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Research article

Stability of non-critical waves to a cooperation-competition model with three species

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Abstract: In this paper, we prove the stability of traveling wave solutions for a three-species cooperation-competition system. To ensure the existence of traveling wave fronts, i.e. monotone traveling wave solutions, for this system, the wave speed c must be greater than or equal to a critical value c_*. In the existing literature, the weighted energy method has been employed to prove that traveling wave fronts are stable when c is sufficiently larger than c_*. This paper improves this result by showing that, under certain conditions on the system parameters, traveling wave fronts with $c > c_*$ (i.e., non-critical waves) are stable. First, by virtue of the asymptotic behavior and positivity of traveling wave fronts, we establish some estimates. Then, we prove the estimates for solutions to the perturbed system. Finally, the exponential stability of non-critical waves is obtained by the weighted energy method and the squeezing technique. Our conclusions expand the range of speeds for which traveling wave fronts are stable, thereby strengthening existing conclusions.

Keywords: cooperative-competitive system; stability; traveling wave solution
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1. Introduction

In recent years, with the advancement of scientific research, single-species and two-species models can no longer adequately explain complex and diverse interspecific relationships. Consequently, the study of the spread of multi-species models has emerged as a new focus of academic inquiry. This paper mainly considers the propagation problem of a class of three-species cooperative-competitive systems as follows:

$$\begin{cases} (u_1)_t = d_1(u_1)_{xx} + r_1u_1(1 - u_1 + a_{12}u_2 - a_{13}u_3), \\ (u_2)_t = d_2(u_2)_{xx} + r_2u_2(1 + a_{21}u_1 - u_2 - a_{23}u_3), \\ (u_3)_t = d_3(u_3)_{xx} + r_3u_3(1 - a_{31}u_1 - a_{32}u_2 - u_3), \end{cases} \tag{1.1}$$

where $u_i(i = 1, 2, 3)$ denotes the density of the i-th species, and all parameters in the system are positive. The coefficients d_i $(i = 1, 2, 3)$ are the diffusion rates, which describe the spatial dispersal of the species; r_i $(i = 1, 2, 3)$ are the intrinsic growth rates; and a_{ij} $(i \neq j)$ characterize the strengths of interspecific interactions. The above system is a three-species Lotka–Volterra type reaction diffusion model, which extends the classical Lotka–Volterra competition model by incorporating both cooperative and competitive effects. In the framework considered here, species u_1 and u_2 cooperate with each other, and both compete with the third species, u_3. The equilibria of (1.1) are given by

$$e_0 = (0, 0, 0), \quad e_1 = (1, 0, 0), \quad e_2 = (0, 1, 0), \quad e_3 = (0, 0, 1),$$

$$e_4 = \left(\frac{1 + a_{12}}{1 - a_{12}a_{21}}, \frac{1 + a_{21}}{1 - a_{12}a_{21}}, 0\right) := (b_1, b_2, 0), \quad e_5 = \left(\frac{1 - a_{13}}{1 - a_{13}a_{31}}, 0, \frac{1 - a_{31}}{1 - a_{13}a_{31}}\right),$$

$$e_6 = \left(0, \frac{1 - a_{23}}{1 - a_{23}a_{32}}, \frac{1 - a_{32}}{1 - a_{23}a_{32}}\right), \quad e_7 = \left(\frac{D_1}{D}, \frac{D_2}{D}, \frac{D_3}{D}\right),$$

where

$$D = \begin{vmatrix} 1 & -a_{12} & a_{13} \\ -a_{21} & 1 & a_{23} \\ a_{31} & a_{32} & 1 \end{vmatrix}, D_1 = \begin{vmatrix} 1 & -a_{12} & a_{13} \\ 1 & 1 & a_{23} \\ 1 & a_{32} & 1 \end{vmatrix}, D_2 = \begin{vmatrix} 1 & 1 & a_{13} \\ -a_{21} & 1 & a_{23} \\ a_{31} & 1 & 1 \end{vmatrix}, D_3 = \begin{vmatrix} 1 & -a_{12} & 1 \\ -a_{21} & 1 & 1 \\ a_{31} & a_{32} & 1 \end{vmatrix}.$$

In this paper, we assume that

(A_1) $0 < a_{12}a_{21} < 1$, $0 < a_{13}, a_{23} < 1$, $1 - a_{31}b_1 - a_{32}b_2 < 0$, $d_3 \leqslant \min\{d_1, d_2\}$,

$$r_3(a_{31}b_1 + a_{32}b_2) \leqslant \min\{r_1(1 - a_{13}), r_2(1 - a_{23})\},$$

where

$$b_1 = \frac{1 + a_{12}}{1 - a_{12}a_{21}}, b_2 = \frac{1 + a_{21}}{1 - a_{12}a_{21}}.$$

Assumption (A1) admits the following biological interpretation. The condition $0 < a_{12}a_{21} < 1$ ensures that the mutual cooperation between species u_1 and u_2 is not excessively strong, which guarantees the existence of a positive and bounded coexistence equilibrium (b_1, b_2). The inequalities $0 < a_{13}, a_{23} < 1$ describe moderate competitive effects exerted by species u_3 on u_1 and u_2. Moreover, the key condition $1 - a_{31}b_1 - a_{32}b_2 < 0$ implies that, when u_1 and u_2 reach their equilibrium states, their combined inhibitory effect on u_3 is sufficiently strong to drive an invasion process, leading to the eventual replacement of u_3. In addition, the condition $d_3 \leq \min\{d_1, d_2\}$ reflects that species u_3 has a relatively weaker spatial dispersal ability, which further facilitates the invasion of u_1 and u_2. Finally, the inequality $r_3(a_{31}b_1 + a_{32}b_2) \leq \min\{r_1(1 - a_{13}), r_2(1 - a_{23})\}$ ensures that u_3 remains at a competitive disadvantage even when its intrinsic growth is taken into account.

It follows from [11] that, under part of Assumption (A1), namely

$$0 < a_{12}a_{21} < 1, \quad 0 < a_{13}, a_{23} < 1, \quad 1 - a_{31}b_1 - a_{32}b_2 < 0,$$

for the corresponding ordinary differential system associated with (1.1), the equilibrium $e_3 = (0, 0, 1)$ is unstable, whereas $e_4 = (b_1, b_2, 0)$ is locally asymptotically stable. This indicates that the competitive ability of species u_3 is relatively weak compared to that of species u_1 and u_2. In this framework, species u_1 and u_2 cooperate with each other, and both compete with species u_3, which naturally leads to the invasion of $(b_1, b_2, 0)$ into $(0, 0, 1)$. This invasion process can be characterized by traveling wave solutions connecting these two equilibria. For the existence, minimal wave speed selection mechanism, uniqueness, and asymptotic behavior of these traveling wave solutions and wavefront solutions (strictly monotonic traveling wave solutions), one may refer to [2–5, 10–12]. In practical problems, we are concerned with the long-time behavior of species, namely the long-time behavior of solutions to the initial value problem of (1.1). Due to the properties of traveling wave solutions, the study of such problems can be transformed into the study of the stability of traveling wave solutions. Recently, for sufficiently large wave speeds, Ma et al. [6] proved the exponential stability of the wavefront solutions of (1.1), connecting $(0, 0, 1)$ and $(b_1, b_2, 0)$ by using the weighted energy method.

Mei et al. [7–9] have conducted extensive work on the stability of traveling wave solutions for scalar equations using the weighted energy method. More precisely, in [7], they proved the stability of non-critical waves (wavefront solutions with a wave speed greater than the critical wave speed) by using L^2 estimates. Later, in [8], they combined L^1 estimates and L^2 estimates to prove the stability of both non-critical waves and critical waves. In [9], they demonstrated the algebraic exponential stability of critical waves through the Fourier transform and the weighted energy method. For reaction-diffusion Lotka–Volterra competition systems with two species, Cao et al. [1] proved the stability of non-critical waves by improving the weighted energy method. Compared with the existing works on scalar equations and two-species systems, the three-species cooperative–competitive system (1.1) considered in this paper involves additional difficulties due to the coupling of cooperative and competitive interactions. In particular, the lack of a comparison principle and the increased dimensionality make the stability analysis more challenging. Inspired by [1], in this paper we further refine the weighted energy method and, by exploiting the strict positivity of wavefront solutions, establish the exponential stability of non-critical traveling waves of (1.1) that connect $(0, 0, 1)$ and $(b_1, b_2, 0)$, thereby relaxing the restriction on the wave speed imposed in [6].

We would like to stress that the three-dimensional cooperative–competitive system considered in this paper is significantly more difficult to analyze than classical one- or two-species models. The main difficulty arises from the strong coupling among the three components, which leads to a more involved structure of the characteristic equation and makes the analysis more delicate. Furthermore, the presence of multiple interaction terms introduces additional challenges in deriving suitable a priori estimates. In particular, the weighted energy method must be carefully adapted to handle the nonlinear interaction terms. Therefore, the extension of stability results from lower-dimensional systems to the present three-species case is nontrivial.

The rest of the paper is organized as follows. In Section 2, we present some preliminaries and state the main result. In Section 3, we complete the proof of the main result by using the improved weighted energy method. Finally, we provide conclusory remarks in Section 4.

2. Preliminaries and main conclusions

In this section, we will introduce the notation, function spaces used in the following text, and main conclusions. In this paper, C denotes a general constant, and $C_i(i = 1, 2, \cdots)$ represents a specific constant. Let I be an interval, in particular, when $I = R$, $L^p(I)(p \geqslant 1)$ denotes the Lebesgue integrable space defined on the interval I, and $W^{k,p}(I)(k \geqslant 0, p \geqslant 1)$ is the space defined on I for which derivative $\frac{d^i}{dx^i} f(x)(i = 1, 2, \cdots, k)$ also belongs to $L^p(I)$. When $p = 2$, $W^{k,2}(I)$ is denoted as $H^k(I)$. $L^p_\omega(I)$ represents the weighted space with weight function $\omega(x) > 0$, whose norm is

$$\|f\|_{L^p_\omega} = \left(\int_I \omega(x) |f(x)|^p dx \right)^{1/p},$$

and $W^{k,p}_\omega(I)$ is a weighted Sobolev space with norm

$$\|f\|_{W^{k,p}_\omega} = (\sum_{i=0}^{k} \int_I \omega(x) | \frac{d^i f(x)}{dx^i} |^p dx)^{\frac{1}{p}}.$$

In order to study the exponential stability of non-critical waves connecting $(0, 0, 1)$ and $(b_1, b_2, 0)$ of (1.1), we let $\hat{u}_i = u_i$, $i = 1, 2$, $\hat{u}_3 = 1 - u_3$. Then, for convenience, we omit the “ˆ” and obtain the following cooperative system from (1.1):

$$\begin{cases} (u_1)_t = d_1(u_1)_{xx} + r_1 u_1(1 - a_{13} - u_1 + a_{12}u_2 + a_{13}u_3), \\ (u_2)_t = d_2(u_2)_{xx} + r_2 u_2(1 - a_{23} + a_{21}u_1 - u_2 + a_{23}u_3), \\ (u_3)_t = d_3(u_3)_{xx} + r_3(1 - u_3)(a_{31}u_1 + a_{32}u_2 - u_3). \end{cases} \tag{2.1}$$

Obviously, the existence of wavefront solutions of (1.1) connecting $(0, 0, 1)$ and $(b_1, b_2, 0)$ which is equivalent to the existence of wavefront solutions of (2.1) connecting $(0, 0, 0)$ and $(b_1, b_2, 1)$. The wavefront solution $\phi(\xi) := (\phi_1(\xi), \phi_2(\xi), \phi_3(\xi))$, $\xi = x + ct$, to (2.1) satisfies the following boundary value problem:

$$\begin{cases} d_1\phi_1'' - c\phi_1' + r_1\phi_1(1 - a_{13} - \phi_1 + a_{12}\phi_2 + a_{13}\phi_3) = 0, \\ d_2\phi_2'' - c\phi_2' + r_2\phi_2(1 - a_{23} + a_{21}\phi_1 - \phi_2 + a_{23}\phi_3) = 0, \\ d_3\phi_3'' - c\phi_3' + r_3(1 - \phi_3)(a_{31}\phi_1 + a_{32}\phi_2 - \phi_3) = 0, \end{cases} \tag{2.2}$$

$$\lim_{\xi \to \to -\infty} (\phi_1(\xi), \phi_2(\xi), \phi_3(\xi)) = (0, 0, 0), \quad \lim_{\xi \to \to +\infty} (\phi_1(\xi), \phi_2(\xi), \phi_3(\xi)) = (b_1, b_2, 1), \tag{2.3}$$

$$\phi_i'(\xi) > 0, \ i = 1, 2, 3. \tag{2.4}$$

In the following, the wavefront solutions of (2.1) we mention are the solutions satisfying (2.2)–(2.4).

Next, we linearize (2.2) at $(0, 0, 0)$ and the corresponding characteristic equation is

$$\Delta_1(\lambda_1, c)\Delta_2(\lambda_2, c)\Delta_3(\lambda_3, c) = 0, \tag{2.5}$$

where

$$\Delta_1(\lambda, c) = d_1\lambda^2 - c\lambda + r_1(1 - a_{13}), \ \Delta_2(\lambda_1, c) = d_2\lambda^2 - c\lambda + r_2(1 - a_{23}), \ \Delta_3(\lambda, c) = d_3\lambda^2 - c\lambda - r_3.$$

Direct computation shows that, when $c > c_* := \max\{2\sqrt{r_1d_1(1-a_{13})}, 2\sqrt{r_2d_2(1-a_{23})}\}$, (2.5) has five positive roots and one negative root:

$$\lambda_1^- = \frac{c - \sqrt{c^2 - 4r_1d_1(1-a_{13})}}{2d_1} > 0, \quad \lambda_1^+ = \frac{c + \sqrt{c^2 - 4r_1d_1(1-a_{13})}}{2d_1} > 0,$$

$$\lambda_2^- = \frac{c - \sqrt{c^2 - 4r_2d_2(1-a_{23})}}{2d_2} > 0, \quad \lambda_2^+ = \frac{c + \sqrt{c^2 - 4r_2d_2(1-a_{23})}}{2d_2} > 0,$$

$$\lambda_3^- = \frac{c - \sqrt{c^2 + 4r_3d_3}}{2d_3} < 0, \qquad \lambda_3^+ = \frac{c + \sqrt{c^2 + 4r_3d_3}}{2d_3} > 0.$$

Obviously $\lambda_i^- < \lambda_i^+$, $i = 1, 2$. We further assume that

(A$_2$) $(\lambda_1^-, \lambda_1^+) \subseteq (\lambda_2^-, \lambda_2^+)$ or $(\lambda_2^-, \lambda_2^+) \subseteq (\lambda_1^-, \lambda_1^+)$.

The choice of λ in the weighted function

$$\omega(x) = \begin{cases} e^{-\lambda(x-\xi_0)}, & x < \xi_0, \\ 1, & x \geqslant \xi_0, \end{cases} \tag{2.6}$$

where ξ_0 is an undetermined constant, depends on the relationships of the roots of (2.5). Before the main results, we classify the roots into categories as follows:

(i) $\lambda_3^+ \geqslant \max\{\lambda_1^+, \lambda_2^+\}$, $r_3 < \min\left\{\frac{-\Delta_1(\lambda, c)}{a_{31}}, \frac{-\Delta_2(\lambda, c)}{a_{32}}\right\}$, $\lambda \in (\lambda_1^-, \lambda_1^+) \cap (\lambda_2^-, \lambda_2^+)$;

(ii) $\min\{\lambda_1^+, \lambda_2^+\} < \lambda_3^+ < \max\{\lambda_1^+, \lambda_2^+\}$, $r_3 < \min\left\{\frac{-\Delta_1(\lambda, c)}{a_{31}}, \frac{-\Delta_2(\lambda, c)}{a_{32}}\right\}$, $\lambda \in (\lambda_1^-, \lambda_1^+) \cap (\lambda_2^-, \lambda_2^+)$;

(iii) $\max\{\lambda_1^-, \lambda_2^-\} < \lambda_3^+ < \min\{\lambda_1^+, \lambda_2^+\}$, $r_3 < \min\left\{\frac{-\Delta_1(\lambda, c)}{a_{31}}, \frac{-\Delta_2(\lambda, c)}{a_{32}}\right\}$, $\lambda \in (\lambda_1^-, \lambda_1^+) \cap (\lambda_2^-, \lambda_2^+) \cap (\lambda_3^-, \lambda_3^+)$;

(iv) $\lambda_2^- > \lambda_1^-$, $\lambda_1^- < \lambda_3^+ < \lambda_2^-$, $r_3 < \min\left\{\frac{-\Delta_1(\lambda, c)}{a_{31}}, \frac{-\Delta_2(\lambda, c)}{a_{32}}\right\}$, $c > \max\{c_*, c_1\}$, $\lambda \in (\lambda_1^-, \lambda_1^+) \cap (\lambda_2^-, \lambda_2^+)$, where $c_1 = \frac{d_3\lambda^2 - r_3}{\lambda}$;

(v) $\lambda_1^- > \lambda_2^-$, $\lambda_2^- < \lambda_3^+ < \lambda_1^-$, $r_3 < \min\left\{\frac{-\Delta_1(\lambda, c)}{a_{31}}, \frac{-\Delta_2(\lambda, c)}{a_{32}}\right\}$, $c > \max\{c_*, c_1\}$, $\lambda \in (\lambda_1^-, \lambda_1^+) \cap (\lambda_2^-, \lambda_2^+)$;

(vi) $0 < \lambda_3 \leqslant \min\{\lambda_1^-, \lambda_2^-\}$, $r_3 < \min\left\{\frac{-\Delta_1(\lambda, c)}{a_{31}}, \frac{-\Delta_2(\lambda, c)}{a_{32}}\right\}$, $c > \max\{c_*, c_1\}$, $\lambda \in (\lambda_1^-, \lambda_1^+) \cap (\lambda_2^-, \lambda_2^+)$.

With the above preliminaries, we introduce the main results of this paper as follows.

Theorem 2.1. *Under Assumptions* (A_1) *and* (A_2), *let* (ϕ_1, ϕ_2, ϕ_3) *be the wavefront solution of* (2.1). *We introduce the initial condition as follows:*

$$u_i(x, 0) = u_{i_0}(x), \quad i = 1, 2, 3. \tag{2.7}$$

If the initial condition satisfies

$$0 \leqslant u_{i_0}(x) \leqslant b_i, \quad i = 1, 2, 3, \ b_3 = 1, \ x \in \mathbb{R},$$

and

$$u_{i_0}(x) - \phi_i(x) \in W^{1,1}_{\omega}(\mathbb{R}) \cap H^2(\mathbb{R}), \quad i = 1,2,3,$$

where ω is given in (2.6)*. Then there exists a unique nonnegative solution*

$$(u_1(x,t), u_2(x,t), u_3(x,t))$$

to (2.1) *with* (2.7)*. This solution satisfies*

$$(0,0,0) \leqslant (u_1(t,x), u_2(t,x), u_3(t,x)) \leqslant (b_1, b_2, b_3), \quad x \in \mathbb{R},\ t > 0,$$

and

$$u_i(t,x) - \phi_i(x+ct) \in C([0,\infty), W^{1,1}_{\omega}(\mathbb{R}) \cap H^2(\mathbb{R})), \quad i = 1,2,3.$$

Furthermore, the solution (u_1, u_2, u_3) *of the initial value problem converges to the wavefront solution* (ϕ_1, ϕ_2, ϕ_3)*, namely*

$$\sup_{x\in\mathbb{R}} |u_i(x,t) - \phi_i(x+ct)| \leqslant Ce^{-\frac{1}{3}\mu t}, \quad t > 0,\ i = 1,2,3,$$

where μ is a sufficiently small positive number, if a_{12}, a_{21}, a_{13}, a_{23}, r_3 are sufficiently small, and $a_{31} + a_{32}$ is sufficiently large, and one of the conditions $(i) - (vi)$ *is satisfied.*

Remark 2.2. *In [6], the wavefront solution is stable if the wave speed c satisfies*

$$c > c^* = 2\max\{c_2, c_3, c_3\},$$

where

$$c_2 = \sqrt{r_1 b_1 + r_2 b_2 a_{21} + r_3 a_{31}}, \quad c_3 = \sqrt{r_1 b_1 a_{12} + r_2(1 + a_{23}) + r_3 a_{32}},$$
$$c_4 = \sqrt{r_1 b_1 a_{13} + r_2 b_2 a_{23} + 2r_3}.$$

Obviously, when $d_1 = d_2 = d_3 = 1$, noting that $b_1 > 1$, we have $r_2(1 + a_{23}) > r_2(1 - a_{23})$, $r_1 b_1 a_{13} > r_1(1 - a_{13})$. According to Theorem 2.1, when the wave speed $c_* < c < c^*$, the wavefront solution remains stable. Therefore, the results of this paper expand the range of wave speeds for which the wavefront solution is stable.

3. Stability of non-critical waves

In this section, we will prove the stability of non-critical waves. According to [8], when the assumptions in Theorem 2.1 hold, the solution (u_1, u_2, u_3) to the initial value problems (2.1) and (2.7) satisfies

$$u_i(x,t) - \phi_i(x+ct) \in C([0,+\infty), W^{1,1}_{\omega}(\mathbb{R}) \cap H^2(\mathbb{R})).$$

Furthermore, by applying the comparison principle, we conclude that $0 \leqslant u_i(x,t) \leqslant b_i$, for $i = 1, 2, 3$.

To prove the stability of non-critical waves, it suffices to show that as time tends to infinity, the difference between the solution to the initial value problems (2.1) and (2.7) and the wavefront solution tends to zero. For this purpose, let $(u_1^\pm, u_2^\pm, u_3^\pm)$ denote the solution to the initial value problems (2.1) and (2.7) with the initial condition

$$\widetilde{u}_{i_0}^{+}(x,0) = \max\{u_{i_0}(x), \phi_i(x)\}, \quad \widetilde{u}_{i_0}^{-}(x,0) = \min\{u_{i_0}(x), \phi_i(x)\}, \qquad i = 1,2,3.$$

It is easy to prove that

$$0 \leqslant \widetilde{u}_{i_0}^-(x) \leqslant u_{i_0}(x) \leqslant \widetilde{u}_{i_0}^+(x) \leqslant b_i, \quad 0 \leqslant \widetilde{u}_{i_0}^-(x) \leqslant \phi_i(x) \leqslant \widetilde{u}_{i_0}^+(x) \leqslant b_i, \qquad i = 1, 2, 3.$$

Due to the lack of smoothness of the initial value, according to [8,9], we can choose a smooth function $u_{i_0}^\pm$ satisfying, for $i = 1, 2, 3$,

$$\begin{aligned} 0 \leqslant u_{i_0}^-(x) \leqslant \widetilde{u}_{i_0}^-(x) \leqslant u_{i_0}(x) \leqslant \widetilde{u}_{i_0}^+(x) \le u_{i_0}^+(x) \leqslant b_i, \\ 0 \leqslant u_{i_0}^-(x) \leqslant \widetilde{u}_{i_0}^-(x) \leqslant \phi_i(x) \leqslant \widetilde{u}_{i_0}^+(x) \leqslant u_{i_0}^+(x) \leqslant b_i. \end{aligned}$$

Let $(u_1^\pm, u_2^\pm, u_3^\pm)$ be the solution to the initial value problem with the initial condition $(u_{1_0}^\pm, u_{2_0}^\pm, u_{3_0}^\pm)$. Then, by the comparison theorem, we deduce that for $i = 1, 2, 3$,

$$0 \leqslant u_i^-(x,t) \leqslant u_i(x,t) \leqslant u_i^+(x,t) \leqslant b_i, \quad 0 \leqslant u_i^-(x,t) \leqslant \phi_i(x+ct) \leqslant u_i^+(x,t) \leqslant b_i.$$

Subsequently, for $i = 1, 2, 3$, we will estimate the difference between $u_i^\pm$ and ϕ_i. The estimation process of the difference between u_i^+ and ϕ_i is similar to that of u_i^- and ϕ_i. Therefore, we only present the estimation of the difference between u_i^+ and ϕ_i. For this purpose, let

$$U_i(\xi,t) = u_i^+(x,t) - \phi_i(\xi), i = 1, 2, 3,$$

and then U_i satisfies the following system:

$$\begin{cases} (U_1)_t + c(U_1)_\xi - d_1(U_1)_{\xi\xi} = & r_1[1 - a_{13} - 2\phi_1 + a_{12}(U_2 + \phi_2) + a_{13}(U_3 + \phi_3)]U_1 \\ & -r_1 U_1^2 + r_1\phi_1(a_{12}U_2 + a_{13}U_3), \\ (U_2)_t + c(U_2)_\xi - d_2(U_2)_{\xi\xi} = & r_2[1 - a_{23} + a_{21}(U_1 + \phi_1) - 2\phi_2 + a_{23}(U_3 + \phi_3)]U_2 \\ & -r_2 U_2^2 + r_2\phi_2(a_{21}U_1 + a_{23}U_3), \\ (U_3)_t + c(U_3)_\xi - d_3(U_3)_{\xi\xi} = & r_3[2\phi_3 - 1 + U_3 - a_{31}(U_1 + \phi_1) - a_{32}(U_2 + \phi_2)]U_3 \\ & +r_3(1 - \phi_3)(a_{31}U_1 + a_{32}U_2). \end{cases} \tag{3.1}$$

To give the estimate of the difference between u_i^+ and ϕ_i, we first present the following estimate.

Lemma 3.1. *Suppose that the assumptions in Theorem 2.1 hold. Let*

$$\begin{aligned} A_1(\xi,t) =& - d_1\frac{\omega''}{\omega} - c\frac{\omega'}{\omega} - r_1(1 - a_{13}) + 2r_1\phi_1 - a_{12}r_1(U_2 + \phi_2) - a_{13}r_1(U_3 + \phi_3) \\ & - a_{21}r_2\phi_2 - a_{31}r_3(1 - \phi_3), \\ A_2(\xi,t) =& - d_2\frac{\omega''}{\omega} - c\frac{\omega'}{\omega} - r_2(1 - a_{23}) + 2r_2\phi_2 - a_{21}r_2(U_1 + \phi_1) - a_{23}r_2(U_3 + \phi_3) \\ & - a_{12}r_1\phi_1 - a_{32}r_3(1 - \phi_3), \\ A_3(\xi,t) =& - d_3\frac{\omega''}{\omega} - c\frac{\omega'}{\omega} + r_3 + a_{31}r_3(U_1 + \phi_1) + a_{32}r_3(U_2 + \phi_2) - 2r_3\phi_3 - r_3U_3 \\ & - a_{13}r_1\phi_1 - a_{23}r_2\phi_2, \\ A_4(\xi,t) =& - 2r_1(1 - a_{13}) + 4r_1\phi_1 - 2a_{12}r_1(U_2 + \phi_2) - 2a_{13}r_1(U_3 + \phi_3) - a_{21}r_2\phi_2 \end{aligned}$$

$$\begin{aligned}
&\quad - a_{31}r_3(1-\phi_3) + 2r_1U_1 - r_1(a_{12}+a_{13})\phi_1,\\
A_5(\xi,t) =& - 2r_2(1-a_{23}) + 4r_2\phi_2 - 2a_{21}r_2(U_1+\phi_1) - 2a_{23}r_2(U_3+\phi_3) - a_{12}r_1\phi_1\\
&\quad - a_{32}r_3(1-\phi_3) + 2r_2U_2 - r_2(a_{21}+a_{23})\phi_2,\\
A_6(\xi,t) =& r_3 + 2a_{31}r_3(U_1+\phi_1) + 2a_{32}r_3(U_2+\phi_2) - 4r_3\phi_3 - 2r_3U_3 - a_{13}r_1\phi_1 - a_{23}r_2\phi_2\\
&\quad + r_3 - r_3(a_{31}+a_{32})(1-\phi_3).
\end{aligned}$$

There exists a positive constant Kand a sufficiently large ξ_0 such that

$$A_i(\xi,t) > K,\ \xi \in (-\infty,+\infty),\ i = 1,2,3,\ A_i(\xi,t) > K,\ \xi \in (\xi_0,+\infty),\ i = 4,5,6.$$

Proof. Because $U_i \in W^{1,1}_{\omega}(\mathbb{R})$, we have

$$\lim_{\xi\to\pm\infty} U_i\omega = 0,\ i = 1,2,3.$$

From (2.6), we know that

$$\lim_{\xi\to+\infty} U_i = \lim_{\xi\to+\infty} U_i\omega\omega^{-1} = 0, \quad \lim_{\xi\to-\infty} U_i = \lim_{\xi\to-\infty} U_i\omega\omega^{-1} = 0.$$

Therefore, a direct computation yields

$$\lim_{\xi\to+\infty} A_1(\xi,t) = -r_1(1-a_{13}) + 2r_1b_1 - a_{12}r_1b_2 - a_{13}r_1 - a_{21}r_2b_2 > \frac{r_1 + r_1a_{12} - 2a_{21}r_2}{1-a_{12}a_{21}}.$$

If $a_{21} < \frac{r_1}{2r_2}$, then

$$\lim_{\xi\to+\infty} A_1(\xi,t) > 0.$$

Similarly, when $a_{12} < \frac{r_2}{2r_1}$,

$$\lim_{\xi\to+\infty} A_2(\xi,t) > 0.$$

Observe that

$$\lim_{\xi\to+\infty} A_3(\xi,t) = r_3(a_{31}b_1 + a_{32}b_2 - 1) - a_{13}r_1b_1 - a_{23}r_2b_2,$$

and in view of (A_1), it follows that when a_{13} and a_{23} are sufficiently small,

$$\lim_{\xi\to+\infty} A_3(\xi,t) > 0.$$

By repeating the foregoing arguments, we obtain that

$$\lim_{\xi\to+\infty} A_4(\xi,t) = r_1[(1-a_{12})b_1 + (1-a_{13})b_1] - a_{21}r_2b_2.$$

Therefore, by (A_1), when a_{21} is sufficiently small,

$$\lim_{\xi\to+\infty} A_4(\xi,t) > 0$$

is valid. Similarly, when a_{12} is sufficiently small, we deduce from (A_1) that

$$\lim_{\xi\to+\infty} A_5(\xi,t) > 0.$$

Analogous to the derivation for

$$\lim_{\xi\to+\infty} A_3(\xi,t) > 0,$$

by (A_1), when a_{13} and a_{23} are sufficiently small,

$$\lim_{\xi\to+\infty} A_6(\xi,t) > 0.$$

Next, we prove that

$$\lim_{\xi\to-\infty} A_i(\xi,t) > 0, \quad i = 1,2,3.$$

Likewise, a direct computation shows that,

$$\lim_{\xi\to-\infty} A_1(\xi,t) = -d_1\lambda^2 + c\lambda - r_1(1-a_{13}) - a_{31}r_3.$$

When one of $(i)-(vi)$ holds, because $c > c_*$ and $\lambda \in (\lambda_1^-, \lambda_1^+) \cap (\lambda_2^-, \lambda_2^+)$, then $\Delta_1(\lambda, c) < 0$. Therefore, for a sufficiently small r_3,

$$\lim_{\xi\to-\infty} A_1(\xi,t) > 0.$$

Similarly,

$$\lim_{\xi\to-\infty} A_2(\xi,t) > 0.$$

Direct computations give

$$\lim_{\xi\to-\infty} A_3(\xi,t) = -d_3\lambda^2 + c\lambda + r_3.$$

If one of $(i)-(iii)$ holds, then $\Delta_3(\lambda, c) < 0$. Therefore,

$$\lim_{\xi\to-\infty} A_3(\xi,t) > 0.$$

On the other hand, if one of $(iv)-(vi)$ holds, because $c > c_1$, we find that

$$\lim_{\xi\to-\infty} A_3(\xi,t) > 0.$$

In summary, there exists a sufficiently large ξ_0 and a positive constant K_1 such that

$$\begin{aligned} &A_i(\xi,t) > K_1, \quad i = 1,2,3,\ \xi \in (-\infty,-\xi_0) \cup (\xi_0,+\infty),\\ &A_i(\xi,t) > K_1, \quad i = 4,5,6,\ \xi \in (\xi_0,+\infty). \end{aligned}$$

When $\xi \in [-\xi_0, \xi_0]$, by the strict monotonicity and positivity of the wavefront solution, we have

$$\begin{aligned} A_1(\xi,t) >& - d_1\lambda^2 + c\lambda - r_1(1-a_{13}) + 2r_1\phi_1(-\xi_0)\\ &- a_{12}r_1 - a_{12}r_1\phi_2(\xi_0) - a_{13}r_1 - a_{13}r_1\phi_3(\xi_0) - a_{21}r_2\phi_2(\xi_0) - a_{31}r_3\\ =& - d_1\lambda^2 + c\lambda - r_1(1-a_{13}) - a_{21}r_2\phi_2(\xi_0) - a_{31}r_3\\ &+ r_1(2\phi_1(-\xi_0) - a_{12} - a_{12}\phi_2(\xi_0) - a_{13} - a_{13}\phi_3(\xi_0)). \end{aligned}$$

From $(i) - (vi)$, we see that $-\Delta_1(\lambda, c) < 0$. Therefore, for sufficiently small a_{12}, a_{21}, a_{13}, r_3, and by the positivity of the wavefront solution, we conclude that $K_{21} > 0$, such that when $\xi \in [-\xi_0, \xi_0]$, $A_1(\xi, t) > K_{21}$.

Similarly, for sufficiently small a_{12}, a_{21}, a_{23}, r_3, there exists $K_{22} > 0$, such that $A_2(\xi, t) > K_{22}$ when $\xi \in [-\xi_0, \xi_0]$. Finally, for sufficiently small a_{13}, a_{23} and sufficiently large $a_{31} + a_{32}$, when $\xi \in [-\xi_0, \xi_0]$, there exists $K_{23} > 0$, such that

$$\begin{aligned}A_3(\xi, t) >& - d_3\lambda^2 + c\lambda - r_3 + r_3(a_{31}\phi_1(-\xi_0) + a_{32}\phi_2(-\xi_0) - 2\phi_3(\xi_0) - 1)\\ &- a_{13}r_1\phi_1(\xi_0) - a_{23}r_2\phi_2(\xi_0)\\ >&K_{23}.\end{aligned}$$

Let $K_2 = \min\{K_{21}, K_{22}, K_{23}\}$ and $K = \min\{K_1, K_2\}$. Thus, the lemma is proved. □

For the L^1-estimates and L^2-estimates of the solutions to (3.1), similar to the proof procedures in [1, 6–9], we can obtain the following lemmas.

Lemma 3.2. *Suppose that the assumptions in Theorem 2.1 hold, then*

$$\sum_{i=1}^{3}(\|U_i(t)\|_{L^1_\omega(\mathbb{R})} + \int_0^t e^{-\mu(t-s)}\|U_i(s)\|_{L^1_\omega(\mathbb{R})}ds) \leqslant Ce^{-\mu t}$$

where $0 < \mu < K$.

Proof. Multiplying both sides of (3.1) by $\omega(\xi)e^{\mu t}$, we obtain the following result.

$$\begin{aligned}\frac{\partial}{\partial t}(\omega e^{\mu t}U_1) + e^{\mu t}\frac{\partial}{\partial \xi}[c\omega U_1 - d_1\omega(U_1)_\xi + d_1\omega' U_1] =& - r_1\omega e^{\mu t}U_1^2 + r_1\phi_1[a_{12}U_2 + a_{13}U_3]\omega e^{\mu t}\\ &+ (B_1 + \mu)\omega e^{\mu t}U_1,\\ \frac{\partial}{\partial t}(\omega e^{\mu t}U_2) + e^{\mu t}\frac{\partial}{\partial \xi}[c\omega U_2 - d_2\omega(U_2)_\xi + d_2\omega' U_2] =& - r_2\omega e^{\mu t}U_2^2 + r_2\phi_2[a_{21}U_1 + a_{23}U_3]\omega e^{\mu t}\\ &+ (B_2 + \mu)\omega e^{\mu t}U_2,\\ \frac{\partial}{\partial t}(\omega e^{\mu t}U_3) + e^{\mu t}\frac{\partial}{\partial \xi}[c\omega U_3 d_3\omega(U_3)_\xi + d_3\omega' U_3] =&r_3(1 - \phi_3)(a_{31}U_1 + a_{32}U_2)\omega e^{\mu t}\\ &+ (B_3 + \mu)\omega e^{\mu t}U_3\end{aligned}$$

where

$$\begin{aligned}B_1(\xi, t) &= d_1\frac{\omega''}{\omega} + c\frac{\omega'}{\omega} + r_1[1 - a_{13} - 2\phi_1 + a_{12}(U_2 + \phi_2) + a_{13}(U_3 + \phi_3)],\\ B_2(\xi, t) &= d_2\frac{\omega''}{\omega} + c\frac{\omega'}{\omega} + r_2[1 - a_{23} - 2\phi_2 + a_{21}(U_1 + \phi_1) + a_{23}(U_3 + \phi_3)],\\ B_3(\xi, t) &= d_3\frac{\omega''}{\omega} + c\frac{\omega'}{\omega} + r_3[2\phi_3 - 1 + U_3 - a_{31}(U_1 + \phi_1) - a_{32}(U_2 + \phi_2)].\end{aligned}$$

Integrating the above identities with respect to ξ over $(-\infty, +\infty)$ and with respect to t over $(0, t)$, and summing the resulting expressions, the conclusion follows from Lemma 3.1 and the arguments in [7]. □

By similar proof process in Lemma 3.2, we can deduce the following estimates.

Lemma 3.3. *Suppose that the assumptions in Theorem 2.1 hold, then*

$$\sum_{i=1}^{3} \|U_i(t)\|^2_{L^2(\mathbb{R})} + \sum_{i=1}^{3} \int_0^t \|U_i(s)\|^2_{H^1(\mathbb{R})} ds \leqslant C.$$

Lemma 3.4. *Suppose that the assumptions in Theorem 2.1 hold, then*

$$\sum_{i=1}^{3} (\|(U_i)_\xi(t)\|^2_{L^2(\mathbb{R})} + \int_0^t \|(U_i)_\xi(s)\|^2_{L^2(\mathbb{R})} ds) \leqslant C.$$

By Lemmas 3.2 and 3.4, it is not difficult to verify the following lemmas hold.

Lemma 3.5. *Suppose that the assumptions in Theorem 2.1 hold, then*

$$\sum_{i=1}^{3} \|U_i(t)\|_{L^\infty(-\infty,\xi_0]} \leqslant Ce^{-\frac{\mu}{3}t}, \quad t > 0.$$

Lemma 3.6. *Suppose that the assumptions in Theorem 2.1 hold, then*

$$\sum_{i=1}^{3} \|U_i(t)\|_{L^\infty(\xi_0,+\infty)} \leqslant Ce^{-\frac{\mu}{3}t}, \quad t > 0.$$

From Lemmas 3.5 and 3.6, we have

$$\sup_{x\in\mathbb{R}} |u_i^+(\xi, t) - \phi(x + ct)| \leqslant Ce^{-\frac{\mu}{3}t}, \quad t > 0, \ i = 1, 2, 3.$$

Similarly,

$$\sup_{x\in\mathbb{R}} |u_i^-(\xi, t) - \phi(x + ct)| \leqslant Ce^{-\frac{\mu}{3}t}, \quad t > 0, \ i = 1, 2, 3.$$

Therefore, by combining the previous two formulas and the squeezing technique, we can finish the proof of Theorem 2.1.

4. Conclusions

In this paper, we investigated a three-species cooperative–competitive reaction–diffusion system and studied the traveling wave solutions connecting the equilibria $(0, 0, 1)$ and $(b_1, b_2, 0)$, which describe the invasion dynamics of the cooperative species. We established the stability properties of the traveling wave solutions and analyzed their asymptotic behavior. Compared with existing results, our work extends the known stability results by relaxing certain restrictions, providing a more general framework for such systems. Our approach can be applied to a broader class of reaction–diffusion models. It would be interesting to further explore the stability of traveling wave solutions under weaker conditions or in higher-dimensional cases, which will be left for future research.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that he has no conflict of interest.

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