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Research article

On the decomposition of perfect arbitrary algebras as a direct sum of indecomposable ideals

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Abstract: We consider perfect algebras A (that is, $A^2 = A$), in their greatest generality. That is, of arbitrary dimension, over an arbitrary base field, and no identity or condition are assumed for the product. We introduce a certain action involving the linear group of A and provide a characterization and a necessary condition for A, in terms of the above action, to be a direct sum of indecomposable ideals.

Keywords: arbitrary algebra; decomposition; indecomposable ideal; perfect algebra; linear group; structure theory
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1. Introduction and previous definitions

Definition 1.1. An arbitrary linear space A, over a base field $\mathbb{K}$, is said to be an **algebra** if it is endowed with a bilinear map

$$A \times A \to A,$$
$$(x, y) \mapsto xy,$$

called the **product** of A.

We note that A is considered in its highest level of generality. That is, A is a linear space of arbitrary dimension, the base field $\mathbb{K}$ is arbitrary, and we do not require the product of A to satisfy any identity. Hence, our results apply to every class of algebras (associative, alternative, Lie, Jordan, Leibniz, evolution, etc.).

An **ideal** of an algebra A is a linear subspace I such that $AI + IA \subset I$.

Definition 1.2. An algebra A is called **indecomposable** if it cannot be expressed as the direct sum of two nonzero ideals.

The problem of whether a given algebra A is the direct sum of indecomposable ideals is an old topic in algebra. The importance of such a decomposition

$$A = \bigoplus_{j \in J} I_j$$

lies in the fact that it provides the finest way of operating in A componentwise (each I_j). In other words, if $\{A_j\}_{j \in J}$ is an arbitrary family of algebras on a same base field, we get a new algebra A by considering the linear space $A = \bigoplus_{j \in J} A_j$ and considering the product by components. The converse problem is whether a given algebra A arises from the aforementioned construction for some family of algebras $\{A_j\}_{j \in J}$. Furthermore, being the finest way of doing, is the problem of asking if A is the direct sum of indecomposable ideals.

Many papers have appeared on this subject since the early works. Indeed, there has been much activity regarding this problem in the last years. However, almost all of them only concern particular classes of algebras (associative, Lie, Leibniz, etc.), and with finiteness conditions. See for instance [1–3].

Furthermore, it is noteworthy that this problem has been investigated within the framework of arbitrary algebras equipped with various types of bilinear forms. Specifically, we highlight the contributions of [6]. In this reference, the authors examine H^*-algebras and establish a decomposition theorem for the case of a zero annihilator.

Definition 1.3. An algebra A is called **perfect** if $A = A^2$. That is,

$$A = \text{span}\{xy : x, y \in A\}.$$

Our aim is to show the role the linear group of A plays in the possible decomposition of A as the direct sum of indecomposable ideals. From now on the **linear group** of a given algebra A will be denoted by

$$\text{GL}(A) = \{f : A \to A : f \text{ is a linear bijection on } A\}.$$

The paper is organized as follows. In Section 2, we will begin by recalling known results about decomposition of algebras induced by bases (mainly from the references [4, 5]). Then we will prove the uniqueness of certain decompositions of an algebra A as a direct sum of ideals when A is perfect. From here, we will be able to introduce an adequate action from the linear group of A onto the set of linear bases $\mathbb{B}$ of A,

$$\text{GL}(A) \times \mathbb{B} \to \mathcal{P}(\mathbb{I}), \tag{1.1}$$

where $\mathcal{P}(\mathbb{I})$ denotes the power set of $\mathbb{I}$, $\mathbb{I}$ being the set of nonzero ideals of A. In Section 3, we will show how the action (1.1) allows us to provide a characterization for A to be the direct sum of indecomposable ideals. In Section 4, we will also state a useful sufficient condition for A to be a direct sum of indecomposable ideals, by means of the action (1.1).

2. Decompositions depending on bases

In the paper [5], it studies certain decompositions of a pair $(A, \mathcal{B})$, where A is an arbitrary algebra and $\mathcal{B}$ a fixed basis of A:

Fixing an above pair $(A, \mathcal{B})$, it is said that an ideal I of A is a **$\mathcal{B}$-ideal** if I admits a basis $\mathcal{B}_I \subset \mathcal{B}$. Also, A is called **$\mathcal{B}$-indecomposable** if A cannot be expressed as the direct sum of two nonzero $\mathcal{B}$-ideals.

The main result in [5] asserts:

Theorem 2.1. *Let $(A, \mathcal{B})$ be an arbitrary algebra A with a fixed basis $\mathcal{B}$. Then A is the direct sum*

$$A = \bigoplus_{j \in J} I_j,$$

where any I_j is a $\mathcal{B}$-ideal of A. Moreover, each I_j is $\mathcal{B}$-indecomposable.

Let us prove that in case A is perfect and the above decomposition is unique.

Proposition 2.1. *Let $(A, \mathcal{B})$ be an arbitrary algebra A with a fixed basis $\mathcal{B}$ such that A is perfect. Then the decomposition*

$$A = \bigoplus_{j \in J} I_j,$$

provided by Theorem 2.1 is unique (up to permutation of the factors).

Proof. Suppose

$$A = \bigoplus_{k \in K} I_k$$

is another decomposition of A as the direct sum of $\mathcal{B}$-indecomposable $\mathcal{B}$-ideals of A. Since A is perfect, we have

$$A = AA = (\bigoplus_{j \in J} I_j)(\bigoplus_{k \in K} I_k) \subset \bigoplus_{j \in J, k \in K} (I_j \cap I_k),$$

and so

$$A = \bigoplus_{j \in J, k \in K} (I_j \cap I_k).$$

From here, for any $j_0 \in J$ we have that

$$I_{j_0} = \bigoplus_{k \in K} (I_{j_0} \cap I_k).$$

Now observe that any $I_{j_0} \cap I_k$ above is a $\mathcal{B}$-ideal of A. Since I_{j_0} is $\mathcal{B}$-indecomposable, we can assert that $I_{j_0} \subset I_{k_0}$ for some $k_0 \in K$.

In a similar way, we have that $I_{k_0} \subset I_{j_0}$, and so $I_{j_0} = I_{k_0}$. That is, the decomposition in Theorem 2.1 is unique (up to a permutation of the factors). $\square$

Remark 2.1. Of course, the above decomposition of A depends on the fixed basis $\mathcal{B}$ as [5, Example 4.1] shows. This implies, in particular, that an ideal I can be $\mathcal{B}$-indecomposable but not $\mathcal{B}'$-indecomposable for a different linear basis $\mathcal{B}' \neq \mathcal{B}$. Hence, I cannot be indecomposable (in an absolute way).

Now, for an arbitrary perfect algebra A, let us denote by

$$\mathbb{B} = \{\mathcal{B} : \mathcal{B} \text{ is a linear basis of } A\},$$

by

$$\mathbb{I} = \{I \subset A : I \text{ is a nonzero ideal of } A\}$$

and by $\mathcal{P}(\mathbb{I})$ the power set of $\mathbb{I}$.

For any $\mathcal{B} \in \mathbb{B}$, Theorem 2.1 and Proposition 2.1 give us a unique decomposition of A as a direct sum of nonzero ideals

$$A = \bigoplus_{j \in J_{\mathcal{B}}} I_j,$$

where each I_j is $\mathcal{B}$-indecomposable.

In this case, we denote

$$\mathbb{I}_{\mathcal{B}} = \{I_j : j \in J_{\mathcal{B}}\} \in \mathcal{P}(\mathbb{I}).$$

For any $f \in \mathrm{GL}(A)$ and $\mathcal{B} = \{e_i\}_{i \in I}$, we denote by

$$f(\mathcal{B}) = \{f(e_i)\}_{i \in I} \in \mathbb{B},$$

and we define the key map

$$\begin{aligned} \mathrm{GL}(A) \times \mathbb{B} &\to \mathcal{P}(\mathbb{I}), \\ (f, \mathcal{B}) &\mapsto f \cdot \mathcal{B}, \end{aligned} \tag{2.1}$$

as

$$f \cdot \mathcal{B} = \mathbb{I}_{f(\mathcal{B})}$$

for any $f \in \mathrm{GL}(A)$ and $\mathcal{B} \in \mathbb{B}$.

3. Characterization theorem

We will provide in this section a characterization for A to be the direct sum of indecomposable ideals, through the above action (2.1). We will denote by

$$Id : A \to A, \quad Id(x) = x$$

for any $x \in A$, the **identity linear bijection** on A.

The present study is motivated by Proposition 2.1. Given that this proposition yields distinct decompositions of an algebra A depending upon the choice of a linear basis, it is necessary to establish a framework for relating all possible bases of A. To this end, we utilize the linear group $\mathrm{GL}(A)$, which serves as a fundamental component in our characterization theorem.

Theorem 3.1. *Let A be an arbitrary perfect algebra. Then A is the direct sum of indecomposable ideals if and only if there exists $\mathcal{B} \in \mathbb{B}$ such that for any $f \in \mathrm{GL}(A)$, there is an onto map $\sigma_f : Id \cdot \mathcal{B} \to f \cdot \mathcal{B}$ satisfying that any $J \in f \cdot \mathcal{B}$ is of the form*

$$J = \bigoplus_{I_k \in \sigma_f^{-1}(J)} I_k.$$

Proof. Suppose

$$A = \bigoplus_{i \in I} I_i \tag{3.1}$$

is the direct sum of indecomposable ideals.

For any $i \in I$, let us fix a linear basis $\mathcal{B}_i$ of the ideal I_i. Hence,

$$\mathcal{B} = \bigcup_{i\in I} \mathcal{B}_i$$

is a linear basis of A. Let us verify that $\mathcal{B}$ is the linear basis we are looking for:

First, observe that the decomposition (3.1) is in particular a decomposition of A through $\mathcal{B}$-indecomposable $\mathcal{B}$-ideals, and so

$$Id \cdot \mathcal{B} = \{I_i : i \in I\}. \tag{3.2}$$

Second, consider any

$$f \in \mathrm{GL}(A)$$

and write $f \cdot \mathcal{B} = \{J_j\}_{j\in J}$.

Since $A^2 = A$, we have

$$A = A^2 = (\bigoplus_{i\in I} I_j)(\bigoplus_{j\in J} J_j) \subset \bigoplus_{i\in I, j\in J}(I_i J_j) \subset \bigoplus_{i\in I, j\in J}(I_i \cap J_j),$$

and so

$$A = \bigoplus_{i\in I, j\in J}(I_i \cap J_j). \tag{3.3}$$

From here, for any $i_0 \in I$ we have

$$I_{i_0} = \bigoplus_{j\in J}(I_{i_0} \cap J_j).$$

Since the decomposition (3.1) is by means of indecomposable ideals, we get that there exists a unique $j_0 \in J$ such that

$$I_{i_0} \subset J_{j_0} \tag{3.4}$$

and so $I_{i_0} \cap J_j = \{0\}$ for any $j \in J$ such that $j \neq j_0$.

In a similar way, Eq (3.3) allows us to get that for any $j_0 \in J$, we have

$$J_{j_0} = \bigoplus_{i\in I}(I_i \cap J_{j_0}).$$

From the above, either $I_i \cap J_{j_0} = I_i$ or $I_i \cap J_{j_0} = \{0\}$, and so

$$J_{j_0} = \bigoplus_{i\in I_{j_0}} I_i \tag{3.5}$$

for some nonempty $I_{j_0} \subset I$.

If we now define

$$\sigma_f : Id \cdot \mathcal{B} \to f \cdot \mathcal{B}$$

as $\sigma_f(I_{i_0}) = J_{j_0}$, where J_{j_0} is the only element in $f \cdot \mathcal{B}$ such that $I_{i_0} \subset J_{j_0}$ (see Eq (3.4)), we can assert by Eq (3.5) that for any $j_0 \in J$, we have that σ_f is onto and

$$J_{j_0} = \bigoplus_{I_i\in\sigma_f^{-1}(J_{j_0})} I_i$$

as desired.

Conversely, suppose there is $\mathcal{B} \in \mathbb{B}$ such that for any $f \in \mathrm{GL}(A)$, there is an onto map $\sigma_f : Id \cdot \mathcal{B} \to f \cdot \mathcal{B}$ satisfying that any $J \in f \cdot \mathcal{B}$ is of the form

$$J = \bigoplus_{I_k \in \sigma_f^{-1}(J)} I_k.$$

Observing the fact that σ_f is onto ensures $\sigma_f^{-1}(J) \neq \emptyset$ for any $J \in f \cdot \mathcal{B}$.

Denote $Id \cdot \mathcal{B} = \{I_i\}_{i \in I}$ and verify that

$$A = \bigoplus_{i \in I} I_i$$

is a decomposition of A as a direct sum of indecomposable ideals:

In case there exists some $i_0 \in I$ such that

$$I_{i_0} = P \oplus Q$$

with P and Q being nonzero ideals of A, if we fix $\mathcal{B}_P$ a linear basis of P, $\mathcal{B}_Q$ a linear basis of Q and $\mathcal{B}_i$ a linear basis of I_i, for any $i \in I$ with $i \neq i_0$, we have that

$$\mathcal{B}' = \mathcal{B}_P \cup \mathcal{B}_Q \cup \Big(\bigcup_{i \in I, i \neq i_0} \mathcal{B}_i \Big)$$

is a linear basis of A. Hence, there exists $f \in \mathrm{GL}(A)$ such that $f(\mathcal{B}) = \mathcal{B}'$.

Since P, Q, and any I_i, $i \in I, i \neq i_0$ are $\mathcal{B}'$-ideals of A, we get that P and Q are direct sums of some elements in $f \cdot \mathcal{B}$. That is, the map

$$\sigma_f : Id \cdot \mathcal{B} \to f \cdot \mathcal{B}$$

satisfies in particular that

$$0 \neq P = \bigoplus_{I_p \in \sigma_f^{-1}(P_1)} I_p$$

and

$$0 \neq Q = \bigoplus_{I_q \in \sigma_f^{-1}(Q_1)} I_q$$

with $P_1, Q_1 \subset f \cdot \mathcal{B}$, which contradicts the facts $I_{i_0} = P \oplus Q$ and $A = \bigoplus_{i \in I} I_i$. We conclude I_{i_0} is an indecomposable ideal of A. □

Example 3.1. *Consider the complex algebra with basis $B = \{u_n\}_{n \in \mathbb{N}}$ defined by the nonzero products among the elements of B, (the product is extended by bilinearity to the whole algebra), given by*

$$u_n u_n = \begin{cases} \frac{2n+1}{2} u_n - \frac{1}{2} u_{n+1}, & \text{if } n \text{ is odd,} \\ \frac{2n-1}{2} u_{n-1} - \frac{1}{2} u_n, & \text{if } n \text{ is even,} \end{cases}$$

$$u_n u_{n+1} = u_{n+1} u_n = \frac{2n+1}{2} u_{n+1} - \frac{1}{2} u_n, \quad \text{if } n \text{ is odd.}$$

We can verify from the above identities that A is a perfect algebra. Indeed, we have

$$u_n = \frac{2n+1}{2n(n+1)} u_n u_n + \frac{1}{2n(n+1)} u_n u_{n+1}, \quad \text{if } n \text{ is odd}$$

and

$$u_n = \frac{1}{2n(n-1)} u_n u_n + \frac{2n-1}{2n(n-1)} u_{n-1} u_n, \quad \textit{if } n \textit{ is even.}$$

Let us check that A is the direct sum of indecomposable ideals by using Theorem 3.1:
First, consider the new basis $\mathcal{B}$ *of A given by*

$$\mathcal{B} = \{e_n\}_{n \in \mathbb{N}},$$

where

$$e_n = \begin{cases} u_n + u_{n+1}, & \textit{if } n \textit{ is odd,} \\ u_{n-1} - u_n, & \textit{if } n \textit{ is even.} \end{cases}$$

Following [5] (see also Section 2), we have that the decomposition of A as the direct sum of $\mathcal{B}$*-indecomposable* $\mathcal{B}$*-ideals is*

$$A = \bigoplus_{n \in \mathbb{N}} I_n,$$

with

$$I_n = \mathbb{C} e_n \tag{3.6}$$

for any $n \in \mathbb{N}$*. Indeed, we can verify that the nonzero products among the elements of* $\mathcal{B}$ *are* $e_n e_n = 2n e_n$*,* $n \in \mathbb{N}$*.*

From Eq (3.6), we get

$$Id \cdot \mathcal{B} = \{\mathbb{C} e_n : n \in \mathbb{N}\}. \tag{3.7}$$

Given now any $f \in \mathrm{GL}(A)$*, to obtain* $f \cdot \mathcal{B}$*, we have to consider the decomposition of A as the direct sum of* $f(\mathcal{B})$*-indecomposable* $f(\mathcal{B})$*-ideals, where* $f(\mathcal{B})$ *denotes the linear basis* $f(\mathcal{B}) = \{f(e_n) : e_n \in \mathcal{B}\}$*. Let us write*

$$A = \bigoplus_{k \in K} J_k$$

as such a decomposition. Then we can write

$$f \cdot \mathcal{B} = \{J_k : k \in K\}. \tag{3.8}$$

Since $A^2 = A$*, we have*

$$A = A^2 = (\bigoplus_{n \in \mathbb{N}} I_n)(\bigoplus_{k \in K} J_k) \subset \bigoplus_{n \in \mathbb{N}, k \in K} (I_n \cap J_k).$$

From here,

$$A = \bigoplus_{n \in \mathbb{N}, k \in K} (I_n \cap J_k), \tag{3.9}$$

and for any $k_0 \in K$*, we have*

$$J_{k_0} = \bigoplus_{m \in M_0} (J_{k_0} \cap I_m)$$

with $J_{k_0} \cap I_m \neq 0$*.*

Since $\dim(I_m) = 1$*, for any* $m \in M_0$ *(see Eq (3.6)), we get* $J_{i_0} \cap I_m = I_m$*, and so*

$$J_{k_0} = \bigoplus_{m \in M_0} I_m. \tag{3.10}$$

Taking into account Eqs (3.7)–(3.10), we can define the onto map

$$\sigma_f : Id \cdot \mathcal{B} \to f \cdot \mathcal{B},$$

such that $\sigma_f(I_m) = J_{k_0}$ *for any* $m \in M_0$.

Observe that any $J_{k_0} \in f \cdot \mathcal{B}$ *is of the form*

$$J_{k_0} = \bigoplus_{I_m \in \sigma_f^{-1}(J_{k_0})} I_m$$

by Eq (3.10).

Theorem 3.1 allows now to assert that A is the direct sum of indecomposable ideals.

4. A sufficient condition

Recall from the previous section that we are denoting by

$$\mathbb{B} = \{\mathcal{B} : \mathcal{B} \text{ is a linear basis of } A\},$$

and by

$$\mathbb{I} = \{I \subset A : I \text{ is a nonzero ideal of } A\}.$$

In this section, we will give a useful sufficient condition for A to be the direct sum of indecomposable ideals in terms of the below action (see Eq (2.1)):

$$\mathrm{GL}(A) \times \mathbb{B} \to \mathcal{P}(\mathbb{I}),$$
$$(f, \mathcal{B}) \mapsto f \cdot \mathcal{B}.$$

Observe that if we look at Eq (3.2), we see that the set

$$Id \cdot \mathbb{B} = \{Id \cdot \mathcal{B} : \mathcal{B} \in \mathbb{B}\}$$

gives us all of the different possible decompositions of A as a direct sum of $\mathcal{B}$-indecomposable $\mathcal{B}$-ideals of A.

Finally, we will denote by $\#(C)$ the **cardinal** of any set C.

Theorem 4.1. *Let A be an arbitrary perfect algebra. If* $\#(Id \cdot \mathbb{B}) < \infty$, *then*

$$A = \bigoplus_{j \in J} I_j,$$

where any I_j *is an indecomposable ideal of A. Furthermore, this decomposition is unique (up to a permutation of the factors).*

Proof. First, let us introduce the following relation on $\mathbb{B}$: Given $\mathcal{B}, \mathcal{B}' \in \mathbb{B}$, we say that $\mathcal{B} \sim \mathcal{B}'$ if and only if $Id \cdot \mathcal{B} = Id \cdot \mathcal{B}'$. Observe that the relation $\sim$ is an equivalence relation, and so we can consider the quotient set

$$\mathbb{B}/\sim = \{[\mathcal{B}] : \mathcal{B} \in \mathbb{B}\}.$$

It is easy to verify that the map

$$(\mathbb{B}/\sim) \to (Id \cdot \mathbb{B}),$$
$$[\mathcal{B}] \mapsto Id \cdot \mathcal{B}$$

is well-defined and bijective. Hence,

$$\#(\mathbb{B}/\sim) = \#(Id \cdot \mathbb{B}) < \infty. \tag{4.1}$$

Consider now the set of maps

$$\mathbb{F} = \{f : (\mathbb{B}/\sim) \to \mathbb{I} : f([\mathcal{B}]) \in Id \cdot \mathcal{B}\}. \tag{4.2}$$

Second, observe that if

$$A = \bigoplus_{k \in K} I_k \text{ and } A = \bigoplus_{p \in P} I_p$$

are two decompositions of A as the direct sum of ideals, then we get as in the proof of Theorem 3.1 that

$$A = AA = (\bigoplus_{k \in K} I_k)(\bigoplus_{p \in P} I_p) \subset \bigoplus_{k \in K, p \in P} (I_k \cap I_p).$$

That is,

$$A = \bigoplus_{k \in K, p \in P} (I_k \cap I_p). \tag{4.3}$$

Third, recall that for any $\mathcal{B} \in \mathbb{B}$, Theorem 2.1 gives us the decomposition of A as the direct sum of nonzero $\mathcal{B}$-ideals

$$A = \bigoplus_{j \in J_{\mathcal{B}}} I_j, \tag{4.4}$$

where each I_j is $\mathcal{B}$-indecomposable.

From Eqs (4.1)–(4.4), we get

$$A = \bigoplus_{f \in \mathbb{F}} (\bigcap_{[\mathcal{B}] \in (\mathbb{B}/\sim)} f([\mathcal{B}])), \tag{4.5}$$

being any $\bigcap_{[\mathcal{B}] \in (\mathbb{B}/\sim)} f([\mathcal{B}])$ an ideal of A.

Let us show that any $0 \neq \bigcap_{[\mathcal{B}] \in (\mathbb{B}/\sim)} f([\mathcal{B}])$ is indecomposable:

Suppose

$$0 \neq \bigcap_{[\mathcal{B}] \in (\mathbb{B}/\sim)} f([\mathcal{B}]) = P \oplus Q \tag{4.6}$$

with P and Q being ideals of $\bigcap_{[\mathcal{B}] \in (\mathbb{B}/\sim)} f([\mathcal{B}])$. Let us fix $\mathcal{B}_P$ and $\mathcal{B}_Q$ two linear bases of P and Q, respectively. For any $f' \in \mathbb{F}$ with $f' \neq f$, fix also a linear basis $\mathcal{B}_{f'}$ of $\bigcap_{[\mathcal{B}] \in (\mathbb{B}/\sim)} f'([\mathcal{B}])$. Then

$$\mathcal{B}' = \mathcal{B}_P \cup \mathcal{B}_Q \cup \Big(\bigcup_{f' \in \mathbb{F} \setminus \{f\}} \mathcal{B}_{f'} \Big) \tag{4.7}$$

is a linear basis of A. By considering the pair $(A, \mathcal{B}')$, Theorem 2.1 gives us

$$A = \bigoplus_{r \in R} I_r, \tag{4.8}$$

where any I_r is a $\mathcal{B}'$-indecomposable $\mathcal{B}'$-ideal of A.

By Eq (4.5), we also have

$$A = P \oplus Q \oplus \Big(\bigoplus_{f' \in \mathbb{F} \setminus \{f\}} \Big(\bigcap_{[\mathcal{B}] \in (\mathbb{B}/\sim)} f'([B]) \Big)\Big). \tag{4.9}$$

Taking now into account Eqs (4.8) and (4.9), we get

$$A = AA = \Big(\bigoplus_{r \in R} I_r\Big)\Big(P \oplus Q \oplus \Big(\bigoplus_{f' \in \mathbb{F} \setminus \{f\}} \Big(\bigcap_{[\mathcal{B}] \in (\mathbb{B}/\sim)} f'([B]) \Big)\Big)\Big)$$

$$\subset \Big(\bigoplus_{r \in R} (P \cap I_r)\Big) \oplus \Big(\bigoplus_{r \in R} (Q \cap I_r)\Big) \oplus \Big(\bigoplus_{f' \in \mathbb{F} \setminus \{f\}, r \in R} \Big(\bigcap_{[\mathcal{B}] \in (\mathbb{B}/\sim)} f'([\mathcal{B}]) \cap I_r \Big)\Big).$$

Hence,

$$A = \Big(\bigoplus_{r \in R} (P \cap I_r)\Big) \oplus \Big(\bigoplus_{r \in R} (Q \cap I_r)\Big) \oplus \Big(\bigoplus_{f' \in \mathbb{F} \setminus \{f\}, r \in R} \Big(\bigcap_{[\mathcal{B}] \in (\mathbb{B}/\sim)} f'([\mathcal{B}]) \cap I_r \Big)\Big),$$

and so we have

$$I_r = (I_r \cap P) \oplus (I_r \cap Q) \oplus \Big(\bigoplus_{f' \in \mathbb{F} \setminus \{f\}} \Big(\bigcap_{[\mathcal{B}] \in (\mathbb{B}/\sim)} f'([\mathcal{B}]) \cap I_r \Big)\Big) \tag{4.10}$$

for any $r \in R$.

Since Eq (4.8) tells us that any I_r, $r \in R$, is $\mathcal{B}'$-indecomposable, Eqs (4.7) and (4.10) imply that any I_r, $r \in R$ satisfies just one of the next possibilities:

(a) Either $I_r \cap P = I_r$, $I_r \cap Q = 0$ and $\bigoplus_{f' \in \mathbb{F} \setminus \{f\}} \big(\bigcap_{[\mathcal{B}] \in \mathbb{B}/\sim} (f'([\mathcal{B}]) \cap I_r)\big) = 0$.

(b) Or $I_r \cap P = 0$, $I_r \cap Q = I_r$ and $\bigoplus_{f' \in \mathbb{F} \setminus \{f\}} \big(\bigcap_{[\mathcal{B}] \in \mathbb{B}/\sim} (f'([\mathcal{B}]) \cap I_r)\big) = 0$.

(c) Or $I_r \cap P = 0$, $I_r \cap Q = 0$ and $\bigoplus_{f' \in \mathbb{F} \setminus \{f\}} \big(\bigcap_{[\mathcal{B}] \in \mathbb{B}/\sim} (f'([\mathcal{B}]) \cap I_r)\big) = I_r$.

Hence, R is the disjoint union $R = R_1 \cup R_2 \cup R_3$, where

$$R_1 = \{r \in R : I_r \text{ satisfies the above possibility (a) }\},$$

$$R_2 = \{s \in R : I_s \text{ satisfies the above possibility (b) }\},$$

and

$$R_3 = \{t \in R : I_t \text{ satisfies the above possibility (c) }\}.$$

From here, we can write (see Eq (4.8))

$$A = \Big(\bigoplus_{r \in R_1} I_r\Big) \oplus \Big(\bigoplus_{s \in R_2} I_s\Big) \oplus \Big(\bigoplus_{t \in R_3} I_t\Big)$$

with

$$P = \bigoplus_{r \in R_1} I_r \text{ and } Q = \bigoplus_{s \in R_2} I_s.$$

Observe that we can also write

$$Id \cdot \mathcal{B}' = \{I_r : r \in R_1\} \cup \{I_s : s \in R_2\} \cup \{I_t : t \in R_3\}.$$

Now we have three cases to distinguish:

In the first one, $f([\mathcal{B}']) \in \{I_r : r \in R_1\}$. Then

$$0 \neq \bigcap_{[\mathcal{B}] \in (\mathbb{B}/\sim)} f([B]) \subset P \cap (\bigcap_{[\mathcal{B}] \in (\mathbb{B}/\sim) \setminus \{[\mathcal{B}']\}} f([B]) \subset P,$$

which contradicts Eq (4.6).

In the second one, $f([\mathcal{B}']) \in \{I_s : s \in R_2\}$, and we get a contradiction with Eq (4.6) as in the previous case.

Finally, in the third case $f([\mathcal{B}']) \in \{I_t : t \in R_3\}$. Then

$$0 \neq \bigcap_{[\mathcal{B}] \in (\mathbb{B}/\sim)} f([\mathcal{B}]) \subset I_t \cap (\bigcap_{[\mathcal{B}] \in (\mathbb{B}/\sim) \setminus \{[\mathcal{B}']\}} f([\mathcal{B}])) \subset I_t$$

with $I_t \cap P = 0$, $I_t \cap Q = 0$, which also contradicts Eq (4.6).

Consequently, Eq (4.6) does not hold, and any

$$0 \neq \bigcap_{[\mathcal{B}] \in (\mathbb{B}/\sim)} f([\mathcal{B}])$$

is an indecomposable ideal of A.

By defining

$$\mathbb{F}^* = \{f \in \mathbb{F} : \bigcap_{[\mathcal{B}] \in (\mathbb{B}/\sim)} f([\mathcal{B}]) \neq 0\},$$

and denoting $\bigcap_{[\mathcal{B}] \in (\mathbb{B}/\sim)} f([\mathcal{B}])$ as I_f for any $f \in \mathbb{F}^*$, we show that

$$A = \bigoplus_{f \in \mathbb{F}^*} I_f \tag{4.11}$$

with any I_f an indecomposable ideal of A.

Finally, let us prove the uniqueness of the decomposition of Eq (4.11):

Suppose

$$A = \bigoplus_{j \in J} I_j$$

is another decomposition of A as a direct sum of indecomposable ideals. By Eq (4.3), we have that

$$A = \bigoplus_{j \in J, f \in \mathbb{F}^*} (I_j \cap I_f)$$

being so any

$$I_{j_0} = \bigoplus_{f \in \mathbb{F}^*} (I_{j_0} \cap I_f),$$

$j_0 \in J$.

Since I_{j_0} is indecomposable, we get that there exists $f_0 \in \mathbb{F}^*$ such that $I_{j_0} = I_{j_0} \cap I_{f_0}$. That is, $I_{j_0} \subset I_{f_0}$. In a similar way, we have that $I_{f_0} \subset I_{j_0}$, and so $I_{j_0} = I_{f_0}$. That is, the decomposition of Eq (4.11) is unique (up to a permutation of the factors). □

Example 4.1. *Consider the complex algebra with basis*

$$\mathcal{B} = \{e_z : z \in \mathbb{Z}\} \tag{4.12}$$

defined by the nonzero products among the elements of B given by

$$e_1 e_1 = e_1,$$

$$e_{-1} e_{-1} = -e_{-1},$$

$$e_n e_{-n} = e_0 \quad \textit{for any} \quad n \in \mathbb{N} - \{1\},$$

$$e_0 e_z = z e_z \quad \textit{for any} \quad z \in \mathbb{Z} - \{0, 1, -1\}.$$

We can verify from the above identities that A is a perfect algebra. Indeed, we have

$$e_1 = e_1 e_1,$$

$$e_{-1} = -e_{-1} e_{-1},$$

$$e_0 = e_2 e_{-2},$$

and

$$e_z = \frac{1}{z} e_0 e_z$$

for $z \notin \{0, 1, -1\}$.

Let us prove that A is the direct sum of indecomposable ideals by using Theorem 4.1:

Observe the condition to verify: That is, $\#(Id \cdot \mathbb{B}) < \infty$*, means that the set of possible decompositions of A as direct sums of B-indecomposable B-ideals, where B ranges in the set of bases of A, is finite.*

Let $B \in \mathbb{B}$*. Then we can write*

$$B = \{u_z : z \in \mathbb{Z}\}$$

with any u_z*,* $z \in \mathbb{Z}$*, of the form*

$$u_z = \sum_{i \in I_z} \lambda_i e_i, \tag{4.13}$$

where any $\lambda_i \in \mathbb{C} \setminus \{0\}$ *and any* $e_i \in \mathcal{B}$ *(see Eq (4.12)). We can distinguish six possibilities for the basis B. From now,* z_0 *will denote an arbitrary element in* $\mathbb{Z} \setminus \{1, -1\}$*:*

(a) *There exists $z \in \mathbb{Z}$ such that all the elements of $\{e_1, e_{-1}, e_{z_0}\}$ appear in the expression (4.13) of u_z as a linear combination of the elements of $\mathcal{B}$. Then by following [5], (see also Section 2), we have that the decomposition of A as the direct sum of B-indecomposable B-ideals is*

$$A = A.$$

From here, we get

$$Id \cdot B = \{A\}. \tag{4.14}$$

(b) *There do not exist any $z \in \mathbb{Z}$ such that all the elements of $\{e_1, e_{-1}, e_{z_0}\}$ appear in the expression (4.13) of u_z, and there exist some $z_1, z_2 \in \mathbb{Z}$ such that all the elements of $\{e_1, e_{-1}\}$ appear in the expression (4.13) of u_{z_1}; either all the elements of $\{e_1, e_{z_0}\}$ or all the elements of $\{e_{-1}, e_{z_0}\}$ appear in the expression (4.13) of u_{z_2}. Applying [5], (see also Section 2), we find that the decomposition of A as the direct sum of B-indecomposable B-ideals is again*

$$A = A,$$

and so

$$Id \cdot B = \{A\}. \tag{4.15}$$

(c) *There exists some $z_1 \in \mathbb{Z}$ such that all the elements of $\{e_1, e_{-1}\}$ appear in the expression (4.13) of z_1; there not exist any $z_2 \in \mathbb{Z}$ such that either all the elements of $\{e_1, e_{z_0}\}$ or all the elements of $\{e_{-1}, e_{z_0}\}$ appear in the expression (4.13) of z_2.*
By [5], we find that the decomposition of A as the direct sum of B-indecomposable B-ideals is

$$A = I_1 \oplus I_2$$

with $I_1 = \mathbb{C}e_1 \oplus \mathbb{C}e_{-1}$ and $I_2 = \bigoplus_{z \in \mathbb{Z} \setminus \{1,-1\}} \mathbb{C}e_z$. Hence,

$$Id \cdot B = \{I_1, I_2\}. \tag{4.16}$$

(d) *There do not exist any $z_1 \in \mathbb{Z}$ such that all the elements of $\{e_1, e_{-1}\}$ appear in the expression (4.13) of u_{z_1}; there exists $z_2 \in \mathbb{Z}$ such that all the elements of $\{e_1, e_{z_0}\}$ appear in the expression (4.13) of u_{z_2}; and there do not exist any $z_3 \in \mathbb{Z}$ such that all the elements of $\{e_{-1}, e_{z_0}\}$ appear in the expression (4.13) of u_{z_3}.*
We get as above that the decomposition of A as the direct sum of B-indecomposable B-ideals is

$$A = I_3 \oplus I_4$$

with $I_3 = \mathbb{C}e_1 \oplus (\bigoplus_{z \in \mathbb{Z} \setminus \{1,-1\}} \mathbb{C}e_z)$ and $I_4 = \mathbb{C}e_{-1}$, and so

$$Id \cdot B = \{I_3, I_4\}. \tag{4.17}$$

(e) *There do not exist any $z_1 \in \mathbb{Z}$ such that all the elements of $\{e_1, e_{-1}\}$ appear in the expression (4.13) of u_{z_1}; there exists $z_2 \in \mathbb{Z}$ such that all the elements of $\{e_{-1}, e_{z_0}\}$ appear in the expression (4.13)*

of u_{z_2}; and there do not exist any $z_3 \in \mathbb{Z}$ such that all the elements of $\{e_1, e_{z_0}\}$ appear in the expression (4.13) of u_{z_3}.
We obtain that the decomposition of A as the direct sum of B-indecomposable B-ideals is

$$A = I_5 \oplus I_6$$

with $I_5 = \mathbb{C}e_1$ and $I_6 = \mathbb{C}e_{-1} \oplus (\bigoplus_{z\in\mathbb{Z}\setminus\{1,-1\}} \mathbb{C}e_z)$. Hence,

$$Id \cdot B = \{I_5, I_6\}. \tag{4.18}$$

(f) *There are no $z_1 \in \mathbb{Z}$ such that the elements of $\{e_1, e_{-1}\}$ appear in the expression (4.13) of u_{z_1}, nor any $z_2 \in \mathbb{Z}$ such that the elements of $\{e_1, e_{z_0}\}$ appear in the expression (4.13) of u_{z_2}, nor any $z_3 \in \mathbb{Z}$ such that the elements of $\{e_{-1}, e_{z_0}\}$ appear in the expression (4.13) of u_{z_3}.*
We find that the decomposition of A as the direct sum of B-indecomposable B-ideals is $A = I_2 \oplus I_4 \oplus I_5$ with I_2 as in item (c), I_4 as in item (d), and I_5 as in item (e).
From here, hence

$$Id \cdot B = \{I_2, I_4, I_5\}. \tag{4.19}$$

By Eqs (4.14)–(4.19), we can assert

$$Id \cdot \mathbb{B} = \{A, \{I_1 \oplus I_2\}, \{I_3 \oplus I_4\}, \{I_5 \oplus I_6\}, \{I_2 \oplus I_4 \oplus I_5\}\}$$

and so

$$\#(Id \cdot \mathbb{B}) = 5 < \infty.$$

Theorem 4.1 gives us now that A is a direct sum of indecomposable ideals.

Remark 4.1. Observe that the condition on Theorem 4.1 holds for any finite dimensional perfect algebra over a finite base field.

Remark 4.2. The sufficient condition in Theorem 4.1 can be obtained as a consequence of the characterization theorem in Theorem 3.1.

Although the proof is less intrinsic, we provide in this remark a way of proving Theorem 4.1 using Theorem 3.1:

Suppose A is an arbitrary perfect algebra satisfying

$$\#(Id \cdot \mathbb{B}) < \infty. \tag{4.20}$$

We wish to prove that

$$A = \bigoplus_{j\in J} I_j,$$

where any I_j is an indecomposable ideal of A. By Theorem 3.1, this is equivalent to show that A admits a linear basis $\mathcal{B}$ such that for any

$$f \in \mathrm{GL}(A)$$

there is an onto map $\sigma_f : Id \cdot \mathcal{B} \to f \cdot \mathcal{B}$ satisfying that any $J \in f \cdot \mathcal{B}$ is of the form

$$J = \bigoplus_{I_k\in\sigma_f^{-1}(J)} I_k.$$

Since condition (4.20) means that the set of possible decompositions of A as the direct sum of B-indecomposable B-ideals, where B ranges in the set of linear bases of A, is finite, we can write this above set of decompositions of A as direct sums of B-indecomposable B-ideals, $B \in \mathbb{B}$, as

$$\mathcal{D} = \{A = \bigoplus_{j_i \in J_i} I_{(i,j_i)} : i \in \{1, ..., n\}\}, \tag{4.21}$$

and so

$$Id \cdot \mathbb{B} = \{\{I_{(i,j_i)} : j_i \in J_i\} : i \in \{1, ..., n\}\}. \tag{4.22}$$

If we denote by $A^1 = A$ and $A^n = A^{n-1}A$ for any $n \geq 2$, the fact $A^2 = A$ implies

$$A^n = A.$$

From here, Eq (4.21) allows us to assert

$$\begin{aligned} A = A^n &= (\cdots(((\bigoplus_{j_1 \in J_1} I_{(1,j_1)})(\bigoplus_{j_2 \in J_2} I_{(2,j_2)}))(\bigoplus_{j_3 \in J_3} I_{(3,j_3)}))\cdots)(\bigoplus_{j_n \in J_n} I_{(n,j_n)}) \\ &\subset \bigoplus_{(j_1,j_2,...,j_n) \in J_1 \times J_2 \times \cdots \times J_n} (\bigcap_{i \in \{1,...,n\}} I_{(i,j_i)}). \end{aligned}$$

Hence,

$$A = \bigoplus_{(j_1,j_2,...,j_n) \in J_1 \times J_2 \times \cdots \times J_n} (\bigcap_{i \in \{1,...,n\}} I_{(i,j_i)}), \tag{4.23}$$

and we have as a consequence that

$$I_{(i_0,j_{i_0})} = \bigoplus_{(j_1,...,j_n) \in J_1 \times \cdots J_{i_0-1} \times J_{i_0+1} \times \cdots J_n} I_{(i_0,j_{i_0})} \cap (\bigcap_{i \in \{1,...,n\} \setminus \{i_0\}} I_{(i,j_i)}) \tag{4.24}$$

for any (i_0, j_{i_0}) with $i_0 \in \{1, ..., n\}$ and $j_{i_0} \in J_{i_0}$.

Let us denote by

$$\Omega = \{(j_1, j_2, ..., j_n) \in J_1 \times J_2 \cdots \cdots J_n : \bigcap_{i \in \{1,...,n\}} I_{(i,j_i)} \neq 0\},$$

and let us fix any linear basis $\mathcal{B}_{(j_1,j_2,...,j_n)}$ for any $(j_1, j_2, ..., j_n) \in \Omega$.

We denote by

$$\mathcal{B} = \bigcup_{(j_1,j_2,...,j_n) \in \Omega} \mathcal{B}_{(j_1,j_2,...,j_n)}. \tag{4.25}$$

Equation (4.23) allows us to assert that $\mathcal{B}$ is a linear basis of A.

Let us verify that $\mathcal{B}$ is the basis we are looking for: So, let us fix any $f \in \mathrm{GL}(A)$. On the one hand, Eqs (4.23) and (4.25) give us

$$Id \cdot \mathcal{B} = \{\bigcap_{i \in \{1,...,n\}} I_{(i,j_i)} : (j_1, ..., j_n) \in \Omega\}. \tag{4.26}$$

On the other hand, since

$$f \cdot \mathcal{B} = Id \cdot f(\mathcal{B}) \in Id \cdot \mathbb{B},$$

Eq (4.22) gives us that there exists $i_0 \in \{1, ..., n\}$ such that

$$f \cdot \mathcal{B} = \{I_{(i_0, j_{i_0})} : j_{i_0} \in J_{i_0}\}. \tag{4.27}$$

From here (see Eqs (4.24)–(4.27)), we can define the onto map

$$\sigma_f : Id \cdot \mathcal{B} \to f \cdot \mathcal{B}$$

as

$$\sigma_f(\bigcap_{i \in \{1,...,n\}} I_{(i,j_i)}) = I_{(i_0, j_{i_0})} \cap (\bigcap_{i \in \{1,...,n\} \setminus \{i_0\}} I_{(i,j_i)})$$

for any $(j_1, ..., j_n) \in \Omega$.

Now observe that Eq (4.24) gives us

$$I_{(i_0, j_{i_0})} = \bigoplus_{(j_1,...,j_n) \in J_1 \times \cdots J_{i-1} \times J_{i+1} \times \cdots J_n} I_{(i_0, j_{i_0})} \cap (\bigcap_{i \in \{1,...,n\} \setminus \{i_0\}} I_{(i,j_i)}),$$

and so we can assert that any $I_{(i_0, j_{i_0})} \in f \cdot \mathcal{B}$ is of the form

$$I_{(i_0, j_{i_0})} = \bigoplus_{(\bigcap_{i \in \{1,...,n\}} I_{(i,j_i)}) \in \sigma_f^{-1}(I_{(i_0, j_{i_0})})} (\bigcap_{i \in \{1,...,n\}} I_{(i,j_i)}).$$

From here, the necessary condition of Theorem 3.1 holds, and we can assert that A decomposes as a direct sum of indecomposablem ideals.

5. Conclusions

We have introduced a new way of treating the problem of the decomposition of an arbitrary algebra A as a direct sum of indecomposable ideals by showing the importan role the general linear group $GL(A)$ plays in this problem.

Use of Generative-AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Prof. Antonio J. Calderon is the Guest Editor of special issue "Recent advances in algebra, Topology and Categories" for AIMS Mathematics. Prof. Antonio J. Caldero was not involved in the editorial review and the decision to publish this article. The author declares no conflict of interest in this paper.

References

1. Y. C. Casado, M. S. Molina, M. V. Velasco, Evolution algebras of arbitrary dimension and their decompositions, *Linear Algebra Appl.*, **495** (2016), 122–162. https://doi.org/10.1016/j.laa.2016.01.007
2. L. Y. Chen, Y. Z. Zhang, D. J. Meng, Decomposition of restricted Lie superalgebras, *Acta Math. Sci. Ser. A (Chin. Ed.)*, **27** (2007), 577–583.
3. P. Jitjankarn, G. Yamskulna, On indecomposable vertex algebras associated with vertex algebroids, *J. Algebra*, **560** (2020), 791–817. https://doi.org/10.1016/j.jalgebra.2020.06.004
4. A. J. Calderon, I. Kaygorodov, P. Saraiva, Decompositions of linear spaces induced by n-linear maps, *Linear Multilinear Algebra*, **67** (2019) 1250–1268. https://doi.org/10.1080/03081087.2018.1450829
5. A. J. Calderon, B. Gaye, Fine decompositions of algebraic systems induced by bases, *Linear Multilinear Algebra*, **70** (2022), 2804–2817. https://doi.org/10.1080/03081087.2020.1812498
6. J. A. Cuenca Mira, A. R. Rodríguez-Palacios, Structure theory for noncommutative Jordan H^*-algebras, *J. Algebra*, **106** (1987), 1–14. https://doi.org/10.1016/0021-8693(87)90018-4