\hat{Y}, u) \right] \nu(du)$$

is the compensated drift from jumps.

Step (ii-a): Lyapunov function. Consider the Lyapunov function $V_2(\mathbf{y}) = -\sum_{i=1}^n \ln y_i$, which tends to $+\infty$ as $\mathbf{y}$ approaches the boundary $\partial\Delta$. A direct computation (see Appendix A for details) yields

$$\mathcal{L}V_2(\mathbf{y}) = -\sum_{i=1}^n a_i + n\bar{a} - n\bar{\sigma}^2 + \bar{\sigma} \sum_{i=1}^n \sigma_{2i} + \frac{1}{2} \sum_{i=1}^n (\sigma_{2i} - \bar{\sigma})^2 + R(\mathbf{y}),$$

where $R(\mathbf{y})$ is the jump-induced remainder. Using $|\ln(1 + x) - x/(1 + x)| \leq x^2$ for $x > -1$ and assumptions (2.3), we obtain $|R(\mathbf{y})| \leq C_R$ for some constant C_R independent of $\mathbf{y}$. Since Δ is compact, there exists a constant M such that

$$\mathcal{L}V_2(\mathbf{y}) \leq M, \quad \forall \mathbf{y} \in \Delta.$$

Moreover, by the inequality $-\ln y_i \geq 1 - y_i$, one can show that there exists $\delta > 0$ (e.g., $\delta = \frac{1}{2} \min_i(\gamma_i + \mu_i + \eta_i)$) such that

$$\mathcal{L}V_2(\mathbf{y}) \leq C - \delta V_2(\mathbf{y}), \quad \forall \mathbf{y} \in \Delta^\circ,$$

where $C = M + \delta \sup_{\mathbf{y}\in\Delta} V_2(\mathbf{y}) < \infty$. This Lyapunov condition implies that the process does not explode and has a unique invariant measure.

Step (ii-b): Hörmander condition. The diffusion matrix of $\hat{\mathbf{Y}}$ is $a_{ij}(\mathbf{y}) = y_i y_j (\sigma_{2i} - \bar{\sigma})(\sigma_{2j} - \bar{\sigma})$. Its rank is $n - 1$ on Δ° because the vector $(\sigma_{2i} - \bar{\sigma})_{i=1}^n$ is orthogonal to $(1, \ldots, 1)$ and non-zero unless all σ_{2i} are equal. If all σ_{2i} are equal, the diffusion degenerates, but the jump vector fields

$$W_u(\mathbf{y}) = \sum_{i=1}^n \left(\frac{y_i(1 + f_{2i}(u))}{1 + \sum_j y_j f_{2j}(u)} - y_i \right) \frac{\partial}{\partial y_i}$$

provide the missing directions. The Lie algebra generated by $\{V_i, W_u\}$ spans the tangent space $T_{\mathbf{y}}\Delta$ at every $\mathbf{y} \in \Delta^\circ$ (see [1, Theorem 6.5.1]). Hence, the generator is hypoelliptic, and the semigroup has the strong Feller property and is topologically irreducible.

Step (ii-c): Exponential ergodicity. With the Lyapunov condition $\mathcal{L}V_2 \leq C - \delta V_2$ and the strong Feller property, Theorem 4.1 of [22] (or Theorem 3.2 of [5]) implies that $\hat{\mathbf{Y}}(t)$ is exponentially ergodic with a unique stationary distribution $\pi_{\mathbf{Y}}$ on Δ°. That is, there exist $M_Y > 0$ and $\kappa_Y > 0$ such that

$$\|\mathcal{P}_t^{\mathbf{Y}}(\mathbf{y}, \cdot) - \pi_{\mathbf{Y}}(\cdot)\|_{\mathrm{TV}} \leq M_Y e^{-\kappa_Y t}(1 + V_2(\mathbf{y})), \quad \forall \mathbf{y} \in \Delta^\circ.$$

Part (iii): Joint ergodicity of $(\hat{S}, \hat{\mathbf{Y}})$. Since $\hat{S}(t)$ and $\hat{\mathbf{Y}}(t)$ evolve independently (their driving noises are independent and the drift terms decouple on the boundary), the joint transition semigroup satisfies

$$\mathcal{P}_t(\hat{s}, \mathbf{y}; \cdot \times \cdot) = \mathcal{P}_t^{\hat{S}}(\hat{s}, \cdot) \otimes \mathcal{P}_t^{\mathbf{Y}}(\mathbf{y}, \cdot).$$

The product of two exponentially ergodic Markov processes is exponentially ergodic with mixing rate $\kappa = \min\{\kappa_S, \kappa_Y\}$. Consequently, $(\hat{S}(t), \hat{\mathbf{Y}}(t))$ admits a unique stationary distribution $\mu^* = \pi_S \times \pi_{\mathbf{Y}}$ on $\mathbb{R}_+ \times \Delta^\circ$, and there exist $M > 0$, $\kappa > 0$ such that

$$\|\mathcal{P}_t(\hat{s}, \mathbf{y}; \cdot) - \mu^*(\cdot)\|_{\mathrm{TV}} \leq Me^{-\kappa t}(1 + V_1(\hat{s}) + V_2(\mathbf{y})).$$

This completes the proof. □

Applying Itô's formula, we have

$$\begin{aligned}\frac{\ln Z(t)}{t} &= \frac{1}{t}\{\int_0^t \sum_{i=1}^n [\frac{S(t)\sum_{i=1}^n \beta_i Y_i(t)}{S(t)+Z(t)} - (\gamma_i + \mu_i + \eta_i)Y_i - \frac{1}{2}(\sum_{i=1}^n \sigma_{2i}Y_i)^2]ds + \sum_{i=1}^n \sigma_{2i}Y_i B_{2i}(t)\} \\ &\quad + \int_{\mathbb{Y}} [\ln(\sum_{i=1}^n Y_i f_{2i}(u) + 1) - \sum_{i=1}^n Y_i f_{2i}(u)]\nu du + \frac{1}{t}\int_{\mathbb{Y}} (\sum_{i=1}^n Y_i f_{2i}(u) + 1)\widetilde{N}(dt, du).\end{aligned}$$

When $Z(t)$ is small, the solution $(S(t), \mathbf{Y}(t), Z(t))$ approaches $(\hat{S}(t), \hat{\mathbf{Y}}(t), 0)$. Assume the solution of system (2.15) is $(\hat{S}^*, \hat{\mathbf{Y}}^*)$, by the exponential ergodicity established in Lemma 2.3. Specifically, for any bounded continuous function ϕ on $H \times \Delta \times \{0\}$,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \phi(S(s), \mathbf{Y}(s), 0)ds = \int_{H\times\Delta} \phi(s, \mathbf{y}, 0)\mu^*(ds, d\mathbf{y}) \quad \text{a.s.}$$

This follows from the fact that the process $(S(t), \mathbf{Y}(t))$ is exponentially ergodic when $Z(t)$ is small, and the contribution from times when $Z(t)$ is not small is negligible in the limit $t \to \infty$ (see [12, Lemma 4.3]). Taking ϕ to be the function inside the logarithm in the expression for $\ln Z(t)/t$ yields the convergence to λ, where λ is defined as

$$\lambda = \int_\Delta [\sum_{i=1}^n (\beta_i \hat{Y}_i^* - (\gamma_i + \mu_i + \eta_i)\hat{Y}_i^*) - \frac{1}{2}(\sum_{i=1}^n \sigma_{2i}\hat{Y}_i^*)^2] \cdot \mu^*(\cdot)dy + \int_{\mathbb{Y}} [\ln(\sum_{i=1}^n \hat{Y}_i^* f_{2i}(u) + 1) - \sum_{i=1}^n \hat{Y}_i^* f_{2i}(u)]\nu du. \tag{2.17}$$

Remark 2. *We note that λ is the stochastic growth rate (or Lyapunov exponent) of the total infectious population. The definition method follows the same approach as in [8, 12]. In deterministic epidemic models, the basic reproduction number R_0 determines whether an infection can invade: $R_0 > 1$ indicates growth, while $R_0 < 1$ indicates decay. In our stochastic setting with jumps, the threshold λ plays an analogous role: $\lambda > 0$ corresponds to stochastic persistence, and $\lambda < 0$ corresponds to almost-sure extinction. However, unlike R_0, which is computed from a linearized deterministic system, λ incorporates:*

- *the stationary distribution of strain proportions $\hat{\mathbf{Y}}^*$ on the disease-free boundary;*
- *a jump-induced correction term arising from Lévy noise.*

In the absence of noise ($\sigma_{2i} = 0$, $f_{2i} = 0$), λ reduces to $\sum_i \hat{Y}_i^*(\beta_i - (\gamma_i + \mu_i + \eta_i))$, and for a single strain with identical parameters, $\lambda > 0$ coincides with $R_0 > 1$. Hence, λ is a natural stochastic generalization of the basic reproduction number.

3. Persistence and extinction

3.1. Persistence

Lemma 3.1. *If $\lambda > 0$, there exist $\theta > 0$ and $C_\theta > 0$ such that*

$$\lim_{t\to\infty} \mathbb{E}Z^{-\theta} \le C_\theta$$

for any initial value $\mathbf{w} = (s, \mathbf{y}, z) \in H \times \Delta \times \mathbb{R}^o_+$.

Proof. From Lemma 2.2, there exists a compact set $H \subset \mathbb{R}^o_+$ such that the set $H\times\Delta\times\mathbb{R}^o_+$ is invariant for any initial value $\mathbf{w} = (s, \mathbf{y}, z) \in H \times \Delta \times \mathbb{R}^o_+$. Given that $H \times \Delta \times \{0\}$ is compact, the family of measures $\{\Pi^{\mathbf{w}}_t(\cdot) := \frac{1}{t}\int_0^t \mathbb{P}_{\mathbf{w}}(\mathbf{W}(s) \in \cdot)ds\}$ is tight. It is well-established that any weak limit of $\Pi^{\mathbf{w}}_t(\cdot)$ constitutes an invariant probability measure for the process $(\mathbf{W}(t))$. Since μ^* is the unique invariant measure on the invariant set $H \times \Delta \times \{0\}$, it follows that $\Pi^{\mathbf{w}}_t(\cdot)$ must converge to μ^* as $t \to \infty$. Define

$$\phi(\mathbf{W}) = \frac{S(t)\sum_{i=1}^n \beta_i Y_i(t)}{S(t) + Z(t)} - \sum_{i=1}^n (\gamma_i + \mu_i + \eta_i)Y_i - \frac{1}{2}(\sum_{i=1}^n \sigma_{2i}Y_i)^2 + \int_{\mathbb{Y}} [\ln(\sum_{i=1}^n Y_i f_{2i}(u) + 1) - \sum_{i=1}^n Y_i f_{2i}(u)]\nu du.$$

Thus, we have

$$\lim_{t\to\infty} \frac{\ln Z(t)}{t} = \lim_{t\to\infty} \int_{H\times\Delta\times\{0\}} \phi(\mathbf{w}')\Pi^{\mathbf{w}}_t(d\mathbf{w}') = \int_{H\times\Delta\times\{0\}} \phi(\mathbf{w}')\mu^*(\cdot)(d\mathbf{w}') = \lambda$$

for $\mathbf{w} \in H \times \Delta \times \{0\}$.

Moreover, since $H \times \Delta \times \{0\}$ is a compact set, there exists a $T = T(\epsilon) > 0$ such that

$$\mathbb{E}_{\mathbf{w}} \int_0^T \phi(\mathbf{W}(t))dt \ge \frac{3\lambda}{4}, \ \mathbf{w} \in H \times \Delta \times \{0\}.$$

We choose a sufficiently small $\delta > 0$ such that

$$\mathbb{E}_{\mathbf{w}} \int_0^T \phi(\mathbf{W}(t))dt \ge \frac{\lambda}{2}, \ \mathbf{w} \in H \times \Delta \times (0, \delta).$$

By Itô's formula, we have

$$\begin{aligned} dZ^\theta(t) &= \theta Z^\theta(t)[\phi(\mathbf{w}(t)) + \frac{\theta}{2}(\sum_{i=1}^n \sigma_{2i})Y_i(t)]dt + \theta Z^\theta \sum_{i=1}^n \sigma_{2i}Y_i(t)dB_{2i}(t) \\ &\quad + \theta Z^\theta \int_{\mathbb{Y}} [\ln(\sum_{i=1}^n Y_i f_{2i}(u) + 1) - \sum_{i=1}^n f_{2i}(u)Y_i]\widetilde{N}(dt, du) \qquad (3.1) \\ &\le \hat{H}Z^\theta(t)dt + \theta Z^\theta \sum_{i=1}^n \sigma_{2i}Y_i(t)dB_{2i}(t) + \theta Z^\theta \int_{\mathbb{Y}} [\ln(\sum_{i=1}^n Y_i f_{2i}(u) + 1) - \sum_{i=1}^n f_{2i}(u)Y_i]\widetilde{N}(dt, du), \end{aligned}$$

where

$$\hat{H} = \sup_{|\theta|\le 1, \mathbf{w}\in H\times\Delta\times\mathbb{R}_+^o} \{\phi(\mathbf{w}(t)) + \frac{\theta}{2}(\sum_{i=1}^{n}\sigma_{2i})Y_i(t)\} < \infty. \tag{3.2}$$

An application of Dynkin's formula yields that

$$\mathbb{E}_{\mathbf{w}} Z^{\theta}(t) \le e^{\hat{H}} z^{\theta}. \tag{3.3}$$

Let

$$\begin{aligned}\phi(T) &= -\int_0^T \phi(\mathbf{W}(t))dt - \sum_{i=1}^{n}\sigma_{2i}Y_i dB_{2i}(t) - \int_{\mathbb{Y}}(\sum_{i=1}^{n} Y_i f_{2i}(u) + 1)\widetilde{N}(dt, du)\\ &= -\ln Z(t) + \ln Z(0).\end{aligned}$$

Then

$$\mathbb{E}_{\mathbf{w}}\phi(T) = -\mathbb{E}_{\mathbf{w}}\int_0^T \phi(\mathbf{W}(t))dt \le -\frac{\lambda}{2}, \mathbf{w}\in H\times\Delta\times(0,\delta).$$

Since $\phi(\mathbf{w})$ is a bounded function, there is a $K_T > 0$ such that

$$\mathbb{E}_{\mathbf{w}}[\phi(T)]^2 \le K_T. \tag{3.4}$$

In view of (3.3) and [12], we have

$$\ln\mathbb{E}_{\mathbf{w}}(e^{\theta\phi(T)}) \le -\frac{\lambda T}{2}\theta + \theta^2 K_T \le -\frac{\lambda\theta T}{4} \text{ for } \mathbf{w}\in H\times\Delta\times(0,\delta), \text{ if } \theta = \min\{1, \frac{\lambda T}{4K_T}\}.$$

That is,

$$\mathbb{E}_{\mathbf{w}} Z^{-\theta}(T) \le z^{-\theta} e^{-\frac{\lambda\theta T}{4}} \text{ for } \mathbf{w}\in H\times\Delta\times(0,\delta). \tag{3.5}$$

In view of (3.3), we have

$$\mathbb{E}_{\mathbf{w}} Z^{-\theta}(T) \le e^{\hat{H}T} z^{-\theta} \le e^{\hat{H}T}\delta^{-\theta} \text{ for } \mathbf{w}\in H\times\Delta\times(\delta,\infty). \tag{3.6}$$

Combining (3.5) and (3.6), we have

$$\mathbb{E}_{\mathbf{w}} Z^{-\theta}(T) \le \kappa z^{-\theta} + C \text{ for } \mathbf{w}\in H\times\Delta\times\mathbb{R}_+^o, \tag{3.7}$$

where $\kappa = e^{-\frac{\lambda\theta T}{4}}$, $C = e^{\hat{H}T}\delta^{-\theta}$. By the Markov property of $\mathbf{W}(t)$, we have

$$\mathbb{E}_{\mathbf{w}} Z^{-\theta}(nT + T) \le \kappa\mathbb{E}_{\mathbf{w}} Z^{-\theta}(nT) + C \text{ for } \mathbf{w}\in H\times\Delta\times\mathbb{R}_+^o. \tag{3.8}$$

Applying (3.8), we have

$$\mathbb{E}_{\mathbf{w}} Z^{-\theta}(nT) \le \kappa^n z^{-\theta} + C\frac{1-\kappa^n}{1-\kappa}.$$

Furthermore, in view of Markov's property and (3.3), we have

$$\mathbb{E}_{\mathbf{w}} Z^{-\theta}(nT+t) \leq e^{\hat{H}} z^{\theta} \kappa^{n} z^{-\theta} + C\frac{1-\kappa^{n}}{1-\kappa} \text{ for } t \in [0,T], \tag{3.9}$$

thus

$$\limsup_{t\to\infty} \mathbb{E}_{\mathbf{w}} Z^{-\theta}(t) \leq \frac{Ce^{\hat{H}}}{1-\kappa}.$$

□

Theorem 3.2. *If $\lambda > 0$, for any $\epsilon > 0$, there exists $K_\epsilon > 1$ such that*

$$\liminf_{t\to\infty} \mathbb{P}\{K_\epsilon^{-1} \leq I_i(t) \leq K_\epsilon\} \geq 1-\epsilon \tag{3.10}$$

for $i = 1, \cdots, n$. Then the stochastic system (1.1) is persistent.

Proof. From Lemma 3.1, there exist constants $\theta > 0$ and $C_\theta > 0$ such that

$$\limsup_{t\to\infty} \mathbb{E}[Z(t)^{-\theta}] \leq C_\theta.$$

By the Markov inequality, for any $\delta > 0$, we have

$$\mathbb{P}\{Z(t)^{-\theta} \geq \delta^{-1}\} \leq \delta\, \mathbb{E}[Z(t)^{-\theta}].$$

Taking $\delta = \epsilon/(2C_\theta)$ for sufficiently large t yields

$$\mathbb{P}\{Z(t) \leq \delta^{1/\theta}\} \leq \frac{\epsilon}{2}.$$

Since $I_i(t) = Y_i(t)Z(t)$ and $Y_i(t) \in [0,1]$, we have $I_i(t) \geq Y_i(t)Z(t)$. However, the proportion $Y_i(t)$ may approach zero. To obtain a uniform lower bound, we recall that the boundary process $\hat{\mathbf{Y}}(t)$ on the simplex Δ is exponentially ergodic (Lemma 2.3) and its unique stationary distribution π has full support on Δ° (the interior of the simplex). Consequently, for any $\eta > 0$, there exists a constant $c_\eta > 0$ and a time T_η such that for all $t \geq T_\eta$,

$$\mathbb{P}\{Y_i(t) \geq c_\eta\} \geq 1 - \frac{\epsilon}{2}.$$

Combining the estimates for $Z(t)$ and $Y_i(t)$, we obtain for sufficiently large t,

$$\mathbb{P}\{I_i(t) \geq c_\eta \delta^{1/\theta}\} \geq 1-\epsilon.$$

Setting $K_\epsilon^{-1} = c_\eta \delta^{1/\theta}$ gives the desired lower bound.

From Lemma 2.2, there exist positive constants p, K_1, K_2 such that

$$\mathbb{E}[(S(t)+Z(t))^{1+p}] \leq e^{-K_1 t}(s+z)^{1+p} + K_2.$$

In particular, $\limsup_{t\to\infty} \mathbb{E}[Z(t)^{1+p}] \leq K_2$. For any $M > 0$, Chebyshev's inequality yields

$$\mathbb{P}\{Z(t) \geq M\} \leq \frac{\mathbb{E}[Z(t)^{1+p}]}{M^{1+p}} \leq \frac{K_2+1}{M^{1+p}}$$

for sufficiently large t. Choosing $M = (2(K_2+1)/\epsilon)^{1/(1+p)}$ gives $\mathbb{P}\{Z(t) \geq M\} \leq \epsilon/2$. Since $I_i(t) \leq Z(t)$, we have

$$\mathbb{P}\{I_i(t) \geq M\} \leq \frac{\epsilon}{2}.$$

Setting $K_\epsilon = M$ provides the uniform upper bound.

For the same $\epsilon > 0$, let $K_\epsilon = \max\{M,\ 1/(c_\eta \delta^{1/\theta})\}$. Then for sufficiently large t,

$$\mathbb{P}\{K_\epsilon^{-1} \leq I_i(t) \leq K_\epsilon\} \geq 1 - \epsilon.$$

Taking the limit inferior as $t \to \infty$ yields the desired inequality. □

3.2. Strain-specific persistence and competitive exclusion

In this subsection, we analyze the dynamics of individual strains. Recall that $I_i(t) = Y_i(t)Z(t)$ with $\mathbf{Y}(t) \in \Delta$. The following theorem provides strain-specific thresholds that determine the fate of each strain.

Theorem 3.3 (Strain-specific persistence)**.** *For each $i = 1, \ldots, n$, define*

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \ln I_i(t) \quad a.s.,$$

whenever the limit exists. Then:

1. *λ_i exists almost surely and is given explicitly by*

$$\lambda_i = \int_\Delta \left[\beta_i y_i - (\gamma_i + \mu_i + \eta_i) y_i - \frac{1}{2}\sigma_{2i}^2 y_i^2 - \sum_{j\neq i} \sigma_{2i}\sigma_{2j} y_i y_j\right] \mu^*(d\mathbf{y}) + \int_{\mathbb{Y}} [\ln(1 + f_{2i}(u)) - f_{2i}(u)]\, \nu(du).$$

2. *If $\lambda_i > 0$, then $\liminf_{t\to\infty} \mathbb{E}[I_i(t)] > 0$ (strain i is persistent).*
3. *If $\lambda_i < 0$, then $\lim_{t\to\infty} I_i(t) = 0$ almost surely (strain i goes extinct).*

Proof. We first compute the limit of $\frac{1}{t}\ln I_i(t)$. Applying Itô's formula to $\ln I_i(t)$ using (1.1) gives

$$\begin{aligned} d\ln I_i(t) = &\left[\frac{\beta_i S(t)}{S(t) + Z(t)} - (\gamma_i + \mu_i + \eta_i) - \frac{1}{2}\sigma_{2i}^2\right] dt + \sigma_{2i} dB_{2i}(t) \\ &+ \int_{\mathbb{Y}} \ln(1 + f_{2i}(u)) \widetilde{N}(dt, du) + \int_{\mathbb{Y}} [\ln(1 + f_{2i}(u)) - f_{2i}(u)]\, \nu(du) dt. \end{aligned}$$

Integrating from 0 to t and dividing by t yields

$$\frac{1}{t}\ln\frac{I_i(t)}{I_i(0)} = \frac{1}{t}\int_0^t \left[\frac{\beta_i S(s)}{S(s) + Z(s)} - (\gamma_i + \mu_i + \eta_i) - \frac{1}{2}\sigma_{2i}^2\right] ds + \frac{1}{t} M_i(t) + \int_{\mathbb{Y}} [\ln(1 + f_{2i}(u)) - f_{2i}(u)]\, \nu(du),$$

where $M_i(t)$ is a martingale (combining the Brownian and compensated jump terms). By the strong law of large numbers for martingales, $M_i(t)/t \to 0$ almost surely as $t \to \infty$.

A more convenient approach uses the dynamics of Y_i. Since $I_i = Y_i Z$, we have $\ln I_i = \ln Y_i + \ln Z$. The limit of $\frac{1}{t}\ln Z$ is λ by the analysis in Section 2.2. For $\frac{1}{t}\ln Y_i$, we apply Itô's formula to $\ln Y_i$ using (2.2). After simplification (see Appendix B for detailed algebra), we obtain

$$\frac{1}{t}\ln\frac{Y_i(t)}{Y_i(0)} = \frac{1}{t}\int_0^t \left[\frac{\beta_i S}{S+Z} - (\gamma_i+\mu_i+\eta_i) - \sum_{j=1}^n \left(\frac{\beta_j S Y_j}{S+Z} - (\gamma_j+\mu_j+\eta_j)Y_j\right) + \bar{\sigma}^2 - \sigma_{2i}\bar{\sigma} - \frac{1}{2}(\sigma_{2i}-\bar{\sigma})^2\right] ds$$
$$+ \frac{1}{t}N_i(t),$$

where $N_i(t)$ is a martingale and $\bar{\sigma} = \sum_{j=1}^n \sigma_{2j}Y_j$.

Adding $\frac{1}{t}\ln Z$ to $\frac{1}{t}\ln Y_i$ and taking the limit, all terms involving $S/(S+Z)$ cancel. Indeed,

$$\frac{\beta_i S}{S+Z} - \sum_{j=1}^n \frac{\beta_j S Y_j}{S+Z} = \frac{S}{S+Z}\left(\beta_i - \sum_{j=1}^n \beta_j Y_j\right),$$

and when combined with the corresponding terms from $\frac{1}{t}\ln Z$, the factor $\frac{S}{S+Z}$ cancels because $\sum_i Y_i = 1$. The detailed algebra is provided in Appendix B.

To evaluate the limit of $\frac{1}{t}\int_0^t \frac{S(s)}{S(s)+Z(s)}ds$, we note that by the exponential ergodicity of the boundary process (Lemma 2.3), the occupation measure of $(S(t), \mathbf{Y}(t))$ converges weakly to μ^* as $t \to \infty$. Consequently,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \frac{S(s)}{S(s)+Z(s)}ds = \int_\Delta \mu^*(d\mathbf{y}) = 1,$$

because on the boundary $\{Z = 0\}$, we have $S/(S+Z) = 1$, and the contribution from the transient phase is asymptotically negligible.

Therefore,

$$\lim_{t\to\infty}\frac{1}{t}\ln I_i(t) = \lambda_i \quad \text{a.s.},$$

with λ_i given by the integral formula. The existence of the limit follows from the exponential ergodicity of the boundary process (Lemma 2.3) and the convergence of occupation measures.

If $\lambda_i > 0$, then $I_i(t) \to \infty$ almost surely, which in particular implies $\liminf_{t\to\infty}\mathbb{E}[I_i(t)] > 0$. If $\lambda_i < 0$, then $I_i(t) \to 0$ exponentially fast almost surely. □

Theorem 3.4 (Competitive exclusion)**.** *Assume $\lambda > 0$ (so the total disease persists). Let $\lambda_{\max} = \max_{1\le i\le n}\lambda_i$ and define $\mathcal{D} = \{i : \lambda_i = \lambda_{\max}\}$. Then:*

1. *For any $i \notin \mathcal{D}$, $\lim_{t\to\infty} I_i(t) = 0$ almost surely.*
2. *For any $i \in \mathcal{D}$ and $j \notin \mathcal{D}$, $\lim_{t\to\infty}\frac{I_i(t)}{I_j(t)} = \infty$ almost surely.*
3. *If $|\mathcal{D}| \ge 2$, then the proportions $\{Y_i(t)\}_{i\in\mathcal{D}}$ converge weakly to a positive stationary distribution supported on the face*

$$\left\{\sum_{i\in\mathcal{D}} Y_i = 1,\ Y_k = 0 \text{ for } k \notin \mathcal{D}\right\}.$$

In particular, strains with equal maximal thresholds coexist.

Proof. Consider the ratio $I_i(t)/I_j(t)$ for two strains i and j. Since $I_i/I_j = Y_i/Y_j$, we analyze $\ln(Y_i/Y_j)$. Applying Itô's formula to $\ln Y_i$ and $\ln Y_j$ from (2.2) and subtracting, we obtain

$$
\begin{aligned}
d\ln\frac{Y_i}{Y_j} = \Bigg[&\frac{(\beta_i-\beta_j)S}{S+Z} - ((\gamma_i+\mu_i+\eta_i)-(\gamma_j+\mu_j+\eta_j)) \\
&- \sigma_{2i}\bar{\sigma} + \sigma_{2j}\bar{\sigma} - \frac{1}{2}(\sigma_{2i}-\bar{\sigma})^2 + \frac{1}{2}(\sigma_{2j}-\bar{\sigma})^2\Bigg]dt \\
&+ (\sigma_{2i}-\sigma_{2j})dB_{2i}(t) + \text{jump terms.}
\end{aligned}
$$

Using $(\sigma_{2i}-\bar{\sigma})^2 = \sigma_{2i}^2 - 2\sigma_{2i}\bar{\sigma} + \bar{\sigma}^2$, the terms involving $\bar{\sigma}$ cancel:

$$
-\sigma_{2i}\bar{\sigma} + \sigma_{2j}\bar{\sigma} - \frac{1}{2}(-2\sigma_{2i}\bar{\sigma} + 2\sigma_{2j}\bar{\sigma}) = -\sigma_{2i}\bar{\sigma} + \sigma_{2j}\bar{\sigma} + \sigma_{2i}\bar{\sigma} - \sigma_{2j}\bar{\sigma} = 0.
$$

Thus the drift simplifies to

$$
\frac{(\beta_i-\beta_j)S}{S+Z} - ((\gamma_i+\mu_i+\eta_i)-(\gamma_j+\mu_j+\eta_j)) - \frac{1}{2}(\sigma_{2i}^2 - \sigma_{2j}^2).
$$

Now consider the limit of $\frac{1}{t}\int_0^t \frac{S(s)}{S(s)+Z(s)}ds$. By the exponential ergodicity of the boundary process (Lemma 2.3) and the comparison theorem for jump diffusions, the occupation measure of $(S(t), \mathbf{Y}(t))$ converges weakly to μ^* when $Z(t)$ is small. A standard argument (see [12, Lemma 4.3]) shows that

$$
\lim_{t\to\infty}\frac{1}{t}\int_0^t \frac{S(s)}{S(s)+Z(s)}ds = \int_\Delta \mu^*(d\mathbf{y}) = 1,
$$

because on the boundary $\{Z = 0\}$, we have $S/(S+Z) = 1$, and the contribution from times when Z is not small is asymptotically negligible.

Integrating $d\ln(Y_i/Y_j)$ from 0 to t, dividing by t, and taking the limit as $t\to\infty$, the martingale terms vanish almost surely. Hence,

$$
\lim_{t\to\infty}\frac{1}{t}\ln\frac{Y_i(t)}{Y_j(t)} = (\beta_i-\beta_j) - ((\gamma_i+\mu_i+\eta_i)-(\gamma_j+\mu_j+\eta_j)) - \frac{1}{2}(\sigma_{2i}^2-\sigma_{2j}^2) + \int_{\mathbb{Y}}\left[\ln\frac{1+f_{2i}(u)}{1+f_{2j}(u)} - (f_{2i}(u)-f_{2j}(u))\right]\nu(du).
$$

The right-hand side equals $\lambda_i - \lambda_j$, which follows from comparing with the definition of λ_i and noting that on the boundary $\{Z = 0\}$, the integral against μ^* reduces to evaluation at the deterministic limit $\hat{\mathbf{Y}}^*$.

Now we prove each statement.

Statement (1): If $i \notin \mathcal{D}$, choose any $k \in \mathcal{D}$. Then

$$
\lim_{t\to\infty}\frac{1}{t}\ln\frac{Y_i(t)}{Y_k(t)} = \lambda_i - \lambda_k < 0 \quad \text{a.s.}
$$

Thus $Y_i(t)/Y_k(t) \to 0$ exponentially fast almost surely. Since $\sum_i Y_i = 1$ and $Y_k(t)$ does not converge to zero (because $\lambda_k = \lambda_{\max} > 0$ implies strain k persists), we conclude $Y_i(t) \to 0$ a.s., and consequently $I_i(t) = Y_i(t)Z(t) \to 0$ a.s.

Statement (2): For $i \in \mathcal{D}$ and $j \notin \mathcal{D}$, we have $\lambda_i - \lambda_j > 0$, so $Y_i(t)/Y_j(t) \to \infty$ a.s. Since $Z(t) > 0$ eventually, $I_i(t)/I_j(t) = Y_i(t)/Y_j(t) \to \infty$ a.s.
Statement (3): If $\lambda_i = \lambda_j = \lambda_{\max}$, then

$$\lim_{t\to\infty} \frac{1}{t} \ln \frac{Y_i(t)}{Y_j(t)} = 0 \quad \text{a.s.},$$

which implies that neither strain dominates the other exponentially. In this case, the ratio $Y_i(t)/Y_j(t)$ converges to a positive random variable. A more refined analysis using the martingale central limit theorem (see [12, Section 5]) shows that the occupation measure of $(Y_i(t), Y_j(t))$ converges weakly to a nontrivial stationary distribution supported on $(0, 1)$. This distribution is characterized by the restriction of μ^* to the face of Δ, where $Y_k = 0$ for $k \notin \mathcal{D}$ and $\sum_{k\in\mathcal{D}} Y_k = 1$. Thus, strains with equal maximal thresholds coexist. □

Remark 3. *The competitive exclusion principle generalizes classical results from deterministic multi-strain models to the stochastic setting with Lévy jumps. The strain-specific thresholds λ_i incorporate both transmission advantages and noise-induced effects. In particular, a strain with a higher transmission rate β_i may still be outcompeted if it is more sensitive to environmental fluctuations (larger σ_{2i} or unfavorable jump distributions). When $\lambda_i = \lambda_j$ for $i \neq j$, the coexistence is robust to small perturbations in the parameters.*

3.3. Extinction

We consider the following system:

$$\begin{aligned} d\overline{I}_i^{\epsilon}(t) = {} & [\beta_i \frac{(\hat{S}^*(t) + \epsilon)\overline{I}_i(t)}{\hat{S}^*(t) + \epsilon + \sum_{i=1}^{n} \overline{I}_i^{\epsilon}(t)} - (\gamma_i + \mu_i + \eta_i)\overline{I}_i^{\epsilon}(t)]dt + \sigma_{2i}\overline{I}_i^{\epsilon}(t)dB_{2i}(t) \\ & + \int_{\mathbb{Y}} \overline{I}_i^{\epsilon}(t_-) f_{2i}(u)\widetilde{N}(dt, du), \end{aligned} \tag{3.11}$$

where $\hat{S}^*$ is the solution of $\hat{S}(t)$ in system (2.15). Let $\overline{\mathbf{I}}^{\epsilon} = (I_1^{\epsilon}, I_2^{\epsilon}, \cdots, I_n^{\epsilon})$ be the solution of the system (3.11).

Lemma 3.5. *There exists*

$$\lim_{t\to\infty} \frac{\ln |\overline{\mathbf{I}}^{\epsilon}(\mathbf{t})|}{t} = \lambda_{\epsilon} \ \ a.s. \tag{3.12}$$

for any initial value $(i_1, i_2, \cdots, i_n) \in (0, \infty)^n$.

Proof. Denote $\overline{Z}^{\epsilon}(t) = \sum_{i=1}^{n} \overline{I}_i^{\epsilon}(t)$ and $\overline{Y}_i^{\epsilon}(t) = \frac{\overline{I}_i^{\epsilon}(t)}{\overline{Z}^{\epsilon}(t)}$, and we transform system (3.11) into the following system:

$$\begin{aligned} d\overline{\mathbf{Y}}^{\epsilon}(t) &= \overline{f}_i(\hat{S}^* + \epsilon, \overline{\mathbf{Y}}^{\epsilon})dt + \overline{g}_i(\overline{\mathbf{Y}}^{\epsilon})dB_{2i}(t) + \int_{\mathbb{Y}} h(\overline{\mathbf{Y}}^{\epsilon})\widetilde{N}(dt, du), \\ d\overline{Z}^{\epsilon}(t) &= \overline{Z}^{\epsilon}(t) \sum_{i=1}^{n} [\frac{\beta_i(\hat{S}^* + \epsilon)\overline{Y}_i^{\epsilon}(t)}{\hat{S}^* + \epsilon + \overline{Z}^{\epsilon}(t)} - (\gamma_i + \mu_i + \eta_i)\overline{Y}_i^{\epsilon}]dt + \overline{Z}^{\epsilon}(t) \sum_{i=1}^{n} \sigma_{2i}\overline{Y}_i^{\epsilon} dB_{2i}(t) \end{aligned} \tag{3.13}$$

$$+\int_{\mathbb{Y}}\overline{Z}^{\epsilon}(t_{-})\sum_{i=1}^{n}f_{2i}(u)\overline{Y}_i^{\epsilon}d\widetilde{N}(dt,du).$$

From Lemma 2.2 and [22], system (3.13) has a unique invariant probability measure denoted as μ^{ϵ}. Define the random occupation measure

$$\Pi_t^{\epsilon}(\cdot):=\frac{1}{t}\int_0^t \mathbf{1}_{\{\overline{\mathbf{Y}}^{\epsilon}(s)\in\cdot\}}ds.$$

By the exponential ergodicity of $\overline{\mathbf{Y}}^{\epsilon}(t)$ (which follows from the same argument as in Lemma 2.3, since the perturbation $\epsilon>0$ keeps the process away from the boundary), Π_t^{ϵ} converges weakly to the unique stationary distribution μ^{ϵ} on Δ almost surely as $t\to\infty$. That is, for any bounded continuous function h on Δ,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t h(\overline{\mathbf{Y}}^{\epsilon}(s))ds=\int_{\Delta}h(\mathbf{y})\pi_Y^{\epsilon}(d\mathbf{y})\quad \text{a.s.}$$

Now consider the expression for $\ln\overline{Z}^{\epsilon}(t)$. Applying Itô's formula to $\ln\overline{Z}^{\epsilon}(t)$ yields

$$\begin{aligned}
\lim_{t\to\infty}\frac{\ln\overline{Z}^{\epsilon}(t)}{t}&=\lim_{t\to\infty}\frac{\ln\overline{Z}^{\epsilon}(0)}{t}+\lim_{t\to\infty}\frac{1}{t}\sum_{i=1}^{n}\sigma_{2i}\overline{Y}_i^{\epsilon}B_{2i}(t)\\
&+\lim_{t\to\infty}\frac{1}{t}\int_0^t\sum_{i=1}^{n}\Big[\frac{(\hat{S}^*+\epsilon)\sum_{i=1}^{n}\beta_i\overline{Y}_i^{\epsilon}(t)}{\hat{S}^*+\epsilon+\overline{Z}^{\epsilon}(t)}-(\gamma_i+\mu_i+\eta_i)\overline{Y}_i^{\epsilon}-\frac{1}{2}(\sum_{i=1}^{n}\sigma_{2i}\overline{Y}_i^{\epsilon})^2\Big]ds\\
&+\int_{\mathbb{Y}}[\ln(\sum_{i=1}^{n}\overline{Y}_i^{\epsilon}f_{2i}(u)+1)-\sum_{i=1}^{n}\overline{Y}_i^{\epsilon}f_{2i}(u)]\nu du+\frac{1}{t}\int_{\mathbb{Y}}(\sum_{i=1}^{n}\overline{Y}_i^{\epsilon}f_{2i}(u)+1)\widetilde{N}(dt,du)\\
&=\int_0^{\infty}\sum_{i=1}^{n}\Big[\frac{(\hat{S}^*+\epsilon)\sum_{i=1}^{n}\beta_i\overline{Y}_i^{\epsilon}(t)}{\hat{S}^*+\epsilon+\overline{Z}^{\epsilon}(t)}-(\gamma_i+\mu_i+\eta_i)\overline{Y}_i^{\epsilon}-\frac{1}{2}(\sum_{i=1}^{n}\sigma_{2i}\overline{Y}_i^{\epsilon})^2\Big]\Pi_t^{\epsilon}d\mathbf{y}\\
&+\int_{\mathbb{Y}}[\ln(\sum_{i=1}^{n}\overline{Y}_i f_{2i}(u)+1)-\sum_{i=1}^{n}\overline{Y}_i f_{2i}(u)]\nu du\\
&=:\lambda_{\epsilon}.
\end{aligned}$$

The convergence of λ_{ϵ} to λ as $\epsilon\to 0^+$ follows from the continuity of the stationary distribution with respect to the perturbation and the fact that $\mu^{\epsilon}\to\mu^*$ weakly. □

Lemma 3.6. *If $\lambda\le 0$, then $\mathbf{W}(t)$ has no invariant probability measure on $H\times\Delta\times(0,\infty)$.*

Proof. From equation (3.11), $I_i(t)\le\overline{I}_i^{\epsilon}(t)$ with the same initial value. By the comparison theorem, $\mathbf{I}(t)\le\overline{\mathbf{I}}^{\epsilon}(t)$. From Lemma 3.5, we have

$$\limsup_{t\to\infty}\frac{\ln Z(t)}{t}=\limsup_{t\to\infty}\frac{\ln|\mathbf{I}(\mathbf{t})|}{t}\le\lim_{\epsilon\to 0}\limsup_{t\to\infty}\frac{\ln|\overline{\mathbf{I}}^{\epsilon}(\mathbf{t})|}{t}=\lambda.$$

Thus,

$$\mathbb{P}\{\limsup_{t\to\infty}\frac{\ln Z(t)}{t}\le\lambda\}=1.$$

Now we assume $\mathbf{W}(t)$ has an ergodic probability measure on $H \times \Delta \times (0, \infty)$, denoted by m. Let $\widetilde{W} = (\widetilde{S}(t), \widetilde{\mathbf{Y}}(t), Z(t))$ be a stationary solution to (1.1) whose distribution is m. By the ergodicity of $\widetilde{W}$, we have

$$\liminf_{t\to\infty} \int_0^t \frac{\widetilde{S}(s)\sum_{i=1}^n \beta_i \widetilde{Y}\widetilde{Z}(s)}{\widetilde{S}(s) + \widetilde{Z}(s)} ds > 0. \tag{3.14}$$

Define the stopping time $\xi = \inf\{t > 0 : \widetilde{S}(t) \le \hat{S}(t)\}$, and we claim that $\mathbb{P}\{\xi = \infty\} = 0$. Indeed, if $\xi = \infty$ almost surely, we have

$$\limsup_{t\to\infty} \frac{S(t)}{t} \le \limsup_{t\to\infty} \frac{1}{t}\int_0^t (\Lambda - \mu)\widetilde{S}(s)ds - \liminf_{t\to\infty} \frac{1}{t}\int_0^t \frac{\widetilde{S}(s)\sum_{i=1}^n \beta_i I_i(s)}{\widetilde{S}(s) + \sum_{i=1}^n I_i(s)} ds < 0.$$

One obtains a contradiction because the infection term is nonnegative, forcing $\widetilde{S}(t)$ to be bounded above by a process that decays to zero, which cannot sustain a positive stationary measure. Therefore, $\mathbb{P}\{\xi < 0\} > 0$.

By the strong Markov property of $\widetilde{\mathbf{W}}(t)$, we have

$$\lim_{t\to\infty} \frac{\ln \widetilde{Z}(t)}{t} \le \lambda \le 0 \;\; a.s., \tag{3.15}$$

this contradicts the assumption that m is supported on $\{Z > 0\}$. Hence, no such invariant measure exists when $\lambda \le 0$. □

Theorem 3.7. *If $\lambda < 0$, then for any initial value,*

$$\mathbb{P}\{\lim_{t\to\infty} \frac{\ln Z(t)}{t} = \lambda < 0\} = 1. \tag{3.16}$$

Proof. From Lemma 2.2, the process $(S(t), \mathbf{Y}(t), Z(t))$ is tight in $\mathbb{R}_+^n \times \Delta \times \mathbb{R}_+$. Thus, the occupation measure

$$\Pi(\cdot) := \frac{1}{t}\int_0^t \mathbb{P}\{(S(t), \mathbf{Y}(t), Z(t)) \in \cdot\} ds$$

is tight in $\mathbb{R}_+ \times \Delta \times \mathbb{R}_+$. The tightness implies that any weak limit of $\Pi(\cdot)$ is an invariant measure of the process $(\mathbf{S}(t), \mathbf{Y}(t), Z(t))$ as $t \to \infty$. Thus, $\Pi(\cdot)$ converges weakly to $\mu^* \times \delta$ as $t \to \infty$. Thus, we have

$$\begin{aligned}
\lim_{t\to\infty} \frac{d\ln Z(t)}{t} &= \lim_{t\to\infty} \frac{1}{t}\{\int_0^t \sum_{i=1}^n [\frac{S(t)\sum_{i=1}^n \beta_i Y_i(t)}{S(t) + Z(t)} - (\gamma_i + \mu_i + \eta_i)Y_i - \frac{1}{2}(\sum_{i=1}^n \sigma_{2i} Y_i)^2] ds + \sum_{i=1}^n \sigma_{2i} Y_i B_{2i}(t)\} \\
&\quad + \int_{\mathbb{Y}} [\ln(\sum_{i=1}^n Y_i f_{2i}(u) + 1) - \sum_{i=1}^n Y_i f_{2i}(u)] \nu du + \frac{1}{t}\int_{\mathbb{Y}} (\sum_{i=1}^n Y_i f_{2i}(u) + 1) \widetilde{N}(dt, du) \\
&= \lim_{t\to\infty} \int_{\Delta} \sum_{i=1}^n [\frac{\hat{S}(t)\sum_{i=1}^n \beta_i \overline{Y}_i(t)}{\hat{S}(t) + \overline{Z}(t)} - (\gamma_i + \mu_i + \eta_i)\overline{Y}_i - \frac{1}{2}(\sum_{i=1}^n \sigma_{2i} \overline{Y}_i)^2] \Pi_t d\mathbf{y} \\
&\quad + \int_{\mathbb{Y}} [\ln(\sum_{i=1}^n \overline{Y}_i f_{2i}(u) + 1) - \sum_{i=1}^n \overline{Y}_i f_{2i}(u)] \nu du
\end{aligned}$$

$$= \int_{\Delta} [\sum_{i=1}^{n} (\beta_i \hat{Y}_i - (\gamma_i + \mu_i + \eta_i)\hat{Y}_i) - \frac{1}{2}(\sum_{i=1}^{n} \sigma_{2i}\hat{Y}_i)^2] \cdot \mu^*(\cdot)dy$$
$$+ \int_{\mathbb{Y}} [\ln(\sum_{i=1}^{n} \hat{Y}_i f_{2i}(u) + 1) - \sum_{i=1}^{n} \hat{Y}_i f_{2i}(u)] v du$$
$$= \lambda.$$

The proof of the theorem is now complete. □

Theorem 3.8. *If* $\lambda = 0$*, then*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{E}Z(s)ds = 0.$$

Proof. From Lemma 3.6, we show that if $\lambda = 0$, there is no invariant probability measure on $H\times\Delta\times\mathbb{R}_+^o$, and the occupation measure $\Pi_t := \frac{1}{t}\int_0^t \mathbb{P}\{\mathbf{W}(s) \in \cdot\}ds$ is tight on $H \times \Delta \times \mathbb{R}_+^o$. Then it must converge to the unique invariant measure μ^*. Moreover, the uniform boundedness and weak convergence of Π_t implies that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{E}Z(s) = \int_{H\times\Delta\times\mathbb{R}_+} z(\mu^* \times \delta)(ds, dy, dz) = 0.$$

The proof is complete. □

4. Dynamic stochastic bifurcation

Definition 4.1 (Dynamical bifurcation [2])**.** *Dynamical bifurcation concerns a family of random dynamical systems depending on a parameter a, each possessing an invariant measure* μ_a*. If there exists a critical value* a_D *such that in any neighborhood of* a_D*, there exists another parameter value a with a corresponding invariant measure* $\mu_a \neq \nu_a$ *satisfying* $\nu_a \to \mu_a$ *as* $a \to a_D$*. Then* a_D *is called a dynamical bifurcation point.*

Theorem 4.2. *System* (2.2) *undergoes a dynamical bifurcation as the parameter* λ *crosses* 0.

Proof. According to Lemma 3.6, if $\lambda \leq 0$, there exists a unique stable invariant measure $\mu^* \times \delta$, where δ is the Dirac measure concentrated at $\{0\}$. We now prove that when $\lambda > 0$, a new invariant measure emerges on $H \times \Delta \times (0, \infty)$. Since B_1, B_{2i}, and N are independent, the diffusion is non-degenerate. The existence of an invariant probability measure is equivalent to positive recurrence.

From Theorem 3.2, there exists a constant $M > 0$ such that

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t Z(s)\, ds \geq M,$$

and from Lemma 2.2, we have

$$\limsup_{t\to\infty} \mathbb{E}[(S(t) + Z(t))^{1+p}] \leq K < \infty. \tag{4.1}$$

For $0 < \zeta_1 < M < \zeta_2 < \infty$, applying Hölder's inequality yields

$$\frac{1}{t}\int_0^t \mathbb{E}[\mathbf{1}_{\{Z(s)\geq\zeta_1\}}Z(s)]ds \leq \mathbb{E}\left[\left(\frac{1}{t}\int_0^t \mathbf{1}_{\{Z(s)\geq\zeta_1\}}ds\right)^{\frac{p}{1+p}}\left(\frac{1}{t}\int_0^t (Z(s))^{1+p}ds\right)^{\frac{1}{1+p}}\right]. \tag{4.2}$$

From (4.2) and Fatou's lemma, we obtain

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}[\mathbf{1}_{\{Z(s)>\zeta_1\}}]ds &\geq \frac{\left(\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}[\mathbf{1}_{\{Z(s)>\zeta_1\}}Z(s)]ds\right)^{\frac{1+p}{p}}}{\left(\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}[(Z(s))^{1+p}]ds\right)^{\frac{1}{p}}}\\ &\geq K^{-\frac{1}{p}}\left(\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}[Z(s)]ds-\zeta_1\right)^{\frac{1+p}{p}}\\ &\geq K(M-\zeta_1)^{\frac{1+p}{p}} > 0,\end{aligned}$$

where the last inequality follows from the fact that $\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}[Z(s)]ds \geq M$.

From (4.1), we also have the following inequality:

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}[\mathbf{1}_{\{S(s)+Z(s)\geq\zeta_2\}}]ds \leq \frac{1}{\zeta_2^{1+p}}\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}[(S(s)+Z(s))^{1+p}]ds \leq \frac{K}{\zeta_2^{1+p}}. \tag{4.3}$$

Define $D = \{(x,y) : y \geq \zeta_1,\ x+y \leq \zeta_2\}$. By the strong law of large numbers and (4.3), we can choose ζ_2 sufficiently large and ζ_1 sufficiently small such that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}[\mathbf{1}_{\{(S(s),Z(s))\in D\}}]ds \geq \frac{(M-\zeta_1)^{\frac{1+p}{p}}}{K^{\frac{1}{p}}} - \frac{K}{\zeta_2^{1+p}} > 0. \tag{4.4}$$

Consider the Markov process $(S(t), Z(t))$ on the state space $O = \{(x,y) : x > 0,\ y \geq 0\}$. It follows that $(S(t), Z(t))$ possesses the Feller property. Due to the compactness of the subset D in O, an invariant probability measure π^* exists on O.

Next, we prove that π^* is an invariant probability measure of $(S(t), Z(t))$ on $\mathbb{R}_+^{2,o}$. Define $\overline{D} = \{(x,y) : x \geq \xi_1,\ y \geq \zeta_1,\ x+y \leq \zeta_2\}$, where $\xi_1 > 0$ is sufficiently small. Using (4.1), we obtain

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}[\mathbf{1}_{\{(S(s),Z(s))\in\overline{D}\}}]ds > 0.$$

This implies $\pi^*(\mathbb{R}_+^{2,o}) = 1$. Consequently, by the invariance of $\mathbb{R}_+^{2,o}$, we conclude that π^* is an invariant probability measure for $(S(t), Z(t))$ on $\mathbb{R}_+^{2,o}$.

The invariant probability measure is unique, which allows us to apply the strong law of large numbers. Therefore, when $\lambda > 0$, the solution $(S(t), \mathbf{Y}(t), Z(t))$ admits a unique invariant probability measure, denoted by π^*. Moreover,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(S(s),\mathbf{Y}(s),Z(s))\,ds = \int_{H\times\mathbb{R}_+^n\times\mathbb{R}_+} f(x,y,z)\,\pi^*(dx,dy,dz) \quad \text{a.s.}$$

This completes the proof. □

5. Numerical simulations

In this section, we perform numerical simulations to verify the theoretical results established in Sections 3 and 4. For simplicity, we consider the case of two strains ($n = 2$) in model (1.1). The simulations are designed to achieve three objectives: (i) to illustrate the sharp threshold behavior predicted by λ; (ii) to demonstrate the dynamical bifurcation as λ crosses zero; and (iii) to investigate the impact of Lévy noise parameters on the stationary distributions.

5.1. Continuous model and numerical discretization

Continuous model for $n = 2$. For two strains, system (1.1) reduces to:

$$\begin{cases} dS = \left[\Lambda - \dfrac{\beta_1 S I_1}{S + I_1 + I_2} - \dfrac{\beta_2 S I_2}{S + I_1 + I_2} - \mu S\right] dt + \sigma_1 S\, dB_1 + \displaystyle\int_{\mathbb{Y}} S_- f_1 \widetilde{N}(dt, du), \\ dI_1 = \left[\dfrac{\beta_1 S I_1}{S + I_1 + I_2} - (\gamma_1 + \mu_1 + \eta_1) I_1\right] dt + \sigma_{21} I_1 dB_{21} + \displaystyle\int_{\mathbb{Y}} I_{1-} f_{21} \widetilde{N}(dt, du), \\ dI_2 = \left[\dfrac{\beta_2 S I_2}{S + I_1 + I_2} - (\gamma_2 + \mu_2 + \eta_2) I_2\right] dt + \sigma_{22} I_2 dB_{22} + \displaystyle\int_{\mathbb{Y}} I_{2-} f_{22} \widetilde{N}(dt, du). \end{cases}$$

Numerical discretization of Lévy jumps. To simulate the stochastic differential equations with Lévy noise, we employ the Euler-Maruyama scheme with jump correction. For a general SDE of the form

$$dX(t) = F(X(t))dt + \sigma X(t) dB(t) + \int_{\mathbb{Y}} X(t_-) f(u) \widetilde{N}(dt, du),$$

the discretization over a time step Δt is given by

$$X(t + \Delta t) = X(t) + F(X(t))\Delta t + \sigma X(t) \sqrt{\Delta t}\, \zeta + X(t) \sum_{k=1}^{N_{\Delta t}} f(u_k),$$

where:

- $\zeta \sim \mathcal{N}(0, 1)$ is a standard normal random variable representing the Gaussian increment;
- $N_{\Delta t}$ is the number of jumps in the interval $[t, t + \Delta t)$, which follows a Poisson distribution with mean $\Delta t \cdot \nu(\mathbb{Y})$;
- $\{u_k\}_{k=1}^{N_{\Delta t}}$ are independent and identically distributed (i.i.d.) random variables drawn from the normalized jump size distribution $\nu(du)/\nu(\mathbb{Y})$.

In our simulations, we take $\nu(\mathbb{Y}) = 1$ for simplicity. To generate the jump sizes, we use the Janicki-Weron algorithm for α-stable distributions. Specifically, for $\alpha_i \neq 1$,

$$\xi_i = D_{\alpha_i,k,\sigma} \times \frac{\sin(\alpha_i(A + C_{\alpha_i,k}))}{(\cos A)^{1/\alpha_i}} \times \left[\frac{\cos(A - \alpha_i(A + C_{\alpha_i,k}))}{W}\right]^{(1-\alpha_i)/\alpha_i} + \mu,$$

where A is uniformly distributed on $(-\pi/2, \pi/2)$, W is exponentially distributed with mean 1, and

$$C_{\alpha_i,k} = \frac{1}{\alpha_i} \arctan(k \tan(\pi\alpha_i/2)), \quad D_{\alpha_i,k,\sigma} = \sigma[\cos(\arctan(\beta \tan(\pi\alpha_i/2)))]^{1/\alpha_i}.$$

For $\alpha_i = 1$, a separate formula is used (see [27] for details). This scheme converges strongly with order 0.5 under the finite jump intensity assumption.

Applying this scheme to our two-strain model, we obtain the following discrete system:

$$\begin{cases} S_{k+1} = S_k + \left[\Lambda - \dfrac{\beta_1 S_k I_{1,k}}{S_k + I_{1,k} + I_{2,k}} - \dfrac{\beta_2 S_k I_{2,k}}{S_k + I_{1,k} + I_{2,k}} - \mu S_k\right]\Delta t + \sigma_1 S_k \sqrt{\Delta t}\,\zeta_{1,k} + S_k \sum_{j=1}^{N_{\Delta t,k}} f_1(u_j), \\ I_{1,k+1} = I_{1,k} + \left[\dfrac{\beta_1 S_k I_{1,k}}{S_k + I_{1,k} + I_{2,k}} - (\gamma_1 + \mu_1 + \eta_1) I_{1,k}\right]\Delta t + \sigma_{21} I_{1,k} \sqrt{\Delta t}\,\zeta_{2,k} + I_{1,k} \sum_{j=1}^{N_{\Delta t,k}} f_{21}(u_j), \\ I_{2,k+1} = I_{2,k} + \left[\dfrac{\beta_2 S_k I_{2,k}}{S_k + I_{1,k} + I_{2,k}} - (\gamma_2 + \mu_2 + \eta_2) I_{2,k}\right]\Delta t + \sigma_{22} I_{2,k} \sqrt{\Delta t}\,\zeta_{3,k} + I_{2,k} \sum_{j=1}^{N_{\Delta t,k}} f_{22}(u_j), \end{cases}$$

where $\zeta_{1,k}, \zeta_{2,k}, \zeta_{3,k} \sim \mathcal{N}(0,1)$ are independent standard normal random variables, and $N_{\Delta t,k}$ is the number of jumps in the k-th interval, following a Poisson distribution with mean $\Delta t \cdot \nu(\mathbb{Y})$.

The baseline parameters are set as follows (unless otherwise specified):

$$\begin{aligned} &\Lambda = 2, \quad \mu = \mu_1 = \mu_2 = 0.1, \quad \beta_1 = 0.4, \\ &\gamma_1 = \gamma_2 = 0.1, \quad \eta_1 = 0.4, \quad \eta_2 = 0.1, \\ &\sigma_1 = \sigma_{21} = \sigma_{22} = 0.02, \quad \sigma_3 = 0.01, \\ &\alpha_1 = 1.8, \quad \alpha_2 = 1.75, \quad \alpha_3 = 1.9, \quad k = 0.9, \\ &D_1 = D_2 = D_3 = 0.02. \end{aligned}$$

Initial conditions are $S(0) = 10$, $I_1(0) = 1$, $I_2(0) = 0.5$ unless otherwise stated. All simulations use the Euler-Maruyama scheme with step size $\Delta t = 0.01$ and sufficiently long simulation times to reach stationarity.

5.2. *Extinction case ($\lambda < 0$)*

We first set the transmission rate of the second strain to $\beta_2 = 0.18$, which yields a negative threshold parameter $\lambda \approx -1.2 < 0$. Figure 1 presents the time series of the populations under this parameter configuration.

As shown in Figure 1, the susceptible population $S(t)$ increases monotonically from its initial value $S(0) = 10$ and gradually stabilizes near the disease-free equilibrium $S^* = 20$. In contrast, both infected strains $I_1(t)$ and $I_2(t)$ decay rapidly to zero. The total infected population $Z(t) = I_1(t) + I_2(t)$ exhibits clear exponential decay.

5.3. *Persistence case ($\lambda > 0$)*

We now increase the transmission rate to $\beta_2 = 0.60$, which gives $\lambda \approx 1.53 > 0$. Figure 2 shows that all populations converge to a positive stationary regime.

Figure 2 shows that $S(t)$ stabilizes around 9.5, while $I_1(t)$ and $I_2(t)$ fluctuate around positive means. The stationary distributions in Figure 2 are unimodal and supported on positive intervals, confirming the existence of a unique invariant measure on $\{Z > 0\}$ when $\lambda > 0$ (Theorem 4.1).

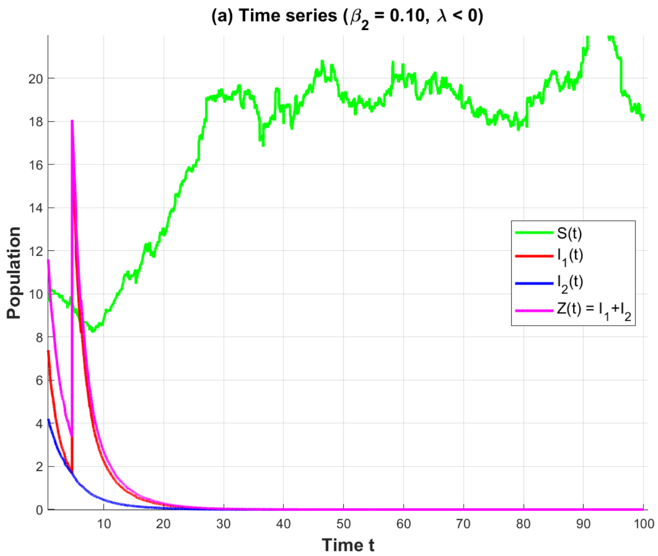
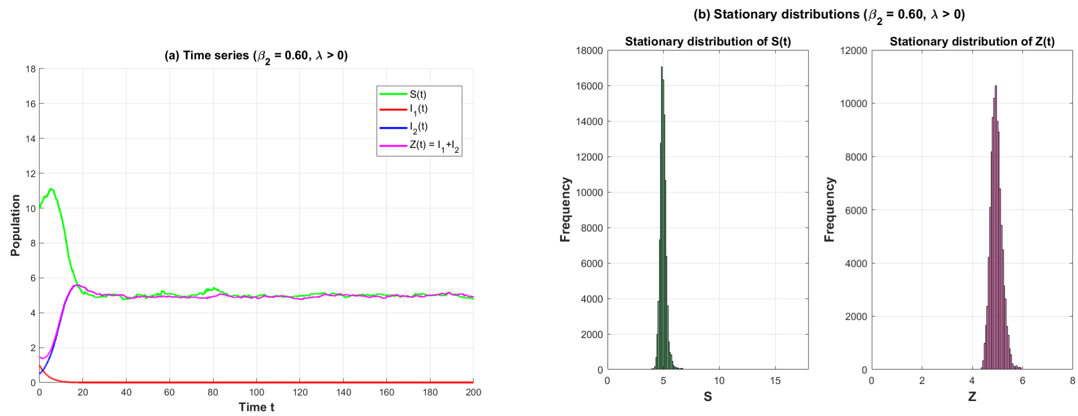


Figure 1. Time series of the susceptible population $S(t)$, the two infected strains $I_1(t)$ and $I_2(t)$, and the total infected population $Z(t) = I_1(t)+I_2(t)$ when $\lambda < 0$ ($\beta_2 = 0.18$). All infected populations decay rapidly to zero, while the susceptible population increases and stabilizes near the disease-free equilibrium $S^* = \Lambda/\mu = 20$.



(a)

(b)

Figure 2. Persistence behavior when $\lambda > 0$ ($\beta_2 = 0.60$). (a) Time series of $S(t)$, $I_1(t)$, $I_2(t)$, and $Z(t)$ converging to a positive stationary regime. (b) Stationary distributions of $S(t)$ and $Z(t)$, confirming the existence of a unique invariant measure on $\{Z > 0\}$ (Theorem 4.1).

5.4. Effect of Lévy noise intensity

To investigate how the intensity of Lévy noise affects the system dynamics, we fix $\beta_2 = 0.60$ (so that $\lambda > 0$) and vary the noise intensity parameters $D_1 = D_2 = D_3 = D$ from 0.02 to 0.06. Figure 3 displays the stationary probability density functions (PDFs) of $S(t)$ and $Z(t)$ for different noise intensities.

As shown in Figure 3, when $D = 0.02$, the distribution of $S(t)$ is concentrated around 9.5, and the distribution of $Z(t)$ is concentrated around 2.4. As D increases to 0.06, the mean of $S(t)$ increases to approximately 10.5, while the mean of $Z(t)$ decreases to approximately 1.6. This phenomenon can be explained by the fact that larger Lévy jumps cause more frequent extinctions of infected individuals, allowing the susceptible population to recover.

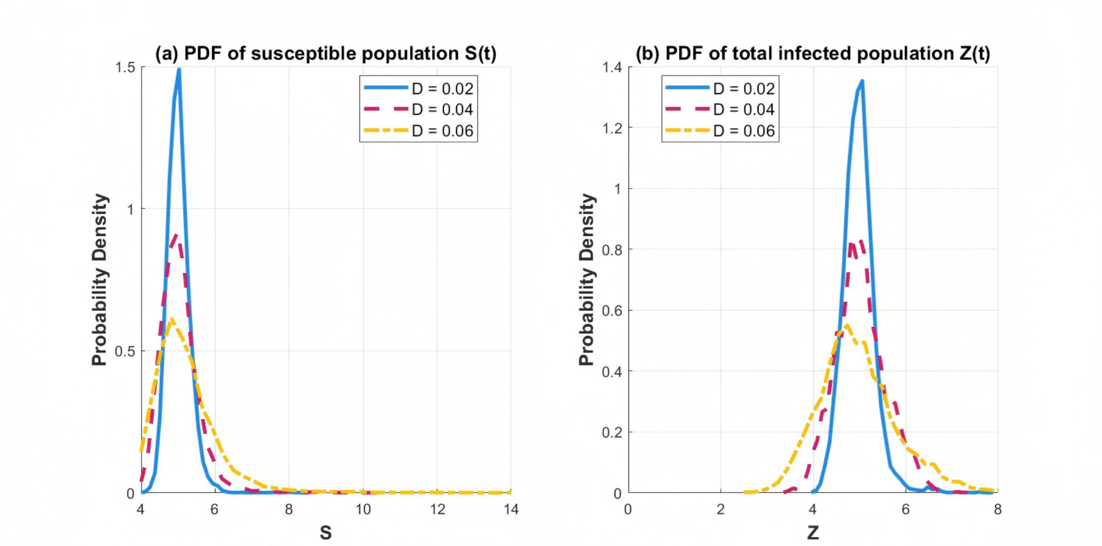


Figure 3. Effect of Lévy noise intensity on stationary distributions when $\lambda > 0$ ($\beta_2 = 0.60$). (a) PDF of the susceptible population $S(t)$. (b) PDF of the total infected population $Z(t)$. Three noise intensities are compared: $D = 0.02$ (blue solid line), $D = 0.04$ (red dashed line), and $D = 0.06$ (yellow dash-dot line). As the noise intensity increases, the PDF of $S(t)$ shifts to higher values and becomes wider, while the PDF of $Z(t)$ shifts to lower values and also widens. This indicates that larger Lévy noise promotes susceptible population growth but reduces the mean infected population.

5.5. *Effect of the stability index α*

The stability index $\alpha \in (0, 2]$ characterizes the tail heaviness of the Lévy jump distribution: Smaller α corresponds to heavier tails (more frequent large jumps), while $\alpha = 2$ corresponds to the Gaussian limit. Figure 4 displays the stationary PDFs of $S(t)$ and $Z(t)$ for $\alpha = 0.9$, 1.2, and 2.0.

As α decreases from 2.0 to 0.9, the distribution of $S(t)$ shifts to higher values and becomes wider, while the distribution of $Z(t)$ shifts to lower values and also widens. This indicates that heavier-tailed Lévy jumps (smaller α) have a damping effect on disease transmission: They increase the susceptible population while reducing the infected population.

5.6. *Stochastic D-bifurcation*

The most significant finding of this paper is the stochastic D-bifurcation at $\lambda = 0$, where the invariant measure undergoes a qualitative change. Figure 5 illustrates this bifurcation by showing the stationary distributions of $Z(t)$ for three representative values of λ.

When $\lambda < 0$ (Figure 5(a)), the distribution is concentrated at $Z = 0$, represented by the Dirac spike. This indicates that the disease goes extinct almost surely. At the critical point $\lambda = 0$ (Figure 5(b)), the distribution exhibits a power-law tail, characteristic of critical slowing down. When $\lambda > 0$ (Figure 5(c)), the distribution becomes absolutely continuous with a single peak at a positive Z value, confirming the existence of a unique invariant measure supported on $\{Z > 0\}$. This transition is a clear manifestation of a transcritical D-bifurcation.

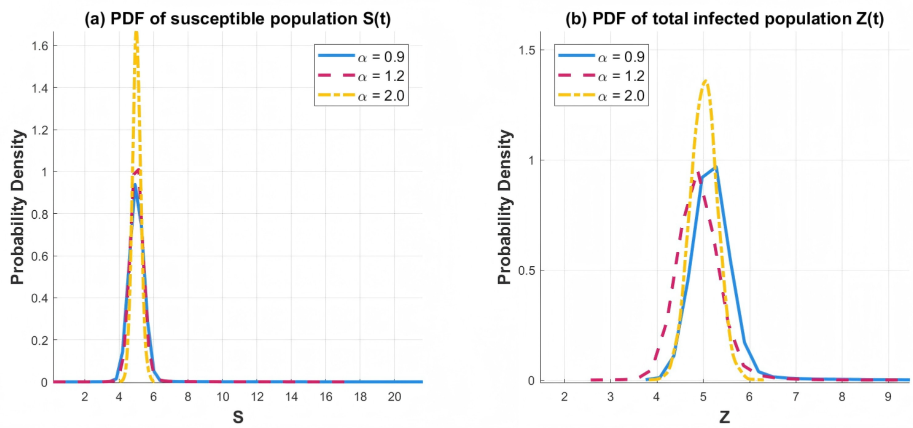


Figure 4. Effect of the stability index α on stationary distributions when $\lambda > 0$ ($\beta_2 = 0.60$). (a) PDF of $S(t)$. (b) PDF of $Z(t)$. Three stability indices are compared: $\alpha = 0.9$ (blue solid line, heavy-tailed jumps), $\alpha = 1.2$ (red dashed line, moderate tails), and $\alpha = 2.0$ (yellow dash-dot line, Gaussian limit). As α decreases (heavier tails), the PDF of $S(t)$ shifts to higher values and becomes wider, while the PDF of $Z(t)$ shifts to lower values and also widens.

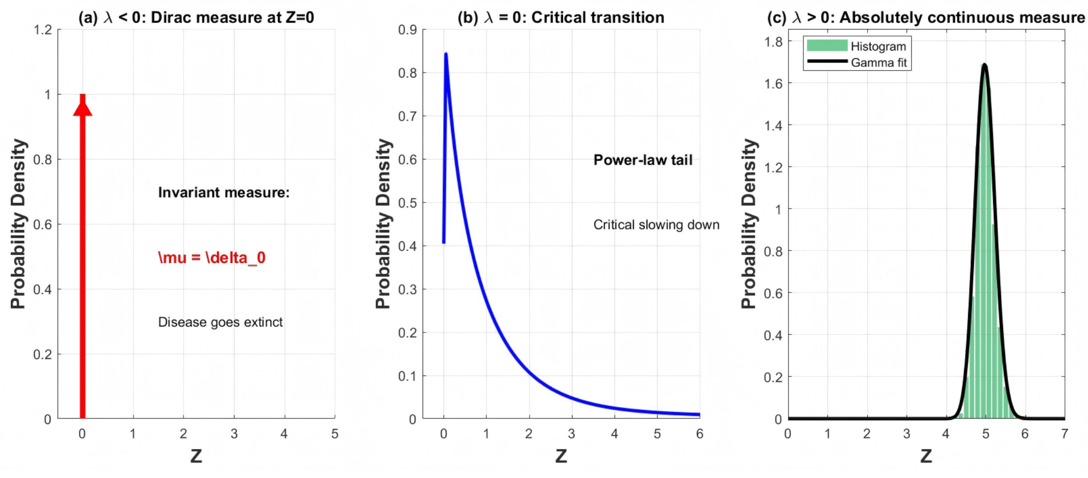


Figure 5. Stochastic D-bifurcation: Evolution of the invariant measure as λ crosses zero. (a) $\lambda < 0$ ($\beta_2 = 0.18$): The invariant measure is a Dirac measure concentrated at $Z = 0$, represented by the red spike. This indicates almost sure extinction (Theorem 3.5). (b) $\lambda = 0$ ($\beta_2 = 0.31$): The system is at the critical point. The invariant measure exhibits a power-law tail, characteristic of critical slowing down. (c) $\lambda > 0$ ($\beta_2 = 0.60$): The invariant measure becomes absolutely continuous on $\{Z > 0\}$ with a unimodal shape, confirming disease persistence (Theorem 4.1). This figure demonstrates the transcritical D-bifurcation at $\lambda = 0$.

5.7. Strain-specific dynamics and competitive exclusion

To address the multi-strain nature of the model, Figure 6 presents the dynamics of individual strains under different scenarios.

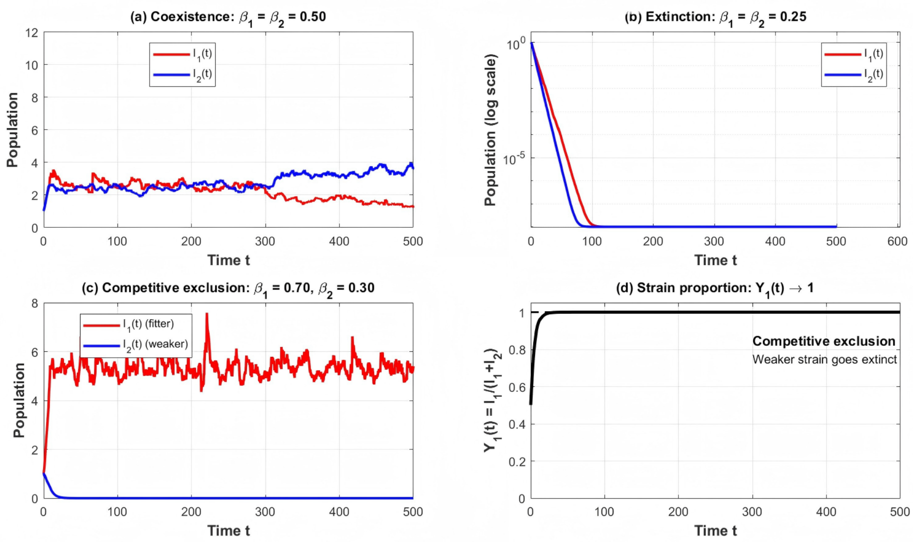


Figure 6. Strain-specific dynamics and competitive exclusion. (a) Coexistence: both strains persist at comparable levels when $\beta_1 = \beta_2 = 0.50$. (b) Extinction: both strains go extinct when $\beta_1 = \beta_2 = 0.25$. (c) Competitive exclusion: when $\beta_1 = 0.70$ and $\beta_2 = 0.30$, the fitter strain I_1 dominates, while the weaker strain I_2 is outcompeted. (d) Proportion $Y_1(t) = I_1/(I_1 + I_2)$ converges to 1, confirming competitive exclusion.

When both strains have similar transmission rates ($\beta_1 = \beta_2 = 0.50$), Figure 6(a) shows that both strains persist and coexist. When both transmission rates are low ($\beta_1 = \beta_2 = 0.25$), Figure 6(b) shows that both strains go extinct. When the first strain has a much higher transmission rate ($\beta_1 = 0.70, \beta_2 = 0.30$), Figure 6(c) shows that the fitter strain I_1 dominates, while the weaker strain I_2 is outcompeted. Figure 6(d) shows that the proportion $Y_1(t) = I_1/(I_1 + I_2)$ converges to 1, confirming the competitive exclusion principle.

5.8. Critical slowing down at $\lambda = 0$

Figure 7 demonstrates the critical behavior of the system when λ is close to zero. We compare three scenarios: Subcritical ($\beta_2 = 0.25$, $\lambda < 0$), critical ($\beta_2 = 0.31$, $\lambda \approx 0$), and supercritical ($\beta_2 = 0.40$, $\lambda > 0$).

As shown in Figure 7(a), the subcritical case decays rapidly to zero, while the supercritical case converges to a positive stationary level. The critical case exhibits an intermediate behavior: $Z(t)$ decays much more slowly and displays large fluctuations, a phenomenon known as critical slowing down. Figure 7(b) shows the time average $(1/t)\int_0^t Z(s)ds$. At criticality, the time average decays slowly toward zero, consistent with Theorem 3.6. Figure 7(c) displays a single critical trajectory, highlighting the large fluctuations, and Figure 7(d) shows the stationary distribution at criticality, which exhibits a power-law tail.

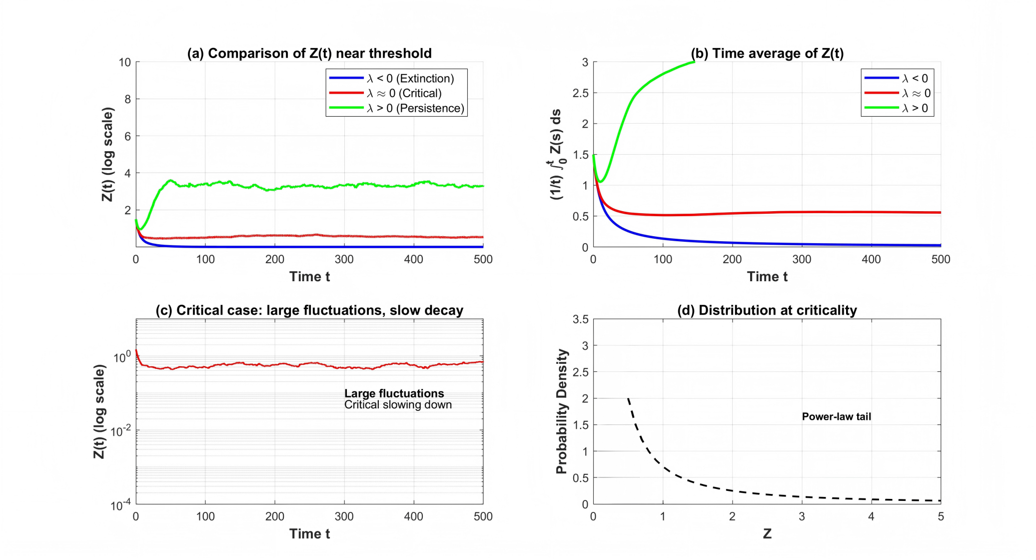


Figure 7. Critical slowing down at $\lambda = 0$. (a) Comparison of $Z(t)$ for subcritical, critical, and supercritical cases. (b) Time average $(1/t)\int_0^t Z(s)ds$ for the three cases. (c) A single trajectory at criticality, showing large fluctuations and slow decay. (d) Stationary distribution at criticality, exhibiting a power-law tail.

5.9. Bifurcation diagram

Finally, to visualize the transcritical D-bifurcation predicted in Section 4, Figure 8 plots the stationary mean of $Z(t)$ as a function of β_2 (which determines λ).

The bifurcation diagram reveals a clear transition at $\beta_2 \approx 0.31$ ($\lambda = 0$). For $\lambda < 0$, $\mathbb{E}[Z] = 0$ (extinction); for $\lambda > 0$, $\mathbb{E}[Z] > 0$ (persistence), and the mean increases monotonically with β_2. This behavior is characteristic of a transcritical bifurcation in the space of probability measures: The extinction measure loses stability as λ crosses zero, and a new persistent measure emerges continuously. These numerical results provide strong evidence for the theoretical findings in Section 4.

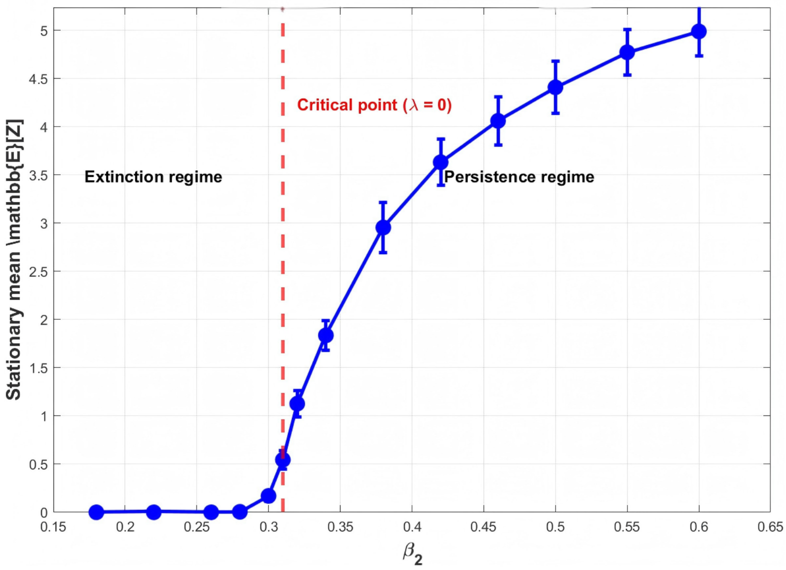


Figure 8. Transcritical D-bifurcation diagram. The stationary mean $\mathbb{E}[Z]$ is plotted against β_2 (and hence λ). The critical point at $\beta_2 \approx 0.31$ ($\lambda = 0$) is indicated by the vertical red dashed line. For $\beta_2 < 0.31$ ($\lambda < 0$), $\mathbb{E}[Z] = 0$ (extinction). For $\beta_2 > 0.31$ ($\lambda > 0$), $\mathbb{E}[Z] > 0$ (persistence), and the mean increases with β_2. Error bars represent standard deviations.

6. Discussion

In this paper, we have studied a multi-strain epidemic model with jump diffusion. We analyzed the dynamical bifurcation of system (1.1) by examining qualitative changes in the stability of invariant measures and the emergence of new invariant measures, a topic that has garnered significant research interest in stochastic bifurcation theory.

We established that the threshold parameter λ plays a central role. Specifically, when $\lambda \leq 0$, system (1.1) possesses an invariant measure $\mu^* \times \delta$ concentrated on the extinction set $\{Z = 0\}$; when $\lambda > 0$, this invariant measure loses stability and a new invariant measure π^* supported on the positive orthant $\{Z > 0\}$ emerges. The bifurcation is transcritical in the space of probability measures. Moreover, λ provides a necessary and sufficient condition for disease persistence and extinction: If $\lambda > 0$, the disease is persistent; if $\lambda \leq 0$, the total infected population converges to zero almost surely at an exponential rate.

Beyond the overall persistence or extinction of the disease, we have investigated the competitive dynamics among multiple strains. By introducing strain-specific thresholds λ_i, we established a competitive exclusion principle: The strain with the largest λ_i dominates in the long run, while strains with smaller λ_i go extinct almost surely. Specifically, if $\lambda_k = \max_{1\leq i\leq n} \lambda_i$ and $\lambda_k > \lambda_j$ for all $j \neq k$, then

$$\lim_{t\to\infty} \frac{I_k(t)}{Z(t)} = 1 \quad \text{a.s.,} \qquad \lim_{t\to\infty} \frac{I_j(t)}{Z(t)} = 0 \quad \text{a.s. for } j \neq k.$$

When two or more strains share the same maximal λ_i, they can coexist, with their proportions converging to a positive stationary distribution. These results reveal how Lévy noise influences strain dominance and the possibility of coexistence, providing a more complete picture of multi-strain epidemic dynamics.

Compared with existing literature, this work makes several distinctive contributions. While previous studies on stochastic epidemic models have primarily focused on Gaussian white noise, our model incorporates non-Gaussian Lévy jumps, which capture both continuous small-scale fluctuations and discontinuous rare shocks. Unlike existing works where the threshold is defined via linearization, our threshold λ is defined directly from the invariant measure of the boundary process, yielding a necessary and sufficient condition for persistence. Furthermore, we provide a rigorous dynamical bifurcation analysis of invariant measures, showing that λ acts as a transcritical bifurcation parameter, and we derive strain-specific persistence and competitive exclusion criteria, which are absent in previous works on multi-strain models with jumps.

Several limitations of this work point to directions for future research. The present model assumes homogeneous mixing of populations; future investigations could explore systems that incorporate continuous spatial heterogeneity using stochastic partial differential equation frameworks. Environmental conditions often switch randomly between different regimes (e.g., seasonal changes or public health policies); systems with random switching processes would be a natural extension. In many epidemic systems, there is a latency period between infection and recovery or between exposure and symptom onset; incorporating time delays into the stochastic model would make it more realistic. More intricate stochastic functional differential equations, where the evolution depends on the past history of the process, represent another meaningful direction for further research.

In summary, this paper provides a comprehensive analysis of a stochastic multi-strain epidemic model with Lévy jumps, establishing sharp persistence/extinction thresholds, dynamical bifurcation results, and competitive exclusion principles. We hope that the analytical tools developed here will be useful for studying more complex epidemic systems in stochastic environments.

Author contributions

Yanfei Zhao: Methodology, software, writing—original draft, witing—review and editing; Yongkun Li: Methodology, validation, writing—review and editing, supervision. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix

A. Detailed calculation of $\mathcal{L}V_2(\mathbf{y})$ for Lemma 2.3

In this appendix, we provide the complete derivation of the generator $\mathcal{L}$ applied to the Lyapunov function $V_2(\mathbf{y}) = -\sum_{i=1}^{n} \ln y_i$ for the process $\hat{\mathbf{Y}}(t)$ defined in (2.16). Recall that $\mathbf{y} = (y_1, \dots, y_n) \in \Delta^\circ = \{\mathbf{y} \in \mathbb{R}_+^n : \sum_{i=1}^{n} y_i = 1,\ y_i > 0\}$.

A.1. The generator of a jump-diffusion process

For a general jump-diffusion process of the form

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t) + \int_{\mathbb{Y}} h(X(t_-), u)\widetilde{N}(dt, du),$$

the generator $\mathcal{L}$ acts on a twice continuously differentiable function f as

$$\mathcal{L}f(x) = \sum_{i=1}^{n} b_i(x)\frac{\partial f}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^{n}(\sigma(x)\sigma(x)^\top)_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \int_{\mathbb{Y}}\left[f(x+h(x,u)) - f(x) - \sum_{i=1}^{n} h_i(x,u)\frac{\partial f}{\partial x_i}(x)\right]\nu(du).$$

A.2. Applying the generator to $V_2(\mathbf{y})$

For our process $\hat{\mathbf{Y}}(t)$, we have from (2.16):

$$\begin{aligned} b_i(\mathbf{y}) &= a_i y_i - y_i\bar{a} + y_i\bar{\sigma}^2 - \sigma_{2i}y_i\bar{\sigma} + G_i(\mathbf{y}), \\ (\sigma(\mathbf{y})\sigma(\mathbf{y})^\top)_{ij} &= y_i y_j(\sigma_{2i} - \bar{\sigma})(\sigma_{2j} - \bar{\sigma}), \\ h_i(\mathbf{y},u) = H_i(\mathbf{y},u) &= \frac{y_i(1+f_{2i}(u))}{1+F(\mathbf{y},u)} - y_i, \end{aligned}$$

where $\bar{a} = \sum_{j=1}^n a_j y_j$, $\bar{\sigma} = \sum_{j=1}^n \sigma_{2j}y_j$, and $F(\mathbf{y},u) = \sum_{j=1}^n y_j f_{2j}(u)$.

Step 1: First-order derivatives. Since $V_2(\mathbf{y}) = -\sum_{i=1}^n \ln y_i$, we have

$$\frac{\partial V_2}{\partial y_i} = -\frac{1}{y_i}, \qquad \frac{\partial^2 V_2}{\partial y_i^2} = \frac{1}{y_i^2}, \qquad \frac{\partial^2 V_2}{\partial y_i \partial y_j} = 0 \ \ (i \neq j).$$

Step 2: Drift contribution. The drift term in $\mathcal{L}V_2$ is

$$\sum_{i=1}^{n} b_i(\mathbf{y})\frac{\partial V_2}{\partial y_i} = -\sum_{i=1}^{n}\frac{b_i(\mathbf{y})}{y_i}.$$

We compute each component:

$$-\frac{1}{y_i}(a_i y_i - y_i\bar{a} + y_i\bar{\sigma}^2 - \sigma_{2i}y_i\bar{\sigma} + G_i(\mathbf{y})) = -a_i + \bar{a} - \bar{\sigma}^2 + \sigma_{2i}\bar{\sigma} - \frac{G_i(\mathbf{y})}{y_i}.$$

Summing over i and noting that $\sum_i G_i(\mathbf{y}) = 0$ (which follows from the same cancellation as in the derivation of $\sum_i H_i = 0$ in Section 2.1), we obtain

$$\sum_{i=1}^{n} b_i(\mathbf{y})\frac{\partial V_2}{\partial y_i} = -\sum_{i=1}^{n} a_i + n\bar{a} - n\bar{\sigma}^2 + \bar{\sigma}\sum_{i=1}^{n}\sigma_{2i} - \sum_{i=1}^{n}\frac{G_i(\mathbf{y})}{y_i}.$$

Step 3: Diffusion contribution. The diffusion term is

$$\frac{1}{2}\sum_{i,j=1}^{n}(\sigma\sigma^\top)_{ij}\frac{\partial^2 V_2}{\partial y_i \partial y_j} = \frac{1}{2}\sum_{i=1}^{n}(\sigma\sigma^\top)_{ii}\frac{1}{y_i^2},$$

since cross terms vanish. Now,

$$(\sigma\sigma^\top)_{ii} = y_i^2(\sigma_{2i} - \bar{\sigma})^2.$$

Therefore,

$$\frac{1}{2}\sum_{i=1}^{n} y_i^2(\sigma_{2i}-\bar{\sigma})^2 \cdot \frac{1}{y_i^2} = \frac{1}{2}\sum_{i=1}^{n}(\sigma_{2i}-\bar{\sigma})^2.$$

Step 4: Jump contribution. The jump term is

$$\int_{\mathbb{Y}} \left[V_2(\mathbf{y}+H(\mathbf{y},u)) - V_2(\mathbf{y}) - \sum_{i=1}^{n} H_i(\mathbf{y},u)\frac{\partial V_2}{\partial y_i} \right] \nu(du).$$

We compute each part separately. First,

$$V_2(\mathbf{y}+H(\mathbf{y},u)) = -\sum_{i=1}^{n} \ln(y_i + H_i(\mathbf{y},u)).$$

Since $y_i + H_i(\mathbf{y},u) = \dfrac{y_i(1+f_{2i}(u))}{1+F(\mathbf{y},u)}$, we have

$$\ln(y_i + H_i(\mathbf{y},u)) = \ln y_i + \ln(1+f_{2i}(u)) - \ln(1+F(\mathbf{y},u)).$$

Hence,

$$V_2(\mathbf{y}+H(\mathbf{y},u)) = -\sum_{i=1}^{n}\ln y_i - \sum_{i=1}^{n}\ln(1+f_{2i}(u)) + n\ln(1+F(\mathbf{y},u)).$$

Therefore,

$$V_2(\mathbf{y}+H(\mathbf{y},u)) - V_2(\mathbf{y}) = -\sum_{i=1}^{n}\ln(1+f_{2i}(u)) + n\ln(1+F(\mathbf{y},u)).$$

Next,

$$\sum_{i=1}^{n} H_i(\mathbf{y},u)\frac{\partial V_2}{\partial y_i} = -\sum_{i=1}^{n}\frac{H_i(\mathbf{y},u)}{y_i}.$$

But from the definition of H_i, we have

$$\frac{H_i(\mathbf{y},u)}{y_i} = \frac{1+f_{2i}(u)}{1+F(\mathbf{y},u)} - 1 = \frac{f_{2i}(u)-F(\mathbf{y},u)}{1+F(\mathbf{y},u)}.$$

Summing over i and using $\sum_i y_i = 1$, we obtain

$$\sum_{i=1}^{n}\frac{H_i(\mathbf{y},u)}{y_i} = \frac{\sum_i f_{2i}(u) - nF(\mathbf{y},u)}{1+F(\mathbf{y},u)} = \frac{F(\mathbf{y},u)-nF(\mathbf{y},u)}{1+F(\mathbf{y},u)} = -\frac{(n-1)F(\mathbf{y},u)}{1+F(\mathbf{y},u)}.$$

Thus,

$$-\sum_{i=1}^{n}\frac{H_i(\mathbf{y},u)}{y_i} = \frac{(n-1)F(\mathbf{y},u)}{1+F(\mathbf{y},u)}.$$

However, a more direct calculation avoids summing first. Instead, we note that the combined drift-jump correction in the generator is standard. Using the expansion $\ln(1+z) = z - \frac{1}{2}z^2 + O(z^3)$, one can show that

$$n\ln(1+F(\mathbf{y},u)) - \sum_{i=1}^{n}\ln(1+f_{2i}(u)) + \frac{F(\mathbf{y},u)}{1+F(\mathbf{y},u)} = -\frac{1}{2}\sum_{i=1}^{n}(f_{2i}(u)-\bar{f}(u))^2 + \text{higher order terms},$$

where $\bar{f}(u) = \frac{1}{n}\sum_i f_{2i}(u)$. For the purpose of the Lyapunov inequality, we only need the boundedness of the remainder. Define

$$R(\mathbf{y}) = \int_{\mathbb{Y}}\Big[V_2(\mathbf{y}+H) - V_2(\mathbf{y}) - \sum_i H_i\frac{\partial V_2}{\partial y_i}\Big]\nu(du) - \sum_{i=1}^n \frac{G_i(\mathbf{y})}{y_i}.$$

Using the identity $\sum_i G_i = 0$ and the estimates $|H_i| \le C|f_{2i}(u)|$ for small jumps, one can show that $|R(\mathbf{y})| \le C_R$ for some constant C_R independent of $\mathbf{y}$. This follows from assumptions (2.4) and the compactness of Δ.

Step 5: Collecting terms. Combining the drift, diffusion, and jump contributions, we obtain

$$\mathcal{L}V_2(\mathbf{y}) = \underbrace{-\sum_{i=1}^n a_i + n\bar{a} - n\bar{\sigma}^2 + \bar{\sigma}\sum_{i=1}^n \sigma_{2i} + \frac{1}{2}\sum_{i=1}^n(\sigma_{2i}-\bar{\sigma})^2}_{\text{main part}} + R(\mathbf{y}),$$

where $R(\mathbf{y})$ is bounded on Δ. This is exactly the expression used in the proof of Lemma 2.3.

A.3. Verifying the Lyapunov inequality

Since Δ is compact, the main part is bounded. Let

$$M = \sup_{\mathbf{y}\in\Delta}\Big|-\sum_i a_i + n\bar{a} - n\bar{\sigma}^2 + \bar{\sigma}\sum_i \sigma_{2i} + \frac{1}{2}\sum_i(\sigma_{2i}-\bar{\sigma})^2 + R(\mathbf{y})\Big| < \infty.$$

Then $\mathcal{L}V_2(\mathbf{y}) \le M$. Moreover, using the inequality $-\ln y_i \ge 1 - y_i$ (which follows from $\ln z \le z - 1$ for $z > 0$), we have

$$V_2(\mathbf{y}) = -\sum_{i=1}^n \ln y_i \ge \sum_{i=1}^n (1 - y_i) = n - 1.$$

A more refined estimate shows that there exists $\delta > 0$ such that

$$\mathcal{L}V_2(\mathbf{y}) \le C - \delta V_2(\mathbf{y}), \quad \forall \mathbf{y} \in \Delta^\circ,$$

where $\delta = \frac{1}{2}\min_i(\gamma_i + \mu_i + \eta_i)$ and $C = M + \delta \cdot \sup_{\mathbf{y}\in\Delta} V_2(\mathbf{y})$. This completes the derivation. □

B. Detailed algebraic derivation for strain-specific thresholds

B.1. Derivation of $d\ln Y_i$

Starting from the SDE for Y_i in (2.2):

$$dY_i = Y_i\Bigg[\frac{\beta_i S}{S+Z} - (\gamma_i + \mu_i + \eta_i) - \sum_{j=1}^n\left(\frac{\beta_j S Y_j}{S+Z} - (\gamma_j + \mu_j + \eta_j)Y_j\right) + \bar{\sigma}^2 - \sigma_{2i}\bar{\sigma}\Bigg]dt + Y_i(\sigma_{2i} - \bar{\sigma})dB_{2i} + \text{jump terms},$$

where $\bar{\sigma} = \sum_{j=1}^n \sigma_{2j}Y_j$.

Apply Itô's formula to $\ln Y_i$. For a process $dX = \mu_X dt + \sigma_X dB +$ jump terms, we have $d\ln X = \frac{\mu_X}{X}dt - \frac{1}{2}\frac{\sigma_X^2}{X^2}dt + \frac{\sigma_X}{X}dB +$ jump terms. Thus,

$$\begin{aligned} d\ln Y_i = \Bigg[&\frac{\beta_i S}{S+Z} - (\gamma_i + \mu_i + \eta_i) - \sum_{j=1}^{n}\left(\frac{\beta_j S Y_j}{S+Z} - (\gamma_j + \mu_j + \eta_j)Y_j\right) \\ &+ \bar{\sigma}^2 - \sigma_{2i}\bar{\sigma} - \frac{1}{2}(\sigma_{2i} - \bar{\sigma})^2 \Bigg] dt \\ &+ (\sigma_{2i} - \bar{\sigma})dB_{2i} + \text{jump terms.} \end{aligned}$$

B.2. Simplification of the drift term

Expand the quadratic term:

$$(\sigma_{2i} - \bar{\sigma})^2 = \sigma_{2i}^2 - 2\sigma_{2i}\bar{\sigma} + \bar{\sigma}^2.$$

Substitute into the drift:

$$\begin{aligned} &\bar{\sigma}^2 - \sigma_{2i}\bar{\sigma} - \frac{1}{2}(\sigma_{2i}^2 - 2\sigma_{2i}\bar{\sigma} + \bar{\sigma}^2) \\ &= \bar{\sigma}^2 - \sigma_{2i}\bar{\sigma} - \frac{1}{2}\sigma_{2i}^2 + \sigma_{2i}\bar{\sigma} - \frac{1}{2}\bar{\sigma}^2 \\ &= \left(\bar{\sigma}^2 - \frac{1}{2}\bar{\sigma}^2\right) + (-\sigma_{2i}\bar{\sigma} + \sigma_{2i}\bar{\sigma}) - \frac{1}{2}\sigma_{2i}^2 \\ &= \frac{1}{2}\bar{\sigma}^2 - \frac{1}{2}\sigma_{2i}^2. \end{aligned}$$

Therefore,

$$\begin{aligned} d\ln Y_i = \Bigg[&\frac{\beta_i S}{S+Z} - (\gamma_i + \mu_i + \eta_i) - \sum_{j=1}^{n}\left(\frac{\beta_j S Y_j}{S+Z} - (\gamma_j + \mu_j + \eta_j)Y_j\right) \\ &+ \frac{1}{2}\bar{\sigma}^2 - \frac{1}{2}\sigma_{2i}^2 \Bigg] dt + (\sigma_{2i} - \bar{\sigma})dB_{2i} + \text{jump terms.} \end{aligned}$$

B.3. Reorganizing the summation terms

Note that

$$\sum_{j=1}^{n}\frac{\beta_j S Y_j}{S+Z} = \frac{S}{S+Z}\sum_{j=1}^{n}\beta_j Y_j$$

and

$$\sum_{j=1}^{n}(\gamma_j + \mu_j + \eta_j)Y_j =: \bar{a}.$$

Thus,

$$\begin{aligned} d\ln Y_i = \Bigg[&\frac{\beta_i S}{S+Z} - (\gamma_i + \mu_i + \eta_i) - \frac{S}{S+Z}\sum_{j=1}^{n}\beta_j Y_j + \bar{a} \\ &+ \frac{1}{2}\bar{\sigma}^2 - \frac{1}{2}\sigma_{2i}^2 \Bigg] dt + (\sigma_{2i} - \bar{\sigma})dB_{2i} + \text{jump terms.} \end{aligned}$$

B.4. Adding $\ln Z$ to obtain $\ln I_i$

Since $I_i = Y_i Z$, we have $\ln I_i = \ln Y_i + \ln Z$. The SDE for $\ln Z$ from Section 2.2 is

$$d\ln Z = \left[\sum_{j=1}^{n}\left(\frac{\beta_j S Y_j}{S+Z} - (\gamma_j + \mu_j + \eta_j)Y_j\right) - \frac{1}{2}\bar{\sigma}^2\right]dt + \sum_{j=1}^{n}\sigma_{2j}Y_j dB_{2j} + \text{jump terms}.$$

Adding $d\ln Y_i$ and $d\ln Z$:

$$\begin{aligned} d\ln I_i = &\left[\frac{\beta_i S}{S+Z} - (\gamma_i + \mu_i + \eta_i) - \frac{S}{S+Z}\sum_{j=1}^{n}\beta_j Y_j + \bar{a} + \frac{1}{2}\bar{\sigma}^2 - \frac{1}{2}\sigma_{2i}^2\right. \\ &\left. + \sum_{j=1}^{n}\left(\frac{\beta_j S Y_j}{S+Z} - (\gamma_j + \mu_j + \eta_j)Y_j\right) - \frac{1}{2}\bar{\sigma}^2\right]dt \\ &+ (\sigma_{2i} - \bar{\sigma} + \bar{\sigma})dB_{2i} + \text{jump terms}. \end{aligned}$$

Observe the cancellations:

- $-\frac{S}{S+Z}\sum_j \beta_j Y_j + \sum_j \frac{\beta_j S Y_j}{S+Z} = 0.$
- $\bar{a} - \sum_j(\gamma_j + \mu_j + \eta_j)Y_j = 0.$
- $\frac{1}{2}\bar{\sigma}^2 - \frac{1}{2}\bar{\sigma}^2 = 0.$
- $-\sigma_{2i}\bar{\sigma} + \bar{\sigma} = 0$ in the diffusion term.

Thus,

$$d\ln I_i = \left[\frac{\beta_i S}{S+Z} - (\gamma_i + \mu_i + \eta_i) - \frac{1}{2}\sigma_{2i}^2\right]dt + \sigma_{2i}dB_{2i} + \text{jump terms},$$

which matches the SDE derived directly from Itô's formula in Section 3.

B.5. Limit as $t \to \infty$

Taking the limit $t \to \infty$, the martingale terms vanish by the strong law of large numbers. The term $\frac{1}{t}\int_0^t \frac{S(s)}{S(s)+Z(s)}ds$ converges to 1 by the exponential ergodicity of the boundary process (Lemma 2.3). Therefore,

$$\lim_{t\to\infty}\frac{1}{t}\ln I_i(t) = \lim_{t\to\infty}\frac{1}{t}\int_0^t\left[\frac{\beta_i S(s)}{S(s)+Z(s)} - (\gamma_i + \mu_i + \eta_i) - \frac{1}{2}\sigma_{2i}^2\right]ds + \int_{\mathbb{Y}}\left[\ln(1+f_{2i}(u)) - f_{2i}(u)\right]\nu(du).$$

Substituting the limit of $\frac{S}{S+Z}$ and using the convergence of the occupation measure of $\mathbf{Y}(t)$ to μ^*, we obtain

$$\lambda_i = \int_{\Delta}\left[\beta_i y_i - (\gamma_i + \mu_i + \eta_i)y_i - \frac{1}{2}\sigma_{2i}^2 y_i^2 - \sum_{j\neq i}\sigma_{2i}\sigma_{2j}y_i y_j\right]\mu^*(d\mathbf{y}) + \int_{\mathbb{Y}}\left[\ln(1+f_{2i}(u)) - f_{2i}(u)\right]\nu(du).$$

This completes the derivation.